Eikonal amplitudes on the Celestial Sphere

Eurostrings 2024, Southampton, UK
Session on Flat Holography

Piotr Tourkine,
LAPTh, Annecy, France

In collaboration with: *Tim Adamo, Wei Bu* 卜微, *Bin Zhu* 朱彬
also earlier work with *Eduardo Casali & Tim*

Based on :

[arXiv:2405.15594]
Eikonal amplitudes on the celestial sphere
T. Adamo, W. Bu, P. Tourkine, B. Zhu

UNIVERSITÉ SAVOIE MONT BLANC
cnrs PHYSIQUE
LAPTh
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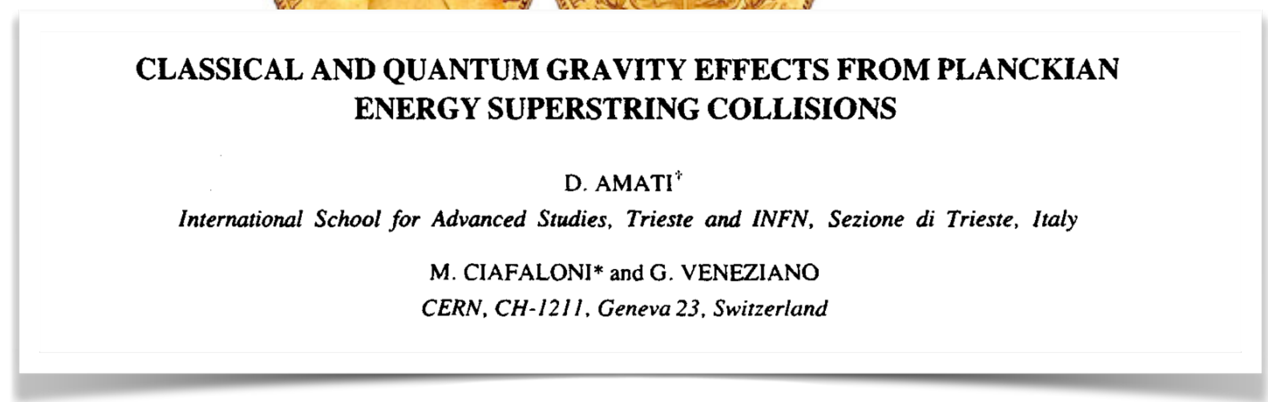
Motivations

- Ultimate goal: the S-matrix (of gravity in particular).
- Gold standard: ACV, 90’s



CLASSICAL AND QUANTUM GRAVITY EFFECTS FROM PLANCKIAN ENERGY SUPERSTRING COLLISIONS

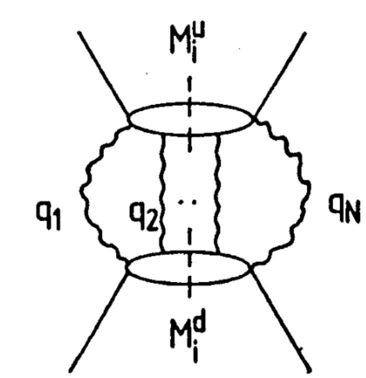
D. AMATI[†]

International School for Advanced Studies, Trieste and INFN, Sezione di Trieste, Italy

