3d Topological Order Labeled by Seifert Manifolds

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with Federico Bonetti and Sakura Schafer-Nameki

An Intriguing Correspondence

We propose an interesting correspondence between

- Modular tensor categories

$$\mathcal{C}\Big[\mathfrak{sl}_N, p_1, q_1\Big] \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_2, q_2\Big] \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_n, q_n\Big]$$

  Simple object: simple anyon

- $(M_3, SL(N,\mathbb{C}))$

$$M_3 = [\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}], \quad n \geq 3$$

  Simple object: *anyonic* $SL(N,\mathbb{C})$ flat connections on M_3

Heavily inspired by Cho, Gang, Kim, '20, Cui, Qiu, Wang, '21, Cui, Gustafson, Qiu, Zhang '21, Choi, Gang, Kim, '22

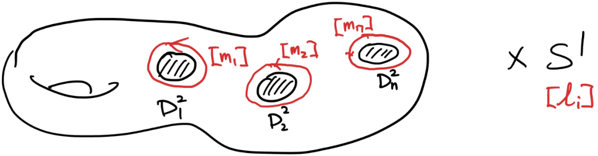
Seifert Manifolds

A Seifert manifold is a circle bundle over punctured Riemann surface $\Sigma_{g,n}$

$$[g; \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}] := \begin{array}{ccc} S^1 & \longrightarrow & M_3 \\ & & \downarrow \\ & & \Sigma_{g,n} \end{array} \tag{1}$$

- Remove n disks from Σ_g

$$\left(\Sigma_g \backslash D_1^2 \cup D_2^2 \cup \cdots \cup D_n^2\right) \times S^1 \tag{2}$$



- fill in the disk bounded by $p_i[m_i] + q_i[\ell_i]$, with $\gcd(p_i, q_i) = 1$
- Many 3-manifolds are Seifert, and they account for all compact oriented manifolds in 6 of the 8 Thurston geometries
- We will only look at $g = 0$, i.e. base is S^2

Modular Tensor Categories

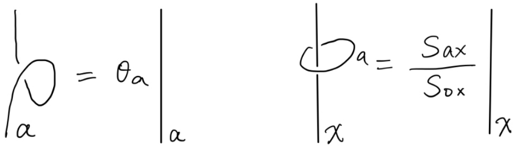
A MTC $\mathcal{C}$ is a ribbon fusion category that has nondegenerate braiding

- Physically, it is known to describe 3d bosonic TQFT Kitaev, Bonderson
- Roughly, it is given by the invariant data $(\mathcal{A}, S, T, c)$
- Needs (F, R) for full specification Mignard, Schauenburg '17

where

- $\mathcal{A} := K_0(\mathcal{C})$ is the fusion ring of the anyons
- S matrix is symmetric and unitary
- T matrix

$$T = e^{2\pi i\left(-\frac{c}{24}\right)} \operatorname{diag}\left(\theta_1, \ldots, \theta_m\right) \tag{3}$$



- Referred to as **premodular** categories if S is not necessariry nondegenerate.

- Crucial in understanding 3d gapped phases Wilczek'90, Moore, Read'91, Wen'95, Kitaev'05, Levin, Walker, Fradkin, Nayak, Tsvelik, Bonderson, Kong, Lan ....
- intimate relation with 2d CFTs, quantum groups, topological quantum computations etc Moore, Seiberg'89, Reshetikhin Turaev'91, Freedman, Rowell, Gannon, Andersen, Wang, Kauffman, Preskill, ...
- Ocneanu rigidity: discrete
- Rank-finiteness theorem: finitely many at any fixed rank Bruillard, Ng, Rowell, Wang, '13

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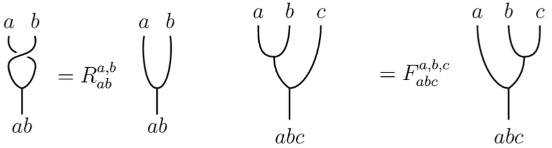
We should be able to classify MTCs!

