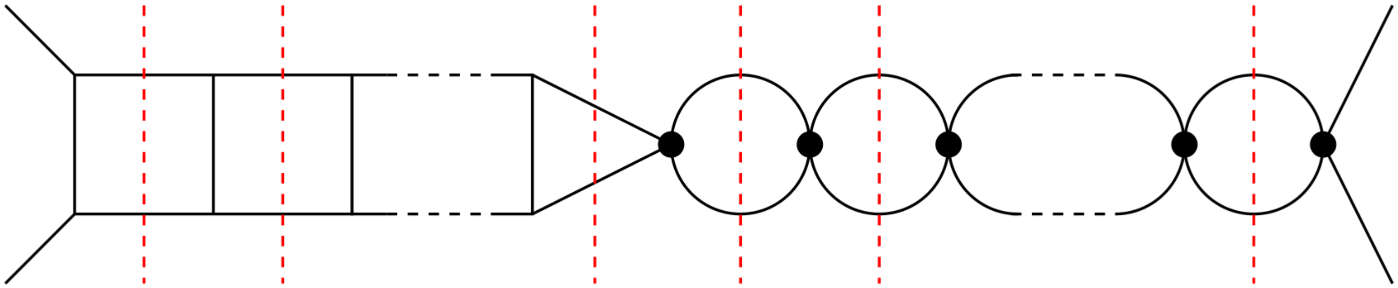
Non-analytic terms of string amplitudes from partial waves



Hynek Paul

Based on [2407.15925] with
Yu-tin Huang & Michele Santagata

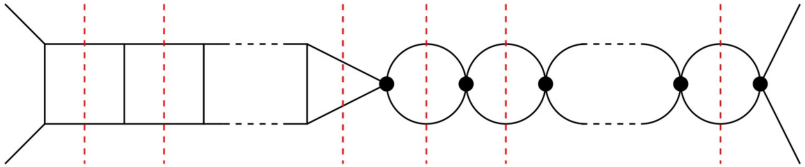
Eurostrings Conference
04/09/2024

KU LEUVEN

Outline

i) General Part: **partial-wave expansion for unitarity cuts**

→ how to easily compute iterative s-channel cuts



ii) Application: **the type IIB closed string amplitude**

→ predict logarithmic structure of the low-energy expansion at higher genus

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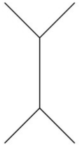
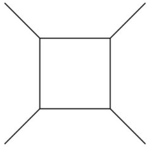
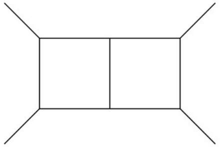
$\mathcal{A}(p_1, p_2, p_3, p_4) = g$  $+\ g^2$  $+\ g^3$  $+ \ldots$

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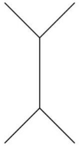
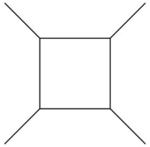
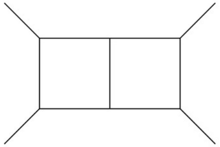
$\mathcal{A}(p_1, p_2, p_3, p_4) = \mathrm{g}$  $+\, \mathrm{g}^2$  $+\, \mathrm{g}^3$  $+ \ldots$

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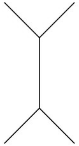
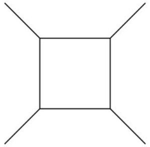
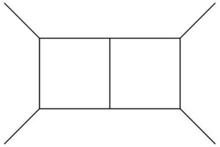
$\mathcal{A}(p_1, p_2, p_3, p_4) \;=\; g$  $+\, g^2$  $+\, g^3$  $+ \ldots$

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Mandelstam invariants

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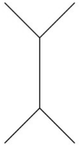
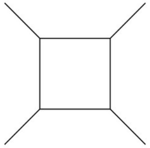
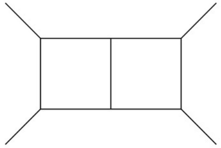
$\mathcal{A}(p_1, p_2, p_3, p_4) = g$  $+ g^2$  $+ g^3$  $+ \ldots$

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We call this the (s-channel) leading log.

It's coefficient can be computed rather easily...

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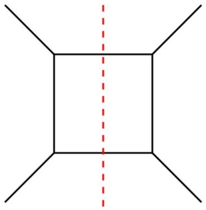

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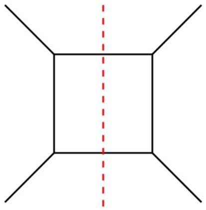

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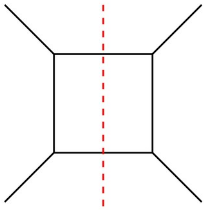

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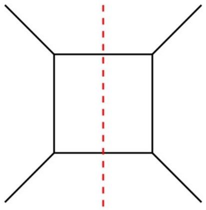

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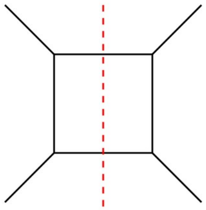