M. CIAFALONI[*] and G. VENEZIANO

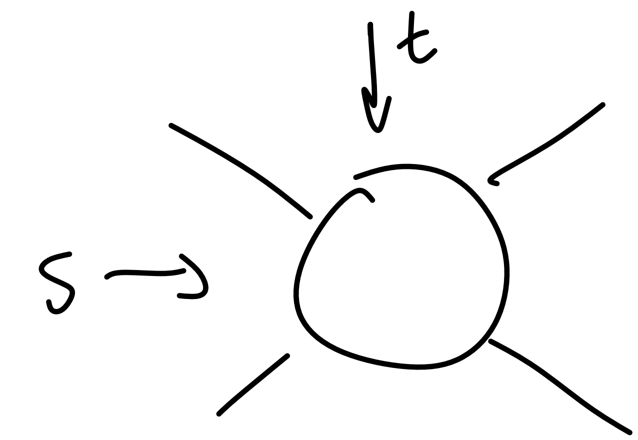
CERN, CH-1211, Geneva 23, Switzerland

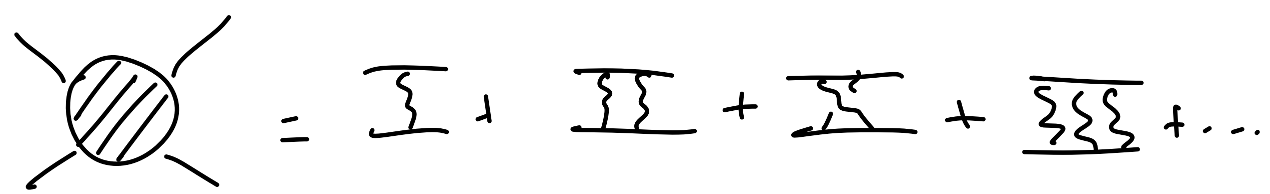
- In one sense, our paper = a step to understand transplanckian scattering on the celestial sphere
- And a proposal to obtain well-defined, analytic, gravitational Celestial amplitudes



Motivations

- Examples in 2 to 2 scattering. Usual variables momentum space variables : *s*,*t*
  - Angular momentum diagonalises full non-perturbative unitarity $S(s,t) \to |S_J(s)|^2 \leq 1$
  - Impact parameter shows eikonal exponentiation:





$$A_{eik}(s,b) = e^{i\delta(s,b)}$$

talk by [Vanhove]

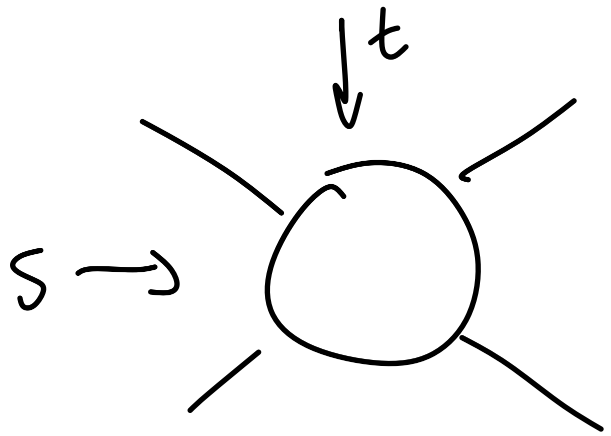
Simplifications occur because of symmetries and kinematical regimes

Celestial amplitudes = S-matrix for definite boost states.

Might tell us something nontrivial.

Celestial amplitudes

2-to-2 massless scattering



$$s = \omega^2, t = -z\omega^2$$

Energy basis → Boost basis.
Nice transformation under SL(2,C)

$$|\Delta, z\rangle = \int \frac{d\omega}{\omega} \omega^\Delta |\omega, z\rangle$$

talk by [Donnay]

$$A(\beta, z) = \int_0^\infty d\omega\, \omega^{\beta-1} M(\omega^2, -z\omega^2)$$

[Pasterski, Shao, Strominger; Stieberger Taylor; Gonzales Puhm Rojas; Arkani-Hamed, Pate, Raclariu, Strominger;]

Celestial amplitudes

Our concern here: are the resulting objects defined at all?

[Donnay]

$$A(\beta, z) = \int_0^\infty d\omega\, \omega^{\beta-1} M(\omega^2, -z\omega^2)$$

Take a tree-level amplitude, $M(s,t) = -\frac{s^J}{t}$

$$\frac{1}{z}\int_0^\infty d\omega\, \omega^{\beta-1}\omega^{2J-2} \overset{!}{=} \delta(\gamma)$$

is either non-defined, or badly non-analytic: at best a delta-function if $\beta = -(2J-2) + i\gamma$

"Celestial amplitudes are anti-Wilsonian"

[arXiv:2012.04208] JHEP **08** (2021) 062
Celestial Amplitudes from UV to IR
N. Arkani-Hamed, M. Pate, A. Raclariu, A. Strominger
43 pages **[hep-th]**
[APRS]

Celestial amplitudes

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[Donnay]

$$A(\beta, z) = \int_0^\infty d\omega\, \omega^{\beta-1} M(\omega^2, -z\omega^2)$$

see however

[arXiv:2401.08877]
Distributional Celestial Amplitudes
M. Borji, Y. Pano
27 pages, 1 figure **[hep-th]**

Interesting attempt at formalising distributional nature of Celestial amplitudes

Take a tree-level amplitude, $M(s,t) = -\frac{s^J}{t}$

$$\frac{1}{z}\int_0^\infty d\omega\, \omega^{\beta-1}\omega^{2J-2} \stackrel{!}{=} \delta(\gamma)$$

