# Trace Anomalies, RG Flow, and Scattering Amplitudes

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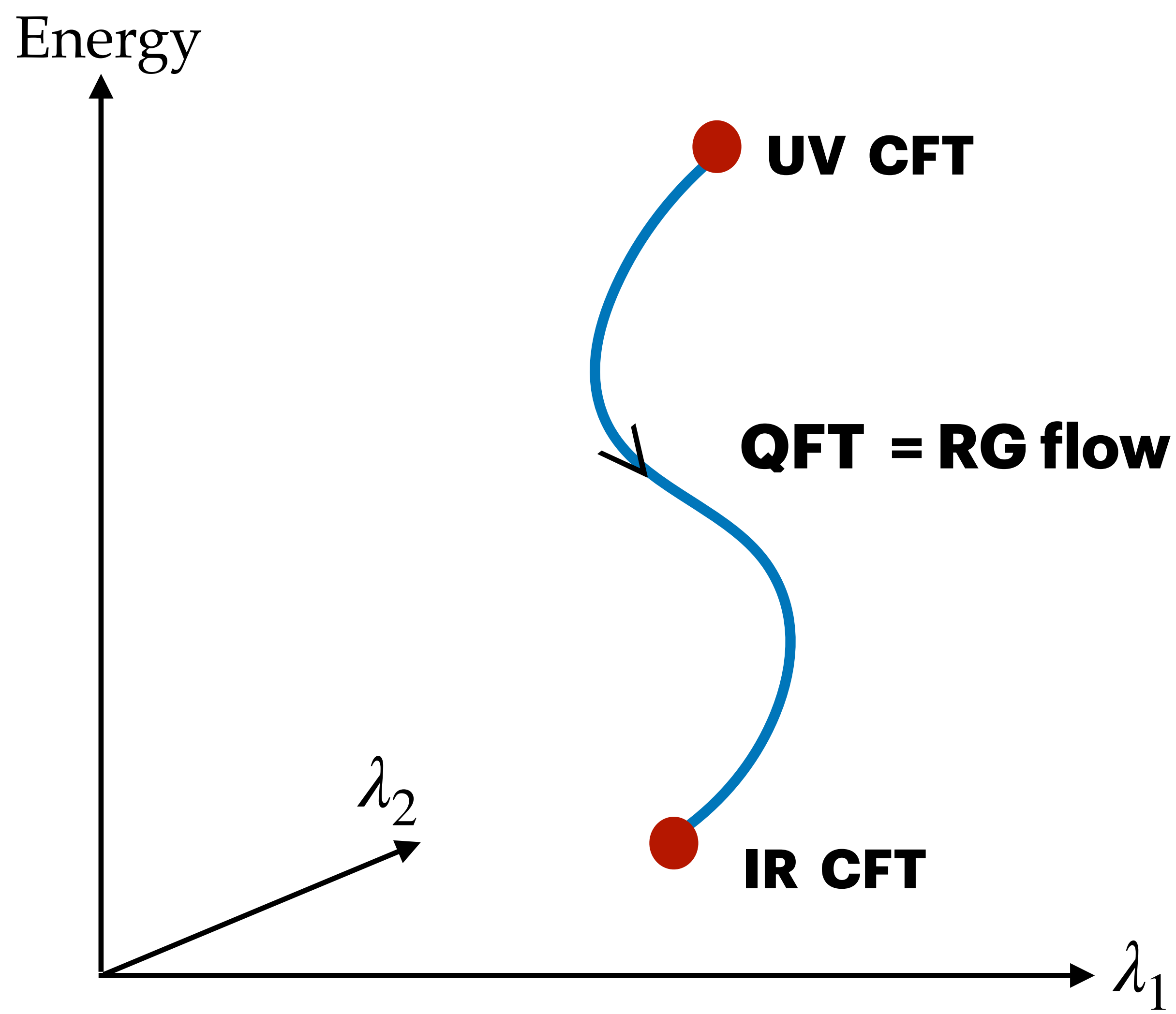
# Based on the papers

(1) arXiv: 2204.01786: Bootstrapping the a-anomaly in 4d QFTs, with **Denis Karateev , Jan Marucha** & **Joao Penedones.**

(2) arXiv: 2312.09308: Trace Anomalies and the Graviton-Dilaton Amplitude, with **Denis Karateev , Zohar Komargodski** & **Joao Penedones.**

(3) arXiv: appeared today: Correlation Functions and Trace Anomalies in Weakly Relevant Flows, with **Denis Karateev.**

**Quantum Field Theories (QFTs) can be non-perturbatively defined as a renormalization group flow between UV fixed point and IR fixed point which are assumed to enjoy conformal symmetry.**




