Operator product expansion in carrollian CFT

based on work to appear with Jakob Salzer

Kevin Nguyen

September 3, 2024

Eurostrings 2024, Southampton

Conformal Field Theory

When theorists can feel safe

Conformal Field Theory is well-defined.

- No divergences
- Non-perturbative
- Fully calculable (in principle)
- Only known way to fully formulate quantum gravity (in AdS)
- Can be completed from the bottom-up through conformal boostrap!

Say no more... Where do I sign?

A Carrollian conformal bootstrap?

Expectations

[Bagchi-Banerjee-Basu-Dutta '22, Donnay-Fiorucci-Herfray-Ruzziconi '22, ...]

Massless scattering amplitudes are correlators in Carrollian CFT

- Indeed the Poincaré group is the conformal Carrollian group!

A Carrollian conformal bootstrap?

Expectations

[Bagchi-Banerjee-Basu-Dutta '22, Donnay-Fiorucci-Herfray-Ruzziconi '22, ...]

Massless scattering amplitudes are correlators in Carrollian CFT

- Indeed the Poincaré group is the conformal Carrollian group!

Open question

Can we construct scattering amplitudes through conformal bootstrap?

Wish list:

- No divergences
- Non-perturbative
- Fully calculable (in principle)
- Predictivity
- Bottom-up completion

Carrollian OPE = collinear factorization?

Gauge theory tree-level amplitudes satisfy collinear factorization:

$$A_n(p_1, p_2, ...) \xrightarrow{1||2} A_3(p_1, p_2, -P) \frac{1}{P^2} A_{n-1}(P, ...) , \quad P \equiv p_1 + p_2 .$$

Carrollian OPE = collinear factorization?

Gauge theory tree-level amplitudes satisfy collinear factorization:

$$A_n(p_1, p_2, ...) \xrightarrow{1||2} A_3(p_1, p_2, -P)\frac{1}{P^2}A_{n-1}(P, ...)\,, \quad P \equiv p_1 + p_2\,.$$

After applying the (modified) Mellin transform, it looks like the leading term of a holomorphic OPE [Fan-Fotopoulos-Taylor '19, Mason-Ruzziconi-Srikant '23],

$$O_1(z_1)O_2(z_2) \overset{z_{12}\sim 0}{\sim} \frac{1}{z_{12}}O_3(z_2)$$

Carrollian OPE = collinear factorization?

Gauge theory tree-level amplitudes satisfy collinear factorization:

$$A_n(p_1, p_2, \ldots) \xrightarrow{1||2} A_3(p_1, p_2, -P)\frac{1}{P^2}A_{n-1}(P, \ldots)\,, \quad P \equiv p_1 + p_2\,.$$

After applying the (modified) Mellin transform, it looks like the leading term of a holomorphic OPE [Fan-Fotopoulos-Taylor '19, Mason-Ruzziconi-Srikant '23],

$$O_1(z_1)O_2(z_2) \overset{z_{12}\sim 0}{\sim} \frac{1}{z_{12}}O_3(z_2)$$

Open questions

- Is the OPE limit always associated to a collinear limit?
- Is there a consistent OPE at subleading orders in z_{12}?
- Is there an OPE for generic massless amplitudes? (ex: $\lambda\phi^4$)

From massless particles to carrollian fields

By $\mathscr{I}$ we simply mean a null surface $\mathbb{R} \times S^2$ covered with complex stereographic coordinates $\mathbf{x} = (u, z, \bar{z})$ and conformal metric

$$ds^2_{\mathscr{I}} = 0\, du^2 + dz d\bar{z}\,.$$

From massless particles to carrollian fields

By $\mathscr{I}$ we simply mean a null surface $\mathbb{R} \times S^2$ covered with complex stereographic coordinates $\mathbf{x} = (u, z, \bar{z})$ and conformal metric

$$ds^2_{\mathscr{I}} = 0\, du^2 + dz d\bar{z}\,.$$

Given a massless particle state $|p\rangle_J$ of helicity J and momentum p^μ parametrised as

$$p^\mu = \omega(1 + z\bar{z}, z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z})\,,$$

we can define the carrollian conformal field through the *modified* Mellin transform [Banerjee '18]

$$O_{\Delta,J}(u, x^i)|0\rangle = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{-i\omega u} |p\rangle_J\,.$$