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43 pages **[hep-th]**
[APRS]

Celestial amplitudes

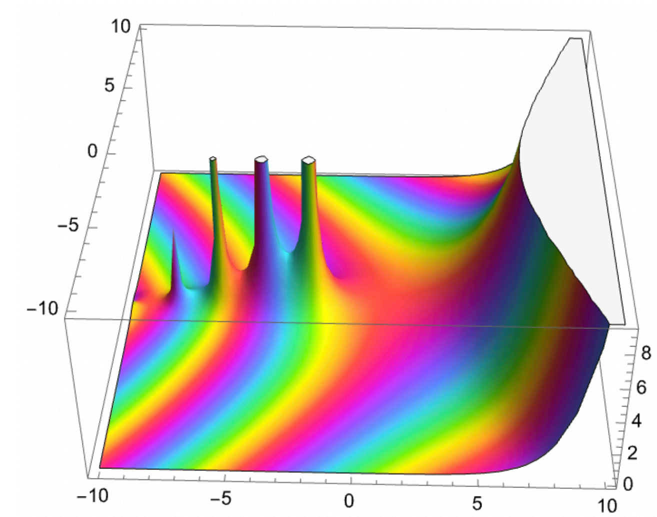
- Same thing happens for any finite order in perturbation theory, which produces $\omega^p(\log(\omega))^q$

- But infinitely many terms can give nice, **analytic** functions. Take

e.g. $\sum_{n=0}^{\infty} \frac{(-\omega^2)^n}{n!} = \exp(-\omega^2)$

J=2

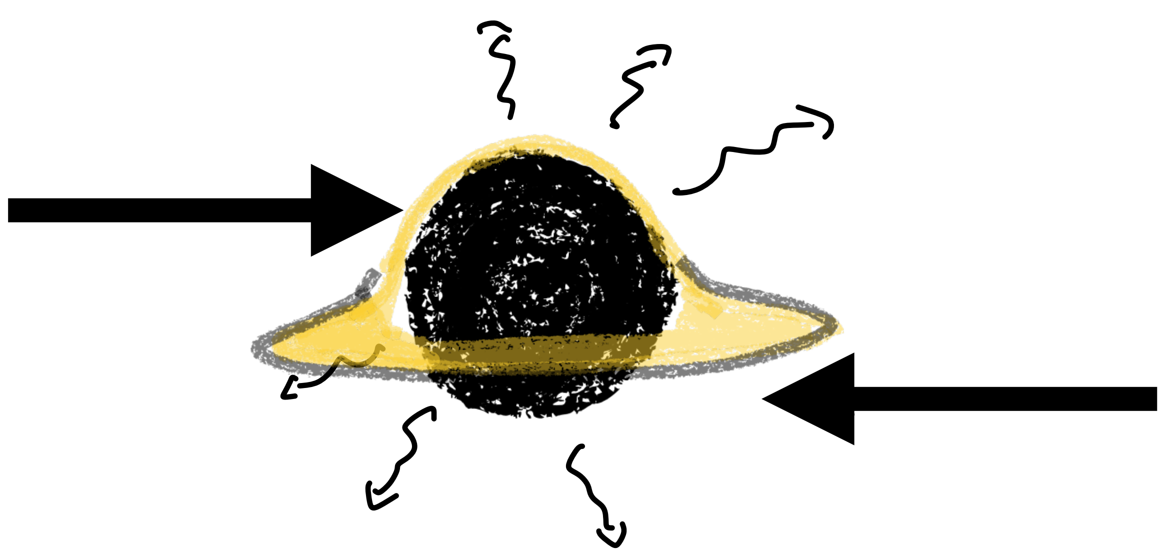
$$\int_0^\infty d\omega\, \omega^{\beta-1+2J-2} e^{-\omega^2} = \frac{1}{2}\Gamma\left(\frac{\beta}{2}+1\right)$$



- Working assumption : this analyticity is desirable, how to restore it?

