Semiclassical black hole microstates

Microcanonical dimension and Hilbert space factorization

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Work in progress with Vijay Balasubramanian, Ben Craps, Mikhail Khramtzov and Maria Knysh
See also
Balasubramanian, Lawrence, Magan and Sasieta [2212.02447] & [2212.08623]
Climent, Emparan, Magan, Sasieta and Vilar Lopez [2401.08775]
Boruch, Iliesiu, Lin and Yan [2406.04396]

Southampton, Sept 5, 2024

Outline

Black Hole entropy

- Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4G}$$

- Entropy counts number of microstates

$$\mathcal{H} = \bigoplus_E \mathcal{H}_E$$

$$S_{BH}(E) \sim \log \dim \mathcal{H}_E$$

- Counting BH microstates

Counting BH microstates

Semiclassical BH microstates in AdS/CFT

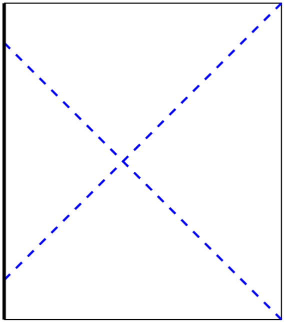
- Geometric duals

Black holes with end-of-the-world branes

- Severe overcounting

Fixed by wormhole contributions

$$\dim \mathcal{H}_E = e^{S_{BH}}$$



(a) Subcritical: $0 \leq T_0 < 1$

Outline

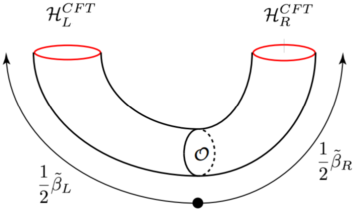
State preparation

CFT side

Two copies of a holographic CFT on $S^{d-1} \times R$

$$|\Psi_\Delta\rangle = |e^{-\frac{\tilde{\beta}_L H_L}{2}} \mathcal{O}_\Delta e^{-\frac{\tilde{\beta}_R H_R}{2}}\rangle$$

$$\sim \sum_{m,n} e^{-\frac{\tilde{\beta}_L E_m + \tilde{\beta}_R E_n}{2}} (\mathcal{O}_\Delta)_{mn} |m, n\rangle$$



State preparation

Eternal black hole, AdS side

Eternal BH + shell of mass m

$$m^2 = \Delta(\Delta - d)$$

Shell trajectory determined by

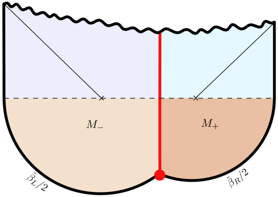
$$f_\pm \dot{\tau}_\pm = \pm\sqrt{-\dot{r}^2 + f_\pm}\,,$$

$$\dot{r}^2 + V_{\text{eff}}(r) = 0\,,$$

where

$$V_{\text{eff}}(r) = -f_+(r) + \left(\frac{M_+ - M_-}{m} - \frac{4\pi G m}{(d-1)V_\Omega r^{d-2}}\right)^2$$

At t=0, the shell is located at r_*
for which $V_{\text{eff}}(r_*) = 0$

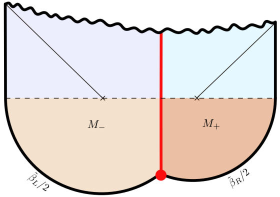


State preparation

Infalling shell, AdS side

Towards microstates for a collapsing black hole

$$V_{\text{eff}}(r) = -f_+(r) + \left(\frac{\Delta M}{m} - \frac{4\pi G m}{(d-1)V_\Omega r^{d-2}}\right)^2$$



For a given ΔM, decreasing m shrinks the wormhole until

$$m_c^2 = (d-1)\frac{\Delta M r_+^{d-2}}{4\pi G} V_\Omega$$

For which $r_* = r_+$: the shell is initially located at the right horizon

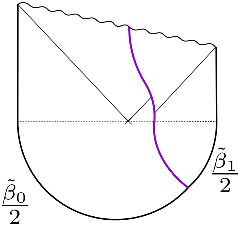
State preparation

Infalling shell, AdS side

$$V_{\rm eff}(r) = -f_+(r) + \left(\frac{\Delta M}{m} - \frac{4\pi G m}{(d-1)V_\Omega r^{d-2}}\right)^2$$