**To specify a particular QFT it is sufficient to provide the UV CFT data, and**

**(1) the relevant deformation triggering the RG flow in the explicit conformal symmetry breaking case,**

**OR**

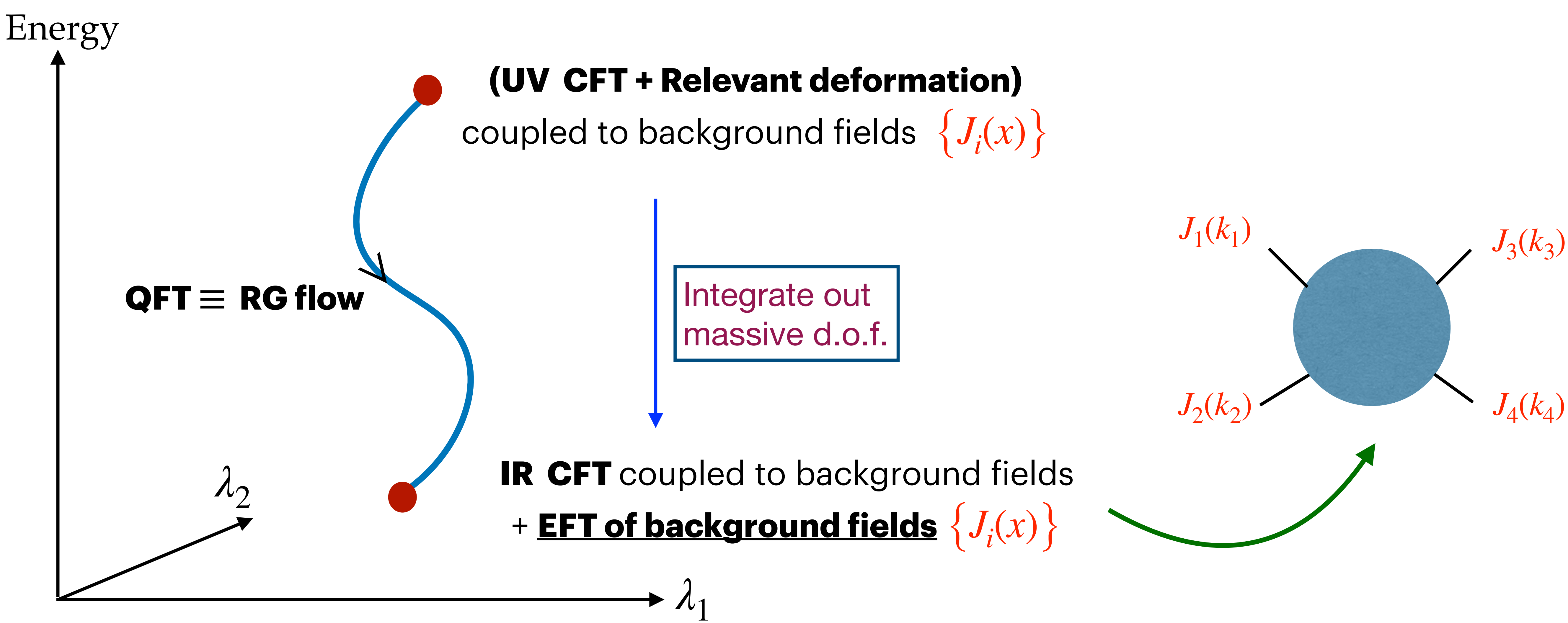
**(2) the VEV of the scalar primary operator in the spontaneous symmetry breaking case.**

# Background field method

# And

# Trace anomaly matching

# Setup and Questions like to answer




1. Can we identify a set of observables involving $\{J_i(x)\}$, which are determined **only by the UV CFT and IR CFT data**, and do not depend on the details of the RG flow?

2. Using **non-perturbative S-matrix bootstrap program** on those observables, can we provide non-trivial bounds on the UV CFT data, under simple assumption on low lying spectrum of the QFT?

# Two sources and anomaly "matching" in 4d

$J_1(x): \ g_{\mu\nu}(x) \Longrightarrow$ **Conformally couple the UV CFT to a non-dynamical curved background**

$J_2(x): \ \Omega(x) = e^{-\tau(x)} \Longrightarrow$ **Rescale all the mass parameters by $\Omega(x)$ to restore explicit conformal symmetry breaking:** $M_i \to M_i(x) \equiv \Omega(x) M_i$

$$g_{\mu\nu}(x) \to e^{2\sigma(x)} g_{\mu\nu}(x)$$

$$\Omega(x) \to e^{-\sigma(x)} \Omega(x)$$

# Two sources and anomaly "matching" in 4d

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$g_{\mu\nu}(x) \to e^{2\sigma(x)} g_{\mu\nu}(x)$

$\Omega(x) \to e^{-\sigma(x)} \Omega(x)$

*Then at any point on the RG flow the Weyl symmetry breaking of the QFT is only due to trace anomaly:*

$$\delta_\sigma W[g_{\mu\nu}, \Omega] = \int d^4x \sqrt{-g}\,\sigma(x)\left(-a_{UV} \times E_4 + c_{UV} \times \mathscr{W}^2\right)$$

- $W[g_{\mu\nu}, \Omega]$: Connected functional
- $\sigma(x)$: infinitesimal Weyl transformation parameter
- $E_4$: Euler density
- $\mathscr{W}^2$: Square of Weyl tensor

# Trace anomaly matching and EFT at IR fixed point

$$A_{IR}[\Phi, g_{\mu\nu}, \tau] = A_{IR\ CFT}[\Phi, g_{\mu\nu}] + A_{EFT}[\tau, g_{\mu\nu}] + \sum_{1\leq\Delta\leq 2} \lambda_\Delta \int d^4x\sqrt{-\hat{g}}\ M^{2-\Delta}R(\hat{g})\ \widehat{\mathbf{O}}_\Delta(x)$$

$$+ \quad \text{irrelevant terms}$$

$$\hat{g}_{\mu\nu}(x) \equiv e^{-2\tau(x)}g_{\mu\nu}(x) \qquad \widehat{\mathbf{O}}_\Delta(x) \equiv e^{\Delta\tau(x)}\mathbf{O}(x)$$

$$A_{EFT}[\tau, g_{\mu\nu}] = -\Delta a \times A_a[\tau, g_{\mu\nu}] + \Delta c \times A_c[\tau, g_{\mu\nu}] + A_{invariant}[\hat{g}_{\mu\nu}]$$

$$\Delta a \equiv a_{UV} - a_{IR} \qquad and \qquad \Delta c \equiv c_{UV} - c_{IR}$$

$$A_a[\tau, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left(\tau E_4 + 4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau + 2(\partial\tau)^4 - 4(\partial\tau)^2\Box\tau\right)$$

$$A_c[\tau, g_{\mu\nu}] = \int d^4x\sqrt{-g}\ \tau\,\mathscr{W}^2$$

$$A_{invariant}[\hat{g}_{\mu\nu}] = \int d^4x\ \sqrt{-\hat{g}}\left(M^4\lambda + M^2r_0\widehat{R} + r_1\widehat{R}^2 + r_2\widehat{W}^2 + r_3\widehat{E}_4\right)$$