Celestial amplitudes

- [APRS] argued that black-hole production at high energies produces an exponential suppression that renders the integral finite and exhibits nice analytic properties.



$$A(\omega, z) \sim e^{-S_{BH}(\omega)/2}$$

$$R_S(\omega) = 2G_N\omega \rightarrow \quad S_{BH}(\omega) = 4\pi G_N\omega^2$$

Entropic suppression

[arXiv:2012.04208] JHEP **08** (2021) 062
Celestial Amplitudes from UV to IR
N. Arkani-Hamed, M. Pate, A. Raclariu, A. Strominger
43 pages **[hep-th]**

[arXiv:0711.5012] Phys.Rev.D **77** (2008) 085025
High-energy gravitational scattering and black hole resonances
S. B. Giddings, M. Srednicki
22 pages, harvmac. v2: minor corrections **[hep-th]**

[arXiv:0908.0004] Phys.Rev. **D81** (2010) 025002
The gravitational S-matrix
S. B. Giddings, R. A. Porto
46 pages, 10 figures **[hep-th]**

Celestial amplitudes

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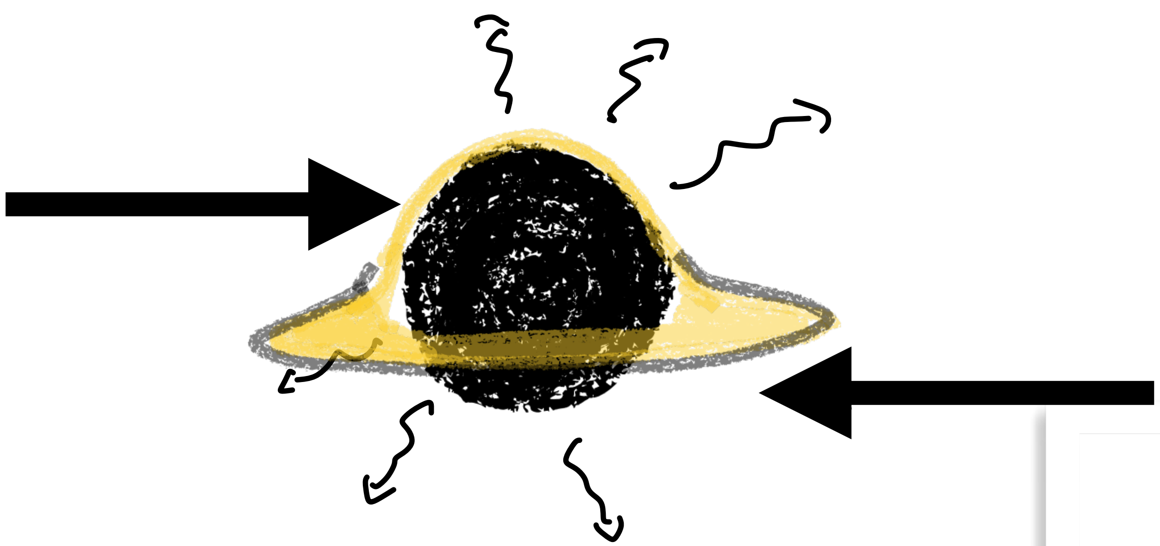


$$\int_0^\infty d\omega\, \omega^{\beta-1+2J-2} e^{-\omega^2} = \frac{1}{2}\Gamma\left(\frac{\beta}{2}+1\right)$$

J=2

$$A(\omega, z) \sim e^{-S_{BH}(\omega)/2}$$

$$S_{BH}(\omega) = 4\pi G_N \omega^2$$

<u>Analytic structure</u>

Poles at negative $\beta = -2n, \quad n \in \mathbb{N}$

Regular for $\beta > 0$

Large $\beta > 0$ limit: $e^{-\beta/2}(\beta/2)^{\beta/2}$

Our paper: eikonal, celestial

Eikonal resummation

- ACV: eikonal captures small but finite angle at leading order
- No time to review transplanckian scattering in gravity, see

