Exploring thermal black holes in a precise AdS_5/CFT_4 setup

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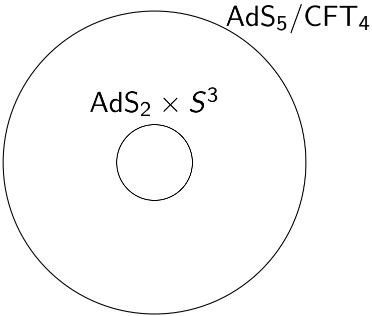
What does AdS_5/CFT_4 know about thermal black holes?



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- Huge list of successful checks: susy indices, HD corrections, correlations functions...

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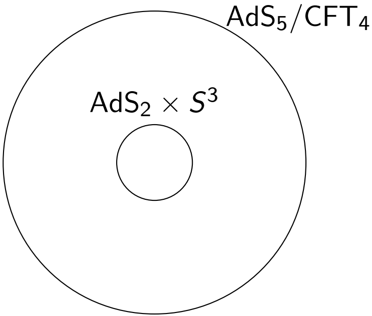

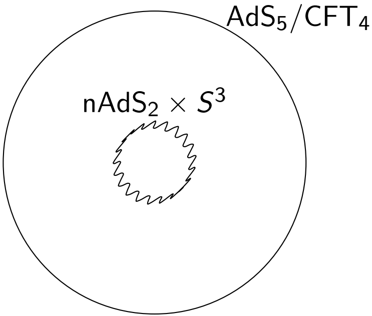

- Susy protects observables between strong and weak coupling
- Huge list of successful checks: susy indices, HD corrections, correlations functions...

- Hard problem: no protection between strong and weak coupling
- Idea: despite that (maybe) near the BPS locus we can retain calculational control

An explicit thermal black hole in 5d supergravity

[Chong, Cvetic, Lu, Pope, 2005]

$$\text{asymptotics: } AdS_5 \times S^5 \longrightarrow \text{near-horizon: } \Big(\underbrace{S^3}_{J_1,J_2} \times_{(\Omega_1,\Omega_2)} \mathbb{R}^2 \Big) \times \underbrace{S^5}_{Q_1,Q_2,Q_3=Q}$$

	charge	BH parameter	conjugate potential	
energy	E	r_+	β	temperature^{-1}
R-charge	Q	q	Φ	electrostatic pot.
ang. mom.	J_1	a	Ω_1	ang. vel.
ang. mom.	J_2	b	Ω_2	ang. vel.
entropy	$S(E,Q,J_i)$		$I(\beta,\Phi,\Omega_i)$	on-shell action

Example: $$\beta(a,b,r_+,q) = \frac{2\pi r_+ \left[(r_+^2+a^2)(r_+^2+b^2)+abq\right]}{r_+^4\left[2r_+^2+a^2+b^2+1\right]-(ab+q)^2}$$

Previously: "susy first, extremal later"

[Cabo-Bizet, Cassani, Martelli, Murthy, 2018]

non-extremal

susy BH:

$$I = \frac{2\pi}{27\,G_5}\,\frac{\varphi^3}{\omega_1\omega_2}$$

$$2\pi\mathrm{i} = \omega_1 + \omega_2 - 2\varphi$$

complex charges!

$$\Delta := E - J_1 - J_2 - \tfrac{3}{2}Q = 0$$

full BH

$$q = -ab + r_+^2(1+a+b) + \mathrm{i}\, r_+(r_+^2 - r_\star^2)$$

$\beta \to \infty$

$$r_+ \to r_\star := \sqrt{a_\star + b_\star + a_\star b_\star}$$

$$\omega_i = \beta(\Omega_i - 1)$$

$$\varphi = \beta\left(\Phi - \tfrac{3}{2}\right)$$

BPS BH:

$$I = \frac{2\pi}{27\,G_5}\,\frac{(\varphi^\star)^3}{\omega_1^\star\omega_2^\star}$$

$$2\pi\mathrm{i} = \omega_1^\star + \omega_2^\star - 2\varphi^\star$$

real charges

non-susy

There are infinitely many ways to reach the BPS locus

- Expand the parameters

$$q(a_\star, b_\star, T, \epsilon) = q_\star(a_\star, b_\star) + q_{0,1}(a_\star, b_\star)\, T + q_{1,0}(a_\star, b_\star)\, \epsilon + \mathcal{O}\left(s^2\right), \quad s := \{T, \epsilon\}$$