At each rank

- find all fusion rings
- for each fusion ring, solve Pentagon/Hexagon relations



$$\sum_n F^{bcd}_{q;pn} F^{and}_{f;qe} F^{abc}_{e;nm} = F^{apb}_{f;qm} F^{mcd}_{f;pe}$$

$$\left(R^{ac}_e\right) F^{bac}_{d;em} \left(R^{ab}_m\right) = \sum_n F^{bca}_{d;en} \left(R^{an}_d\right) F^{abc}_{d;nm}$$

$$\left(R^{ac}_e\right)^{-1} F^{bac}_{d;em} \left(R^{ab}_m\right)^{-1} = \sum_n F^{bca}_{d;en} \left(R^{an}_d\right)^{-1} F^{abc}_{d;nm}$$

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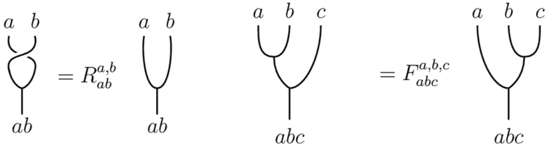

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Pentagon/Hexagon are notoriously hard to solve.
Many more equations than variables: solutions do not always exist

Heroic brute force search:

- rank 2: Ostrik 2002
- rank 3: Ostrik 2005
- rank 4: Rowell, Stong, Wang 2009
- rank 5: Bruillard, Ng, Rowell, Wang 2015
- rank 6: Ng, Rowell, Wang, Wen, 2022 (modular data only)
- rank 11: Ng, Rowell, Wang, Wen 2023 (modular data only)

$$\mathrm{MTC}[M_3] = \mathcal{C}\Big[\mathfrak{sl}_N, p_1, q_1\Big] \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_2, q_2\Big] \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_n, q_n\Big]$$

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Quantum Group Category

The building block $\mathcal{C}[\mathfrak{g}, p, q]$ denotes the semisimplications of category of f.d. modules of $U_\mathfrak{q}(\mathfrak{g})$, with $2p$-root of unity $\mathfrak{q}$. Chari Pressley '95, Sawin '03, ...

- The simplest case is $\mathcal{C}[\mathfrak{sl}_N, p, 1]$ with $\mathfrak{q} = \exp\left(2\pi i \frac{1}{2p}\right)$

  $\mathrm{CS}(\mathfrak{sl}_N)$ *at level* $p - N$

  Fusion ring remains the same but (S, T, F, R) are different.
- $\mathbb{Z}_N$ grading given by 1-form symmetry
- $\mathcal{C}[\mathfrak{g}, p, q]$ is premodular, but not necessarily modular.

$$\mathrm{MTC}[M_3] = \mathcal{C}\Big[\mathfrak{sl}_N, p_1, q_1\Big] \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_2, q_2\Big] \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}\Big[\mathfrak{sl}_N, p_n, q_n\Big]$$

Graded Deligne Product

- Suppose A is an abelian group
- $\mathcal{C} = \oplus_{g\in A}\mathcal{C}_g$ and $\mathcal{D} = \oplus_{g\in A}\mathcal{D}_g$ are two A-graded premodular categories

We define a A-graded product

$$\mathcal{C} \boxtimes_A \mathcal{D} \equiv \oplus_{g\in A}\mathcal{C}_g \boxtimes \mathcal{D}_g \subset \mathcal{C} \boxtimes \mathcal{D}$$

$\mathcal{C} \boxtimes_A \mathcal{D}$ is again a A-graded and premodular. Cui, Qiu, Wang, 21

However, its physical meaning is elusive.

- $\boxtimes_A$ of modular categories could be non-modular
- $\boxtimes_A$ of non-modular categories could be modular

Now back to the proposal

- $\mathrm{MTC} = \mathcal{C}\left[\mathfrak{sl}_N, p_1, q_1\right] \boxtimes_{\mathbb{Z}_N} \mathcal{C}\left[\mathfrak{sl}_N, p_2, q_2\right] \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}\left[\mathfrak{sl}_N, p_n, q_n\right]$
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Remarks:

- intricate structure in space of MTCs yet to discover
  - characterized by $(M_3, G_{\mathbb{C}})$?
  - realizing all MTCs (unitary or non-unitary) with rank $r \leq 5$
- The mysterious graded product $\boxtimes_A$ might play an important roles
- a new (simpler) way to compute F and R symbols

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but, where do we use the geometric properties of M_3?

Character Variety

Flat connections are reps of $\pi_1(M_3)$

$$\rho \in \frac{\hom(\pi_1(M_3), G_{\mathbb{C}})}{\text{conjugation}}$$

said to be **irreducible** if

$$\mathrm{Stab}(\rho) = \{g \in G_{\mathbb{C}} | g\rho(x) = \rho(x)g, \forall x \in \pi_1(x)\} \text{ is at most discrete}$$

For Seifert manifolds, little is known except

$$SL(2, \mathbb{C}), \quad n = 3$$

Anyonic flat connection ↔ Anyons

It turns out

- Part of the answer is surprisingly simple

$$\text{all} \supset \text{irred} \supset \textbf{anyonic}$$

- **anyonic**: no repeated eigenvalues
- non-anyonic ones are interesting too.