From massless particles to carrollian fields

Under Poincaré ISO(1,3) symmetries, they transform like

$$
\begin{aligned}
[H, O(\mathbf{x})] &= -i\partial_u O(\mathbf{x})\,, \\
[K, O(\mathbf{x})] &= -iz\bar{z}\partial_u O(\mathbf{x})\,, \\
[B, O(\mathbf{x})] &= -iz\partial_u O(\mathbf{x})\,, \\
[L_{-1}, O(\mathbf{x})] &= -i\partial_z O(\mathbf{x})\,, \\
[L_0, O(\mathbf{x})] &= -\frac{i}{2}\left(u\partial_u + 2z\partial_z + 2h\right) O(\mathbf{x})\,, \\
[L_1, O(\mathbf{x})] &= -iz\left(u\partial_u + z\partial_z + 2h\right) O(\mathbf{x})\,,
\end{aligned}
$$

with the chiral weights

$$
h = \frac{\Delta + J}{2}\,, \qquad \bar{h} = \frac{\Delta - J}{2}\,.
$$

They can be constructed directly from representation theory [Nguyen-West '23].

Building a simple carrollian OPE

Let's postulate the existence of an OPE of the form

$$O_1(\mathbf{x})\, O_2(0) \overset{\mathbf{x}\sim 0}{\approx} \sum_k f_{12k}(\mathbf{x})\, O_k(0) + subleading\,,$$

and constrain the functions $f_{12k}(\mathbf{x})$ by requiring consistency with Poincaré symmetry.

Building a simple carrollian OPE

Let's postulate the existence of an OPE of the form

$$O_1(\mathbf{x})\,O_2(0) \overset{\mathbf{x}\sim 0}{\approx} \sum_k f_{12k}(\mathbf{x})\,O_k(0) + \textit{subleading}\,,$$

and constrain the functions $f_{12k}(\mathbf{x})$ by requiring consistency with Poincaré symmetry. We find

$$\begin{aligned} f_{12k}(\mathbf{x}) &= \frac{c_0}{u^{2a}\, z^{h_1+h_2-h_k-a}\, \bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_k-a}} + \frac{c_1\,\delta(z)\delta(\bar{z})}{u^{\Delta_1+\Delta_2-\Delta_k-2}} \\ &\quad + \frac{c_2\,\delta(\bar{z})}{u^{\bar{h}_1+\bar{h}_2-\bar{h}_k+b-1}\, z^{h_1+h_2-h_k-b}} + \frac{\bar{c}_2\,\delta(z)}{u^{h_1+h_2-h_k+\bar{b}-1}\, \bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_k-\bar{b}}}\,, \end{aligned}$$

where the coefficients $c_0, c_1, c_2, \bar{c}_2$ as well as the exponents $a, b, \bar{b}$ are arbitrary numbers.

Application: 3-point MHV amplitude

The 3-point MHV amplitude with arbitrary quantum numbers is given by

$$\langle O_1 O_2 O_3 \rangle \sim \frac{\delta(\bar{z}_{12})\delta(\bar{z}_{13})|z_{12}|^{\Delta_3 - J_1 - J_2 - 2}|z_{23}|^{\Delta_1 - J_2 - J_3 - 2}|z_{13}|^{\Delta_2 - J_1 - J_3 - 2}}{\left(z_{23}\, u_1 + z_{31}\, u_2 + z_{12}\, u_3\right)^{2(\bar{h}_1 + \bar{h}_2 + \bar{h}_3 - 2)}}$$

assuming that $J_1 + J_2 + J_3 < 0$.

Application: 3-point MHV amplitude

The 3-point MHV amplitude with arbitrary quantum numbers is given by

$$\langle O_1 O_2 O_3 \rangle \sim \frac{\delta(\bar{z}_{12})\delta(\bar{z}_{13})|z_{12}|^{\Delta_3 - J_1 - J_2 - 2}|z_{23}|^{\Delta_1 - J_2 - J_3 - 2}|z_{13}|^{\Delta_2 - J_1 - J_3 - 2}}{\left(z_{23}\, u_1 + z_{31}\, u_2 + z_{12}\, u_3\right)^{2(\bar{h}_1 + \bar{h}_2 + \bar{h}_3 - 2)}}$$

