Bootstrapping the AdS Veneziano amplitude

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Based on:

2403.13877 with Luis F. Alday, Shai M. Chester, De-liang Zhong

2404.16084 with Luis F. Alday

Idea

Work towards the worldsheet theory of strings for AdS_5/CFT_4 by fixing the amplitudes first!

Outline:

1. Review: The AdS Virasoro-Shapiro amplitude
2. Defining the AdS Veneziano amplitude
3. Spectrum and pole structure
4. Worldsheet representation
5. Checks

1. The AdS Virasoro-Shapiro amplitude

The AdS Virasoro-Shapiro amplitude

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

CFT stress-tensor correlator
$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$

$\longleftrightarrow$ integral transform

5d bulk of AdS:
IIB string theory on $AdS_5 \times S^5$

AdS graviton amplitude
$A_{\text{VS}}(S,T)$

$A_{\text{VS}}(S,T)$ as a worldsheet integral, in a small curvature expansion:

$$A_{\text{VS}}(S,T) = \frac{1}{(S+T)^2}\int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\sum_{k=0}^{\infty}\left(\frac{\alpha'}{R^2}\right)^k G^{(k)}_{\text{VS}}(S,T,z)$$

$G^{(0)}_{\text{VS}}(S,T,z) = 1$ (flat space)

$G^{(1,2)}_{\text{VS}}(S,T,z)$ fixed in
[Alday,TH,Silva;2022,2023],[Alday,TH;2023]

$R =$ AdS radius

$$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}} \longleftarrow \text{t'Hooft coupling}$$

2. Defining the AdS Veneziano amplitude

Open strings in AdS

Type IIB (on $AdS_5 \times S^5$) has only closed strings.

Q: What is the simplest $\mathrm{AdS}_5/\mathrm{CFT}_4$ with weakly coupled open strings?

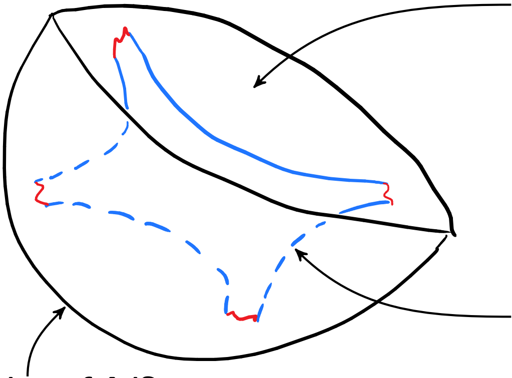
A: An orientifold of type IIB! [Sen;1996],[Banks,Douglas,Seiberg;1996]

$= N$ $D3$-branes near D_4-type F-theory singularity

$=$ type IIB with N $D3$'s, 4 $D7$'s, 1 $O7$

Also: Orbifolds of this $\sim$ other F-theory singularities

One theory, three descriptions



5d bulk of AdS:
type IIB gluons on $AdS_5 \times S^3$

$G = SO(8)$ gauge group

parameters: $g_s \ll 1$, R, α'

dictionary: $\frac{R^4}{\alpha'^2} = \lambda$

2d string worldsheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 2$ $USp(2N)$ gauge theory

$G = SO(8)$ flavour group

parameters: $N \gg 1$, λ

Other possibilities: orbifolds
$G = U(4)$ or $G = SO(4) \times SO(4)$
[Ennes,Lozano,Naculich,Schnitzer;2000]

The AdS amplitude

CFT flavour multiplet correlator

$$\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_3}(x_3)\mathcal{O}^{I_4}(x_4)\rangle \xleftrightarrow[\text{integral transform}]{} A^{I_1 I_2 I_3 I_4}(S,T)$$

AdS gluon amplitude

Colour ordered amplitude:

$$A^{I_1 I_2 I_3 I_4}(S,T) = \mathrm{Tr}(T^{I_1}T^{I_2}T^{I_3}T^{I_4})\,A(S,T) + \text{crossing}$$

Small curvature expansion:

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2}A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \dots$$

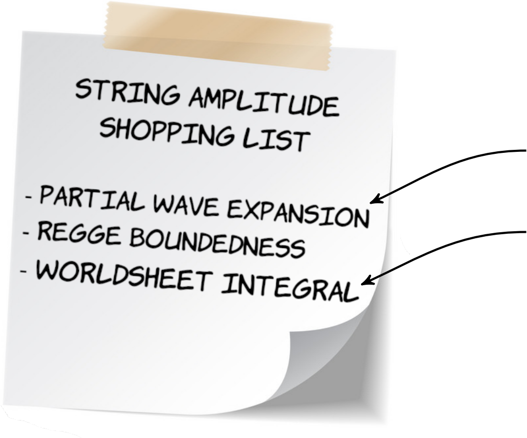
$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)}{\Gamma(1-S-T)}$$