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→ **key property: the partial waves trivialise the phase space integration!**

$$\int d\Omega_2 \, P_\ell^{\frac{d-3}{2}}(\cos\theta_i)\, P_{\ell'}^{\frac{d-3}{2}}(\cos\theta_f) = P_\ell^{\frac{d-3}{2}}(\cos\theta)\delta_{\ell,\ell'}$$

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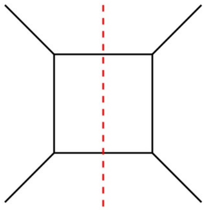

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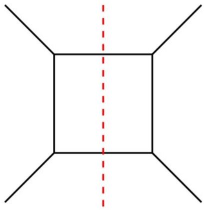

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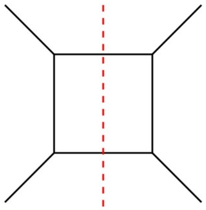

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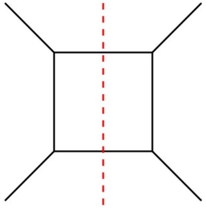

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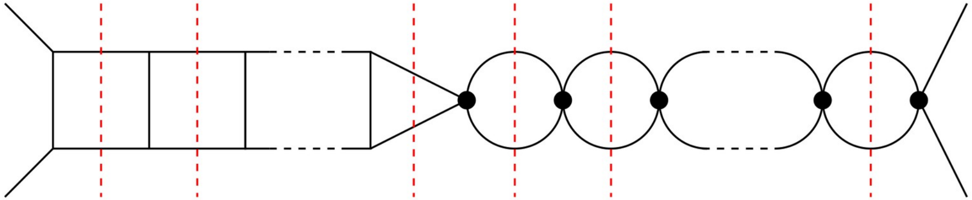

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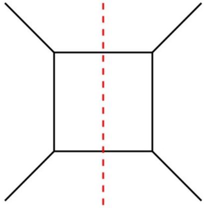

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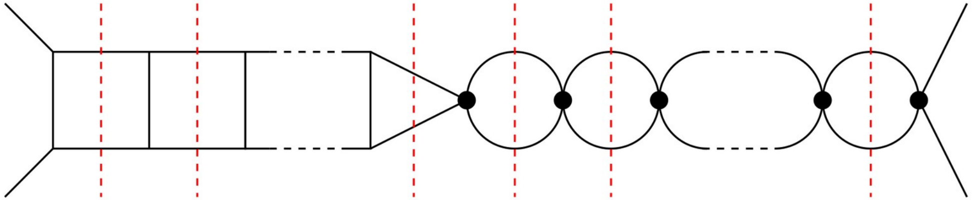

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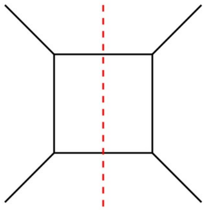

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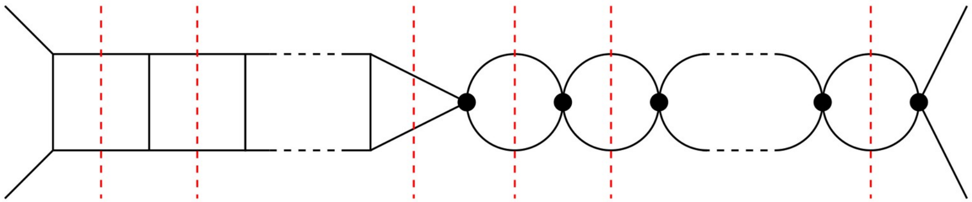

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Comment: this construction is very general and applicable to any EFT, in any dimension!

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string length squared

string coupling

sum over genus h

"gravitational coupling"

$\kappa_{10}^2 = 8\pi G_N = 2^6\pi^7\alpha'^4$

polarisation tensor

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$\rightarrow$ **Virasoro-Shapiro amplitude:** $$\mathcal{A}^{(0)} = \frac{1}{stu} \frac{\Gamma(1-\tilde{s})\Gamma(1-\tilde{t})\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\Gamma(1+\tilde{t})\Gamma(1+\tilde{u})}$$

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→ **Virasoro-Shapiro amplitude:**

$$\mathcal{A}^{(0)} = \frac{1}{stu} \frac{\Gamma(1-\tilde{s})\Gamma(1-\tilde{t})\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\Gamma(1+\tilde{t})\Gamma(1+\tilde{u})}$$

low-energy expansion

$$= \frac{1}{stu} + 2\alpha'^3\zeta_3 + \alpha'^5(s^2+t^2+u^2)\zeta_5 + 2\alpha'^6 stu\zeta_3^2 + \dots$$

sugra $\quad \mathcal{R}^4 \quad \partial^4\mathcal{R}^4 \quad \partial^6\mathcal{R}^4$

(ii) Application: the type IIB closed string amplitude

Recall: all we needed is input from tree-level (genus 0) amplitude

dimensionless Mandelstams

$$\tilde{s} \equiv \alpha' s$$
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Extract tree-level partial-wave coefficients order by order in α':

