Equivariant Localization in Supergravity

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based on [2407.02554] and other works
with P. Benetti Genolini, J.P. Gauntlett, Y. Jiao, and J. Sparks

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Motivation

How to compute SUGRA observables without really trying

$$I \sim \int_M \mathrm{vol}_M (R + \dots)$$

- Looks like we need to know M to perform the integral
- This would involve solving SUGRA EOM and/or KSE
  $\implies$ Difficult task!
- Is there a simpler way to get this result?

Introduction

Use *equivariant localization* to compute these integrals
[Benetti Genolini, Gauntlett, Sparks '23]

$$\text{Obs.} = \int_{M_n} \Phi_n = \sum_{\substack{\text{fixed}\\\text{points}}} \Phi_0$$

Rough idea

- Want to integrate *n*-form over *n*-dim spacetime
- Use fixed point formula to recast this integral as a sum
  - of contributions from lower degree forms
  - on some "special" subspaces

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[Benetti Genolini, Gauntlett, Sparks '23]

$$\text{Obs.} = \int_{M_n} \Phi_n = \sum_{\substack{\text{fixed}\\ \text{points}}} \Phi_0$$

Will make clear in a second

- How are the n- and lower forms related
- The precise form of the fixed point formula
- What I mean by "special" subspaces

Introduction

Use *equivariant localization* to compute these integrals
[Benetti Genolini, Gauntlett, Sparks '23]

$$\text{Obs.} = \int_{M_n} \Phi_n = \sum_{\substack{\text{fixed}\\ \text{points}}} \Phi_0$$

Remarks

- Don't need to know the explicit solution
- Depends only on the *topology*
- Can obtain results for seemingly very different theories in a uniform way

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Equivariant Cohomology

- U(1) action with Killing vector ξ
- Equivariant exterior derivative

$$\mathrm{d}_\xi \equiv \mathrm{d} - \xi \lrcorner$$

- Acts on polyforms

$$\Phi = \Phi_n + \Phi_{n-2} + \cdots + \Phi_0$$

- Φ is *equivariantly closed* if $\mathrm{d}_\xi \Phi = 0$

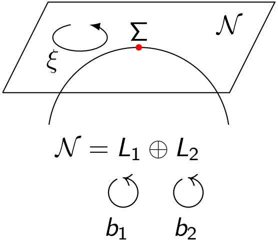
$$\mathrm{d}\Phi_n = 0 \qquad \xi \lrcorner \Phi_n = \mathrm{d}\Phi_{n-2} \qquad \ldots \qquad \xi \lrcorner \Phi_2 = \mathrm{d}\Phi_0$$

Equivariant Localization

BVAB Theorem: [Berline, Vergne '82, Atiyah, Bott '84] The integral of an equivariantly closed form *localizes* to fixed point contributions

$$\int_{M_n} \Phi = \sum_{\Sigma} \frac{(2\pi)^k}{\prod_{i=1}^k b_i} \int_{\Sigma} \frac{f^* \Phi}{[1 + \frac{2\pi}{b_i} c_1(L_i)]}$$

- $\Sigma \subset M_n$ fixed subsp. of codim $2k$: $\xi|_\Sigma = 0$
- f embedding: picks up $\Phi_{\dim\Sigma}$
- b_i weights of $U(1)$ action: $\xi = \sum_{i=1}^k b_i \partial_{\varphi_i}$
- $c_1(L_i)$ first Chern class of the line bundles



BVAB theorem in practice

Expand $1/[1+\frac{2\pi}{b_i}c_1(L_i)] = 1 - \frac{2\pi}{b_i}c_1(L_i) + \dots$

- For a 2d integral

$$\int_{M_2} \Phi_2 = \sum_{\Sigma_0} \frac{2\pi}{b_1} \Phi_0 \Big|_{\Sigma_0}$$

BVAB theorem in practice

Expand $1/[1 + \frac{2\pi}{b_i} c_1(L_i)] = 1 - \frac{2\pi}{b_i} c_1(L_i) + \dots$

