Lessons from light-cone formulation for physics at null infinity

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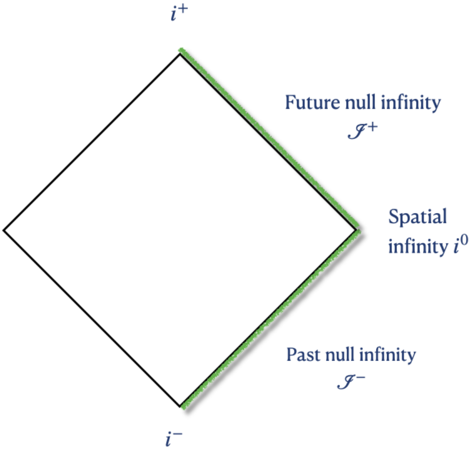
Flat-space holography

Focus: This region of spacetime



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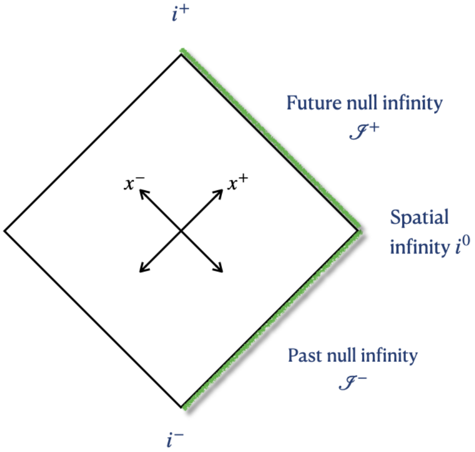


This talk

A different setup with two intersecting null hypersurfaces $\rightarrow$ **Light-cone formulation**

Light-cone formulation

"Forms of relativistic dynamics" [Dirac '49]

- Choice of 'time' for Hamiltonian dynamics

  (a) Instant form: time x^0

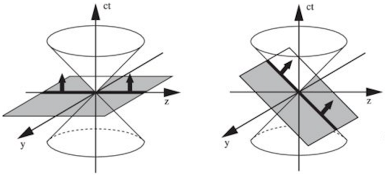
  Spatial foliations (initial data on $x^0 = 0$)

  (b) Front form: time $x^+ = \frac{x^0 + x^3}{\sqrt{2}}$

  Null foliations (initial data on $x^+ = 0$)

- Different IVP in GR: Cauchy vs. Characteristics

(a) Instant form (b) Front form



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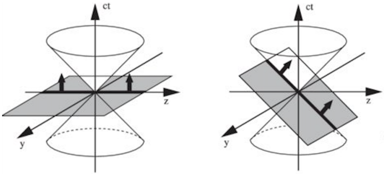

Poincaré algebra (P_μ, $M_{\mu\nu}$)

	Instant Form Cartesian coords: $(x^0, x^a),\ a = 1, 2, 3$
Kinematical	$\mathcal{K} = \{P_a, M_{ab}\}$: 6-dim
Dynamical	$\mathcal{D} = \{P_0, M_{0a}\}$: 4-dim

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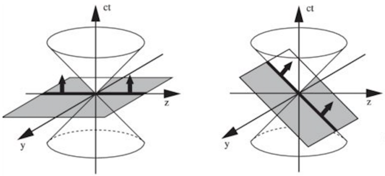

Poincaré algebra $(P_\mu, M_{\mu\nu})$

	Instant Form Cartesian coords: (x^0, x^a), $a = 1, 2, 3$	Front Form Light-cone coords, $(x^+, x^- x^i)$, $i = 1, 2$
• Kinematical	$\mathcal{K} = \{P_a, M_{ab}\}$: 6-dim	$\mathcal{K} = \{P_-, P_i, M_{-i}, M_{ij}, M_{-+}\}$: **7-dim**
• Dynamical	$\mathcal{D} = \{P_0, M_{0a}\}$: 4-dim	$\mathcal{D} = \{P_+, M_{+i}\}$: 3-dim

Largest Kinematical subgroup possible: Easier to derive interacting actions

Why light-cone?

- Reason 1: Largest kinematical subgroup
- Reason 2: Non-relativistic or Galilean aspects [Weinberg '66; Susskind'68]

Light-cone Physics $\longleftrightarrow$ non-relativistic Galilean invariance

$$P_\mu P^\mu = 2P_+ P_- - P_i P^i = 0$$

$\rightarrow$ 3D Galilei subgroup within 4D light-cone Poincaré

$$\Rightarrow \text{ Hamiltonian } P_+ = \frac{P^i P_i}{2P_-}$$

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- Reason 3: Gauge constraints often solvable $\rightarrow$ only physical d.o.f.