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- Demand $T \sim$ physical temperature and $\epsilon \sim$ susy deviation

$$\beta^{-1} = T + \mathcal{O}\big(s^3\big) \qquad 1 + \Omega_1 + \Omega_2 - 2\Phi = 2\pi\mathrm{i}\, T + \epsilon + \mathcal{O}\big(s^3\big) \qquad \operatorname{Im} S = \mathcal{O}\big(s^2\big)$$

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- Result:

$$S = S_\star - \frac{4\pi^2}{M} T + \mathcal{O}\big(s^2\big) \qquad I = I_\star + \frac{2}{M} x + \mathcal{O}(s) \qquad x := \frac{\epsilon}{T}$$

Notice what gets modified at which order

- The "balancing condition" gets modified at first order

$$1 + \Omega_1 + \Omega_2 - 2\Phi = 2\pi\mathrm{i}\, T \quad \longrightarrow \quad 1 + \Omega_1 + \Omega_2 - 2\Phi = 2\pi\mathrm{i}\, T + \epsilon + \mathcal{O}\left(s^2\right)$$

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- Equivalently from the expansion we get

$$\begin{aligned}\omega_i&=\omega_i^\star(a_\star,b_\star)+\sigma_i^\star(a_\star,b_\star)\,x+\mathcal{O}(s)\\ \varphi&=\varphi^\star(a_\star,b_\star)+\phi^\star(a_\star,b_\star)\,x+\mathcal{O}(s)\end{aligned} \quad\Longrightarrow\quad \begin{aligned}\omega_1^\star+\omega_2^\star-2\varphi^\star&=2\pi\mathrm{i}\\ \sigma_1^\star+\sigma_2^\star-2\phi^\star&=1\end{aligned}$$

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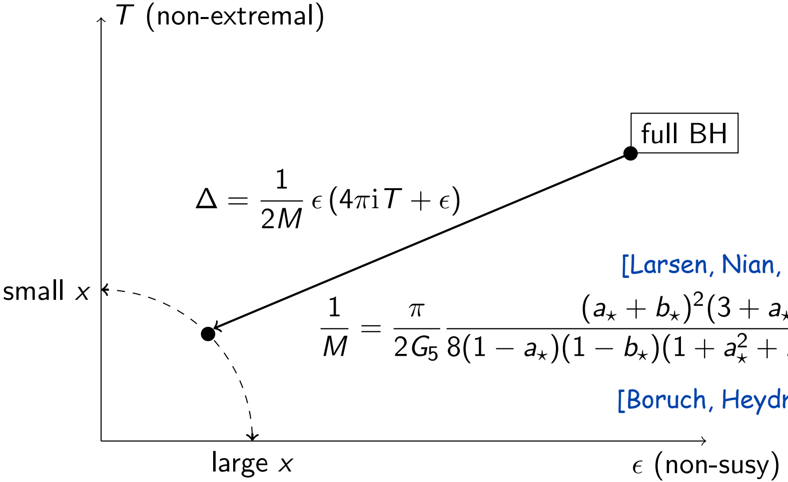
- The BPS bound gets un-saturated at second order

$$\Delta:=E-J_1-J_2-\frac{3}{2}Q=\frac{1}{2M}\,\epsilon\,(4\pi\mathrm{i}\,T+\epsilon)+\mathcal{O}\left(s^3\right)$$

Once near the BPS point tuning $x = \epsilon/T$ we get closer to either susy or extremality



$$\Delta = \frac{1}{2M}\,\epsilon\,(4\pi \mathrm{i}\, T + \epsilon)$$

[Larsen, Nian, Zeng, 2019]

$$\frac{1}{M} = \frac{\pi}{2G_5}\frac{(a_\star + b_\star)^2(3 + a_\star + b_\star - a_\star b_\star)}{8(1-a_\star)(1-b_\star)(1 + a_\star^2 + b_\star^2 + 3a_\star b_\star + 3(a_\star + b_\star))}$$

[Boruch, Heydman, Iliesiu, Turiaci, 2022]

Useful rewritings of the on-shell action $I = I_\star + 2M^{-1}\, x$

- Instead of $(a_\star, b_\star)$, we can express the on-shell action in terms of $(\omega_i^\star, \sigma_i^\star)$

$$I = \frac{2\pi}{27G_5}\left[\frac{(\varphi^\star)^3}{\omega_1^\star\omega_2^\star} + \left(\frac{(\phi^\star)^3}{\sigma_1^\star\sigma_2^\star} + \frac{9\big(1-2(\sigma_1^\star+\sigma_2^\star)+4\big((\sigma_1^\star)^2-\sigma_1^\star\sigma_2^\star+(\sigma_2^\star)^2\big)\big)}{64\sigma_1^\star\sigma_2^\star}\right)x\right]$$

