

EUROSTRINGS 2024

STRONG COUPLING EXPANSION OF DETERMINANT OBSERVABLES IN SUPERSYMMETRIC GAUGE THEORIES

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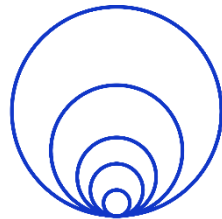
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THEORY
CHALLENGES



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EuroStrings
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AdS/CFT correspondence

AdS/CFT correspondence:

- CFT on the flat boundary
- Strings in the bulk

$$Z_{str} \left[\Phi \Big|_{\partial AdS} = J \right] = Z_{CFT} [J]$$

String observables \leftrightarrow CFT_d observables

Our case:

- IIB superstrings on $AdS_5 \times S_5$
- $\mathcal{N} = 4$ SYM on the boundary
- Gauge symmetry: $SU(N_c)$

Parameters:

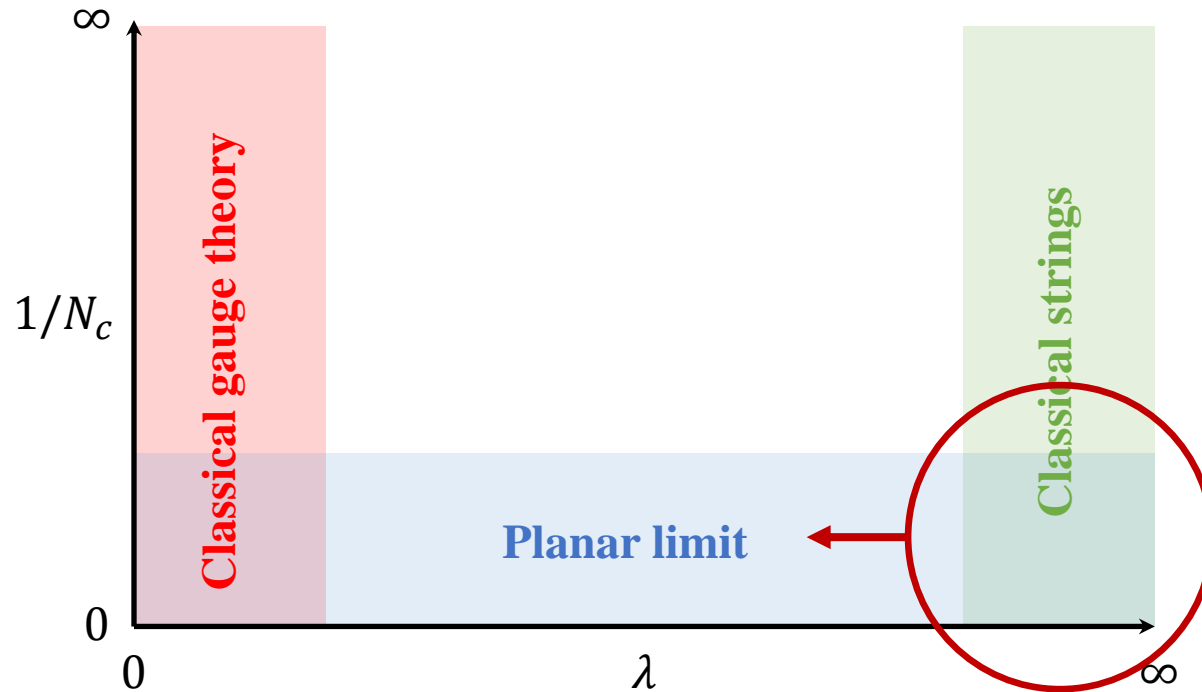
- $\lambda = 4\pi^2 T^2$
 - $\frac{1}{N_c} = \frac{g_{str}}{4\pi^2 T^2}$
- Planar limit ($N_c \rightarrow \infty$)
 \updownarrow
Weakly coupled strings ($g_{st} \rightarrow 0$)

Planar limit:

- $N_c \rightarrow \infty$
- t'Hooft coupling: $\lambda = g_{YM}^2 N_c$ is constant

Strong coupling expansion: $\lambda \rightarrow \infty$ or $g_{YM} \rightarrow \infty$

