Four-point correlators in $\mathcal{N} = 4$ SYM from AdS$_5$ bubbling geometries

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Motivation

Why study 4-pt functions in holography?

- Contain dynamical information
- Non-protected by SUSY
- Strongly coupled CFT data (e.g. anomalous dimensions)

Standard holographic technique is to use Witten diagrams.

- Need to know relevant 3-pt and 4-pt couplings of bulk fields
- For some theories *not all Witten diagrams are known* (e.g. IIB on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ which is dual to D1-D5 CFT)

New technique that bypasses using Witten diagrams

For $\mathrm{AdS}_3/\mathrm{CFT}_2$ HHLL and LLLL correlators were obtained.

Giusto, Russo, Bombini, Galliani, Moscato, Wen, AT, Ceplak, Hughes; 16',17',18',19',20',21'

$\mathrm{AdS}_5/\mathrm{CFT}_4$ **holography**

String theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ SYM, $\lambda = N\, g^2_{YM}$.

Maldacena 1997

We use the supergravity approximation, $N >> 1$ and $\lambda >> 1$: leading order in $1/N$ and $1/\lambda$.

Half-BPS operators:

- $\mathcal{N} = 4$ SYM contains 6 scalars Φ^I, $I = 1, \ldots, 6$ in adjoint
- We consider holomorphic combination $Z = \Phi^1 + i\Phi^2$
- Single-trace CPOs are $\mathcal{O}_n = \mathrm{tr} Z^n$ with $\Delta = J$

In this regime local operators in $\mathcal{N} = 4$ SYM are mapped to quadratic SUGRA fluctuations around $AdS_5 \times S^5$.

Kim, Romans, van Nieuwenhuizen 1985

Outline

Compute 2-pt function of probe field in the non-trivial supergravity background and extract 4-pt function in the vacuum.

- Background is dual to heavy state created by CPOs
- Fluctuations are dual to light operators (descendants)
- Compute HHLL correlator
- Consider light limit and obtain LLLL

Bubbling LLM geometry

10d metric of the LLM solution

$$
\begin{aligned}
ds^2 &= -h^2(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2\,, \\
h^{-2} &= 2y \cosh G\,, \qquad z = \frac{1}{2} \tanh G\,,
\end{aligned}
$$

$$
y\partial_y V_i = \epsilon_{ij}\partial_j z\,, \qquad y\,(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z\,,
$$

where $i = 1, 2$.

$$
\partial_i\partial_i z + y\partial_y \left(\frac{\partial_y z}{y}\right) = 0
$$

Regularity condition on $y = 0$ plane

$$
z(x_1, x_2, y = 0) = \pm\frac{1}{2},
$$

determine black and white coloring of the two-plane.

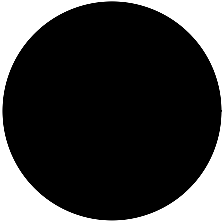
Lin, Lunin, Maldacena 2004

Ripplon deformation

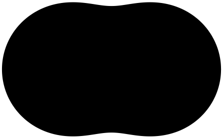
Boundary of black area

$$r(\tilde{\phi}) \,=\, \sqrt{1 + \alpha \cos n\tilde{\phi}}$$

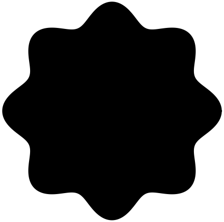
Vacuum, $\alpha = 0$



Ripple, $n = 2$



Ripple, $n = 8$



Ripplon deformation

Geometry is dual to a heavy state $|H_\alpha\rangle$ which is coherent superposition of states created by single- and multi-trace operators.

Skenderis, Taylor 2007

Giusto, Rosso 2024

For $n = 2$:

$$|H_\alpha\rangle = |0\rangle + \alpha\,\mathrm{Tr}(Z^2)|0\rangle + \alpha^2\,\left(\mathrm{Tr}(Z^2)^2 + A\,\mathrm{Tr}(Z^4)\right)|0\rangle + O(\alpha^3)$$

Energy above vacuum and R-charge of the solution:

$$E = J = \frac{1}{4}\alpha^2 N^2$$

- If $\alpha << 1$, $\alpha \sim O(N^0)$ state $|H\rangle$ is *perturbatively heavy*
- If $\alpha \sim O\left(\frac{1}{N}\right)$ the state becomes light

Geometry at $O(\alpha^0)$

At order α^0, the profile is a circle, and the background is empty global AdS

$$z = -\frac{r^2+y^2-1}{2\sqrt{(r^2+1+y^2)^2-4r^2}}\,,$$

$$V_{\tilde{\phi}} = -\frac{1}{2}\left(\frac{r^2+y^2+1}{\sqrt{(r^2+1+y^2)^2-4r^2}}-1\right), \qquad V_r = 0$$