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low-energy expansion ↓

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Extract tree-level partial-wave coefficients order by order in α':

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low-energy expansion $\downarrow$

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Extract tree-level partial-wave coefficients order by order in α':

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higher-derivative corrections: $\mathcal{R}^4$

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$\partial^4\mathcal{R}^4 \quad \epsilon_\ell^{(5)} \sim \zeta_5\left(\delta_{\ell,0} + \frac{1}{154}\delta_{\ell,2}\right)$

$\partial^6\mathcal{R}^4 \quad \epsilon_\ell^{(6)} \sim \zeta_3^2\left(\delta_{\ell,0} - \frac{1}{44}\delta_{\ell,2}\right)$

(ii) Application: the type IIB closed string amplitude

Recall: all we needed is input from tree-level (genus 0) amplitude

dimensionless Mandelstams

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$\partial^4\mathcal{R}^4$ $\quad \epsilon_\ell^{(5)} \sim \zeta_5\left(\delta_{\ell,0} + \dfrac{1}{154}\delta_{\ell,2}\right)$ → **spin truncated:** $\ell = 0, 2, \ldots, \ell_{\max}$

$\partial^6\mathcal{R}^4$ $\quad \epsilon_\ell^{(6)} \sim \zeta_3^2\left(\delta_{\ell,0} - \dfrac{1}{44}\delta_{\ell,2}\right)$

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Leading log is obtained by multiplying tree-level partial-wave coefficients:

$$\epsilon_\ell^{(h,n)} = \left(-\frac{\kappa_{10}^2\, s^4}{4\pi}\right)^h \sum_{\sigma_{n,h}} \epsilon_\ell^{(n_1)} \epsilon_\ell^{(n_2)} \cdots \epsilon_\ell^{(n_{h+1})}$$

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Extra factor of α'^4 for each gluing from 'sewing relation':

$$\sum_{2\,\text{pt states}} \mathcal{R}^4_{1,2\to 2\,\text{pt}}\, \mathcal{R}^4_{2\,\text{pt}\to 3,4} = s^4\, \mathcal{R}^4_{1,2\to 3,4}$$

Sum over all solutions to

$$\sum_{i=1}^{h+1} n_i = n\,, \quad \text{with } n_i \geq 0$$

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(at genus h, product of h+1 tree-level coeffs)

Extra factor of α'^4 **for each gluing from 'sewing relation':**

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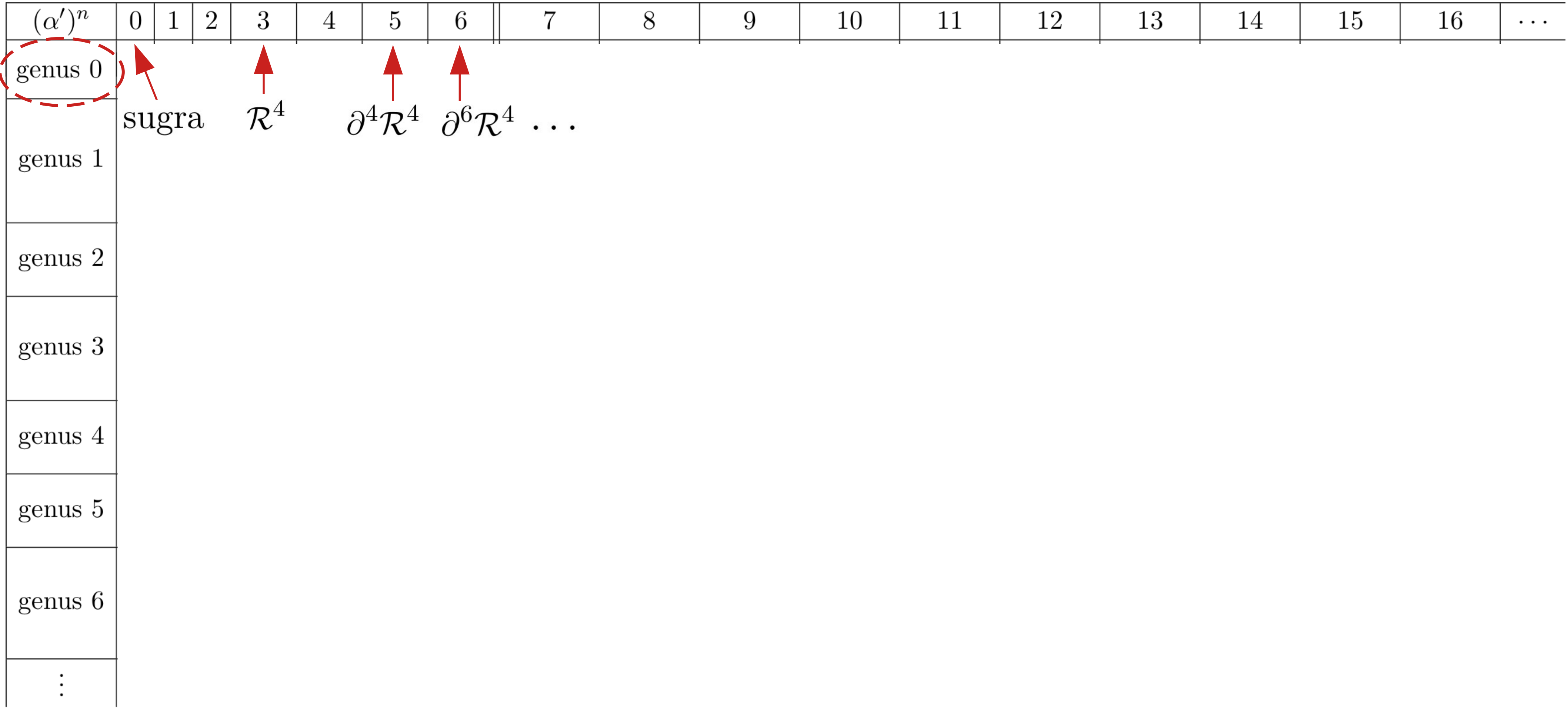
$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0																		
genus 1																		
genus 2																		
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		



