Integrated correlators in a $SU(N)$ $\mathcal{N} = 2$ SYM theory with fundamental flavours

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Motivation

- The study of integrated correlators of primary operators in 4d superconformal gauge theories has received an increasing attention as one the most suitable framework to explore **non-perturbative** physics.
- Various interesting tools: supersymmetric localization, conformal bootstrap, modularity, resurgence.
- In $\mathcal{N} = 4$ SYM theory they have been deeply investigated: large-N expansion, exact and modular properties, general gauge groups, large-charge limit, in the presence of a Wilson line, ...

$$\partial_m^4 \log \mathcal{Z}_{\mathcal{N}=2*}|_{m=0} = \int \prod_{i=1}^{4} dx_i \, \mu(\{x_i\}) \, \langle O_2(x_1) \dots O_2(x_4) \rangle_{\mathcal{N}=4}$$

[Chester, Pufu, 2020] ...

- Holography: scattering of **closed** strings in Type IIB string theory on $AdS_5 \times \mathbb{S}^5$.

Motivation

- Much progress has also been made in $\mathcal{N} = 2$ superconformal gauge theories.
- In particular, integrated correlators were studied in a $\mathcal{N} = 2$ SCFT with Sp(N) gauge group, one anti-symmetric hypermultiplet and four fundamental ones with SO(8) flavor symmetry. [Behan, Chester, Ferrero, 2022]
- This theory is dual to N D3 branes, 4 D7 branes, and an $O7$ plane in Type IIB string theory.
- Specifically, in the large-N limit the four-point function of flavour multiplets is dual to the scattering of **SO(8) open** string gluons on $AdS_5 \times \mathbb{S}^3$. [Alday, Chester, Hansen, Zhong, 2024] [Alday, Hansen, 2024]

The **D** theory

- We consider a $\mathcal{N} = 2$ SCFT, dubbed **D** theory, with **SU(N)** gauge group, **two** anti-symmetric hypers, **four** fundamental and **U(4)** flavour symmetry.
- In Type IIB string theory this model can be engineered with N fractional D3-branes in a $\mathbb{Z}_2$-orbifold probing an O7-orientifold background and with four D7 branes plus their orientifold images.
- Flavour group of the four fundamental hypers
- Gauge group of the D7 branes world-volume theory

→ $U(4)$

- Also in this case the **D7-sector** consists of open string states which in the large-N limit propagate on $AdS_5 \times \mathbb{S}^3$. Among these states there are the **U(4) gluons**.

The **D** theory

- The U(4) gluons are dual to operators of dimension 2 quadratic in the scalars of the fundamental hypers of the SCFT, i.e. **moment-map** operators $\mathcal{J}$ belonging to the flavor current multiplet.
- Integrated 4-point functions can be studied exploiting localization

$$\partial_{m_A}\partial_{m_B}\partial_{m_C}\partial_{m_D}\log\mathcal{Z}_{D^*}\big|_{m=0} = \int\prod_{i=1}^{4}dx_i\;\mu'(\{x_i\})\;\langle\mathcal{J}^A(x_1)\ldots\mathcal{J}^D(x_4)\rangle_D$$

$\mu'(\{x_i\})$ fixed by superconformal symmetry [Chester, 2022]

Holographically similar to Sp(N) theory, but **different** SCFTs. We expect a similar behaviour at **strong coupling** for these correlators, but this is very tough to verify ➔ much more involved **matrix model**!

The massless matrix model

In the large-N limit at fixed $\lambda \equiv g_{YM}^2 N$ we have

$$\mathcal{Z}_{\mathbf{D}} = \int da\, \mathrm{e}^{-\mathrm{tr}\, a^2 - S_{\mathrm{int}}} \cancel{|Z_{inst}|^2}$$

[Pestun, 2007]

where a are $N \times N$ Hermitian matrices and

$$S_{int} = 4 \sum_{k=1}^{\infty} \Big(-\frac{\lambda}{8\pi^2 N}\Big)^{k+1} (2^{2k} - 1) \frac{\zeta_{2k+1}}{k+1} \, \mathrm{tr}\, a^{2k+2} \Longleftarrow \text{Sp(N) theory}$$

$$\underbrace{+ 2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{k-1} (-1)^k \Big(\frac{\lambda}{8\pi^2 N}\Big)^{k+1} \binom{2k+2}{2\ell+1} \frac{\zeta_{2k+1}}{k+1} \, \mathrm{tr}\, a^{2\ell+1} \mathrm{tr}\, a^{2k-2\ell+1}}$$