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- It is not quite possible to write it in terms of what will eventually become field theory fugacities $\omega_i = \omega_i^\star + \sigma_i^\star\,x$ and $\varphi = \frac{1}{2}(\omega_1+\omega_2-2\pi\mathrm{i}-x)$, but we get close

$$I = \frac{2\pi}{27G_5}\left[\frac{\varphi^3}{\omega_1\omega_2} + \frac{9x\big(x^2-2x(\omega_1+\omega_2+g_2(a_\star,b_\star))+4\big(\omega_1^2-\omega_1\omega_2+\omega_2^2+g_1(a_\star,b_\star)\big)\big)}{64\omega_1\omega_2}\right]$$

Classical statements about the partition function, Casimir energy and index

- Classically, we can approximate the partition function as $Z \approx e^{I}$. We also split Z into Casimir energy (E_0) and "index" ($\mathcal{I}$) contributions

$$Z = \mathrm{e}^{-\beta E_0}\mathcal{I} \quad \Longrightarrow \quad I \approx -\beta E_0 + \log \mathcal{I}$$

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$$\beta E_0 = \frac{2\pi}{27 G_5}\left[-\frac{(\omega_1+\omega_2)^3}{8\omega_1\omega_2} - \frac{x\left(x^2+6x(\omega_1+\omega_2)+12(\omega_1^2-7\omega_1\omega_2+\omega_2^2)\right)}{64\omega_1\omega_2}\right]$$

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- While the "index" contains the functions $g_{1,2}(a_\star, b_\star)$

$$\log \mathcal{I} = \frac{2\pi}{27 G_5}\left[\frac{\pi\mathrm{i}\left(4\pi^2 + 6\pi\mathrm{i}(\omega_1+\omega_2-x) - 3(\omega_1+\omega_2-x)^2\right)}{4\omega_1\omega_2} - \frac{9x(g_2 x - 2g_1)}{32\omega_1\omega_2}\right]$$

The near-horizon geometry of the near-BPS black hole

- Reached by **simultaneously** bringing the outer and inner horizons together ($T \to 0$) and driving an observer towards this point ($r \to r_+$). Throwing ϵ in the game

$$t = \frac{\widetilde{t}}{2\pi T} \quad r = r_+(\epsilon, T) + 2\pi T\, c(a_\star, b_\star)\,(\widetilde{r} - 1) \quad \text{then} \quad (\epsilon, T) \to 0$$

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- The resulting metric is

$$\mathrm{d}s^2_{\text{nH-nBPS}} = f_1(\widetilde{r}, \epsilon, T) \underbrace{\left(-\left[(\widetilde{r}^2 - 1) + \mathcal{O}(s) \right] \mathrm{d}\widetilde{t}^2 + \frac{1}{\widetilde{r}^2 - 1}\, \mathrm{d}\widetilde{r}^2 \right)}_{\text{nAdS}_2} \times_{\Omega_1, \Omega_2} f_2(\widetilde{r}, \epsilon, T)\, \mathrm{d}s^2_{S^3}$$

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- The equations of motion and Killing spinor variations of the 5d sugra hold as

$$E_{\mu\nu} = \mathcal{O}\left(s^2\right) \qquad \delta_{\text{susy}} \psi_\mu = \mathcal{O}\left(s^2\right)$$

Susy breaking in the holographic dual

The 4d background, with $n_i = (\sin\theta, \cos\theta)$

$$\mathrm{d}s_4^2 = \mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sum\nolimits_i n_i^2 (\mathrm{d}\phi_i - \mathrm{i}\Omega_i\,\mathrm{d}\tau)^2\,, \quad A = \mathrm{i}\left(\Phi - \frac{3}{2}\right)\mathrm{d}\tau\ , \quad V = -\mathrm{i}\,\mathrm{d}\tau$$

locally solves the Killing spinor equation

$$\left(\nabla_M - \mathrm{i}A_M + \mathrm{i}V_M + \mathrm{i}V^N\sigma_{MN}\right)\zeta = 0$$

for **any** value of ϵ. However, supersymmetry is broken by the amended boundary condition

$$\zeta(\tau+\beta) = \mathrm{e}^{\pi\mathrm{i}+\beta\epsilon/2}\zeta(\tau)$$

The 4d background is a Hopf surface with complex "twisting" parameters

The background is a Hopf surface of the form $S^3 \times_{\Omega_1,\Omega_2} S^1$

$$\mathrm{d}s_4^2 = \Omega(\theta)^2(\mathrm{d}\tau + c)^2 + \underbrace{\mathrm{d}\theta^2 + \sum\nolimits_i n_i^2\,\mathrm{d}\phi_i^2 - \Omega(\theta)^2 c^2}_{\mathrm{d}s_3^2},$$