$(\alpha')^n$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
$\cdots$
genus 0
genus 1
genus 2
genus 3
genus 4
genus 5
genus 6
$\vdots$
sugra
$\mathcal{R}^4$
$\partial^4\mathcal{R}^4$
$\partial^6\mathcal{R}^4$
$\ldots$

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	⋯
genus 0	S			$\mathcal{R}_0$		$_4\mathcal{R}_0$	$_6\mathcal{R}_0$	$_8\mathcal{R}_0$	$_{10}\mathcal{R}_0$	$_{12}\mathcal{R}_0$	$_{14}\mathcal{R}_0$	$_{16}\mathcal{R}_0$	$_{18}\mathcal{R}_0$	$_{20}\mathcal{R}_0$	$_{22}\mathcal{R}_0$	$_{24}\mathcal{R}_0$	$_{26}\mathcal{R}_0$	⋯
genus 1																		
genus 2																		
genus 3																		
genus 4																		
genus 5																		
genus 6																		
⋮																		

forest green: **known analytic terms**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert_4\mathcal{R}_0$	$S\vert_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert_8\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert_6\mathcal{R}_0$	$\cdots$
genus 2																		
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|{}_4\mathcal{R}_0$	$S\|{}_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|{}_8\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|{}_6\mathcal{R}_0$	$\cdots$
genus 2																		
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$\vdots$																		

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$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|_4\mathcal{R}_0$	$S\|_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|_8\mathcal{R}_0$	$S\|_{10}\mathcal{R}_0$ $\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{12}\mathcal{R}_0$ $\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{14}\mathcal{R}_0$ $\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{16}\mathcal{R}_0$ $\mathcal{R}_0\|_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{18}\mathcal{R}_0$ $\mathcal{R}_0\|_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|_6\mathcal{R}_0$	$\cdots$
genus 2																		
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|_4\mathcal{R}_0$	$S\|_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|_8\mathcal{R}_0$	$S\|_{10}\mathcal{R}_0$ $\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{12}\mathcal{R}_0$ $\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{14}\mathcal{R}_0$ $\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{16}\mathcal{R}_0$ $\mathcal{R}_0\|_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{18}\mathcal{R}_0$ $\mathcal{R}_0\|_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|_6\mathcal{R}_0$	$\cdots$
genus 2																		
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

$$\mathcal{A}^{(1)}|_{\log(-\tilde{s})} = f_{\mathrm{S|S}} - \frac{2\pi^2\zeta_3}{45}\,\tilde{s}^4 - \frac{\pi^2\zeta_5}{10080}\,\tilde{s}^4\left(22\tilde{s}^2 - \tilde{t}\tilde{u}\right) - \frac{\pi^2\zeta_3^2}{40320}\,\tilde{s}^5\left(12\tilde{s}^2 + \tilde{t}\tilde{u}\right) + \ldots \checkmark$$

[Green,Russo,Vanhove'08],[Edison,Guillen,Johansson,Schlotterer,Teng'21],[Eberhardt,Mizera'22]