  Zhiboedov's lectures at Bootstrap conference, Madrid, 2024

- Eikonal captures exactly semi-classical motion in linearised backgrounds

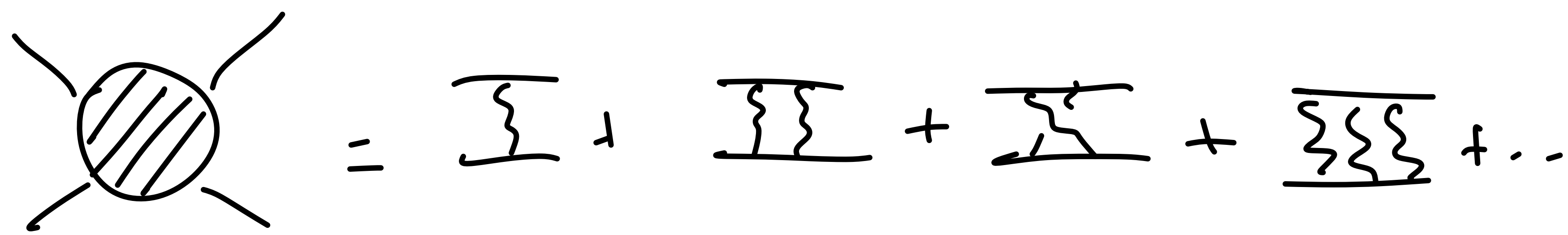
  't Hooft

  Kabat Ortiz

  Shockwaves, BH, Kerr, ..

  Adamo, Cristofoli, PT

Eikonal resummation

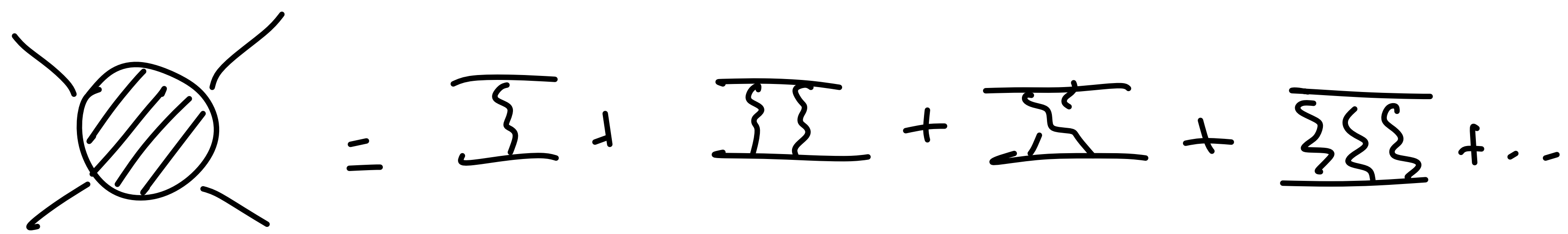
- Amplitude: 

$$\mathscr{M}_{\rm eik} = \frac{8\pi G s^2}{t} \frac{\Gamma(-i G s)}{\Gamma(i G s)} \left(\frac{4\mu^2}{-t}\right)^{-i G s}$$

't Hooft

Eikonal resummation

- Amplitude:



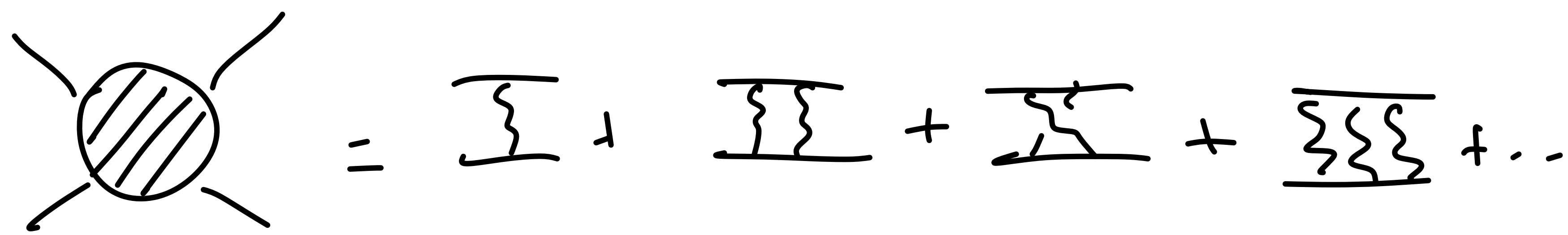
$$\mathcal{M}_{\text{eik}} = \frac{8\pi G s^2}{t} \frac{\Gamma(-i G s)}{\Gamma(i G s)} \left(\frac{4 \mu^2}{-t} \right)^{-i G s}$$

Born   Eikonal phase

μ^2 mass scale for IR regularisation

Eikonal resummation

- Amplitude:



$$\mathscr{M}_{\rm eik} = \frac{8\pi G s^2}{t} \frac{\Gamma(-i G s)}{\Gamma(i G s)} \left(\frac{4\mu^2}{-t}\right)^{-i G s}$$