# Vertices to probe anomaly coefficients

$$e^{-\tau(x)} \equiv 1 - \frac{\varphi(x)}{\sqrt{2}f}$$

**dilaton field**

**graviton field** (traceless and transverse)

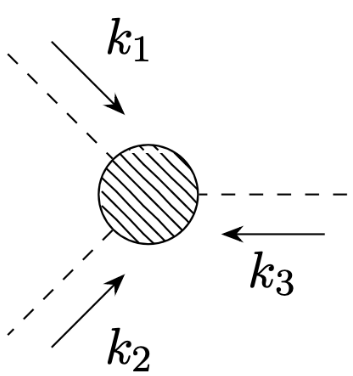
$$g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + 2\kappa\, h_{\mu\nu}(x)$$

**graviton-dilation vertex**

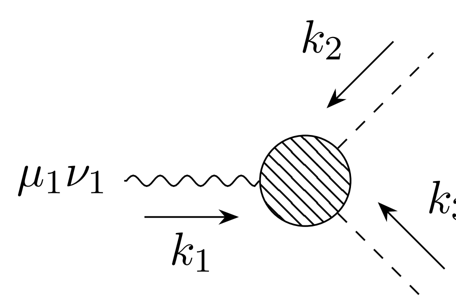
$$(2\pi)^4 \delta^{(4)}(p_1 + \ldots + p_m + q_1 + \ldots + q_n) \times V^{\mu_1\nu_1,\ldots,\mu_m\nu_m}_{(h\ldots h\varphi\ldots\varphi)}(p_1, \ldots, p_m, q_1, \ldots, q_n)$$

$$\equiv \left. \frac{i\,\delta^{m+n} A_{EFT}[\tau, g_{\mu\nu}]}{\delta h_{\mu_1\nu_1}(p_1)\ldots\delta h_{\mu_m\nu_m}(p_m)\delta\varphi(q_1)\ldots\delta\varphi(q_n)} \right|_{h,\varphi=0}$$

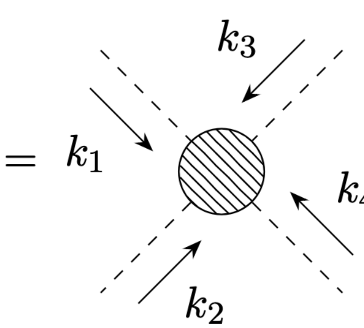
$V_{(\varphi\varphi\varphi)}(k_1, k_2, k_3) =$



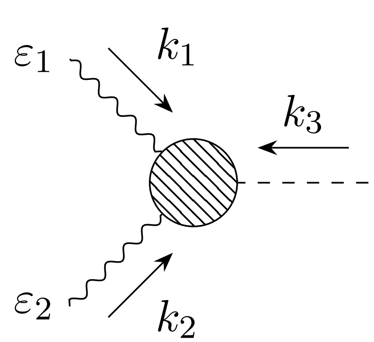

$V^{\mu_1\nu_1}_{(h\varphi\varphi)}(k_1, k_2, k_3) =$



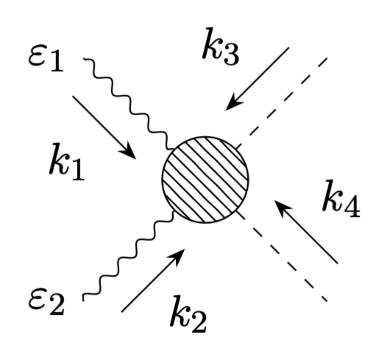

$V_{(\varphi\varphi\varphi\varphi)}(k_1, k_2, k_3, k_4) =$




$V_{(hh\varphi)}(k_1, k_2, k_3; \varepsilon_1, \varepsilon_2) =$




$V_{(hh\varphi\varphi)}(k_1, k_2, k_3, k_4; \varepsilon_1, \varepsilon_2) =$



# Protected amplitudes

# And

# S-matrix bootstrap

# Providing dynamics to graviton and dilaton

$$A^{\varphi}_{kinetic} = -\frac{f^2}{6}\int d^4x\sqrt{-\hat{g}}\; R(\hat{g})$$

$$= \int d^4x\sqrt{-g}\;\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{f^2}{6}R + \frac{\sqrt{2}f}{6}R\varphi - \frac{1}{12}R\varphi^2\right]$$

$$A^{h}_{kinetic} = \left(\frac{1}{2\kappa^2} + \frac{f^2}{6}\right)\int d^4x\sqrt{-g}\; R$$

Weyl symmetry is broken in the Planck scale $\kappa^{-1}$ with the **decoupling limit**: $\kappa \to 0, \qquad f \to \infty, \qquad \kappa \ll \frac{1}{f}$

**Compute scattering amplitudes for the given action:**

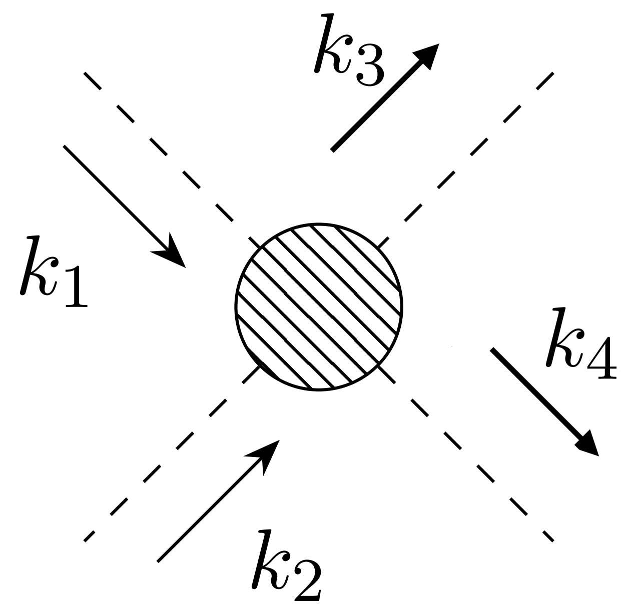
$$A = A^{\varphi}_{kinetic} + A^{h}_{kinetic} + A_{EFT}[\tau, g_{\mu\nu}] + \sum_{1\le\Delta\le 2}\lambda_\Delta\int d^4x\sqrt{-\hat{g}}\; M^{2-\Delta}R(\hat{g})\;\widehat{\mathbf{O}}_\Delta(x)$$