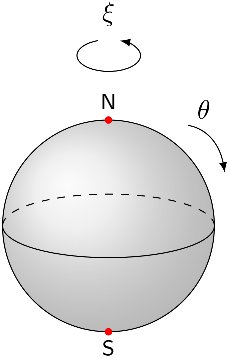
- For a 2d integral

$$\int_{M_2} \Phi_2 = \sum_{\Sigma_0} \frac{2\pi}{b_1} \Phi_0 \Big|_{\Sigma_0}$$

- For a 4d integral

$$\int_{M_4} \Phi_4 = \sum_{\Sigma_0} \frac{(2\pi)^2}{b_1 b_2} \Phi_0 \Big|_{\Sigma_0} + \sum_{\Sigma_2} \int_{\Sigma_2} \left[\frac{2\pi}{b_1} \Phi_2 - \frac{(2\pi)^2}{b_1^2} c_1(L) \Phi_0 \right]$$

Baby Example



$b_N = -b_S = b$

- Metric : $\mathrm{d}s^2 = d\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$

$$\mathrm{Vol} = \int_{S^2} \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi \equiv \int_{S^2} \Phi_2$$

- Killing vector: $\xi = b\partial_\varphi$

$$\xi \lrcorner \Phi_2 = -b\sin\theta\mathrm{d}\theta = \mathrm{d}(b\cos\theta) \equiv \mathrm{d}\Phi_0$$

- Localization: $\Phi = \Phi_2 + \Phi_0$ eq. closed

$$\mathrm{Vol} = \int_{S^2} \Phi = 2\pi\left(\frac{\Phi_0|_N}{b_N} + \frac{\Phi_0|_S}{b_S}\right)$$

$$= 2\pi\left(\frac{b\cos 0}{b} - \frac{b\cos\pi}{b}\right) = 4\pi$$

Supergravity Applications

Applications to supergravity [Benetti Genolini, Gauntlett, Sparks '23]

- SUSY implies existence of the U(1) symmetry
- Use KSE and bilinears to build equiv. closed forms → once
- Localize on various topologies → different sets of fixed points

$$\text{Obs.} = \int_{M_n} \Phi_n = \sum_{\substack{\text{fixed} \\ \text{points}}} \Phi_0$$

Bilinears

- Killing spinor ϵ with KSE $(D + \dots)\epsilon = 0$
- Bilinears follow constraints coming from KSE

$$S = \bar{\epsilon}\epsilon \quad P = \bar{\epsilon}\gamma_* \epsilon \quad K = \bar{\epsilon}\gamma_\mu \epsilon \mathrm{d}x^\mu \quad \xi^\flat = \bar{\epsilon}\gamma_\mu \gamma_* \epsilon \mathrm{d}x^\mu$$

- Killing vector ξ dual to $\xi^\flat$
- Tool to build equiv. closed forms e.g.

$$\mathrm{d}P = \xi \lrcorner F \implies \Phi = F - P \quad \text{is equiv. closed}$$

- Game: build such forms whose top form part is an obs.

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4d $\mathcal{N} = 2$ Gauged SUGRA

$$I = -\frac{1}{16\pi G_4}\int\Big[\big(R - 2\mathcal{G}_{i\tilde{j}}\partial^\mu z^i \partial_\mu \tilde{z}^{\tilde{j}} - \mathcal{V}(z,\tilde{z})\big)\mathrm{vol}_4 + \frac{1}{2}\mathcal{I}_{IJ}F^I \wedge *F^J - \frac{\mathrm{i}}{2}\mathcal{R}_{IJ}F^I \wedge F^J\Big]$$

- 4d gauged SUGRA coupled to n vector multiplets
- Everything specified by the prepotential $\mathcal{F}(z^i)$
- Euclidean theory: z and $\tilde{z}$ indep.