- Celestial transform:

$$\mathscr{A}_{\rm eik}(\beta, z) = -\frac{G}{z} \int_0^\infty d\omega\, \omega^{\beta+1} \frac{\Gamma(-\mathrm{i}G\omega^2)}{\Gamma(\mathrm{i}G\omega^2)} \left(\frac{4\mu^2}{\omega^2 z}\right)^{-\mathrm{i}G\omega^2}$$

Celestial eikonal amplitude

$$\mathscr{A}_{\rm eik}(\beta, z) = -\frac{G}{z}\int_0^\infty d\omega\, \omega^{\beta+1}\, \frac{\Gamma(-{\rm i}G\,\omega^2)}{\Gamma({\rm i}G\,\omega^2)} \left(\frac{4\,\mu^2}{\omega^2\, z}\right)^{-{\rm i}\,G\,\omega^2}$$

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1. Dimensional analysis factors out *G:* $\mathscr{A}_{eik}(\beta, z) \sim G^{-\beta/2} \times f(\beta, z)$

2. Change variables from $\omega^2 \to \omega$

Celestial eikonal amplitude

$$\mathscr{A}_{\text{eik}}(\beta, z) = -\frac{1}{zG^{\beta/2}}\int_0^\infty d\omega\,\omega^{\beta/2}\,\frac{\Gamma(-\mathrm{i}\,\omega)}{\Gamma(\mathrm{i}\,\omega)}\left(\frac{4\,\tilde{\mu}^2}{\omega\, z}\right)^{-\mathrm{i}\,\omega}$$

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1. Dimensional analysis factors out *G:* $\mathscr{A}_{eik}(\beta, z) \sim G^{-\beta/2} \times f(\beta, z)$

2. Change variables from $\omega^2 \to \omega$

3. Of the form $\int_0^\infty d\omega\, \omega^{\beta/2}\, e^{i\phi(\omega)}$

4. Analyticity in β : two regimes, $\beta \to 0$ and $\beta \to \infty$

$\omega \to 0$

$$\int_0^1 d\omega\, \omega^{\beta/2}\, \frac{\Gamma(-\mathrm{i}\,\omega)}{\Gamma(\mathrm{i}\,\omega)} e^{\mathrm{i}\;\omega \log(\omega)} = \sum_{n,m} c_{n,m} \int_0^1 d\omega\, \omega^{\beta/2}\, \omega^n \log(\omega)^m$$

$$\sim \sum_{n,m} c_{n,m} \frac{1}{\left(\frac{\beta}{2} + n + 1\right)^m}$$

Poles at negative integers of higher and higher degree

IR effects, expected to receive perturbative corrections.

Residues at poles involve only single valued zeta-functions

$$\frac{\Gamma(-\mathrm{i}\,\omega)}{\Gamma(\mathrm{i}\,\omega)} = -\exp(2\mathrm{i}\,\gamma_E\,\omega)\,\exp\left[\sum_{k\geq 1} \frac{2\,\zeta(2k+1)}{2k+1}\,(\mathrm{i}\,\omega)^{2k+1}\right]$$

$\omega \to \infty$

- $\displaystyle \int_1^\infty d\omega\, \omega^{\beta/2}\, \frac{\Gamma(-\mathrm{i}\,\omega)}{\Gamma(\mathrm{i}\,\omega)} \omega^{\mathrm{i}\ \omega}$
- Universal part
- Convergence: for large ω, use Stirling to simplify integral to

$$\int^\infty d\omega\, \omega^{\beta/2}\, e^{\mathrm{i}\omega} \omega^{2\mathrm{i}\ \omega}$$

- Elementary calculation shows the convergence of the integral
- Compare with Gamma function $\Gamma(x) = \displaystyle\int_0^\infty d\omega\, \omega^{\beta-1} e^{-\omega}$
- We called our function the *Eikonal Gamma function.*

Summary so far

- Eikonal phase regularises the Mellin integral
- Since the true, full gravitational amplitude contains this eikonal factor (for z small but finite), this is the first indication from an actual calculation that the celestial gravitational amplitude actually exist.
- Properties:
  - Analytic in β, appart from poles at $\beta = -2\mathbb{N} - 2$ corresponding to IR physics (non-universal)
  - Residues at poles involve only single valued zeta-functions
  - Comparison to APRS and BH toy-amplitude? -> Large β

Large positive β of Celestial eikonal

- Saddle point at $\omega_* \sim -\frac{i\beta}{\log(-i\beta)}$: semi-classical boost / conformal dimension of exchanged operator ?