$S_{int}^{\mathbf{E}} \Rightarrow \mathcal{N} = 2$ SU(N) SCFT with 1 symm+1 antisymm hypers

The massless matrix model

In this case an efficient strategy is to perform the change of basis

$$\mathrm{tr}\,a^k = \left(\frac{N}{2}\right)^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor} \sqrt{k-2\ell}\begin{pmatrix} k \\ \ell \end{pmatrix} \underbrace{\mathcal{P}_{k-2\ell}}_{\text{orthonormal for } N\to\infty} + \underbrace{\langle \mathrm{tr}\,a^k\rangle_0}_{\text{VEV in free matrix model}}$$

[Beccaria, Billò, Galvagno, Hasan, Lerda, 2020]

so that one gets an **exact** expression for S_{int} for all values of λ

$$S_{int} = -\frac{1}{2}\sum_{k,\ell=1}^{\infty} \mathcal{P}_{2k+1}\, \mathsf{X}_{2k+1,2\ell+1}\, \mathcal{P}_{2\ell+1} - \sum_{k=1}^{\infty} \mathsf{Y}_{2k}\, \mathcal{P}_{2k}$$

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\frac{e^t}{(e^t-1)^2} J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big) J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

$$\mathsf{Y}_{2k} = (-1)^{k+1}\, 2\sqrt{2k}\int_0^\infty \frac{dt}{t}\frac{e^t}{(e^t+1)^2} J_{2k}\Big(\frac{\sqrt{\lambda}\,t}{\pi}\Big) - \delta_{k,1}\frac{\sqrt{2}\log 2}{4\pi^2}\,\lambda$$

1- and 2-point functions

This result allows us to find

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \mathsf{Y}_{2n} + \frac{\sqrt{2n}}{2N}\Big(\mathsf{Y}^{\,2} - 2\lambda\,\partial_\lambda\, \mathcal{F}_{\mathbf{E}} \Big) + O\Big(\frac{1}{N^2}\Big)$$

$$\langle \mathcal{P}_{2n}\,\mathcal{P}_{2m} \rangle_{\mathbf{D}} - \langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} \langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \delta_{n,m} + \frac{\sqrt{2n}\,\sqrt{2m}\;\; \mathsf{Y}}{N} + O\Big(\frac{1}{N^2}\Big)$$

with

$$\mathsf{Y} \equiv \sum_{k=1}^{\infty} \sqrt{2k}\,\mathsf{Y}_{2k} = \int_0^\infty \frac{dt}{t}\,\frac{e^t}{(e^t+1)^2}\left[\frac{\sqrt{\lambda}\,t}{\pi}\,J_1\Big(\frac{\sqrt{\lambda}\,t}{\pi}\Big)\right] - \frac{\log 2}{2\pi^2}\,\lambda$$

$$\mathcal{F}_{\mathbf{E}} = \frac{1}{2}\mathrm{tr}\log\big(1-\mathsf{X}\big) + O\Big(\frac{1}{N^2}\Big)$$

They will be useful in a moment!

The massive matrix model

We consider a mass-deformation of the **D** theory, giving mass to the **four fundamental** hypers. The small-mass expansion of the massive matrix model in the large-N limit becomes

$$\mathcal{Z}_{\mathbf{D}^*} = \int da \; e^{-\mathrm{tr}a^2} \; e^{-S_{int} - \sum_{i=1}^{4} m_i^2 \, S_2 - \sum_{i=1}^{4} m_i^4 \, S_4 + O(m^6)}$$

where S_2 and S_4 are single-trace deformations. We can have three different mass combinations for the fourth order derivatives of $\mathcal{F}_{\mathbf{D}^*} = -\log \mathcal{Z}_{\mathbf{D}^*}$

$$-\partial^4_{m_i} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = -24 \left\langle S_4 \right\rangle_{\mathbf{D}} + 12 \left\langle {S_2}^2 \right\rangle_{\mathbf{D}} - 12 \left\langle S_2 \right\rangle^2_{\mathbf{D}}$$

$$-\partial^2_{m_i} \partial^2_{m_j} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = 4 \left\langle {S_2}^2 \right\rangle_{\mathbf{D}} - 4 \left\langle S_2 \right\rangle^2_{\mathbf{D}}$$

$$-\partial_{m_1} \partial_{m_2} \partial_{m_3} \partial_{m_4} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = 0$$