assuming that $J_1 + J_2 + J_3 < 0$. Taking the OPE limit we get

$$\langle O_1 O_2 O_3 \rangle \overset{z_{12} \sim 0}{\sim} \frac{z_{12}^{\Delta_3 - J_1 - J_2 - 2}\delta(\bar{z}_{12})}{u_{12}^{2(\bar{h}_1 + \bar{h}_2 + \bar{h}_3 - 2)}} \delta(\bar{z}_{13}) z_{13}^{-2h_3} = f_{124}(\mathbf{x}_{12}) \langle O_4 O_3 \rangle (\mathbf{x}_{13}) \,,$$

where the quantum numbers of O_4 are given by

$$\bar{h}_4 = 1 - \bar{h}_3 \,, \qquad h_4 = h_3 \,.$$

Application: $\lambda\phi^4$ theory

The tree-level amplitude is just $M_4 = 1$. After modified Mellin transform,

$$C_4 \sim \frac{z^{\Delta_1-\Delta_2}\delta(z-\bar{z})}{(1-z)^{\Delta_3-\Delta_2}} \left|\frac{z_{24}}{z_{12}}\right|^{2(\Delta_1-1)} \left|\frac{z_{34}}{z_{23}}\right|^{2(\Delta_2-1)} \left|\frac{z_{14}}{z_{13}}\right|^{2(\Delta_3-1)} \frac{1}{|z_{13}z_{24}|^2}$$

$$\times \frac{1}{\left(u_4 - u_1 z \left|\frac{z_{24}}{z_{12}}\right|^2 + u_2 \frac{1-z}{z} \left|\frac{z_{34}}{z_{23}}\right|^2 - u_3 \frac{1}{1-z} \left|\frac{z_{14}}{z_{13}}\right|^2\right)^{\Sigma\Delta-4}}.$$

Application: $\lambda\phi^4$ theory

The tree-level amplitude is just $M_4 = 1$. After modified Mellin transform,

$$C_4 \sim \frac{z^{\Delta_1-\Delta_2}\delta(z-\bar{z})}{(1-z)^{\Delta_3-\Delta_2}} \left|\frac{z_{24}}{z_{12}}\right|^{2(\Delta_1-1)} \left|\frac{z_{34}}{z_{23}}\right|^{2(\Delta_2-1)} \left|\frac{z_{14}}{z_{13}}\right|^{2(\Delta_3-1)} \frac{1}{|z_{13}z_{24}|^2}$$
$$\times \frac{1}{\left(u_4 - u_1 z \left|\frac{z_{24}}{z_{12}}\right|^2 + u_2 \frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2 - u_3 \frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2\right)^{\Sigma\Delta-4}} .$$

In the OPE limit $\bar{z}_{12} \sim z_{12} \sim 0$ we find

$$C_4 \sim \frac{z_{12}^{\Delta_3+\Delta_4-2}\delta(\bar{z}_{12})}{u_{12}^{\Sigma\Delta-4}} \frac{1}{(\bar{z}_{23})^{1-\Delta_4}(\bar{z}_{24})^{1-\Delta_3}(\bar{z}_{34})^{\Delta_3+\Delta_4-1}} \frac{1}{(z_{23})^{\Delta_3}(z_{24})^{\Delta_4}}$$
$$= f_{125}(\mathbf{x}_{12})\langle O_5(z_2)O_3(z_3)O_4(z_4)\rangle$$

with

$$\bar{h}_5 = 1 - \frac{\Delta_3+\Delta_4}{2}, \qquad h_5 = \frac{\Delta_3+\Delta_4}{2} .$$

The collinear OPE

Let's now consider the weaker limit $z \to 0$ keeping $\bar{z}$ and u arbitrary. Then it makes sense to integrate the position of O_k,

$$O_1(\mathbf{x})O_2(0) \overset{z\sim 0}{\approx} \sum_k z^{\alpha_{12k}} \int_0^1 dt\, ds\, F_{12k}(u, \bar{z}; t, s) O_k(tu, 0, s\bar{z}) \,.$$

The collinear OPE

Let's now consider the weaker limit $z \to 0$ keeping $\bar{z}$ and u arbitrary. Then it makes sense to integrate the position of O_k,

$$O_1(\mathbf{x})O_2(0) \overset{z\sim 0}{\approx} \sum_k z^{\alpha_{12k}} \int_0^1 dt\, ds\, F_{12k}(u, \bar{z}; t, s) O_k(tu, 0, s\bar{z})\,.$$