# Four dilaton amplitude

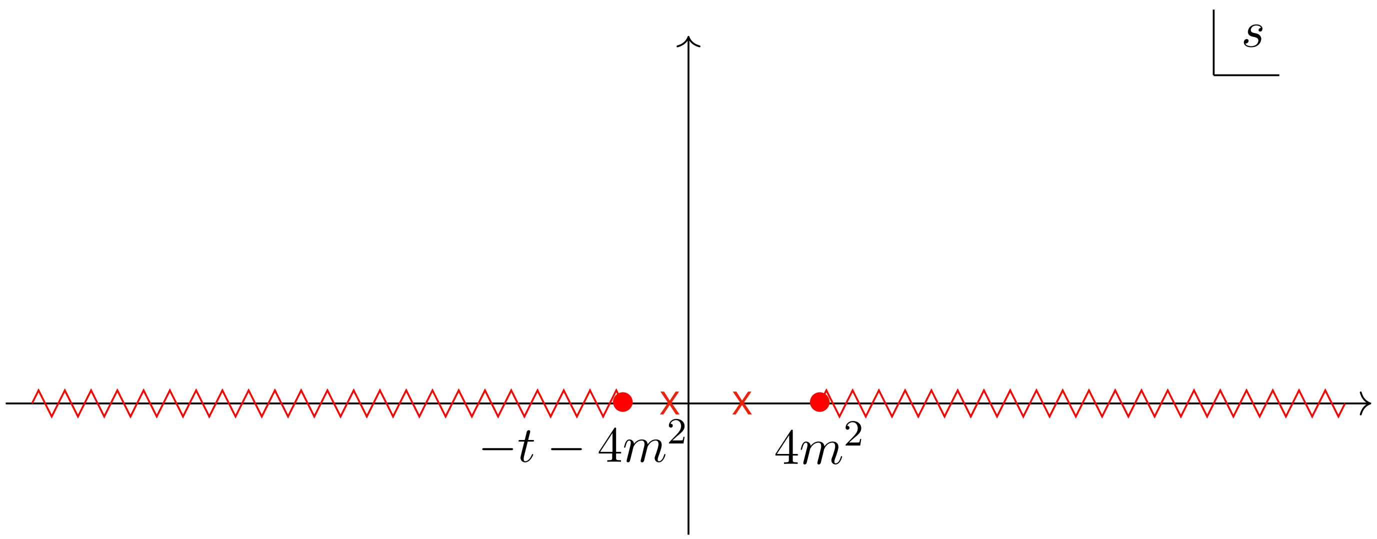
Komargodski & Schwimmer

At leading order in decoupling limit:

$$\mathcal{T}_{\varphi\varphi\to\varphi\varphi}(s,t,u) = \frac{\Delta a}{f^4}(s^2+t^2+u^2)+\cdots$$




$$s = -(k_1+k_2)^2$$
$$t = -(k_1-k_3)^2$$
$$u = -(k_1-k_4)^2$$




Dispersion relation with assumption $\lim_{|s|\to\infty}\frac{\mathcal{T}_{\varphi\varphi\to\varphi\varphi}(s,0,-s)}{s^2}=0$ :

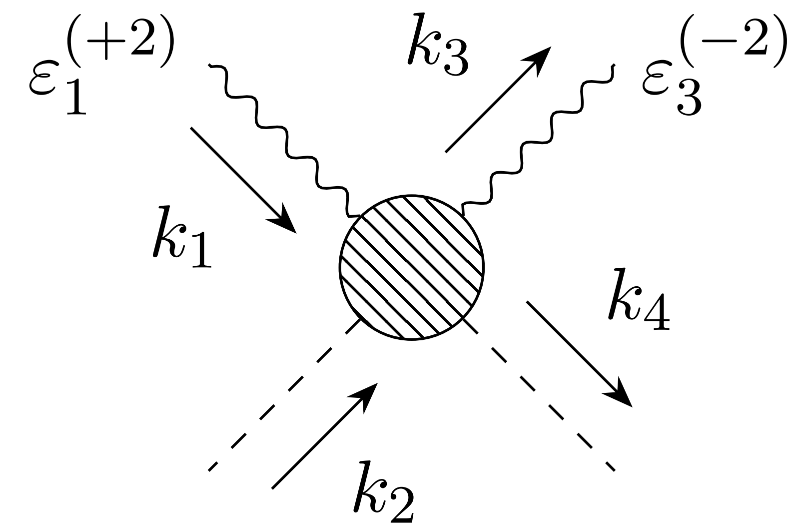
$$\Delta a = f^4\int_{s>0}\frac{ds}{\pi}\frac{\text{Im}\,\mathcal{T}_{\varphi\varphi\to\varphi\varphi}(s,0,-s)}{s^3}\geq 0$$

$a-$theorem

# Graviton-dilaton amplitude

In the COM Fram the helicity flipping amplitude

$$\mathcal{T}^{-2}_{+2}(s,t,u) = \mathcal{T}^{+2}_{-2}(s,t,u) = \frac{\kappa^2}{f^2}\left(\Delta c - \Delta a\right) t^2$$