$R =$ AdS radius

$$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$$

t'Hooft coupling

Plan of attack



We attack the problem from 2 sides:

Poles in terms of OPE data

Single-valued worldsheet ansatz

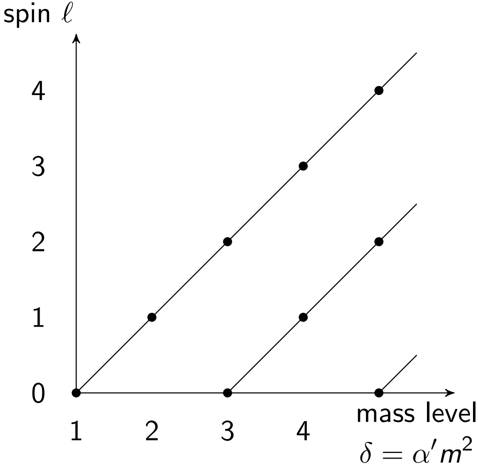
Both have unfixed data.
Equating the two expressions fixes the answer!

3. Spectrum and pole structure

Massive string operators

Flat space:
resonances = massive string modes

$$\lim_{T\to\delta} A^{(0)}(S,T) = \sum_{\ell} \frac{a_{\delta,\ell} P_\ell(\cos\theta)}{T-\delta}$$



AdS/CFT:
conformal partial wave expansion
(OPE)

$$\Delta = Rm + \ldots = R\sqrt{\frac{\delta}{\alpha'}}\left(1 + O\left(\frac{\alpha'}{R^2}\right)\right)$$

flat space mass

CFT picture: mesons
(= bound states of 2 quarks
(= hypermultipets in the
fundamental of gauge group))
[Aharony,Fayyazuddin,Maldacena;1998]

analogue in $\mathcal{N} = 4$ SYM: Konishi etc.

Pole structure from the OPE

We can expand $\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_3}(x_3)\mathcal{O}^{I_4}(x_4)\rangle$ into conformal blocks using:

Operator product expansion (OPE)

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_{\Delta,\ell}\ \text{primaries}} C_{\Delta,\ell}\ c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\Big|_{y=0}$$

OPE data

- Δ = dimension
- ℓ = spin
- $C_{\Delta,\ell}$ = OPE coefficients

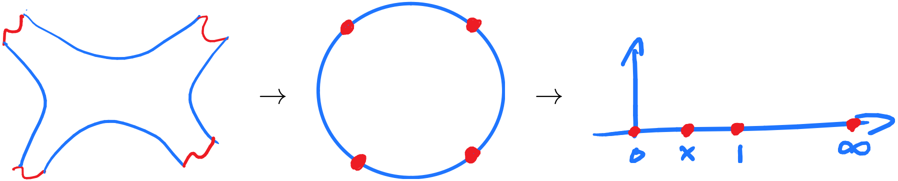
This fixes the pole structure of the AdS amplitude:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\text{OPE data})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T,\text{OPE data})}{S-\delta} + O((S-\delta)^0)$$

The numerator functions are known explicitly.

4. Worldsheet representation

Worldsheet integral



Flat space:

$$A^{(0)}(S,T) = \frac{1}{S+T}\int_0^1 dx\; x^{-S-1}(1-x)^{-T-1}$$

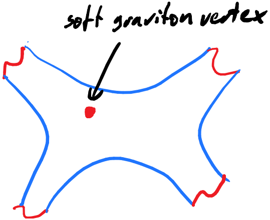
Curvature corrections:

$$A^{(k)}(S,T) = \frac{1}{S+T}\int_0^1 dx\; x^{-S-1}(1-x)^{-T-1}G^{(k)}(S,T,x)$$

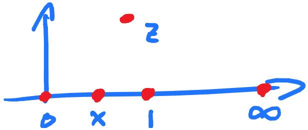
What kind of functions are $G^{(k)}(S,T,x)$?