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|{}_4\mathcal{R}_0$	$S\|{}_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|{}_8\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|{}_6\mathcal{R}_0$	$\cdots$
genus 2									$S\|S\|S$			$S\|S\|\mathcal{R}_0$		$S\|S\|{}_4\mathcal{R}_0$	$S\|S\|{}_6\mathcal{R}_0$ $S\|\mathcal{R}_0\|\mathcal{R}_0$	$S\|S\|{}_8\mathcal{R}_0$	$S\|S\|{}_{10}\mathcal{R}_0$ $S\|\mathcal{R}_0\|{}_4\mathcal{R}_0$	$\cdots$
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2									$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|_4\mathcal{R}_0$	$S\|_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|_8\mathcal{R}_0$	$S\|_{10}\mathcal{R}_0$ $\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{12}\mathcal{R}_0$ $\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{14}\mathcal{R}_0$ $\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{16}\mathcal{R}_0$ $\mathcal{R}_0\|_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{18}\mathcal{R}_0$ $\mathcal{R}_0\|_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|_6\mathcal{R}_0$	$\cdots$
genus 2									$S\|S\|S$			$S\|S\|\mathcal{R}_0$		$S\|S\|_4\mathcal{R}_0$	$S\|S\|_6\mathcal{R}_0$ $S\|\mathcal{R}_0\|\mathcal{R}_0$	$S\|S\|_8\mathcal{R}_0$	$S\|S\|_{10}\mathcal{R}_0$ $S\|\mathcal{R}_0\|_4\mathcal{R}_0$	$\cdots$
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\|S$			$S\|\mathcal{R}_0$		$S\|_4\mathcal{R}_0$	$S\|_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|_8\mathcal{R}_0$	$S\|_{10}\mathcal{R}_0$ $\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{12}\mathcal{R}_0$ $\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{14}\mathcal{R}_0$ $\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_4\mathcal{R}_0$	$S\|_{16}\mathcal{R}_0$ $\mathcal{R}_0\|_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_6\mathcal{R}_0$	$S\|_{18}\mathcal{R}_0$ $\mathcal{R}_0\|_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|_6\mathcal{R}_0$	$\cdots$
genus 2									$S\|S\|S$			$S\|S\|\mathcal{R}_0$		$S\|S\|_4\mathcal{R}_0$	$S\|S\|_6\mathcal{R}_0$ $S\|\mathcal{R}_0\|\mathcal{R}_0$	$S\|S\|_8\mathcal{R}_0$	$S\|S\|_{10}\mathcal{R}_0$ $S\|\mathcal{R}_0\|_4\mathcal{R}_0$	$\cdots$
genus 3																		
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

$$\mathcal{A}^{(2)}|_{\log^2(-\tilde{s})} = f_{\mathrm{S|S|S}} + \frac{\pi^4\zeta_3}{1350}\,\tilde{s}^8 + \frac{\pi^4\zeta_5}{2^9\cdot 33075}\,\tilde{s}^8(610\tilde{s}^2 - \tilde{t}\tilde{u}) + \frac{\pi^4\zeta_3^2}{2^{11}\cdot 33075}\,\tilde{s}^9(510\tilde{s}^2 + \tilde{t}\tilde{u}) + \ldots$$

$\rightarrow$ **new predictions for leading logarithmic terms!**

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2									$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3													$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert_4\mathcal{R}_0$	$S\vert_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert_8\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert_6\mathcal{R}_0$	$\cdots$
genus 2									$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert_4\mathcal{R}_0$	$S\vert S\vert_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert_8\mathcal{R}_0$	$S\vert S\vert_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert_4\mathcal{R}_0$	$\cdots$
genus 3													$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																		
genus 5																		
genus 6																		
$\vdots$																		

$$\mathcal{A}^{(3)}\big|_{\log^3(-\tilde{s})} = f_{\mathrm{S|S|S|S}} - \frac{\pi^6\zeta_3}{91125}\,\tilde{s}^{12} - \frac{\pi^6\zeta_5}{2^{10}\cdot 31255875}\,\tilde{s}^{12}\left(1707\tilde{s}^2 - \tilde{t}\tilde{u}\right) + \ldots$$

→ **new predictions for leading logarithmic terms!**

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2									$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3													$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

forest green: **known analytic terms**

candy apple red: **leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1					$S\vert S$			$S\vert\mathcal{R}_0$		$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2									$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3													$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

However, this is not the end of the story...

... further higher-genus analytic terms exits!

forest green**: known analytic terms**

candy apple red**: leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$		$S\vert S\vert S$			$S\vert S\vert\mathcal{R}_0$		$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$						$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

However, this is not the end of the story...
... further higher-genus analytic terms exits!

forest green: **known analytic terms**
candy apple red: **leading logs**

Remark: these are modular invariant functions, with the protected terms given by (generalised) Eisenstein functions $E_{\frac{3}{2}}$ $E_{\frac{5}{2}}$ $\mathcal{E}_{\frac{3}{2},\frac{3}{2}}$