Requiring consistency with Poincaré symmetry at this order, we find

$$\alpha_{12k} = h_k - h_2 - h_1\,,$$

$$F_{12k} = c_{12k}\, \bar{z}^{\bar{h}_k - \bar{h}_2 - \bar{h}_1}\, t^{\bar{h}_k - \bar{h}_2 + \bar{h}_1 - 1}(1-t)^{\bar{h}_k + \bar{h}_2 - \bar{h}_1 - 1}\delta(t-s).$$

The collinear OPE

Let's now consider the weaker limit $z \to 0$ keeping $\bar{z}$ and u arbitrary. Then it makes sense to integrate the position of O_k,

$$O_1(\mathbf{x})O_2(0) \overset{z\sim 0}{\approx} \sum_k z^{\alpha_{12k}} \int_0^1 dt\, ds\, F_{12k}(u, \bar{z}; t, s) O_k(tu, 0, s\bar{z})\,.$$

Requiring consistency with Poincaré symmetry at this order, we find

$$\alpha_{12k} = h_k - h_2 - h_1\,,$$
$$F_{12k} = c_{12k}\, \bar{z}^{\bar{h}_k - \bar{h}_2 - \bar{h}_1}\, t^{\bar{h}_k - \bar{h}_2 + \bar{h}_1 - 1}(1-t)^{\bar{h}_k + \bar{h}_2 - \bar{h}_1 - 1}\delta(t-s).$$

In the special case $\Delta_{1,2} = 1$ and $\Delta_3 = 1 + p$ with $p = J_1 + J_2 - J_3 - 1$, we can write

$$O_1(\mathbf{x})O_2(0) \overset{z\sim 0}{\approx} z^{-1}\, \bar{z}^p \int_0^1 dt\, t^{J_2 - J_3 - 1}(1-t)^{J_1 - J_3 - 1} \partial_u^p O_3(tu, 0, t\bar{z}),$$

thereby recovering the collinear OPE limit [Mason-Ruzziconi-Srikant '23].

The collinear OPE: comments

- At the level of 4-point MHV amplitudes, this yields

$$C_4(\mathbf{x},0;...) \overset{z\sim 0}{\approx} z^{-1}\,\bar{z}_1^p \int_0^1 dt\, t^{J_2-J_k-1}(1-t)^{J_1-J_k-1}\partial_u^p C_3(tu,0,t\bar{z},...),$$

with C_3 the three-point MHV amplitude with support

$$C_3 \propto \delta(\bar{z}_{12})\delta(\bar{z}_{23})\,.$$

Hence this collinear OPE controls the regime $\bar{z}_{12}, \bar{z}_{13} \ll z \equiv z_{14} \ll 1$.

The collinear OPE: comments

- At the level of 4-point MHV amplitudes, this yields

$$C_4(\mathbf{x}, 0; \dots) \overset{z\sim 0}{\approx} z^{-1}\, \bar{z}_1^p \int_0^1 dt\, t^{J_2-J_k-1}(1-t)^{J_1-J_k-1} \partial_u^p C_3(tu, 0, t\bar{z}, \dots),$$

with C_3 the three-point MHV amplitude with support

$$C_3 \propto \delta(\bar{z}_{12})\delta(\bar{z}_{23})\,.$$

Hence this collinear OPE controls the regime $\bar{z}_{12}, \bar{z}_{13} \ll z \equiv z_{14} \ll 1$.

- At subleading order we would naturally consider

$$O_1(\mathbf{x})O_2(0) \overset{z\sim 0}{\approx} z^{\alpha_{12k}} \int_0^1 dt\, ds\, F_{12k}(u, \bar{z}; t, s) O_k(tu, 0, s\bar{z}) + z^{\alpha_{12k}+1} \int_0^1 dt\, ds\, G_{12k}(u, \bar{z}; t, s) \partial_z O_k(tu, 0, s\bar{z})\,,$$

but this fails to solve the $ISO(1,3)$ constraints...

Perspectives

To appear soon:

- Classification of 2-,3- and 4-point functions with complex kinematics
- Carrollian OPE limits
- Carrollian OPE blocks
- Explicit examples using MHV amplitudes
- Carrollian manifestation of the double copy $GR = (YM)^2$

Open questions:

- Role of massive particles in the carrollian OPE?
  $(p_1 + p_2)^2 = 2\, p_1 \cdot p_2 \neq 0$
- How to extend the collinear OPE beyond leading order? Relation to non-factorisation of subleading collinear terms?
  [Nandan-Plefka-Wormsbecher '16]
- 4-point carrollian blocks? crossing equations? bootstrap? ...