Equivariant Localization

- Action

$$I = \frac{\pi}{2G_4}\frac{1}{(2\pi)^2}\int_{M_4}\Phi_4$$

- Equivariantly closed form

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0 \qquad \mathrm{d}_\xi \Phi = 0$$

Equivariantly Closed Form

$$\Phi_4 \equiv -\frac{1}{2}\mathcal{V}\mathrm{vol}_4 - \frac{1}{4}\mathcal{I}_{IJ}F^I \wedge *F^J + \frac{\mathrm{i}}{4}\mathcal{R}_{IJ}F^I \wedge F^J$$

$$\Phi_2 \equiv \frac{1}{\sqrt{2}}\mathrm{e}^{\mathcal{K}/2}(WU_{[+]} + \widetilde{W}U_{[-]}) - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}(C^I F^J_{[+]} + \widetilde{C}^I F^J_{[-]}) + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{R}_{IJ}F^J(C^I - \widetilde{C}^I)$$

$$\Phi_0 \equiv \mathrm{i}\big[\mathcal{F}(C) - \mathcal{F}(\widetilde{C}) - \partial_I\mathcal{F}(C)\widetilde{C}^I + \partial_I\mathcal{F}(\widetilde{C})C^I\big]$$

Equivariant Localization

- Action

$$I = \frac{\pi}{2G_4} \frac{1}{(2\pi)^2} \int_{M_4} \Phi_4$$

- Equivariantly closed form

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0 \qquad \mathrm{d}_\xi \Phi = 0$$

- Equivariant localization

$$I = \frac{\pi}{2G_4} \left\{ \sum_{\text{nuts}} \frac{\Phi_0}{b_1 b_2} + \sum_{\text{bolts}} \int_\Sigma \frac{\Phi_2}{2\pi b} - \frac{\Phi_0 c_1(L)}{b^2} \right\}$$

$$+ \text{ boundary terms}$$

Main Result

Free energy of 4d gauged SUGRA coupled to vector matter

$$F = \frac{\pi}{G_4}\left[\sum_{\text{nuts}_\pm} \mp \frac{(b_1 \mp b_2)^2}{b_1 b_2} \mathrm{i}\mathcal{F}(u_\pm^J) + \sum_{\text{bolts}_\pm} \left(-\partial_I \mathrm{i}\mathcal{F}(u_\pm^J)\mathfrak{p}_\pm^I \pm \mathrm{i}\mathcal{F}(u_\pm^J) \int_{\Sigma_\pm} c_1(L) \right) \right]$$

- $\mathcal{F}$ general prepotential
- $u_\pm$ function of the scalar fields at the fixed points
- $\pm$ chirality of the spinor
- Don't need explicit solution!
- Pick topology and evaluate $\longrightarrow$ Examples

Examples

Free energy of 4d gauged SUGRA coupled to vector matter

$$F = \frac{\pi}{G_4}\left[\sum_{\mathrm{nuts}_\pm} \mp \frac{(b_1 \mp b_2)^2}{b_1 b_2} \mathrm{i}\mathcal{F}(u^J_\pm) + \sum_{\mathrm{bolts}_\pm}\left(-\partial_I \mathrm{i}\mathcal{F}(u^J_\pm)\mathfrak{p}^I_\pm \pm \mathrm{i}\mathcal{F}(u^J_\pm)\int_{\Sigma_\pm} c_1(L)\right)\right]$$