Dispersion relation with assumption $\lim_{|s|\to\infty} \partial_t^2 \mathcal{T}^{-2}_{+2}(s,0,-s) = 0$ :

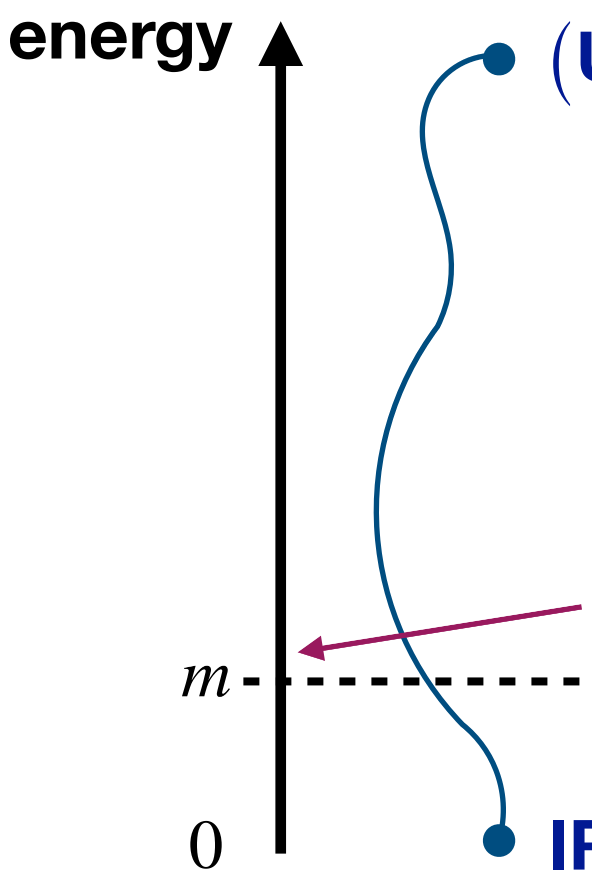
$$\Delta c - \Delta a = \frac{f^2}{\kappa^2}\int_{s>0} \frac{ds}{\pi}\, \frac{\mathrm{Im}\, \partial_t^2 \mathcal{T}^{-2}_{+2}(s,0,-s)}{s}$$

**NOT** sign definite

- $(\Delta c - \Delta a)$ probes spinning massive states with partial wave spin $\geq 2$

# Bootstrap Setup

Karateev, Marucha, Penedones, **B.S.**




**(UV CFT + deformation)$_{\text{compensated}}$**

$(a_{UV}, c_{UV})$

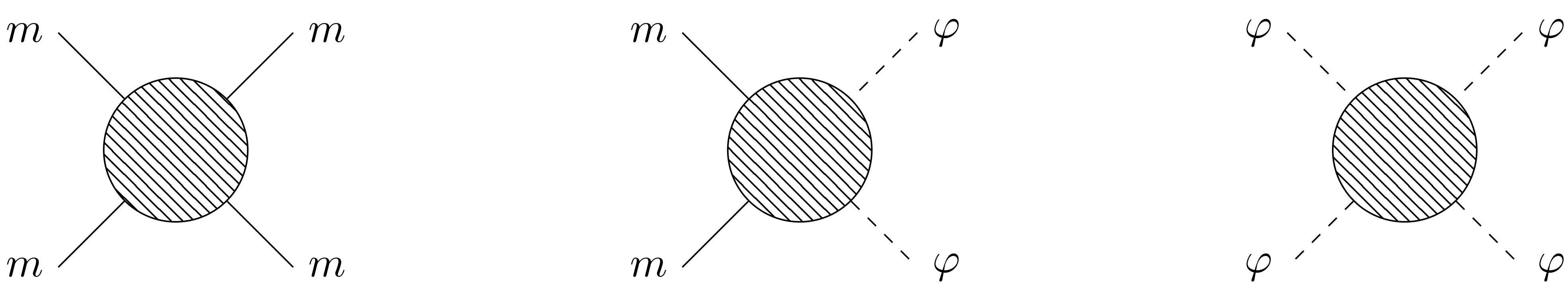
mass gap ($\mathbb{Z}_2$ odd scalar particle)

**IR CFT(empty) + dilaton EFT**

$(a_{IR} = 0, c_{IR} = 0)$

**Question:** what is the minimum value of $a_{UV}$ ?

Set of amplitudes considered in the non-perturbative S-matrix bootstrap program:




$\mathcal{T}_{mm\to mm}$ $\mathcal{T}_{mm\to\varphi\varphi}$ $\mathcal{T}_{\varphi\varphi\to\varphi\varphi}$