  "Constrained Hamiltonian Systems" [Dirac 1959; Bergmann 1959]

  $$S_H[\phi, \pi_\phi, \lambda_i] = \int dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H} - \lambda_i \mathcal{G}^i\right) , \quad \mathcal{G}^i \text{ gauge constraints}$$

  Closer to on-shell physics: Scattering amplitudes, Helicity states, GW waves...

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  Many successes: DLCQ, Loop computations in QCD,
  UV finiteness of $\mathcal{N} = 4$ Super Yang-Mills,
  Light-cone quantization of strings,
  Higher-spins and supersymmetric theories
  Scattering amps.: KLT, MHV Lagrangians, Self-dual YM, ...

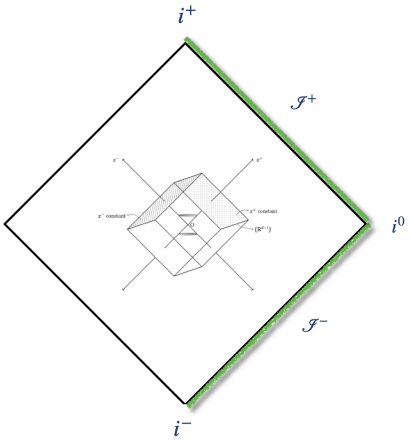
Disclaimer: Non-covariant and ugly!

Brief Outline

- Light-cone formulation of QFT
- LC approach to Asymptotic symmetries
- Some recent works:

  a) Null-front canonical analysis

  b) Links to Carrollian Physics



Light-cone Electromagnetism

Gauge-fixing the Maxwell action

Light-cone gauge : $A_- = -A^+ = -\frac{A^0 + A^3}{\sqrt{2}} = 0$

Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$

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- Constraint ($\nu = +$): $\partial_{-}^{2} A^{-} + \partial_i \partial_{-} A^i = 0$

$$A^{-} = -\frac{\partial_i A^i}{\partial_{-}} + a_1(x^+, x^i)\, x^{-} + a_0(x^+, x^i)$$

"Inverse derivative" [Mandelstam '83, Leibbrandt '83]

$$\frac{1}{\partial_{-}} g(x^{-}) = -\int \epsilon(x^{-} - y^{-})\, g(y^{-})\, dy^{-} + \text{“}const.\text{”}$$

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$$\frac{1}{\partial_-} g(x^-) = -\int \epsilon(x^- - y^-)\, g(y^-)\, dy^- + \text{"const."}$$

- Trivial equation ($\nu = -$) relates a_0 and a_1 $\Rightarrow$ one arbitrary constant   Let's set them to zero
- Dynamical equations ($\nu = i$)

$$(2\partial_- \partial_+ - \partial_i \partial^i) A^j = \Box_{lc} A^j = 0 \quad \to \quad \text{two propagating modes}$$

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Light-cone action for Electromagnetism

Complexify the $A^i \rightarrow (A, \bar{A})$: ± 1 helicity states of the photon

$$\mathcal{S}_{lc}[A, \bar{A}] = \int d^4x\ \bar{A}\,(\partial_+\partial_- - \partial\bar{\partial})\,A \quad \rightarrow \quad lc_2 \text{ formalism}$$

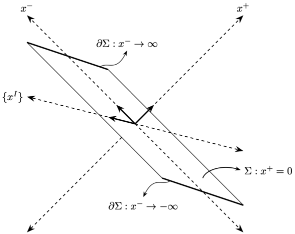
Boundary conditions are sneaky!

Boundary conditions in light-cone formalism

$$\text{Constraint:} \quad A^- = -\frac{\partial_i A^i}{\partial_-} + a_1(x^+, x^i)\, x^- + a_0(x^+, x^i)$$

$$\triangle a_0 = \partial_+ a_1 = \triangle \Phi \; ; \quad \triangle = 2\partial\bar{\partial}$$

$$\text{Fall-off at large } x^- : \; A^i = \partial^i \Phi + \frac{A^i_{(0)}}{(x^-)} + \frac{A^i_{(1)}}{(x^-)^2} + \dots$$



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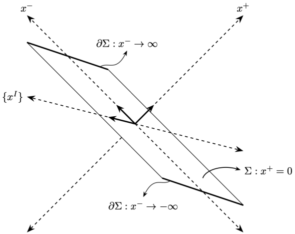

Large gauge transformations $A_\mu \to A_\mu + \partial_\mu \epsilon(x)$

Case I: Zero modes a_0, a_1 set to zero

- Fall-off: $A^i = \mathcal{O}\left(\frac{1}{x^-}\right)$
- LC action:

$$\mathcal{S}_{lc}[A, \bar{A}] = \int d^4x \; \bar{A}\, (\partial_+ \partial_- - \partial\bar{\partial})\, A$$