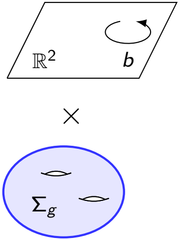
- Pick a prepotential e.g. STU $\to$ dual to ABJM

$$\mathcal{F}(X^I) = -2\mathrm{i}\sqrt{X^0 X^1 X^2 X^3}$$

- Pick a topology for M_4: nuts and/or bolts

Black Saddle

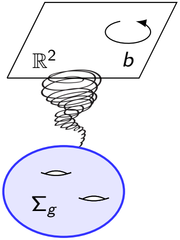
- "BH" sol [Bobev, Charles, Min '20]
- $M_4 = \mathbb{R}^2 \times \Sigma_g$
- Fixed point set: Σ_g
  $\rightarrow$ 1 bolt
- Field th. [Benini, Hristov, Zaffaroni '15]
- Two "branches" of solutions $\pm$



$$F = -\frac{\pi}{G_4}\sqrt{u_\pm^0 u_\pm^1 u_\pm^2 u_\pm^3}\sum_{I=0}^{3}\frac{\mathfrak{p}^I}{u_\pm^I}$$

Taub-Bolt Saddle

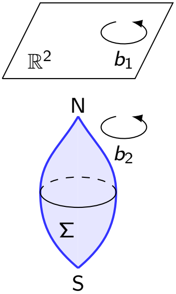
- New rot. "BH" result
- $M_4 = \mathcal{O}(-p) \to \Sigma_g$
- Fixed point set: Σ_g
  $\to$ 1 bolt & $\int c_1(L) = -p$
- Field th. [Toldo, Willett '17]
- Also expect two branches



$$F = -\frac{\pi}{G_4}\sqrt{u_\pm^0 u_\pm^1 u_\pm^2 u_\pm^3}\left(\sum_{I=0}^{3}\frac{\mathfrak{p}_\pm^I}{u_\pm^I} \pm 2p\right)$$

Spindle

- New[?!] acc. "BH" result
- $M_4 = \mathbb{R}^2 \times \Sigma$ with Σ a spindle
- Fixed point set: 2 poles of the spindle $\rightarrow$ 2 nuts
- Field th. [Colombo, Hosseini, Martelli, Pittelli, Zaffaroni '24]
- $\sigma = \pm 1$ twist, anti-twist



$$F = -\frac{2\pi}{G_4}\frac{1}{b_2}\left[\sqrt{y_N^0 y_N^1 y_N^2 y_N^3} - \sigma\sqrt{y_S^0 y_S^1 y_S^2 y_S^3}\right]$$

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Summary

- Used equivariant localization to obtain a general formula for the free energy of Euclidean 4d $\mathcal{N} = 2$ gauged SUGRA coupled to vector matter
- Can then pick a topology and recover known results in a uniform and straightforward way and/or obtain new results where the explicit solution is unknown

Summary

Take home message

Equivariant localization is a powerful tool to extract SUGRA observables without knowing the explicit solution.

- Hard work is to build the equivariantly closed forms
- Then straightforward to obtain results for various families of solution by picking the topology

Outlook

Other applications in supergravity

- BH near horizon geometry
  [Benetti Genolini, Gauntlett, Jiao, AL, Sparks '24][Suh '24]
- Localization on the internal space
  [Benetti Genolini, Gauntlett, Sparks '23][Couzens, AL '24]
- Another approach to geometrical localization
  [Colombo, Faedo, [Martelli, Zaffaroni '23] '24]

Future directions

- No SUSY
- Odd dimension
- Higher derivative corrections $\rightarrow$ beyond large N

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THANK YOU

Scalar field variables

- z^i "physical" fields
- $X^I = X^I(z^i)$ holomorphic
- $C^I \equiv \mathrm{e}^{\mathcal{K}/2} X^I (S - P)$, $\widetilde{C}^I \equiv \mathrm{e}^{\mathcal{K}/2} \widetilde{X}^I (S + P)$
- $u^I_+ \equiv \left.\frac{\widetilde{X}^I}{\zeta_J \widetilde{X}^J}\right|_+ , \quad u^I_- \equiv \left.\frac{X^I}{\zeta_J X^J}\right|_-$
- $y^I_\pm = \pm(1 \mp b_1/b_2) u^I_\pm$