Single-valuedness of $G^{(k)}(S, T, x)$

$$\boxed{\text{non-linear } \sigma\text{-model}} \xrightarrow[\text{expansion}]{\text{small curvature}} \boxed{\begin{array}{c}\text{flat space amplitude}\\ \text{w. extra soft gravitons}\end{array}}$$



$\rightarrow$



$$\rightarrow \int_0^1 dx\, x^{-S-1}(1-x)^{-T-1} \underbrace{\int_{H^+} d^2z \frac{|z|^{2p_1\cdot q}|1-z|^{2p_3\cdot q}|x-z|^{2p_2\cdot q}}{|z|^2|1-z|^2}}_{q \ll 1 \quad \rightarrow \quad \mathrm{SVMPLs}(x)}$$

$\rightarrow G^{(k)}(S, T, x)$ contains SVMPLs of weight $\leq 3k$

Multiple polylogarithms (MPLs)

Definition ($|z_1 \dots z_r| = r =$ weight) $z_i \in \{0, 1\}$

$$L_{z_1 \dots z_r}(z) = \int_{0 \le t_r \le \dots \le t_1 \le z} \frac{dt_1}{t_1 - z_1} \cdots \frac{dt_r}{t_r - z_r}$$

Properties:

- $\partial_z L_{z_i w}(z) = \frac{1}{z - z_i} L_w(z)$
- multi-valued
- holomorphic
- $L_w(1) =$ multiple zeta values

Examples:

- $L_{1^p}(z) = \frac{1}{p!} \log^p(1 - z)$
- $L_{0^p 1}(z) = -\mathrm{Li}_{p+1}(z)$

Single-valued multiple polylogarithms (SVMPLs)

SVMPLs [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{|w_1|+|w_2|=|w|} c_{w_1 w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

Properties:

- $\partial_z \mathcal{L}_{z_i w}(z) = \frac{1}{z - z_i} \mathcal{L}_w(z)$
- single-valued
- non-holomorphic
- $\mathcal{L}_w(1) \equiv$ single-valued multiple zeta values

Examples:

- $\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$
- $\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$

Ansatz for worldsheet integrand

Ansatz:

$$G^{(k)}(S,T,x) = \frac{1}{(S+T)^k} \sum_{n=0}^{3k} \sum_{j} P_{n,j}(S,T)\ T_{n,j}(x)$$

homogeneous degree n polynomials   weight n (SV)MPLs

weight	0	1	2	3	4	5	6
# of MPLs(x)	1	2	5	11	23	48	98
# of SVMPLs($\bar{x} = x$)	1	2	3	7	11	22	39

$G^{(1)}$: pole structure fixes MPL ansatz → result also matches SVMPL ansatz

$G^{(2)}$: pole structure fixes SVMPL ansatz up to 1 coefficient

Solution for worldsheet integrand

First correction: mv/sv ansatz has 33/22 rational parameters

Solution

$$G^{(1)}(S,T,x) = (S+T)^{-1}\Big[3 + (4T-S)\log(x) + (4S-T)\log(1-x) - S(3S+4T)\log^2(x)$$
$$-\,T(3T+4S)\log^2(1-x) + \left(5S^2+12ST+5T^2\right)\left(\zeta(2)-\mathrm{Li}_2(x)-\mathrm{Li}_2(1-x)\right)\Big]$$
$$+\,T(2S+T)\left(\log(1-x)\mathrm{Li}_2(1-x)-\mathrm{Li}_3(1-x)\right) + S^2\log^2(x)\left(\tfrac{2}{3}\log(x)-\log(1-x)\right)$$
$$+\,S(2T+S)\left(\log(x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\right) + T^2\log^2(1-x)\left(\tfrac{2}{3}\log(1-x)-\log(x)\right) + \zeta(3)(S+T)^2$$

Second correction: sv ansatz has 254 rational parameters

Solution

$$G^{(2)}(S,T,x) = \ldots$$

5. Checks

Low energy expansion

Relates to low energy effective action (SYM + derivative interactions)

$$A(S,T) = -\frac{1}{ST} + \sum_{a,b,k=0}^{\infty} \frac{\sigma_1^a \sigma_2^b}{\sqrt{\lambda}^k} \alpha_{a,b}^{(k)}\,, \qquad \sigma_1 = -U,\ \sigma_2 = -ST$$

$$= \underbrace{-\frac{1}{ST}}_{\text{SYM}} + \underbrace{\alpha_{0,0}^{(0)}}_{F^4} + \underbrace{\sigma_1\, \alpha_{1,0}^{(0)} + \frac{\alpha_{0,0}^{(1)}}{\sqrt{\lambda}}}_{D^2 F^4} + \underbrace{\sigma_1^2\, \alpha_{2,0}^{(0)} + \sigma_2\, \alpha_{0,1}^{(0)} + \frac{\sigma_1\, \alpha_{1,0}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{0,0}^{(2)}}{\lambda}}_{D^4 F^4} + \dots$$

$\alpha_{a,b}^{(0)}$ = flat space, we fix all $\alpha_{a,b}^{(1)}$ and $\alpha_{a,b}^{(2)}$ and the full $D^6 F^4$ term.