$$\Omega(\theta)^2 = 1 - \sum\nolimits_i n_i^2\Omega_i^2\,, \quad c = -\frac{\mathrm{i}}{\Omega(\theta)^2}\sum\nolimits_i n_i^2\Omega_i\,\mathrm{d}\phi_i$$

with complex Killing vector

$$K = \zeta\sigma^M\widetilde{\zeta}\partial_M = \frac{1}{2}\Big[\sum\nolimits_i(\Omega_i - 1)\partial_{\phi_i} - \mathrm{i}\partial_\tau\Big]$$

In the Cardy limit size$(S^1) \ll$ size(S^3) one obtains an effective 3d CS theory

[Assel, Cassani, Martelli, 2014]

(1) Relate the 4d background (A, V, ω) to the 3d background fields $(\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{V}, \boldsymbol{\omega})$

(2) Evaluate the classical building blocks (supersymmetrized CS actions)

$$I_1 = \frac{\mathrm{i}}{4\pi}\int_{S^3} \boldsymbol{c}\wedge \mathrm{d}\boldsymbol{c} \quad I_2 = \frac{\mathrm{i}}{4\pi}\int_{S^3} \widetilde{\boldsymbol{A}}\wedge \mathrm{d}\boldsymbol{c} \quad I_3 = \frac{\mathrm{i}}{4\pi}\int_{S^3} \widetilde{\boldsymbol{A}}\wedge \mathrm{d}\widetilde{\boldsymbol{A}} \quad I_4 = \frac{\mathrm{i}}{192\pi}\int_{S^3} \boldsymbol{\omega}\wedge \mathrm{d}\boldsymbol{\omega}$$

(3) Determine their coefficients by integrating a tower of massive KK modes, involving sums like (note the imprint of the susy breaking in the 3d theory)

$$\mathrm{sum}^{(k)} = \sum\nolimits_n (m_n)^k\, \mathrm{sgn}(m_n)\,, \quad m_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\left(1 + \frac{\beta\epsilon}{2\pi\mathrm{i}}\right) r - (\rho\cdot u)\right)$$

[Di Pietro, Komargodski, 2014] [Di Pietro, Honda, 2015] [Ardehali, Murthy, 2021]

(4) Sum over all 4d fermion towers and evaluate the classical 3d CS action (I_{CS}) on the dominant saddle $u = 0$

Steps 1 & 2: The 3d background has no explicit dependence on the susy breaking parameter ϵ

- The matching is performed by ensuring that the reduced 4d susy variations coincide with the 3d susy variations

$$\Big(\nabla_M - \mathrm{i}A_M + \mathrm{i}V_M + \mathrm{i}V^N\sigma_{MN}\Big)\zeta \quad \begin{cases} \big[\nabla_\mu - \mathrm{i}(\boldsymbol{A}_\mu - \boldsymbol{V}_\mu) + \frac{1}{2}\varepsilon_{\mu\nu\rho}\boldsymbol{V}^\nu\gamma^\rho + \frac{1}{2}\boldsymbol{H}\gamma_\mu\big]\eta \\ \Big[\Big((-\mathrm{i}\star\boldsymbol{c})_\mu - \mathrm{i}\partial_\mu\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{V}_\mu\Big)\gamma^\mu + \mathrm{i}(\boldsymbol{D} + \boldsymbol{\sigma}\boldsymbol{H})\Big]\eta \end{cases}$$

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- In previous analysis the 4d gauge field was assumed real. Lifting this assumption and taking into account the susy breaking BC's

$$\boldsymbol{A} = -\frac{\mathrm{i}}{2}\left[\left(\Omega_1 + \Omega_2 - 2 - \cancel{\left(\frac{2\pi\mathrm{i}}{\beta} + \epsilon\right)} + \cancel{\left(\frac{2\pi\mathrm{i}}{\beta} + \epsilon\right)}\right)c + \Omega\star c\right], \quad \boldsymbol{c} = c$$

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- Thus the CS actions evaluate as in [Cassani, Komargodski, 2021]

$$I_1 = -\frac{\mathrm{i}\pi}{\omega_1\omega_2}\beta^2 \quad I_2 = -\frac{\pi(\omega_1+\omega_2)}{\omega_1\omega_2}\beta \quad I_3 = \frac{\mathrm{i}\pi(\omega_1+\omega_2)^2}{4\omega_1\omega_2} \quad I_4 = \frac{\mathrm{i}\pi(\omega_1-\omega_2)^2}{48\omega_1\omega_2}$$