# Bootstrap bound on $a_{UV}$

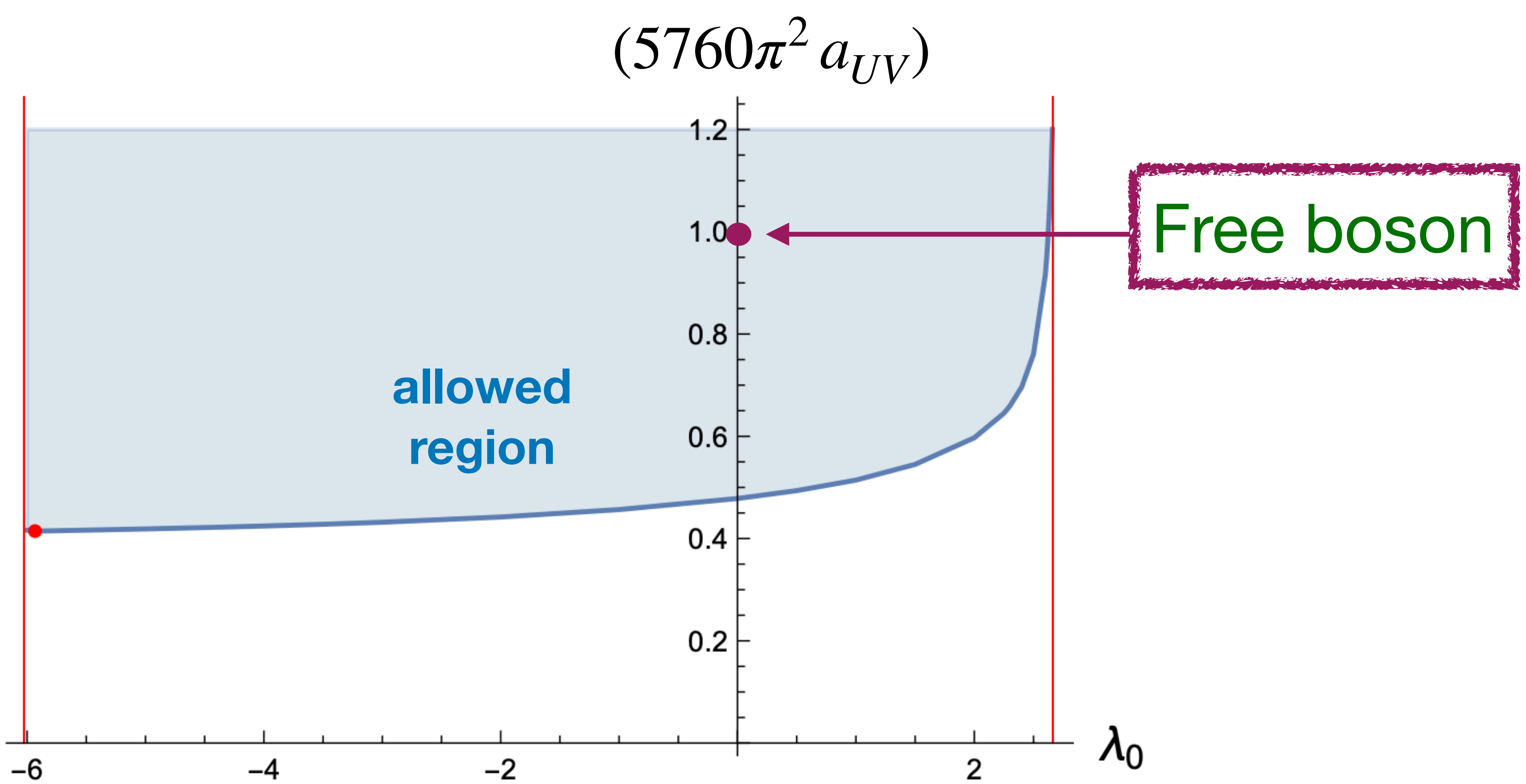
**Non-perturbative observables:**

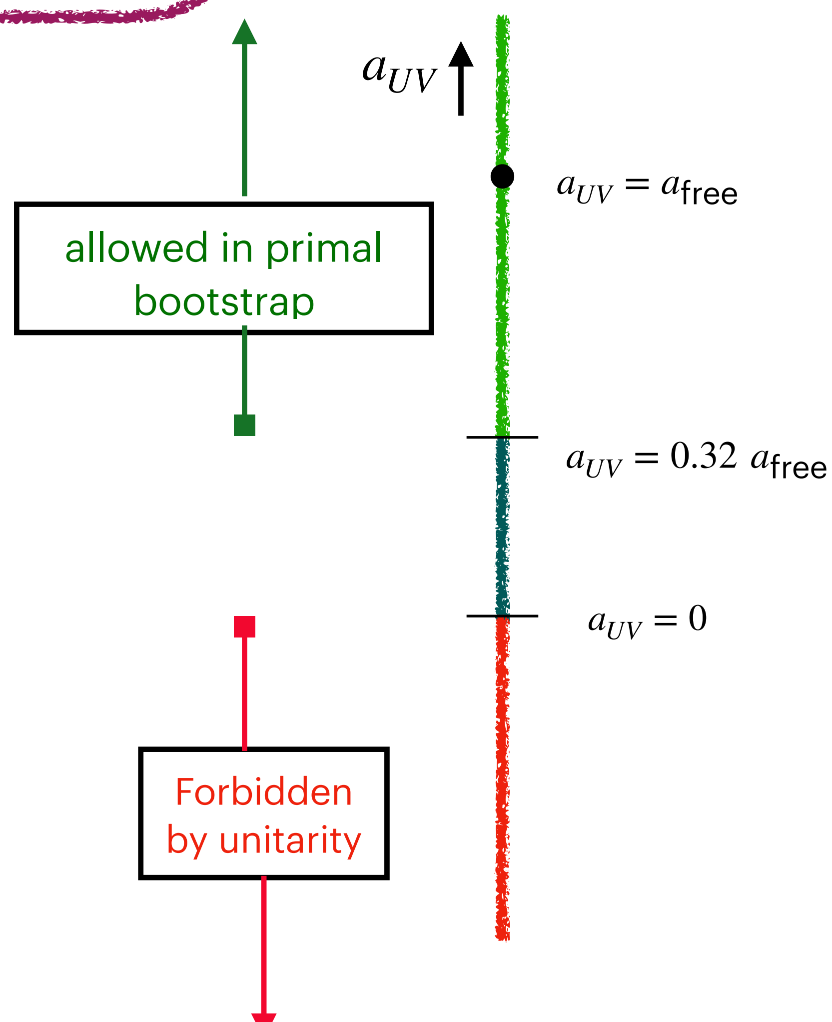
$$\lambda_0 \equiv \frac{1}{32\pi}\mathcal{T}_{mm\to mm}(4m^2/3, 4m^2/3, 4m^2/3)$$

$$\lambda_2 \equiv \frac{1}{32\pi}m^4\partial_s^2\mathcal{T}_{mm\to mm}(4m^2/3, 4m^2/3, 4m^2/3)$$