The shell is located at r_* for which $V_{\rm eff}(r_*) = 0$

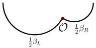
Further decreasing m give shells initially positioned outside the right horizon that fall into the BH



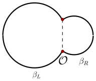
Outline

Norm of the states

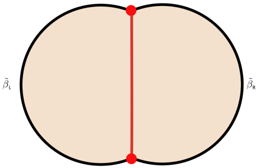
The norm can be computed by the euclidean path integral with fixed boundary conditions

$$|\Psi_m\rangle =$$ 

,

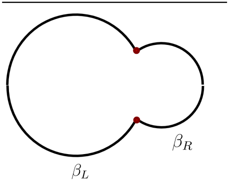
$$\langle\Psi_m|\Psi_m\rangle =$$ 

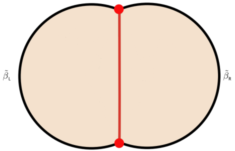
In the semiclassical approximation, given by $\sum e^{-I_{\rm on-shell}}$ over classical saddle point geometries

$$\overline{\langle\Psi_m|\Psi_m\rangle} =$$ 

State overlaps

Consider two such states with masses m and n. What are their overlaps?

$\overline{\langle \Psi_m | \Psi_n \rangle} =$ 

$= \delta_{mn}$ 

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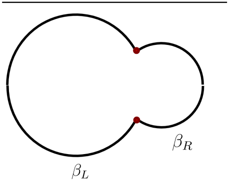

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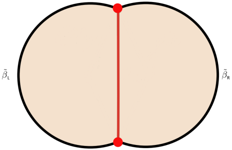


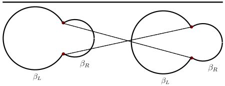
At leading order in semiclassical expansion, states are orthogonal

But there are infinite such states, tension with entropy of black hole being finite

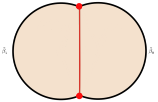
State overlaps

There is small amount of overlap between the states, captured by wormhole contributions

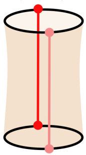
$$\overline{\langle \Psi_m | \Psi_n \rangle \langle \Psi_n | \Psi_m \rangle} =$$



$$= \delta_{mn}$$



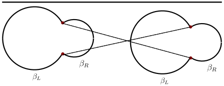
$$^{2} \; +$$



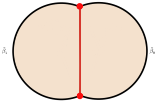
State overlaps

There is small amount of overlap between the states, captured by wormhole contributions

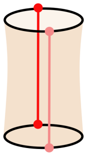
$$\overline{\langle\Psi_m|\Psi_n\rangle\langle\Psi_n|\Psi_m\rangle} =$$



$$= \delta_{mn}$$

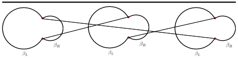


$$^2 \; +$$

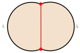


Even more information about overlaps from n-boundary wormholes

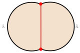
$$\overline{\langle\Psi_m|\Psi_n\rangle\langle\Psi_n|\Psi_k\rangle\langle\Psi_k|\Psi_m\rangle} =$$

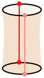


$$= \delta_{mnk}$$

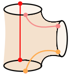


$$^3 + (\delta_{mn} + \delta_{mk} + \delta_{nk})$$





$$+$$



Factorization puzzle

And microscopic interpretation

Importantly, the semiclassical approximation of producs of overlaps don't factorize

$$\overline{\langle\Psi_m|\Psi_n\rangle\langle\Psi_n|\Psi_m\rangle} \neq \overline{\langle\Psi_m|\Psi_n\rangle}\;\overline{\langle\Psi_n|\Psi_m\rangle}$$