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\vert\mathcal{R}_1$	$S\vert S\vert S$		$S\vert{}_6\mathcal{R}_1$ $\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$						$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By the same reasoning, these give rise to sub-leading log's

forest green: **known analytic terms**

candy apple red: **leading logs**

deep sky blue: **sub-leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert_4\mathcal{R}_0$	$S\vert_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert_8\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\vert\mathcal{R}_1$	$S\vert S\vert S$		$S\vert_6\mathcal{R}_1$ $\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert_4\mathcal{R}_0$	$S\vert S\vert_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert_8\mathcal{R}_0$	$S\vert S\vert_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$						$S\vert S\vert S\vert S$			$S\vert S\vert S\vert\mathcal{R}_0$		$\cdots$
genus 4																	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By the same reasoning, these give rise to sub-leading log's

forest green: **known analytic terms**

candy apple red: **leading logs**

deep sky blue: **sub-leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\|S$	0	${}_6\mathcal{R}_1$	$S\|\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\|{}_4\mathcal{R}_0$	$S\|{}_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|{}_8\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|{}_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\|\mathcal{R}_1$	$S\|S\|S$		$S\|{}_6\mathcal{R}_1$ $\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|{}_4\mathcal{R}_0$	$S\|S\|{}_6\mathcal{R}_0$ $S\|\mathcal{R}_0\|\mathcal{R}_0$	$S\|S\|{}_8\mathcal{R}_0$	$S\|S\|{}_{10}\mathcal{R}_0$ $S\|\mathcal{R}_0\|{}_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$					$S\|S\|\mathcal{R}_1$	$S\|S\|S\|S$		$S\|S\|{}_6\mathcal{R}_1$ $S\|\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|S\|\mathcal{R}_0$	$S\|S\|{}_{10}\mathcal{R}_1$ $S\|{}_4\mathcal{R}_0\|\mathcal{R}_1$	$\cdots$
genus 4																	$S\|S\|S\|S\|S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By the same reasoning, these give rise to sub-leading log's

forest green: **known analytic terms**

candy apple red: **leading logs**

deep sky blue: **sub-leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\vert\mathcal{R}_1$	$S\vert S\vert S$		$S\vert{}_6\mathcal{R}_1$ $\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$					$S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S$		$S\vert S\vert{}_6\mathcal{R}_1$ $S\vert\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert S\vert\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_1$ $S\vert{}_4\mathcal{R}_0\vert\mathcal{R}_1$	$\cdots$
genus 4																$S\vert S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6																		$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By the same reasoning, these give rise to sub-leading log's

forest green: **known analytic terms**

candy apple red: **leading logs**

deep sky blue: **sub-leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert_4\mathcal{R}_0$	$S\vert_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert_8\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_4\mathcal{R}_0$	$S\vert_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_6\mathcal{R}_0$	$S\vert_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\vert\mathcal{R}_1$	$S\vert S\vert S$	${}_{12}\mathcal{R}_2$	$S\vert_6\mathcal{R}_1$ $\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_0$	$S\vert_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert_4\mathcal{R}_0$	$S\vert S\vert_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert_8\mathcal{R}_0$	$S\vert S\vert_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$	${}_8\mathcal{R}_3$	${}_{10}\mathcal{R}_3$	$S\vert_4\mathcal{R}_2$	$S\vert_6\mathcal{R}_2$ $\mathcal{R}_1\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S$	$S\vert_{12}\mathcal{R}_2$ $\mathcal{R}_0\vert_6\mathcal{R}_2$ $\mathcal{R}_1\vert_6\mathcal{R}_1$	$S\vert S\vert_6\mathcal{R}_1$ $S\vert\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert S\vert\mathcal{R}_0$	$S\vert S\vert_{10}\mathcal{R}_1$ $S\vert_4\mathcal{R}_0\vert\mathcal{R}_1$	$\cdots$
genus 4								${}_8\mathcal{R}_4$	${}_{10}\mathcal{R}_4$	${}_{12}\mathcal{R}_4$	$S\vert_6\mathcal{R}_3$	$S\vert_8\mathcal{R}_3$	$S\vert_{10}\mathcal{R}_3$ $\mathcal{R}_1\vert_4\mathcal{R}_2$	$S\vert S\vert_4\mathcal{R}_2$	$S\vert S\vert_6\mathcal{R}_2$ $S\vert\mathcal{R}_1\vert\mathcal{R}_1$	$S\vert S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5																		$\cdots$
genus 6								and sub-sub-leading log's etc.										$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