Steps 3 & 4: Due to the susy breaking the “real mass” is complex

- The sum over KK towers involve $\mathrm{sgn}(z \in \mathbb{C})$, we attempt a simple extension of the sgn function

$$\mathrm{sgn}(m_n) = \mathrm{sgn}(\mathrm{Re}\, m_n) \qquad m_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\left(1 + \frac{x}{2\pi\mathrm{i}}\right) r - (\rho \cdot u)\right)$$

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- With this the final classical CS action is

$$I_{\mathrm{CS}} = \frac{\mathrm{Tr}\, R^3}{6} \frac{\pi \mathrm{i}\left[4\pi^2 + 6\pi \mathrm{i}(\omega_1 + \omega_2 - x) - 3(\omega_1 + \omega_2 - x)^2\right]}{4\omega_1\omega_2} + \underbrace{\cdots}_{\text{subleading in } N}$$

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- Identify $\log \mathcal{I} = I_{\mathrm{CS}}$ and **import from gravity** the "field theory like" result for the corrected Casimir energy, then

$$\log Z_{\mathrm{QFT}} = \frac{\mathrm{Tr}\, R^3}{6}\left[\frac{\varphi^3}{\omega_1\omega_2} + \frac{9x\left(x^2 - 2x(\omega_1 + \omega_2) + 4\left(\omega_1^2 - \omega_1\omega_2 + \omega_2^2\right)\right)}{64\omega_1\omega_2}\right]$$

Comparison between field theory and gravity

$$\log Z_{\mathrm{QFT}} = \frac{\mathrm{Tr}\, R^3}{6}\left[\frac{\varphi^3}{\omega_1\omega_2} + \frac{9x\big(x^2 - 2x(\omega_1+\omega_2) + 4\big(\omega_1^2 - \omega_1\omega_2 + \omega_2^2\big)\big)}{64\omega_1\omega_2}\right]$$

$$\frac{\mathrm{Tr}\, R^3}{6} = \frac{2\pi}{27G_5}$$

$$I_{\mathrm{grav}} = \frac{2\pi}{27G_5}\left[\frac{\varphi^3}{\omega_1\omega_2} + \frac{9x\big(x^2 - 2x(\omega_1+\omega_2+g_2) + 4\big(\omega_1^2 - \omega_1\omega_2 + \omega_2^2 + g_1\big)\big)}{64\omega_1\omega_2}\right]$$

We have a match for small and for large $x = \epsilon/T$

- Using the relations $\omega_i = \omega_i^\star + \sigma_i^\star x$, for small x ($\epsilon \ll T$ aka closer to the susy locus):

$$\log Z_{\rm QFT} = \frac{\mathrm{Tr}\, R^3}{6}\left[\frac{(\omega_1^\star + \omega_2^\star - 2\pi\mathrm{i})^3}{8\omega_1^\star\omega_2^\star} + \mathcal{O}(x)\right] \qquad I_{\rm grav} = \frac{2\pi}{27 G_5}\left[\frac{(\omega_1^\star + \omega_2^\star - 2\pi\mathrm{i})^3}{8\omega_1^\star\omega_2^\star} + \mathcal{O}(x)\right]$$

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where M is the Schwarzian mass scale

$$\frac{1}{M} = \frac{\pi}{27 G_5}\left[\frac{(\sigma_1^\star + \sigma_2^\star - 1)^3}{8\sigma_1^\star\sigma_2^\star} + \frac{9\big(1 - 2(\sigma_1^\star + \sigma_2^\star) + 4\big((\sigma_1^\star)^2 - \sigma_1^\star\sigma_2^\star + (\sigma_2^\star)^2\big)\big)}{64\sigma_1^\star\sigma_2^\star}\right]$$

Conclusions and open problems

- Need an independent solely QFT calculation of the Casimir energy βE_0 when $\epsilon \neq 0$

- Need to re-derive the coefficients of the CS actions from scratch
  - Is $\mathrm{sgn}(z) = \mathrm{sgn}(\mathrm{Re}\, z)$ legit?
  - Note: any change in them would not affect the $x \to \infty$ and $x \to 0$ analysis
  - Freedom to choose renormalization scheme of the on-shell action in gravity?

- Should we even hope for a precise match on the classical level?
  - Gravity: $T \to 0$ vs. field theory: $\beta = \frac{1}{T} \to 0$
  - A priory nothing is protected when $\epsilon \neq 0$, nevertheless we see indications that something is protected when ~~susy~~ but extremal

- Relation to recent work of [Cabo-Bizet, 2024]?

Thank you!