- Yields:

$$|\mathscr{A}_{\mathrm{eik}}|^2 \sim \left(\frac{\beta}{\log(\beta)}\right)^{\beta} e^{-\beta}$$

- Compare with BH ansatz of APRS:

$$|\mathscr{A}_{\mathrm{BH}}(\beta, z)|^2 \sim \Gamma(\beta) \sim \beta^{\beta} e^{-\beta} \qquad |\mathscr{A}_{\mathrm{eik}}(\beta, z)|^2 \sim \left(\frac{\beta}{\log(\beta)}\right)^{\beta} e^{-\beta}$$

Comments

$$|\mathscr{A}_{\rm BH}(\beta,z)|^2 \sim \Gamma(\beta) \sim \beta^{\beta}\, e^{-\beta} \qquad\qquad |\mathscr{A}_{\rm eik}(\beta,z)|^2 \sim \left(\frac{\beta}{\log(\beta)}\right)^{\beta} e^{-\beta}$$

1. Exponential BH suppression → Exponentially large in β
   Oscillating eikonal phase → Exponentially large in β

2. $\mathscr{A}_{BH}(\beta) \gg \mathscr{A}_{eik}(\beta)$

3. Implication for the spectrum & Regge trajectories of a putative CCFT ?

Dispersion relation

Can we compute the full integral? Not yet, but:

Dispersion relation

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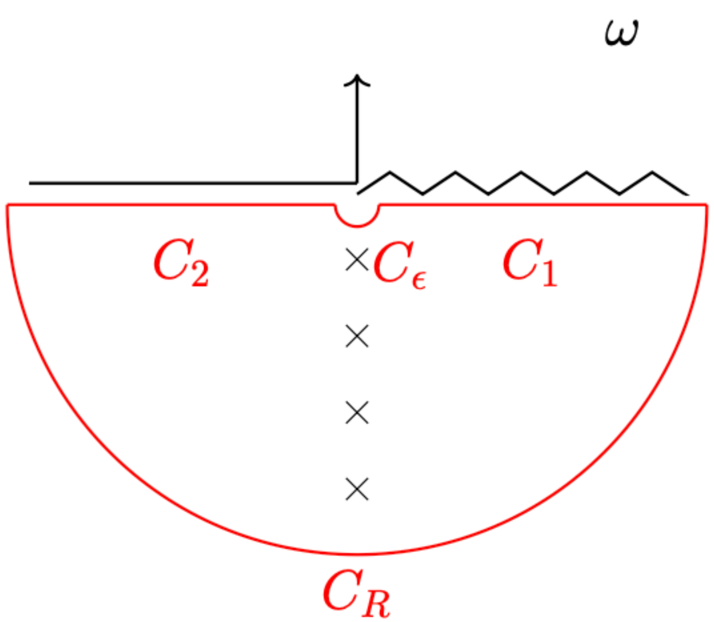


$$\frac{-1}{2\,G^{\beta/2}\,z}\oint_C \mathrm{d}\omega\,\omega^{\frac{\beta}{2}}\,\frac{\Gamma(-\mathrm{i}\,\omega)}{\Gamma(\mathrm{i}\,\omega)}\left(\frac{4\,\tilde{\mu}^2}{\omega\,z}\right)^{-\mathrm{i}\,\omega} = 0$$