The massive matrix model

We need to compute the r.h.s. of these equations ➔ write S_2 and S_4 in terms of the $\mathcal{P}$ operators. We find **exact** expression in the coupling λ for the first three $1/N$ orders

$$S_4 = -\frac{N}{12}\,\frac{4\pi}{\sqrt{\lambda}}\,\mathsf{Z}_1^{(3)} - \frac{1}{12}\sum_{k=1}^{\infty}(-1)^k\sqrt{2k}\;\mathsf{Z}_{2k}^{(4)}\,\mathcal{P}_{2k}$$

$$-\frac{1}{24\,N}\left[\frac{\sqrt{\lambda}}{4\pi}\,\mathsf{Z}_1^{(5)} + \frac{1}{6}\Big(\frac{\sqrt{\lambda}}{4\pi}\Big)^2\,\mathsf{Z}_2^{(6)}\right] + O\Big(\frac{1}{N^3}\Big)$$

$$S_2 = \sum_{k=1}^{\infty}(-1)^k\sqrt{2k}\;\mathsf{Z}_{2k}^{(2)}\,\mathcal{P}_{2k}$$

with

$$\mathsf{Z}_n^{(p)} = \int_0^{\infty}\frac{dt}{t}\,\frac{e^t\,t^{\,p}}{(e^t-1)^2}\,J_n\Big(\frac{\sqrt{\lambda}\,t}{2\pi}\Big)$$

Results

We finally evaluate their VEVs and get

$$-\partial^4_{m_i}\mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda\to\infty}{\sim} \frac{16\pi^2}{\lambda}N + 3\log\lambda + 6\gamma - 6\log(4\pi) - 3\,\zeta_3 + 11 + \frac{3}{4N}\Big(1 - \frac{2\log 2}{\pi^2}\lambda\Big) + O\Big(\frac{1}{N^2}\Big)$$

$$-\partial^2_{m_i}\partial^2_{m_j}\mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda\to\infty}{\sim} \log\lambda + 2\gamma - 2\log(4\pi) - 2\,\zeta_3 + \frac{11}{3} + \frac{1}{4N}\Big(1 - \frac{2\log 2}{\pi^2}\lambda\Big) + O\Big(\frac{1}{N^2}\Big)$$

[Billò, Frau, Lerda, Pini, PV, 2024]

- Weak coupling $\Longrightarrow$ Completely **different** from Sp(N) theory
- Strong coupling $\Longrightarrow$ **Similar** to Sp(N) theory (the asymptotic expansion of $Z_n^{(p)}$ cancels out the **double-trace effect**)

These results furnish constraints for the dual gluon amplitudes in AdS

[Alday, Chester, Hansen, Zhong, 2024]
[Alday, Hansen, 2024]

Conclusions and outlook

We studied the derivatives of the free energy of the **D*** theory in the large-N expansion, obtaining exact expressions in λ and derived their strong coupling limit.

- It would be interesting to find a systematic way to compute **higher orders in the 1/N expansion**.
- It would be important to explore the **large-N limit at fixed Yang-Mills coupling**, where the instantons cannot be neglected. It would be very interesting to check whether they provide the completion of the perturbative results into **modular functions**.
- Studying the strong coupling expansions in terms of Bessel functions we have shown that they include only a finite number of terms. It would be interesting to apply the **Cheshire cat resurgence** methods to determine the non-perturbative corrections $O\left(e^{-\sqrt{\lambda}}\right)$.

Thanks for your attention!