Conjecture (supported by extensive numerical checks)

exists a natural isomorphism

geometry — conn components of **anyonic** $SL(N,\mathbb{C})$ connections in sector ℓ on $M_3 = \left[\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}\right]$

$\sum_{\ell=1}^{N-1} \mathring{\mathcal{R}}_{p_1,q_1,N,\ell} \otimes \mathring{\mathcal{R}}_{p_1,q_1,N,\ell} \otimes \cdots \otimes \mathring{\mathcal{R}}_{p_n,q_n,N,\ell}$

1-to-1 correspondent to

MTC — anyons with 1-form charge ℓ

$\mathcal{C}\left[\mathfrak{sl}_N, p_1, q_1\right] \boxtimes_{\mathbb{Z}_N} \mathcal{C}\left[\mathfrak{sl}_N, p_2, q_2\right] \boxtimes_{\mathbb{Z}_N} \cdots \boxtimes_{\mathbb{Z}_N} \mathcal{C}\left[\mathfrak{sl}_N, p_n, q_n\right]$

$\sum_{\ell=1}^{N-1} \Delta_{\mathfrak{sl}_N,p_1-N,\ell} \otimes \Delta_{\mathfrak{sl}_N,p_2-N,\ell} \otimes \cdots \otimes \Delta_{\mathfrak{sl}_N,p_n-N,\ell}$

Check I: Invariance Under the Diffeomorphism

Given two Seifert manifolds

$$M_3 = [\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}]$$

$$M_3' = [\frac{p_1}{q_1'}, \frac{p_2}{q_2'}, \ldots, \frac{p_n}{q_n'}]$$

if there exist integers $\{m_k\}_{k=1}^n$ with

$$q_k' = q_k + p_k m_k \ , \quad k = 1, \ldots, n, \quad \sum_{k=1}^{n} m_k = 0 \tag{4}$$

then it is known M_3 and M_3' are diffeomorphic.

We can check MTC$(\mathfrak{sl}(N), M_3)$ and MTC$(\mathfrak{sl}(N), M_3')$ are the same

Check II: Modularity

The modularity condition seems very random

- $\mathrm{SU}(2)_{k_1} \boxtimes_{\mathbb{Z}_2} \mathrm{SU}(2)_{k_2}$ is modular if and only if $k_1 + k_2$ is odd.
- $\mathcal{C}\left[\mathfrak{sl}_2, p_1, q_1\right] \boxtimes_{\mathbb{Z}_2} \mathcal{C}\left[\mathfrak{sl}_2, p_2, q_2\right] \boxtimes_{\mathbb{Z}_2} \mathcal{C}\left[\mathfrak{sl}_2, p_3, q_3\right]$ is modular if and only if

$$p_1 p_2 p_3 \left| \sum_{k=1}^{3} \frac{q_k}{p_k} \right| \text{ is odd}$$

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Conjecture

$\mathrm{MTC}[M_3, SL(N,\mathbb{C})]$ is modular if and only if $H_1(M_3, \mathbb{Z}_N)$ is trivial.

$N = 2$ and $n = 3$ case is proved in Cui, Qiu, Wang 21'

Check III: Spins

One of the most important topological invariant is the classical Chern Simons invariant

$$\mathrm{CS}_\rho = \int_{M_3} AdA + \frac{2}{3}A[A, A] \quad (5)$$

As initially suggested in Cho, Gang, Kim 20'

$$e^{2\pi i \mathrm{CS}_\rho} \sim \theta_\rho \quad (6)$$

$SU(N)$ CS invariant on Seifert M_3 has been computed by Nishi 98' which we can match precisely to the spins.

Outlook

Summary:

- an interesting correspondence between MTCs and Seifert geometries
- a powerful conjecture on the character variety
- new constructions of MTCs using graded product
- hints on the hidden structure of the space of MTCs

Future directions:

- more general $G_{\mathbb{C}}$, higher genus $g > 0$
- what is F, R symbol in geometry side?
- behavior of $\mathrm{MTC}[M_3]$ under cutting and gluing of M_3
- unitarity

Thank you!