This can be understood from the ETH applied to $\mathcal{O}$

$$\mathcal{O}_{mn} \equiv \langle E_m|\mathcal{O}|E_n\rangle = f(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} g(\bar{E},\omega)^{1/2} R_{mn}$$

where $\bar{E} = \frac{E_m+E_n}{2}$, $\omega = E_m - E_n$, and

$$\overline{R_{mn}} = 0\,, \quad \overline{|R_{mn}|^2} = 1$$

Consistency with the semiclassical approximation sets $f = 0$ and $g = \cdots$

Factorization puzzle

And microscopic interpretation

Recall

$$|\Psi_\Delta\rangle \sim \sum_{m,n} e^{-\frac{\tilde{\beta}_L E_m + \tilde{\beta}_R E_n}{2}} \left(\mathcal{O}_\Delta\right)_{mn} |m,n\rangle$$

Applying ETH to our states we find

$$\overline{\langle\Psi_{\Delta'}|\Psi_\Delta\rangle} \sim \sum_{n,m} e^{-\bar{\beta}\bar{E} - \frac{\Delta\beta\omega}{2} - S(\bar{E})} g(\bar{E},\omega)\overline{R^{\Delta}_{mn}}\ \overline{R^{\Delta'}_{mn}} = 0$$

where $\bar{\beta} = \tilde{\beta}_L + \tilde{\beta}_R$ and $\Delta\beta = \tilde{\beta}_L - \tilde{\beta}_R$. On the other hand,

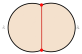
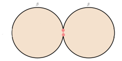
$$\overline{|\langle\Psi_{\Delta'}|\Psi_\Delta\rangle|^2} \sim \sum_{n,m} e^{-2\bar{\beta}\bar{E} - \Delta\beta\omega - 2S(\bar{E})} g(\bar{E},\omega)^2 \overline{|R^{\Delta}_{mn}|^2}\ \overline{|R^{\Delta'}_{mn}|^2} \neq 0$$

The semiclassical gravitational path integral averages over the erratic R_{mn}, computes only the smooth part of the approximated quantities

Pinching limit

Large shell mass

To easily compute the on-shell actions, use the $m \to \infty$ limit, in which geometries pinch off

 $\to$  $= Z(\beta)^2$

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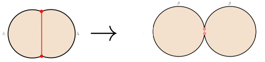
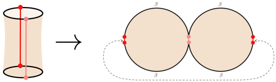


$= Z(\beta)^2$



$= Z(2\beta)^2$

Pinching limit

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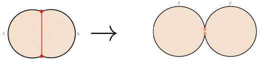
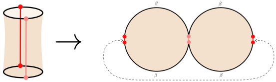
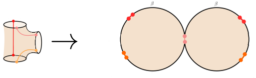


$$= Z(\beta)^2$$



$$= Z(2\beta)^2$$

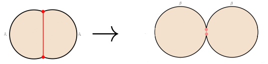


$$= Z(3\beta)^2$$

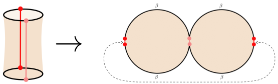
Pinching limit

Large shell mass

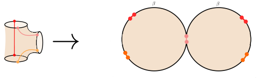
To easily compute the on-shell actions, use the $m \to \infty$ limit, in which geometries pinch off



$$= Z(\beta)^2$$



$$= Z(2\beta)^2$$



$$= Z(3\beta)^2$$

$$\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle \cdots \langle \Psi_{m_n} | \Psi_{m_1} \rangle}\Big|_{\text{con}} \to Z(n\beta)^2$$

Different temperatures, large mass

Different temperatures and finite mass corrections

For states prepared with different temperatures β_L and β_R, we find

$$\overline{\langle\Psi_{m_1}|\Psi_{m_2}\rangle\cdots\langle\Psi_{m_n}|\Psi_{m_1}\rangle}\Big|_{\mathrm{con}} \to Z(n\beta_L)Z(n\beta_R)$$

Furthermore, keeping finite m corrections we find

$$\overline{\langle\Psi_{m_1}|\Psi_{m_2}\rangle\cdots\langle\Psi_{m_n}|\Psi_{m_1}\rangle}\Big|_{\mathrm{con}} \to Z(n\beta_L)Z(n\beta_R)\exp\left(\frac{a_1}{m}+\frac{a_2}{m^2}+\cdots\right)$$

Outline

Gram matrix and span of $\{|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_\Omega\rangle\}$

Focus on a family of states $|\psi_n\rangle = |\Psi_{nm_0}\rangle$ for some $m_0 \gg 1$. What is their span?