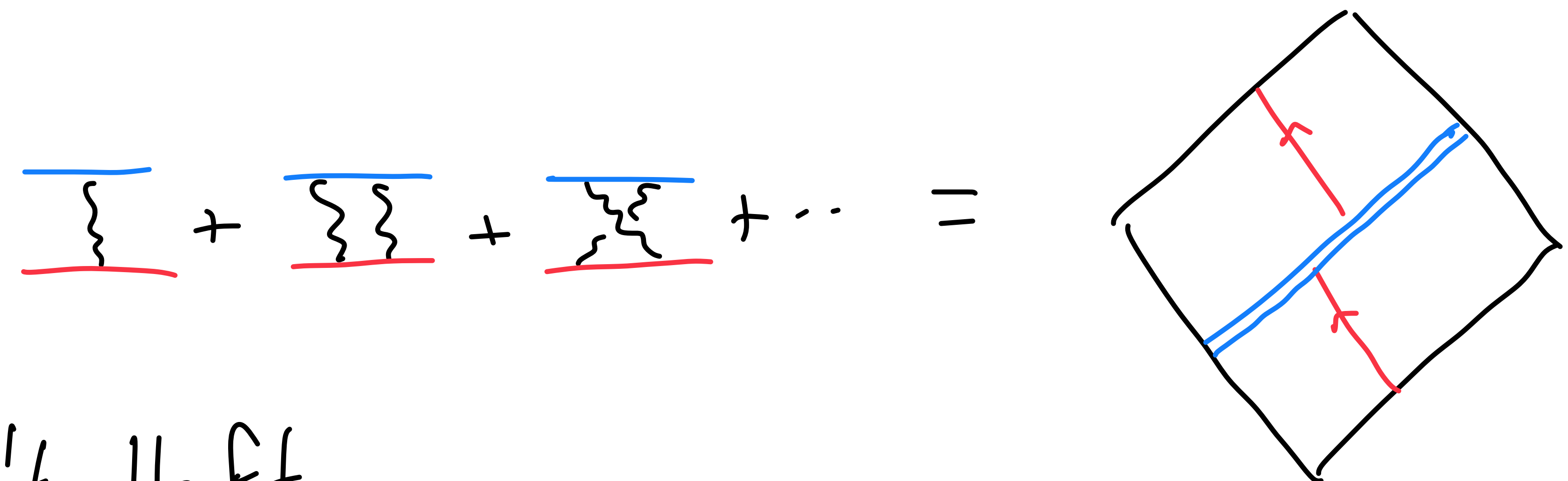
$$\mathcal{A}_{\text{eik}}(\beta,z)+\mathrm{e}^{-\frac{\mathrm{i}\pi\beta}{2}}\,\overline{\mathcal{A}_{\text{eik}}}(\bar{\beta},-z) = -\frac{\pi}{z}\,\mathrm{e}^{-\mathrm{i}\pi\beta}\sum_{n\geq 1}\frac{\mathrm{i}^n}{n!\,(n-1)!}\left(\frac{-\mathrm{i}\,n}{G}\right)^{\frac{\beta}{2}}\left(\frac{n\,z}{4\,\tilde{\mu}^2}\right)^n$$

Two unknown functions, but can’t close the contour above !

Miss one relation !?

1-to-1 probe scattering

usual story in flat space:



't Hooft

Correspondence between 2-2 eikonal scattering

and

probe 1-1 scattering in background generated by the other particle

1 to 1 probe scattering

- In the paper we also study the 1-to-1 amplitudes in various linearised (eikonalising) spacetimes: shockwave, linearised BHs, etc.
- Important <u>unnoticed</u> subtlety on Mellin transform in these backgrounds
- **We define a map to relate both results**
- Important avenue because those kind of line operators play important role in AdS holographic duality

Other works on eikonal amplitude

- Focussed on link with shockwave spacetimes. Dropped non-trivial terms of the eikonal phase.

[arXiv:2206.10547] JHEP **03** (2023) 030
Eikonal Approximation in Celestial CFT
L. P. de Gioia, A. Raclariu

[arXiv:2207.13719] JHEP **10** (2022) 073
Celestial holography on Kerr-Schild backgrounds
R. Gonzo, T. McLoughlin, A. Puhm

Conclusion

- Eikonal phase makes Celestial amplitudes analytic and well-defined
- Proposal: should be part of the definition of any non-perturbative Celestial gauge or gravitational amplitude
- Also: our paper = first step to translate ACV to the celestial sphere
- Important overlooked aspects of Mellin transform of Shockwave spacetimes

Prospects

- Can we extract any information on the spectrum of the putative CCFT ?

- Can one get a closed form for the Celestial Eikonal amplitude ? Improve the dispersion relations ?

- Longer term goal: build full model for amplitude which encompass eikonal+radiation all the way to BH formation ?

thank you!

extras

Derivation of 4-pt Celestial amplitude

- PSS: $A(\Delta_i, z_i, \bar{z}_i) = \int \prod \frac{d\omega_i}{\omega_i} \omega^{\Delta_i - 1} A(\omega_i, z_i, \bar{z}_i)$
- Trivial manipulations give kinematic prefactor and $A(\Delta_i, z_i, \bar{z}_i) = XA(\beta, z)$
- where $\beta = \sum \Delta_i - 1$ and $z = \frac{z_{13} z_{24}}{z_{12} z_{34}}$

[Stieberger, Taylor; Pasterski, Shao, Strominger; Gonzales Puhm Rojas; Arkani-Hamed, Pate, Raclariu, Strominger]