Backup slides

U(4) flavour group

Let us show how the $\mathbb{Z}_2$-orbifold projection acts on the initial SO(8) gauge group of the eight D7-branes in the orientifold background. Let Λ be a Hermitian anti-symmetric 8×8 Chan-Paton matrix in the $\mathfrak{so}(8)$ algebra. Under the $\mathbb{Z}_2$-orbifold it transforms as

$$\Lambda \to \gamma \Lambda \gamma^{-1} \quad \text{with} \quad \gamma = \begin{pmatrix} 0 & -\mathrm{i}\,\mathbb{1} \\ \mathrm{i}\,\mathbb{1} & 0 \end{pmatrix}$$

[Gimon, Polchinski, 1996]

where we have written the matrix in 4×4 blocks. Thus, Λ is invariant under the orbifold only if it takes the form

$$\begin{pmatrix} A & \mathrm{i}\,S \\ -\mathrm{i}\,S & A \end{pmatrix} \quad \text{with} \quad A^t = -A \,, \quad A^* = -A \,, \quad S^t = S \,, \quad S^* = S$$

Matrices of this form represent the embedding into $\mathfrak{so}(8)$ of a $\mathfrak{u}(4)$ Hermitian matrix $A + S$.

U(4) mass combinations

In the $\mathbf{D}^*$ theory we restrict the masses to be along the four Cartan directions of U(4) labeled by $i = 1, \ldots, 4$. To find the U(4) invariant mass combinations, recall that the four Cartan generators $\boldsymbol{\lambda}^i$ in the defining representation of U(4) must be embedded into 8×8 matrices as

$$\begin{pmatrix} 0 & \mathrm{i}\,\boldsymbol{\lambda}^i \\ -\mathrm{i}\,\boldsymbol{\lambda}^i & 0 \end{pmatrix}$$

So we can consider the combination of these embedded Cartan generators

$$M = \begin{pmatrix} 0 & \begin{matrix} \mathrm{i}\,m_1 & 0 & 0 & 0 \\ 0 & \mathrm{i}\,m_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}\,m_3 & 0 \\ 0 & 0 & 0 & \mathrm{i}\,m_4 \end{matrix} \\ \begin{matrix} -\mathrm{i}\,m_1 & 0 & 0 & 0 \\ 0 & -\mathrm{i}\,m_2 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,m_3 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,m_4 \end{matrix} & 0 \end{pmatrix}$$

U(4) mass combinations

This matrix satisfies

$$\mathrm{tr} M^{2k+1} = 0 \quad \mathrm{tr} M^{2k} = 2\sum_{i=1}^{4} m_i^{2k} \quad \mathrm{Pfaff}(M) = m_1\, m_2\, m_3\, m_4$$

From this we see that at order 4 in the masses, there are three independent U(4)-invariant structures, which we can take to be

$$\sum_{i=1}^{4} m_i^4 = \frac{1}{2}\,\mathrm{tr} M^4$$

$$\sum_{i<j=1}^{4} m_i^2\, m_j^2 = -\frac{1}{4}\,\mathrm{tr} M^4 + \frac{1}{8}\left(\mathrm{tr} M^2\right)^2$$

$$m_1\, m_2\, m_3\, m_4 = \mathrm{Pfaff}(M)$$

Matrix model $\mathbf{E}$ theory

At leading order in the large-N expansion

- $$\left\langle P_{2n} \right\rangle_{\mathbf{E}} = -\frac{\sqrt{2k}\,\lambda\,\partial_\lambda \mathcal{F}_{\mathbf{E}}}{N}$$
- $$\left\langle P_{2n} P_{2m} \right\rangle_{\mathbf{E}} = \delta_{n,m}$$
- $$\left\langle P_{2n+1} P_{2m+1} \right\rangle_{\mathbf{E}} = \mathsf{D}_{2n+1,2m+1}$$

$$\mathsf{D}_{n,m} \equiv \left(\frac{1}{1-\mathsf{X}}\right)_{n,m}$$

[Beccaria, Billò, Frau, Lerda, Pini, 2021]

- $$\left\langle P_{2n+1} P_{2m+1} P_{2n+2m+2} \right\rangle_{\mathbf{E}} = \frac{\sqrt{2n+2m+2}}{N} \mathsf{d}_{2n+1} \mathsf{d}_{2m+1}$$

$\mathsf{d}_k = \sum_{k'} \sqrt{k'} \mathsf{D}_{k,k'}$

[Billò, Frau, Lerda, Pini, PV, 2022]