AdS/CFT correspondence

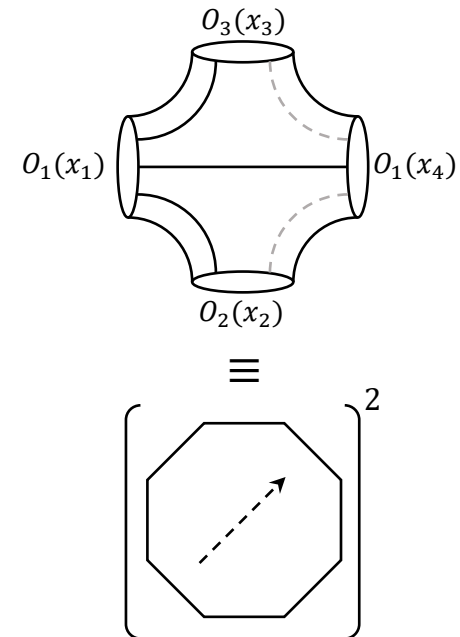
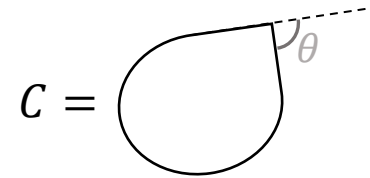


Determinant observables

Examples:

1. Planar $\mathcal{N} = 4$ SYM:

- **Cusp anomalous dimension:** [Polyakov'80]
 - Renormalization of a Wilson loop with a cusp
 - $\langle \text{Tr } \mathcal{P} \exp\{i \int_C dx \cdot A(x)\} \rangle \sim \Lambda^{\Gamma_{\text{cusp}}(\theta)}$
- **Octagon form factor:** [Coronado'18]
 - Four-point correlation function:
$$G_4 = \langle O_1(x_1) O_2(x_2) O_3(x_3) O_1(x_4) \rangle$$
 - O_1, O_2, O_3 : half-BPS operators
 - Dual to scattering amplitude of four closed strings



What is common?

Determinant observables

Determinant observables:

- Observable $\mathcal{F}(g)$:

$$e^{\mathcal{F}(g)} = \det_{1 \leq n, m < \infty} (\delta_{nm} - K_{nm}(g))$$

- Where $g = \frac{\sqrt{\lambda}}{4\pi}$

- The semi-infinite matrix is: $K_{nm}(g) = \int_0^\infty dx \psi_n(x) \chi\left(\frac{\sqrt{x}}{2g}\right) \psi_m(x)$

- The functions $\psi_n(x)$ are: $\psi_n(x) = (-1)^n \sqrt{2n + \ell + 1} J_{2n+\ell+1}(\sqrt{x}) / \sqrt{x}$

- Observables are specified by the symbol $\chi(x)$ and the number ℓ !

Observable	$\chi(x)$	ℓ
$\mathcal{N} = 4$ cusp	$1 - \coth x/2$	0
$\mathcal{N} = 4$ octagon	$1 / \cosh^2 x/2$	0
$\mathcal{N} = 2$ free energy	$-1 / \sinh^2 x/2$	1

Determinant observables

Symbol in general:

- Observables are specified by the symbol $\chi(x)$ and the number ℓ !

- Define:

$$1 - \chi(x) = bx^{2\beta} \phi(x)\phi(-x)$$

- Where:

$$\phi(x) = \prod_{n=1}^{\infty} \frac{1 - \frac{ix}{2\pi x_n}}{1 - \frac{ix}{2\pi y_n}}$$

- Observables are specified by:

- Numbers β, ℓ
- Locations of zeros x_n
- Locations of poles y_n

$$\phi_{cusp}(x) = \sqrt{\pi} \frac{\Gamma\left(1 - \frac{ix}{2\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{ix}{2\pi}\right)}$$

Observable	ℓ	β	ϕ	x_n	y_n
$\mathcal{N} = 4$ cusp	0	$-1/2$	ϕ_{cusp}	$n - 1/2$	n
$\mathcal{N} = 4$ octagon	0	1	ϕ_{cusp}^{-2}	n	$n - 1/2$
$\mathcal{N} = 2$ free energy	1	-1	ϕ_{cusp}^2	n	$n - 1/2$

Strong coupling expansion

Perturbative coefficients:

- $\mathcal{F}(g) = -gA_0 + \frac{1}{2}A_1^2 \log g + B + \sum_{k=1}^{\infty} \frac{A_{k+1}}{2k(k+1)} g^{-k} + \Delta f(g)$, as $g \rightarrow \infty$
- Perturbative coefficients: [Belitsky,Korchemsky'20]

Perturbative coefficient	Value
A_0	$2I_0$
A_1^2	$2\ell\beta + \beta^2$
A_2	$-(4\ell_\beta^2 - 1)/4 I_1$
A_3	$-3(4\ell_\beta^2 - 1)/16 I_1^2$
...	