S-matrix bootstrap bounds:

$$-6.0253 \leq \lambda_0 \leq +2.6613$$
$$0 \leq \lambda_2 \leq +2.2568$$



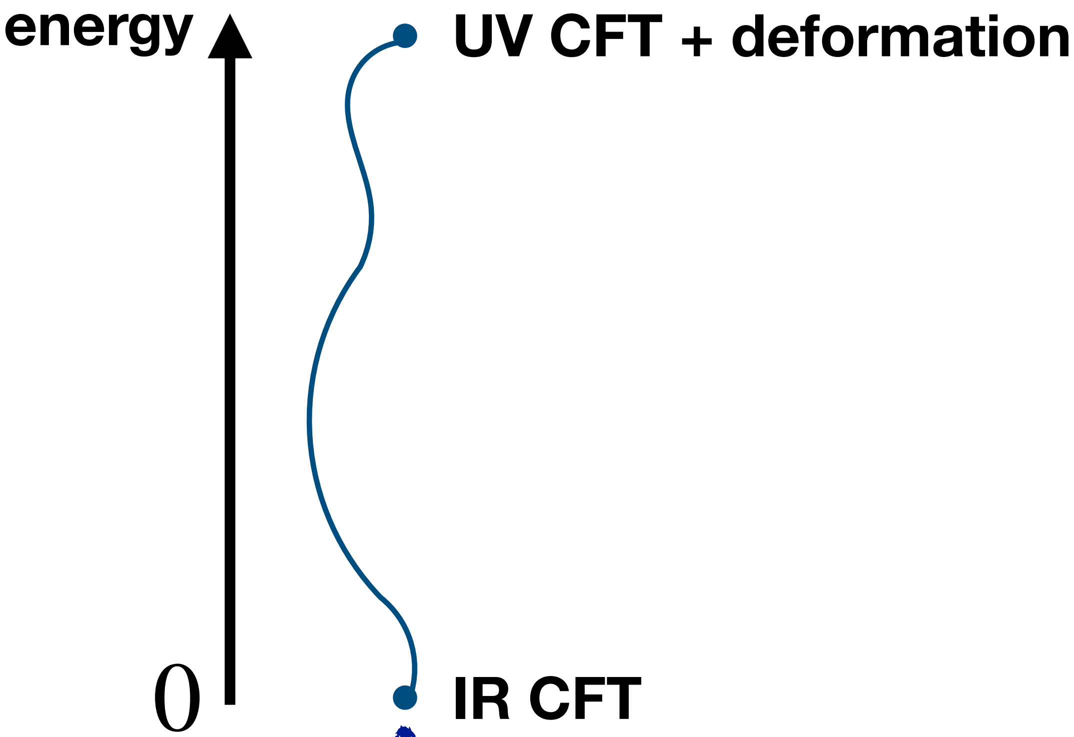
# Example

(Already tested the proposal for free CFTs with mass deformation)

# Weakly Relevant Flow

Based on the paper appeared today with Denis Karateev

# Setup




$$A_{QFT} = A_{UV\,CFT} + \lambda_0\, m^{4-\Delta} \int d^4x\ \mathcal{O}(x)$$

**short flow if** $\Delta = 4 - \delta \qquad 0 < \delta \ll 1$

perturbative parameter $\lambda_0$

"renormalization" at scale $\mu$

$\lambda(\mu) \to$ renormalized coupling

$Z(\mu)^{-\frac{1}{2}}\ \mathcal{O}(x) \equiv \mathbf{O}(x) \to$ renormalized operator

$$\beta(\lambda) = -\,\delta\lambda + \frac{1}{2} C_{\mathcal{O}\mathcal{O}\mathcal{O}} \Omega_3 \lambda^2 + O(\lambda^3)$$

OPE coefficient

$Vol(\mathbb{S}^3) = 2\pi^2$

$$\gamma(\lambda) = C_{\mathcal{O}\mathcal{O}\mathcal{O}} \Omega_3 \lambda + O(\lambda^2)$$

$$\beta(\lambda_\star) = 0$$

$$\lambda_\star = \frac{2\delta}{C_{\mathcal{O}\mathcal{O}\mathcal{O}} \Omega_3}$$

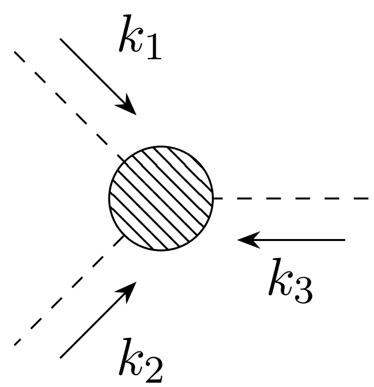
# Goal and Outcome

Compute $\Delta a$ and $\Delta c$ values directly for the weakly relevant RG flow from sphere partition function and stress tensor correlation function.

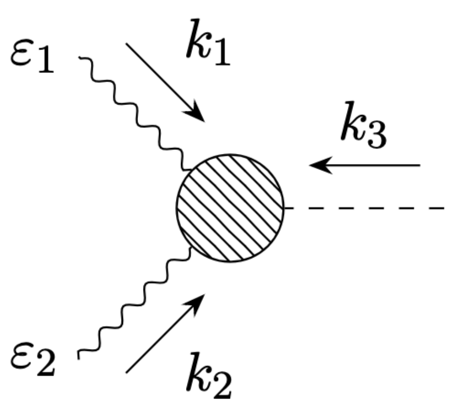
Couple the RG flow to dilaton-graviton background and evaluate relevant vertices involving dilaton-graviton at low energy. Compare them with the general prediction from anomaly matching EFT to read off $\Delta a$ and $\Delta c$ values.