Localization provides 1 constraint for each interaction term
$G = SO(8)$: [Behan,Chester,Ferrero;2023], $G = U(4)$: [Billo,Frau,Lerda,Pini,Vallarino;2024]

$\alpha_{0,0}^{(1)} = 0$ agrees, $\alpha_{0,0}^{(2)} = 48\zeta(2)^2$ fixes final # in $G^{(2)}(S,T,x)$ ✓

High energy limit

High energy limit ↔ classical worldsheet computation, generalizing [Gross,Mende;1987] to AdS [Alday,TH,Nocchi;2023]

$$\lim_{S,T,R\to\infty} A^{\text{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A^{\text{flat}}(S,T)\right) e^{-\varepsilon_{\text{open/closed}}(S,T)}$$

High energy saddle of worldsheet integral:

$$\varepsilon_{\text{open}}(S,T) = -\frac{\alpha'}{R^2} G^{(1)}(S,T,z=\tfrac{S}{S+T}) + O(S)$$

Exponentiation relates different curvature corrections:

$$\frac{1}{2}\left(\varepsilon_{\text{open}}(S,T)\right)^2 = \frac{\alpha'^2}{R^4} G^{(2)}(S,T,z=\tfrac{S}{S+T}) + O(S^3)$$

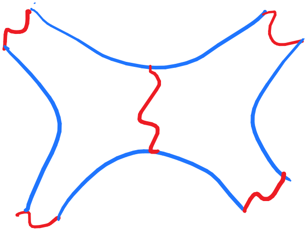
✓

Relation between open and closed strings as expected:

$$\varepsilon_{\text{open}}(S,T) = \frac{1}{2}\varepsilon_{\text{closed}}(4S,4T)$$

✓

OPE data



We extract the OPE data:

$A^{(0)}$ data | $A^{(1)}$ data | $A^{(2)}$ data

$$\Delta_{\delta,\ell} = \boxed{\sqrt{\delta}\lambda^{\frac{1}{4}}} + \boxed{\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}} + \boxed{\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}} + \dots$$

$$C^2_{\delta,\ell} = \boxed{C^{2(0)}_{\delta,\ell}} + \boxed{\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}} + \boxed{\lambda^{-1}C^{2(2)}_{\delta,\ell}} + \dots$$

Leading Regge trajectory:

$$\Delta = \sqrt{\delta}\lambda^{\frac{1}{4}}\left[\underbrace{1}_{0} + \left(\underbrace{\frac{3\delta}{4} + \frac{1}{2\delta}}_{0}\underbrace{-\frac{3}{4}}_{1}\right)\frac{1}{\sqrt{\lambda}} - \left(\underbrace{\frac{21\delta^2}{32} + \frac{1}{8\delta^2}}_{0}\underbrace{+\frac{(3+14\zeta(3))\delta}{4} - \frac{3}{8\delta}}_{1}\underbrace{-\frac{41}{32}}_{2}\right)\frac{1}{\lambda} + \dots\right]$$

0: matches classical solution for glued folded open string! ✓

1: 1-loop fluctuation

2: 2-loop fluctuation

} open problem for semi-classics / integrability!

Summary

STRING AMPLITUDE
SHOPPING LIST

- PARTIAL WAVE EXPANSION
- REGGE BOUNDEDNESS
- WORLDSHEET INTEGRAL

Recipes

poles from OPE

$+$

single-valued ansatz

$=$

AdS Virasoro-Shapiro
& AdS Veneziano

Checks:

- Low energy expansion
- High energy limit
- OPE data for massive strings

Future directions

- Conformal dimensions for massive string operators from integrability?
- Other AdS backgrounds, e.g. type IIA on $AdS_4 \times CP^3$ / ABJM
- Go beyond the small curvature expansion?
- Compute AdS string amplitudes directly from string theory?

Thank you!