$$\ell_\beta = \ell + \beta$$

- Moments: $I_n = \int_0^\infty \frac{dx}{\pi} \frac{(x^{-1}\partial_x)^n}{(2n-1)!!} x \partial_x \log(1 - \chi(x))$
- A_n grow factorially at large $n!$ → Non-perturbative terms ($\Delta f(g)$)!

$$\Delta f(g) = \sum_{n \geq 1} (g e^{-8\pi g x_1})^n \left(A_1^{(n)} + \sum_{k \geq 1} \frac{A_{k+1}^{(n)}}{2k(k+1)} g^{-k} \right)$$

Non-perturbative corrections

Aim: non-perturbative coefficients

Truncated Bessel operator:

• Define:
$$K_{nm}(g) = \int_0^\infty dx \psi_n(x) \chi\left(\frac{\sqrt{x}}{2g}\right) \psi_m(x) \equiv \langle \psi_n | \mathbf{K}_\chi | \psi_m \rangle$$

• Integral operator:
$$\mathbf{K}_\chi f(x) = \int_0^\infty K(x, z) \chi\left(\frac{\sqrt{y}}{2g}\right) f(x)$$

Truncated Bessel kernel

• Where:
$$K(x, y) = \sum_{n \geq 1} \psi_n(x) \psi_n(y)$$

• Then the observable:
$$\mathcal{F}(g) = \log \det(1 - \mathbf{K}_\chi)$$
 Fredholm determinant

Non-perturbative corrections

Auxiliary function:

- Define: $q(x, g) = \langle x | \frac{1}{1-K_x} | \phi_0 \rangle$
- Such that: $\langle x | \phi_0 \rangle = J_\ell(\sqrt{x})$
- **$q(x, g)$ is an entire function!**

1. Integral equation:

- $q(x)$ satisfies: $\int_0^\infty dx (1 - \chi(x)) J_{2n+\ell-1}(2gx) q(x, g) = 0$
- Solution: $q(x, g) = \frac{e^{2igx}}{\Phi(-x)} h(x, g) + (-1)^\ell \frac{e^{-2igx}}{\Phi(x)} h(-x, g)$
- With: $h(x, g) = \sum_{n \geq 0} e^{-8\pi g x_1 n} \sum_{k \geq 0} h_k^{(n)}(x) g^{-k-\frac{1}{2}}$
- **Gives an ansatz for $q(x)$**

Non-perturbative corrections

2. Differential equation:

- Connects the observable $\mathcal{F}(g)$ with the function $q(x)$

$$\left((g\partial_g)^2 + 4g^2x^2 - \ell^2 + 2g^2\partial_g^2\mathcal{F}(g) \right) q(x, g) = 0$$

- The functions $h_k^{(n)}$ can be expressed in terms of the coefficients $A_k^{(n)}$!

3. Integral relation:

- Observable $\mathcal{F}(g)$ and $q(x)$ are also related by:

$$\partial_g\mathcal{F}(g) = -\frac{1}{2}\int_0^\infty dx \partial_x \chi(x) q^2(x) \partial_x \partial_g q(x)$$

- The functions $A_k^{(n)}$ can be expressed in terms of the coefficients $h_k^{(n)}$!

Non-perturbative corrections

- Solution for :
$$q(x, g) = \frac{e^{2igx}}{\Phi(-x)} h(x, g) + (-1)^\ell \frac{e^{-2igx}}{\Phi(x)} h(-x, g)$$

- With:
$$h(x, g) = \sum_{n \geq 0} e^{-8\pi g x_1 n} \sum_{k \geq 0} h_k^{(n)}(x) g^{-k - \frac{1}{2}}$$

4. Quantization condition:

- $q(x, g)$ is an entire function

- Therefore:

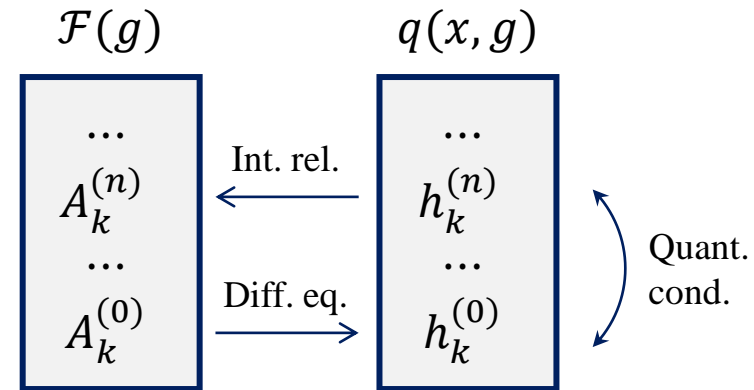
Quantization condition

$$\lim_{x \rightarrow -2\pi i x_n} (1 - \chi(x)) q(x, g) = 0$$

- **Connects the functions $h_k^{(n)}$ with different n 's!**

- $A_k^{(n)}$ can be expressed in terms of $A_k \equiv A_k^{(0)}$!