Compute the rank of the $\Omega \times \Omega$ Gram matrix $G_{mn} = \langle\psi_m|\psi_n\rangle$ for $m, n \in \{1, 2, \cdots, \Omega\}$

We do this using the resolvent method

$$R_{pq}(\lambda) = \left(\frac{1}{\lambda - G}\right)_{pq} = \frac{1}{\lambda}\left(\delta_{pq} + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} (G^n)_{pq}\right)$$

Density of eigenstates is

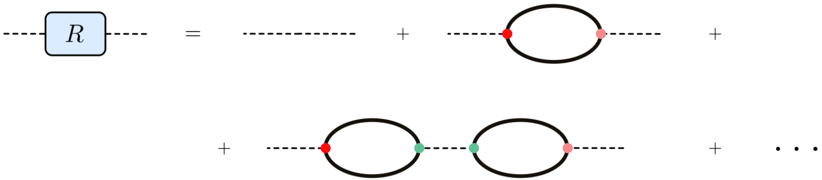
$$D(\lambda) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)\right) ,$$

where $R(\lambda) = \mathrm{tr} R_{pq}(\lambda)$

Resolvent matrix from gravitational path integral

Compute resolvent using gravity path integral

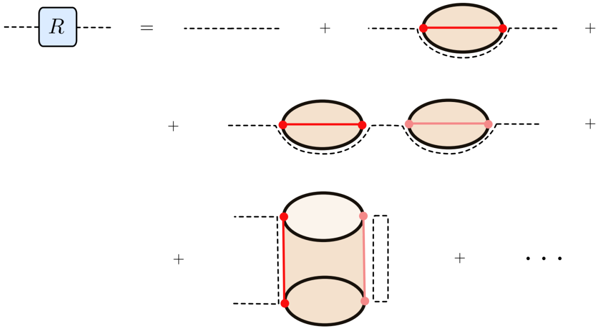
$$R_{pq} = \frac{1}{\lambda}\left(\delta_{pq} + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} \left(G^n\right)_{pq}\right)$$



Semiclassical approximation

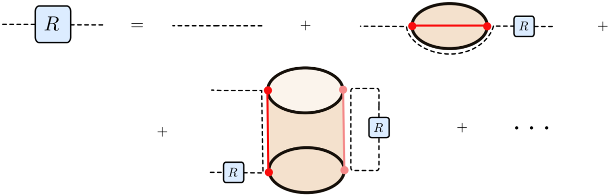
Compute resolvent using gravity path integral in semiclassical approx

$$\overline{R_{pq}} = \frac{1}{\lambda}\left(\delta_{pq} + \sum_{n=1}^{\infty}\frac{1}{\lambda^n}\overline{(G^n)_{pq}}\right)$$



Schwinger-Dyson equation

Rearranging the diagrams



We get a Schwinger-Dyson equation for $\overline{R_{pq}}$

$$\overline{R_{pq}} = \frac{1}{\lambda}\left(\delta_{pq} + \sum_{n=1}^{\infty} \frac{Z(n\beta)^2}{Z(\beta)^{2n}} \overline{R}^{n-1} \overline{R_{pq}}\right)$$

$$\lambda \overline{R} = \Omega + \sum_{n=1}^{\infty} \frac{Z(n\beta)^2}{Z(\beta)^{2n}} \overline{R}^n$$

Microcanonical projection

To solve the Schwinger-Dyson equation

$$\lambda \overline{R} = \Omega + \sum_{n=1}^{\infty} \frac{Z(n\beta)^2}{Z(\beta)^{2n}} \overline{R}^n$$

we invert the Laplace transform

$$Z(n\beta) = \int dE \, z(E) e^{-n\beta E}$$

and project to a microcanonical window $(E, E + \Delta E)$, defining

$$e^S \equiv z(E)\Delta E$$

we find

$$\lambda \overline{R} = \Omega + e^{2S} \sum_{n=1}^{\infty} \left(\frac{\overline{R}}{e^{2S}} \right) = \Omega + \frac{e^{2S}\overline{R}}{e^{2S} - \overline{R}}$$