- LGTs : $\epsilon(x, \bar{x}) = f(x) + \bar{f}(\bar{x}) \;\to$ Constrained

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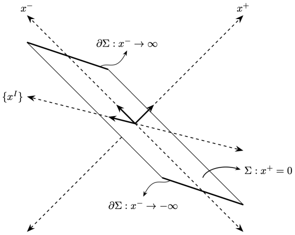

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Case II: Zero modes $a_0, a_1 \neq 0$

- Fall-off: $A^i = \mathcal{O}(1)$
- LC action:

$$\mathcal{S}[A, \bar{A}, \Phi] = \mathcal{S}_{lc}[A, \bar{A}] + \int_{\partial\Sigma} (\Phi\text{-term})$$

- LGTs : Arbitrary function $\epsilon(x, \bar{x})$

[SM (2022)]

Key lesson: Boundary conditions, zero modes in x^- subtle: handle with care!

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- Some recent works:

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  [Barnich, SM, Speziale, Tan, arXiv: 2401.14873]

  b) Links to Carrollian Physics

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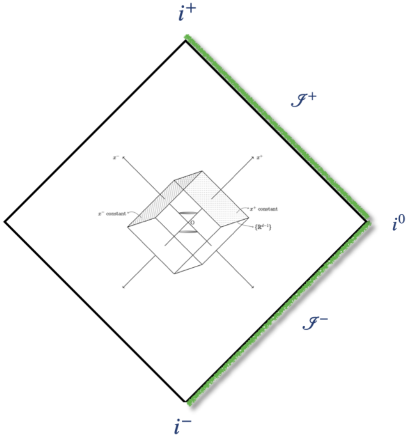

Back to basics: Chiral Bosons in 2D

Issue: Boundary conditions, zero modes in x^- delicate

How to account for boundary conditions or zero modes correctly?

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The problem: 2D chiral boson Partition function

Partition function $Z(\beta, \alpha) = Tr\ e^{-\beta \hat{H} + i\alpha \hat{P}}$

Time x^0, Periodic boundary conditions $x^1 \to x^1 + L$

$$\boxed{Z(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}}$$

τ modular parameter

[Di Francesco-Mathieu-Sènèchal or your favourite CFT textbook]

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Scalar field Lagrangian

$$S = \frac{1}{2}\int d^2x\ \partial_\mu \phi \partial^\mu \phi = \int dx^+ dx^-\ \partial_- \phi \partial_+ \phi \quad \to \quad \text{EOM: } \phi = \phi_+(x^+) + \phi_-(x^-)$$

Left and Right movers

Goal: To reproduce $Z(\tau, \bar{\tau})$ using light-cone qunatization

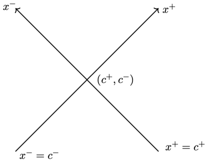
2D Chiral bosons in light-cone approach

From Hamiltonian analysis and IVP

$$S_H[\phi, \pi^+, \lambda^+] = \int dx^+ \int dx^- [\pi^+ \partial_+ \phi - \lambda^+(\pi^+ - \partial_- \phi)]$$

General Solution:

$$\phi(x^+, x^-) = \int_{c^+}^{x^+} dy^+ \underbrace{\bar{\lambda}^+(y^+)}_{\text{at } x^- = c^-} + \int_{c^-}^{x^-} dy^- \underbrace{\pi^+(y^-)}_{\text{at } x^+ = c^+} + \underbrace{\phi(c^+, c^-)}_{\text{matching conditions}}$$



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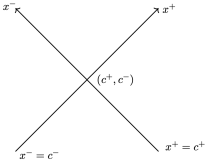

Key results [Barnich, SM, Speziale, Tan (2024)]

- Lagrange multiplier λ^+ carries part of initial data
- Infinite tower of global shift symmetries
- Matching conditions crucial for zero modes
- Must quantize on two intersecting light fronts

  $\longrightarrow$ treat both x^+ and x^- as time

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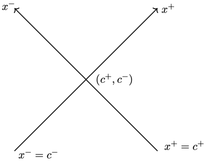

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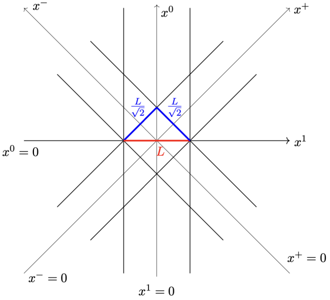

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Does the LC Poincaré algebra know about this?