- Whole expansion of $\mathcal{F}(g)$ can be determined!



Method

Integral equations:

$$\int_0^\infty dx (1 - \chi(x)) J_{2n+\ell-1}(2gx) q(x, g) = 0$$

- Give an **ansatz for $q(x)$** in the form of a transseries.

$q(x)$ and the observable:

$$\partial_g \mathcal{F}(g) = -\frac{1}{2} \int_0^\infty dx \partial_x \chi(x) q^2(x) \partial_x \partial_g q(x)$$

- Connects the observable to the function $q(x)$.

Differential equation:

$$\left((g \partial_g)^2 + 4g^2 x^2 - \ell^2 + 2g^2 \partial_g^2 \mathcal{F}(g) \right) q(x, g) = 0$$

- Coefficients $A_k^{(n)}$ **enter the function $q(x)$** .

Quantization condition:

$$\lim_{x \rightarrow -2\pi i x_n} (1 - \chi(x)) q(x, g) = 0$$

- **Connects different levels of exponential corrections** via the analytic structure of $\chi(x)$.

Whole transseries of $\mathcal{F}(g)$ can be determined in an efficient way!

Results

Cusp anomalous dimension:

- $\mathcal{F}(g) = \frac{3}{8} \log \cosh 2\pi g - \frac{1}{8} \log \frac{\sinh 2\pi g}{2\pi g}$
- Agrees with the known result [Beccaria, Korchemsky, Tseytlin '23]

Octagon form factor:

- $$\Delta f(g) = \frac{i\pi g'}{4} e^{-8\pi g} \left(1 - \frac{7}{4(4\pi g')} - \frac{63}{32(4\pi g')^2} + \dots \right) +$$
$$+ \frac{(\pi g')^2}{32} e^{-16\pi g} \left(1 + \frac{81i-14}{4(4\pi g')} + \frac{-1431i-24}{32(4\pi g')^2} + \dots \right)$$

- Where $g' = g + \log 2 / \pi$

$\mathcal{N} = 2$ SYM:

- $$\Delta f(g) = 2i\pi g'' e^{-4\pi g} \left(1 + \frac{16 \log 2 + 1}{2(4\pi g'')} + \frac{96 \log 2 - 15}{8(4\pi g'')^2} + \dots \right) +$$
$$+ 2(\pi g'')^2 e^{-8\pi g} \left(1 + \frac{16 \log 2 + 1}{(4\pi g'')} + \frac{64 \log^2 2 + 32 \log 2 - 3}{(4\pi g'')^2} + \dots \right)$$

- Where $g'' = g - \log 2 / \pi$

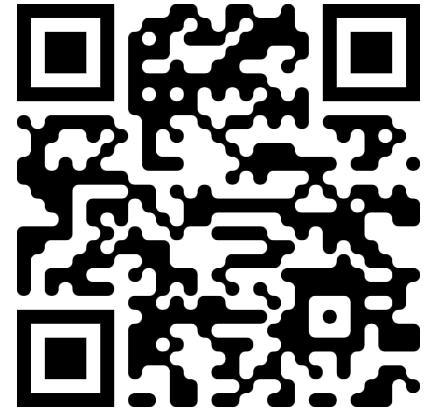
Summary

Results:

- Observables in planar $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM
- Strong coupling expansion
- Systematic method to determine non-perturbative contributions
- Numerical verification
- Resurgence properties checked

Outlook:

- Further observables in supersymmetric gauge theories
- Other limits of the octagon
- Application to other models



[Bajnok,Boldis,Korchemsky'24]

**THANK YOU
FOR YOUR ATTENTION!**

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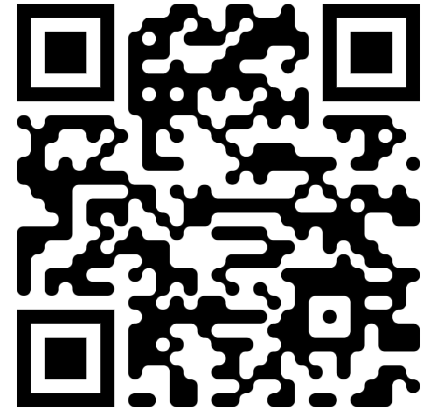
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