Density of eigenvalues and rank of Gram matrix

Solving for $\overline{R}$ and using the definition of $D(\lambda)$ we find

$$\overline{D(\lambda)} = \frac{e^{2S}}{2\pi\lambda}\sqrt{\left(\lambda - \left(1-\Omega^{1/2}e^{-S}\right)^2\right)\left(\left(1+\Omega^{1/2}e^{-S}\right)^2 - \lambda\right)}$$

$$+\delta(\lambda)\left(\Omega - e^{2S}\right)\theta\left(\Omega - e^{2S}\right)$$

Density of eigenvalues and rank of Gram matrix

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$$+\delta(\lambda)\left(\Omega - e^{2S}\right)\theta\left(\Omega - e^{2S}\right)$$

- Has a continuous part for $\left(1 - \Omega^{1/2}e^{-S}\right)^2 < \lambda < \left(1 + \Omega^{1/2}e^{-S}\right)^2$
- For $\Omega > e^{2S}$, there is also a singular part at $\lambda = 0$, indicating degeneracy of the Gram matrix

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$$\overline{D(\lambda)} = \frac{e^{2S}}{2\pi\lambda}\sqrt{\left(\lambda - \left(1 - \Omega^{1/2}e^{-S}\right)^2\right)\left(\left(1 + \Omega^{1/2}e^{-S}\right)^2 - \lambda\right)}$$
$$+\delta(\lambda)\left(\Omega - e^{2S}\right)\theta\left(\Omega - e^{2S}\right)$$

- Has a continuous part for $\left(1 - \Omega^{1/2}e^{-S}\right)^2 < \lambda < \left(1 + \Omega^{1/2}e^{-S}\right)^2$
- For $\Omega > e^{2S}$, there is also a singular part at $\lambda = 0$, indicating degeneracy of the Gram matrix
- Rank of Gram matrix can be computed by integrating the density of non-zero eigenstates

$$\mathrm{rank}\, G_{pq} = \begin{cases} \Omega & \text{for} \quad \Omega < e^{2S}\,, \\ e^{2S} & \text{for} \quad \Omega > e^{2S}\,. \end{cases}$$

Dimension of $\mathcal{H}_E$ and Bekenstein-Hawking entropy

The maximal rank of the Gram matrix indicate that the dimension of the microcanonical subspace $\mathcal{H}_E$ is

$$\dim \mathcal{H}_E = e^{2S}$$

The actual value of S can be computed by evaluating the on-shell action on the thermal disk, and gives the Bekenstein-Hawking entropy of the black hole of mass E

$$S = S_{BH} = \frac{A}{4G}$$

The factor of 2 in the dimension of $\mathcal{H}_E$ is from working on a double copy of the CFT

$$\mathcal{H}_E = \mathcal{H}_E^{CFT_L} \otimes \mathcal{H}_E^{CFT_R}, \quad \dim \mathcal{H}_E^{CFT_{L,R}} = e^S$$

Extending to different E_L, E_R

Microcanonical projection

If we study states with $E_L < E_R$, the inverse Laplace transform reads

$$z(E_L, E_R) = \int d\beta_0 d\beta_1 Z(\beta_L, \beta_R) e^{\beta_L E_L + \beta_R E_R}$$

For a fixed shell mass m, this includes both inside and outside shell states

$$\frac{M_R(\beta_R) - M_L(\beta_L)}{m} - \frac{4\pi G m}{(d-1) V_\Omega R_*^{d-2}}$$

Infalling shell/white hole states are naturally included in the microcanonical Hilbert space

Different E_L and E_R

Density of eigenvalues and rank of Gram matrix

Fix E_L and E_R, expand in internal shells with mass nm_0 with m_0 large

Identical results for $\overline{D(\lambda)}$ and $\dim\mathcal{H}_E$, but with $2S \to S_L + S_R$