forest green: **known analytic terms**

candy apple red: **leading logs**

deep sky blue: **sub-leading logs**

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$	$S\vert S$	0	${}_6\mathcal{R}_1$	$S\vert\mathcal{R}_0$	${}_{10}\mathcal{R}_1$	$S\vert{}_4\mathcal{R}_0$	$S\vert{}_6\mathcal{R}_0$ $\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert{}_8\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$S\vert{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$S\vert{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\vert{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\vert{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\vert{}_6\mathcal{R}_0$	$\cdots$
genus 2						${}_4\mathcal{R}_2$	${}_6\mathcal{R}_2$	$S\vert\mathcal{R}_1$	$S\vert S\vert S$	${}_{12}\mathcal{R}_2$	$S\vert{}_6\mathcal{R}_1$ $\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_0$	$S\vert{}_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert{}_4\mathcal{R}_0$	$S\vert S\vert{}_6\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert\mathcal{R}_0$	$S\vert S\vert{}_8\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_0$ $S\vert\mathcal{R}_0\vert{}_4\mathcal{R}_0$	$\cdots$
genus 3							${}_6\mathcal{R}_3$	${}_8\mathcal{R}_3$	${}_{10}\mathcal{R}_3$	$S\vert{}_4\mathcal{R}_2$	$S\vert{}_6\mathcal{R}_2$ $\mathcal{R}_1\vert\mathcal{R}_1$	$S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S$	$S\vert{}_{12}\mathcal{R}_2$ $\mathcal{R}_0\vert{}_6\mathcal{R}_2$ $\mathcal{R}_1\vert{}_6\mathcal{R}_1$	$S\vert S\vert{}_6\mathcal{R}_1$ $S\vert\mathcal{R}_0\vert\mathcal{R}_1$	$S\vert S\vert S\vert\mathcal{R}_0$	$S\vert S\vert{}_{10}\mathcal{R}_1$ $S\vert{}_4\mathcal{R}_0\vert\mathcal{R}_1$	$\cdots$
genus 4								${}_8\mathcal{R}_4$	${}_{10}\mathcal{R}_4$	${}_{12}\mathcal{R}_4$	$S\vert{}_6\mathcal{R}_3$	$S\vert{}_8\mathcal{R}_3$	$S\vert{}_{10}\mathcal{R}_3$ $\mathcal{R}_1\vert{}_4\mathcal{R}_2$	$S\vert S\vert{}_4\mathcal{R}_2$	$S\vert S\vert{}_6\mathcal{R}_2$ $S\vert\mathcal{R}_1\vert\mathcal{R}_1$	$S\vert S\vert S\vert\mathcal{R}_1$	$S\vert S\vert S\vert S\vert S$	$\cdots$
genus 5								${}_8\mathcal{R}_5$	${}_{10}\mathcal{R}_5$	${}_{12}\mathcal{R}_5$	${}_{14}\mathcal{R}_5$	$S\vert{}_8\mathcal{R}_4$	$S\vert{}_{10}\mathcal{R}_4$	$S\vert{}_{12}\mathcal{R}_4$ $\mathcal{R}_1\vert{}_6\mathcal{R}_3$	$S\vert S\vert{}_6\mathcal{R}_3$	$S\vert S\vert{}_8\mathcal{R}_3$	$S\vert S\vert{}_{10}\mathcal{R}_3$ $S\vert\mathcal{R}_1\vert{}_4\mathcal{R}_2$	$\cdots$
genus 6								${}_8\mathcal{R}_6$	${}_{10}\mathcal{R}_6$	${}_{12}\mathcal{R}_6$	${}_{14}\mathcal{R}_6$	$S\vert{}_8\mathcal{R}_5$	$S\vert{}_{10}\mathcal{R}_5$	$S\vert{}_{12}\mathcal{R}_5$	$S\vert{}_{14}\mathcal{R}_5$ $\mathcal{R}_0\vert{}_8\mathcal{R}_5$ $\mathcal{R}_1\vert{}_8\mathcal{R}_4$	$S\vert S\vert{}_8\mathcal{R}_4$	$S\vert S\vert{}_{10}\mathcal{R}_4$	$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

→ our method reveals an intricate pattern of logarithmic terms in the low energy expansion