$\downarrow$

Carrollian Physics

Light-cone and Carrollian Physics

Two notions of time: Newtonian and Carrollian [Duval, Gibbons, Hovarthy, Zhang 2014]

Subgroups of 4D light-cone Poincaré

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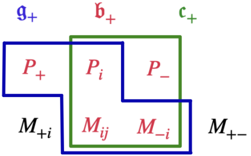

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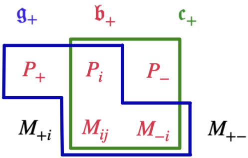

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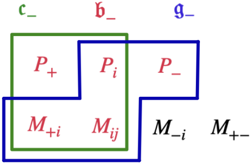

Two copies of 3D Carroll $\mathfrak{c}_\pm$, Bargmann $\mathfrak{b}_\pm$, Galilei $g_\pm$ [SM, arXiv: 2406.10353; Bagchi, Nachiketh, Soni]

$$(\mathfrak{g}_+, \mathfrak{b}_+, \mathfrak{c}_+) \xleftrightarrow{x^+ \leftrightarrow x^-} (\mathfrak{g}_-, \mathfrak{b}_-, \mathfrak{c}_-)$$

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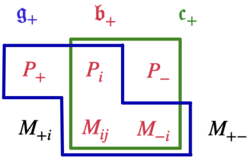

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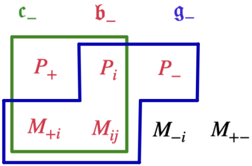

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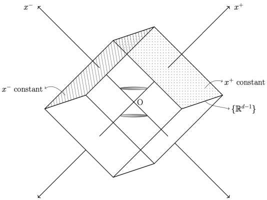
Physical relevance

- 4D Light-cone physics $\leftrightarrow$ 3D Galilean invariance: $\mathfrak{g}_+, \mathfrak{b}_+$ [Susskind]
- $\mathfrak{c}_+ \quad \rightarrow \quad$ stability group of light front at $x^+ =$ constant
- $\mathfrak{c}_- \quad \rightarrow \quad$ stability group of light front at $x^- =$ constant



Carrollian aspects of light-cone field theories

- Scalar field action:

$$\mathcal{S} = \int dx^+ dx^- d^{d-1}x \left(\partial_+\phi\partial_-\phi - \frac{1}{2}\partial_i\phi\partial^i\phi \right)$$

- Conjugate momenta:

$$\pi = \frac{\delta\mathcal{L}}{\delta(\partial_+\phi)} = \partial_-\phi \quad \Rightarrow \quad \text{Constraint} : \chi = \pi - \partial_-\phi$$

- Hamiltonian density

$$\mathcal{H}^{lc} = \pi\partial_+\phi - \mathcal{L} = \frac{1}{2}\partial_i\phi\partial^i\phi \quad \rightarrow \quad \text{No } \partial_+ \text{ or } \pi \text{ terms in } \mathcal{H}$$

Poisson bracket algebra (or more precisely, Dirac bracket)

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0$$ [Henneaux (1979)]

$\rightarrow$ Light-cone Hamiltonians are of the magnetic Carroll type!

[SM, arXiv: 2406.10353

Shortcut to obtaining Carrollian actions from Lorentzian ones

Some concluding remarks

Lessons

- Boundary conditions in LC subtle: Consider both fronts + matching conditions
- LC theories exhibit both Carrollian and Galilean features

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- Dictionary between symmetries in LC approach and asymptotic symmetries at $\mathcal{I}$
  [Barnich, Ciambelli, Gonzalez, Arxiv: 2405.17722]
- Connections to scattering amplitudes, double copy, Self-dual YM and GR, ...
- LC BMS symmetries as conformal Carroll symmetries, ...
- Ambitious goal: Explore flat-space holography

  Two intersecting null surfaces, matching conditions, Carrollian aspects, ... [Bekaert, Raj, Arxiv: 2407.17860]

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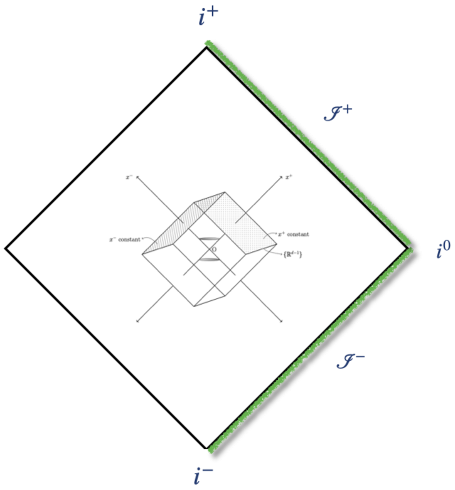
- Modest goals

  Learn about LC theories from a Carrollian perspective

  AND/ OR

  Learn about Carrollian field theories from LC theories



Thank you for your attention!