We then have

$$\dim\mathcal{H}_E = e^{S_L+S_R}$$

The different entropies count the dimension of each microcanonical subspace of $\mathcal{H}_{E_{L,R}}^{CFT_{L,R}}$ at energies $E_{L,R}$

Exact details of list of states is unimportant

Outline

Hilbert space factorization puzzle

In AdS/CFT, consider a double copy of the holgraphic CFT

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Is obviously a product of factors $\mathcal{H}_{L,R}$

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Addressed in JT gravity in [Boruch, Iliesiu, Lin and Yan]. Here we discuss the higher dimensional setting

Factorization via semiclassical microstates

Following [Boruch, Iliesiu, Lin, Yan], define the auxiliary Hilbert space

$$\mathcal{H}_\Omega = \mathrm{Span}\left\{ |\Psi_n\rangle \,, \quad n \in 1, 2, \cdots, \Omega \right\}$$

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Compute

$$\mathrm{Tr}_{\mathcal{H}_\Omega}\left(k_L k_R\right) = \left(G^{-1}\right)_{ij}\langle\Psi_i|k_L k_R|\Psi_j\rangle\,, \quad G_{ij} = \langle\Psi_i|\Psi_j\rangle$$

For $k_{L,R} \in \mathcal{A}_{\mathcal{H}_{L,R}}$, and show that, for large Ω

$$\mathrm{Tr}_{\mathcal{H}_\Omega}\left(k_L k_R\right) = \mathrm{Tr}_{\mathcal{H}_L}(k_L)\mathrm{Tr}_{\mathcal{H}_R}(k_R)$$

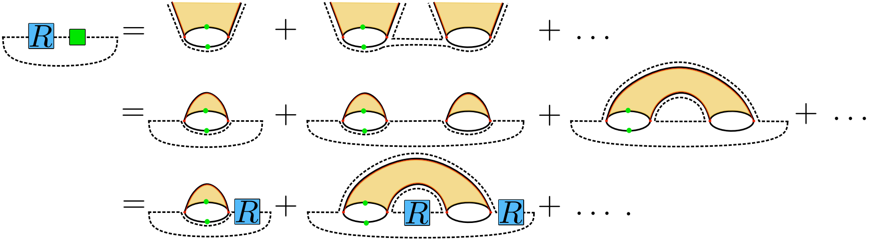
Hilbert space factorization

The computation can be done by analytic continuation of

$$\overline{(G^n)_{ij}\langle\Psi_i|k_L k_R|\Psi_j\rangle} = \oint \frac{d\lambda}{2\pi i}\lambda^n \overline{R_{ij}(\lambda)\langle\Psi_i|k_L k_R|\Psi_j\rangle}$$

to $n \to -1$

Semiclassical approximation



Hilbert space factorization

In general this involves 2 point correlation functions in n-boundary wormhole geometries with shells of matter

In the pinching limit (large m_0), the geometries pinch off and we find microcanonical one point functions $k_{L,R}^{E_{L,R}}$

$$\overline{\mathrm{Tr}_{\mathcal{H}_\Omega}(k_L k_R)} = \oint \frac{d\lambda}{2\pi i} dE_L dE_R \frac{k_L^{E_L} k_R^{E_R}}{\lambda} \frac{R(\lambda) e^{S_L+S_R}}{e^{S_L+S_R} - R(\lambda)}$$

$R(\lambda)$ has a pole at $\lambda = 0$ when there are null states in the list $\{|\Psi_n\rangle\}_{n=1}^{\Omega}$
When $\Omega > \mathrm{dim}\mathcal{H}_{\mathrm{bulk}}$

$$\overline{\mathrm{Tr}_{\mathcal{H}_\Omega}(k_L k_R)} = \overline{\mathrm{Tr}_{\mathcal{H}_L}(k_L)}\; \overline{\mathrm{Tr}_{\mathcal{H}_R}(k_R)}$$

Valid for operators $k_{L,R}$ of dimension much less than m_0

Conclusion

Recap

- Semiclassical black hole microstates
- Small overlaps estimated by wormhole contributions
- Correct Hilbert space dimension
- Null states
- Factorization of bulk Hilbert space at leading order
- Generality

Thank you

Thanks!