$(\alpha')^n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
genus 0	S			$\mathcal{R}_0$		${}_4\mathcal{R}_0$	${}_6\mathcal{R}_0$	${}_8\mathcal{R}_0$	${}_{10}\mathcal{R}_0$	${}_{12}\mathcal{R}_0$	${}_{14}\mathcal{R}_0$	${}_{16}\mathcal{R}_0$	${}_{18}\mathcal{R}_0$	${}_{20}\mathcal{R}_0$	${}_{22}\mathcal{R}_0$	${}_{24}\mathcal{R}_0$	${}_{26}\mathcal{R}_0$	$\cdots$
genus 1				$\mathcal{R}_1$		0	${}_6\mathcal{R}_1$		${}_{10}\mathcal{R}_1$	$S\|{}_4\mathcal{R}_0$	$S\|{}_6\mathcal{R}_0$ $\mathcal{R}_0\|\mathcal{R}_0$	$S\|{}_8\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_0$ $\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{12}\mathcal{R}_0$ $\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{14}\mathcal{R}_0$ $\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_4\mathcal{R}_0$	$S\|{}_{16}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{10}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_6\mathcal{R}_0$	$S\|{}_{18}\mathcal{R}_0$ $\mathcal{R}_0\|{}_{12}\mathcal{R}_0$ ${}_4\mathcal{R}_0\|{}_8\mathcal{R}_0$ ${}_6\mathcal{R}_0\|{}_6\mathcal{R}_0$	$\cdots$
genus 2							${}_6\mathcal{R}_2$			${}_{12}\mathcal{R}_2$	$S\|{}_6\mathcal{R}_1$ $\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|\mathcal{R}_0$	$S\|{}_{10}\mathcal{R}_1$ ${}_4\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|{}_4\mathcal{R}_0$	$S\|S\|{}_6\mathcal{R}_0$ $S\|\mathcal{R}_0\|\mathcal{R}_0$	$S\|S\|{}_8\mathcal{R}_0$	$S\|S\|{}_{10}\mathcal{R}_0$ $S\|\mathcal{R}_0\|{}_4\mathcal{R}_0$	$\cdots$
genus 3								${}_8\mathcal{R}_3$	${}_{10}\mathcal{R}_3$	$S\|{}_4\mathcal{R}_2$	$S\|{}_6\mathcal{R}_2$ $\mathcal{R}_1\|\mathcal{R}_1$	$S\|S\|\mathcal{R}_1$	$S\|S\|S\|S$	$S\|{}_{12}\mathcal{R}_2$ $\mathcal{R}_0\|{}_6\mathcal{R}_2$ $\mathcal{R}_1\|{}_6\mathcal{R}_1$	$S\|S\|{}_6\mathcal{R}_1$ $S\|\mathcal{R}_0\|\mathcal{R}_1$	$S\|S\|S\|\mathcal{R}_0$	$S\|S\|{}_{10}\mathcal{R}_1$ $S\|{}_4\mathcal{R}_0\|\mathcal{R}_1$	$\cdots$
genus 4								${}_8\mathcal{R}_4$	${}_{10}\mathcal{R}_4$	${}_{12}\mathcal{R}_4$	$S\|{}_6\mathcal{R}_3$	$S\|{}_8\mathcal{R}_3$	$S\|{}_{10}\mathcal{R}_3$ $\mathcal{R}_1\|{}_4\mathcal{R}_2$	$S\|S\|{}_4\mathcal{R}_2$	$S\|S\|{}_6\mathcal{R}_2$ $S\|\mathcal{R}_1\|\mathcal{R}_1$	$S\|S\|S\|\mathcal{R}_1$	$S\|S\|S\|S\|S$	$\cdots$
genus 5								${}_8\mathcal{R}_5$	${}_{10}\mathcal{R}_5$	${}_{12}\mathcal{R}_5$	${}_{14}\mathcal{R}_5$	$S\|{}_8\mathcal{R}_4$	$S\|{}_{10}\mathcal{R}_4$	$S\|{}_{12}\mathcal{R}_4$ $\mathcal{R}_1\|{}_6\mathcal{R}_3$	$S\|S\|{}_6\mathcal{R}_3$	$S\|S\|{}_8\mathcal{R}_3$	$S\|S\|{}_{10}\mathcal{R}_3$ $S\|\mathcal{R}_1\|{}_4\mathcal{R}_2$	$\cdots$
genus 6								${}_8\mathcal{R}_6$	${}_{10}\mathcal{R}_6$	${}_{12}\mathcal{R}_6$	${}_{14}\mathcal{R}_6$	$S\|{}_8\mathcal{R}_5$	$S\|{}_{10}\mathcal{R}_5$	$S\|{}_{12}\mathcal{R}_5$	$S\|{}_{14}\mathcal{R}_5$ $\mathcal{R}_0\|{}_8\mathcal{R}_5$ $\mathcal{R}_1\|{}_8\mathcal{R}_4$	$S\|S\|{}_8\mathcal{R}_4$	$S\|S\|{}_{10}\mathcal{R}_4$	$\cdots$
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

weight 0

weight 3

weight 5

'Trajectories' of equal transcendental weight

For experts: assuming uniform transcendentality of sub-leading logs leads to a new conjecture →

The coefficient of the genus-two correction to the (unprotected) $\partial^{12}\mathcal{R}^4$ term is proportional to ζ_3.

Summary and Outlook

i)

ii)

iii)

Summary and Outlook

i) Partial-wave expansion simplifies the calculation of iterated s-channel cut diagrams

ii)

iii)

Summary and Outlook

i) Partial-wave expansion simplifies the calculation of iterated s-channel cut diagrams

→ leading log computed from tree-level coefficients:

$$f^{(L)}(s,t) = \sum_{\ell \text{ even}} \left(a_\ell^{(0)}\right)^L P_\ell^{\frac{d-3}{2}}(\cos\theta)$$

ii)

iii)

Summary and Outlook

i) Partial-wave expansion simplifies the calculation of iterated s-channel cut diagrams

→ leading log computed from tree-level coefficients: $f^{(L)}(s,t) = \sum_{\ell \text{ even}} \left(a_\ell^{(0)}\right)^L P_\ell^{\frac{d-3}{2}}(\cos\theta)$

ii) Application I: the type IIB closed string amplitude

iii)

Summary and Outlook

i) Partial-wave expansion simplifies the calculation of iterated s-channel cut diagrams

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e.g. @ 2 loops: $\int d\Omega_2\, \mathcal{A}_4^{(0)} \times \mathcal{A}_4^{(1)} + \int d\Omega_3\, \mathcal{A}_5^{(0)} \times \mathcal{A}_5^{(0)}$