Evaluate VEVs in the **D** theory matrix model

For instance for the 1-point functions

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \frac{\left\langle \mathcal{P}_{2n} \exp\left(\sum_k \mathsf{Y}_{2k}\, \mathcal{P}_{2k}\right)\right\rangle_{\mathbf{E}}}{\left\langle \exp\left(\sum_k \mathsf{Y}_{2k}\, \mathcal{P}_{2k}\right)\right\rangle_{\mathbf{E}}}$$

Expanding in Y_{2k}, we get

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \langle \mathcal{P}_{2n} \rangle_{\mathbf{E}} + \sum_{k=1}^{\infty} \mathsf{Y}_{2k} \, \langle \mathcal{P}_{2n}\, \mathcal{P}_{2k} \rangle_{\mathbf{E}}^{c} + \frac{1}{2} \sum_{k,\ell=1}^{\infty} \mathsf{Y}_{2k}\, \mathsf{Y}_{2\ell} \, \langle \mathcal{P}_{2n}\, \mathcal{P}_{2k}\, \mathcal{P}_{2\ell} \rangle_{\mathbf{E}}^{c} + \ldots$$

Same strategy for 2-point functions.

Details on the strong coupling

Let us present an example. $Z_n^{(p)}$ is defined as

$$Z_n^{(p)} = \int_0^\infty \frac{dt}{t} \frac{e^t\, t^p}{(e^t - 1)^2} J_n\Big(\frac{\sqrt{\lambda}\, t}{2\pi}\Big)$$

for $n \geq 1$ and $p > 1$. In order to study its strong coupling expansion, we use the Mellin-Barnes integral representation of the **Bessel function**

$$J_n(x) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)}{\Gamma(s+n+1)} \Big(\frac{x}{2}\Big)^{2s+n}$$

and obtain

$$Z_n^{(p)} = \int_0^\infty \frac{dt}{t} \frac{e^t\, t^p}{(e^t - 1)^2} \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)}{\Gamma(s+n+1)} \Big(\frac{\sqrt{\lambda}\, t}{4\pi}\Big)^{2s+n}$$

Details on the strong coupling

Evaluating the t-integral, we get

$$\mathrm{Z}_n^{(p)} = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \, \frac{\Gamma(-s)\,\Gamma(2s+n+p)\,\zeta_{2s+n+p-1}}{\Gamma(s+n+1)} \left(\frac{\sqrt{\lambda}}{4\pi}\right)^{2s+n}$$

When $\lambda \to \infty$ this integral receives contributions from poles on the negative real axis of s. Summing the residues over such poles, one finds

$$\mathrm{Z}_n^{(p)} \underset{\lambda\to\infty}{\sim} -\frac{1}{2}\sum_{k=0}^{\infty} \frac{(2k-1)\,B_{2k}}{(2k)!} \, \frac{\Gamma\left(\frac{n+p}{2}+k-1\right)}{\Gamma\left(\frac{n-p}{2}+2-k\right)} \left(\frac{4\pi}{\sqrt{\lambda}}\right)^{p+2k-2}$$

where B_{2k} are the **Bernoulli numbers**. When n and p are both even or both odd, this asymptotic expansion terminates after a finite number of terms or even disappears as for example in $\mathrm{Z}_1^{(5)}$ or $\mathrm{Z}_2^{(6)}$.

Strong coupling expansions

The $\log 2$ terms can be removed by introducing a shifted 't Hooft coupling defined as

$$\frac{1}{\lambda'} = \frac{1}{\lambda} + \frac{\log 2}{2\pi^2 N} .$$

In terms of λ' we have

$$-\partial^4_{m_i} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda'\to\infty}{\sim} \frac{16\pi^2}{\lambda'} N + 3\log\lambda' + 3f(N) - 8\log 2 + 3\,\zeta_3$$

$$-\partial^2_{m_i}\partial^2_{m_j} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda'\to\infty}{\sim} \log\lambda' + f(N)$$

where

$$f(N) = 2\gamma - 2\log(4\pi) - 2\,\zeta_3 + \frac{11}{3} + \frac{1}{4N} + O\Big(\frac{1}{N^2}\Big)$$