The agreement of the two results establishes the anomaly matching condition in presence of background fields on a firm footing and inferring the connection between background field vertices and QFT correlators.

# Two relevant vertices from anomaly matching EFT

$V_{(\varphi\varphi\varphi)}(k_1,k_2,k_3) =$ 


$$= \frac{i\sqrt{2}}{f^3}\Big(\Delta a\left(\left(k_1^2\right)^2+\left(k_2^2\right)^2+\left(k_3^2\right)^2\right) + 2(18r_1-\Delta a)\left(k_1^2k_2^2+k_2^2k_3^2+k_3^2k_1^2\right)+\dots\Big)$$




$$= f_1(k_1,k_2)\times(\varepsilon_1\,.\,\varepsilon_2) + f_2(k_1,k_2)\times(k_1\,.\,\varepsilon_2\,.\,k_1)(k_2\,.\,\varepsilon_1\,.\,k_2) + f_3(k_1,k_2)\times(k_1\,.\,\varepsilon_2\,.\,\varepsilon_1\,.\,k_2)$$

$$f_1(k_1,k_2) = \frac{4i\kappa^2}{\sqrt{2}f}\Big(2(-\Delta a+\Delta c+18r_1)(k_1\,.\,k_2)^2 + (2\Delta a-\Delta c+24r_1)k_1^2k_2^2 + 12r_1(k_1^4+k_2^4) + 42r_1(k_1\,.\,k_2)(k_1^2+k_2^2)\Big)$$

$$f_2(k_1,k_2) = \frac{8i\kappa^2}{\sqrt{2}f}\,(-\Delta a+\Delta c)$$

$$f_3(k_1,k_2) = \frac{8i\kappa^2}{\sqrt{2}f}\,\big(2(\Delta a-\Delta c-6r_1)(k_1\,.\,k_2) - 6r_1(k_1^2+k_2^2)\big)$$

# Vertices in Weakly Relevant Flow

$$\mathrm{A}^{\text{compensated}}_{\text{QFT}} = \mathrm{A}_{\text{UV CFT}}[\phi, g_{\mu\nu}] + \lambda_0 \int d^4x \sqrt{g}\, \left(m\Omega(x)\right)^{\delta}\ \mathcal{O}_g(x)$$

$$A_{\text{UV CFT}}[\phi] + \kappa \int d^4x\, h_{\mu\nu}(x)\, T^{\mu\nu}(x) + O(\kappa^2) \qquad \Omega(x) = 1 - \frac{\varphi(x)}{\sqrt{2}f}$$

$$A_{\text{EFT}}[\varphi, h] \equiv -\log \int [d\phi]_g\, e^{-\mathrm{A}^{\text{compensated}}_{\text{QFT}}}$$

$V_{(\varphi\varphi\varphi)}$ :

$$A_{\text{EFT}}[\varphi, h] = -\frac{1}{4\sqrt{2}f^3}\,\delta^2 \left(m^{\delta}\lambda_0\right)^2 \int d^dx_1 \int d^dx_2\ \varphi(x_2)^2\varphi(x_1) \times \langle \mathcal{O}(x_{12})\mathcal{O}(0)\rangle_{\text{QFT}} + \cdots$$

$V_{(hh\varphi)}$ :

$$A_{\text{EFT}}[\varphi,\, h] = -\frac{\delta\kappa^2}{2\sqrt{2}f}\lambda_0 m^{\delta} \int d^dx_1 \int d^dx_2 \int d^dx_3\ \ h_{\mu\nu}(x_1)h_{\rho\sigma}(x_2)\varphi(x_3) \times \langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)\mathcal{O}(x_3)\rangle_{\text{QFT}} + \ldots$$

Conformal perturbation theory together with solution of Callan-Symanzik equation provides:

$$\langle \mathbf{O}(x_1)\mathbf{O}(x_2)\rangle_{\rm QFT} = \frac{1}{r^{2\Delta}} \times \Big(\chi\left(\mu r, \lambda\right)\Big)^4$$

$$\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)\mathbf{O}(x_3)\rangle_{\rm QFT} =$$
$$\langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)\mathcal{O}(x_3)\rangle_{\rm UV\ CFT} \times \Big(\chi\left(\mu\rho, \lambda\right)\Big)^2 + H^{\mu\nu\rho\sigma}(x_1, x_2, x_3)$$

with

$$\chi(s,\lambda) \equiv \frac{1}{1+\frac{\lambda}{\lambda_\star}\left(s^\delta - 1\right)}$$

$$r \equiv \left(x_{12}^2\right)^{\frac{1}{2}} \qquad \rho \equiv \left(x_{12}^2 x_{13}^2 x_{23}^2\right)^{\frac{1}{6}}$$

# Results

$$\Delta a = \frac{\delta^3}{2304\pi^2 C^2_{\mathcal{O}\mathcal{O}\mathcal{O}}} + O\left(\delta^4\right)$$

— in agreement with direct computation of Komargodski (2011) and Klebanov-Pufu-Safdi (2011)

$$\Delta c = \frac{\delta\,\pi^2}{2304}\frac{C_{TT\mathcal{O}}}{C_{\mathcal{O}\mathcal{O}\mathcal{O}}} + O(\delta^2)$$

— in agreement with our independent computation of change of central charge from two-point stress tensor correlator $\langle T^{\mu\nu}(x)T^{\rho\sigma}(0)\rangle_{\text{QFT}}$

# Take home message

- Anomaly matching condition constrain low energy EFT observables when the RG flow is carefully coupled to background fields.
- S-matrix bootstrap can be useful to bound the space of CFTs once we trigger an RG flow and introduce background field method.
- It would be interesting to play the same game for other anomalies associated with various generalized symmetries.

**Thank You for your attention!**