Upon making the following change of coordinates,

$$y = R\cos\theta\,, \quad r=\sqrt{R^2+1}\sin\theta\,, \quad \tilde{\phi}=\phi-t\,,$$

the metric take the form of empty global $AdS_5 \times S^5$:

$$ds^2 = -(R^2+1)dt^2+\frac{dR^2}{R^2+1}+R^2 d\Omega_3^2+d\theta^2+\sin^2\theta d\phi^2+\cos^2\theta d\tilde{\Omega}_3^2$$

Metric expansion

We work perturbatively in small α to second order:

$$g = g^{(0)} + \alpha g^{(1)} + \alpha^2 g^{(2)}$$

To match the fields at $O(\alpha)$ with supergravity fluctuations one needs to bring the metric into de Donder-Lorentz gauge:

$$D^a h_{(ab)} = D^a h_{a\mu} = 0$$

where $\mu, \nu, \ldots$ are AdS_5 indices and $a, b, \ldots$ are S^5 indices.

Kim, Romans, van Nieuwenhuizen 1985

In de Donder-Lorentz gauge the supergravity solution contains only physical degrees of freedom and one can map linear fluctuations to single-particle operators of $\mathcal{N} = 4$ SYM.

The order α metric $g^{(1)}$ take the form

$$g^{(1)}_{\mu\nu} = \sum_{n=\pm 2} \left(-\frac{6}{5}|n| s_n Y_n \, g^{(0)}_{\mu\nu} + \frac{4}{|n|+1} Y_n \nabla_{(\mu} \nabla_{\nu)} s_n \right) ,$$

$$g^{(1)}_{\alpha\beta} \; = \; \sum_{n=\pm 2} 2|n| s_n Y_n \, g^{(0)}_{\alpha\beta} \, ,$$

where the functions s_n and Y_n are given by

$$s_n \; = \; \frac{|n|+1}{8|n|(R^2+1)^{|n|/2}} e^{int} \, , \qquad Y_n \; = \; e^{in\phi} \sin^{|n|} \theta$$

These are eigenfunctions of the Laplacians on AdS_5 and S^5:

$$\Box_{\rm A} s_n = n(n-4) s_n \, , \qquad \Box_{\rm S} Y_n = -n(n+4) Y_n \, .$$

Field s_n is dual to the CPO $\mathcal{O}_n$.

Geometry at $O(\alpha^2)$

We compute background in closed form at order α^2 for the specified profile.

Again it is convenient to construct a diffeomorphism to put the second-order metric $g^{(2)}$ into de Donder-Lorentz gauge.

The expressions for components of the metric are rather complicated, but they can be used to compute the two-point function of a probe field.

Perturbation theory for the dilaton

The equation of motion for the linearized fluctuation of the dilaton/axion is the 10D minimally coupled massless scalar wave equation on the curved background of interest:

$$\Box\Phi = 0\,.$$

We expand the d'Alembertian and the scalar field Φ perturbatively as

$$\begin{aligned}\Phi &= \Phi^{(0)} + \alpha\,\Phi^{(1)} + \alpha^2\,\Phi^{(2)}\,,\\ \Box &= \Box^{(0)} + \alpha\,\Box^{(1)} + \alpha^2\,\Box^{(2)}\,.\end{aligned}$$

The equation of motion expands as

$$\begin{aligned}\Box^{(0)}\Phi^{(0)} &= 0\,,\\ \Box^{(0)}\Phi^{(1)} &= -\Box^{(1)}\Phi^{(0)}\,,\\ \Box^{(0)}\Phi^{(2)} &= -\Box^{(2)}\Phi^{(0)} - \Box^{(1)}\Phi^{(1)}\,.\end{aligned}$$

Boundary condition

We expand the scalar field Φ in scalar spherical harmonics on $S^{(5)}$,

$$\Phi = \sum_{I_1} B^{I_1} Y^{I_1} ,$$

and expand the coefficients perturbatively in α as

$$B^{I_1} = B^{(0)I_1} + \alpha B^{(1)I_1} + \alpha^2 B^{(2)I_1}$$

The boundary condition for the correlator at large R:

$$\lim_{R\to\infty} \Phi = \frac{\delta(\vec{x}-\vec{n}) Y^I(y)}{R^{d-\Delta}} + \frac{b^I(\vec{x}) Y^I(y)}{R^{\Delta}} + \cdots ,$$

where $\vec{n}$ is a point on the boundary $\mathbb{R} \times S^3$.

We also impose smoothness of Φ in the interior.

The response $b^I(\vec{x})$ appears at order α^2 and encodes the HHLL correlator of interest.

Solution method

We are interested in light probes dual to operators $\bar{\mathcal{D}}_k \sim \bar{Q}^4 \bar{\mathcal{O}}_{k+2}$ with dimension $\Delta = k + 4$ and R-charge $J = -k$, which are descendants of anti-CPOs $\bar{\mathcal{O}}_k \sim \mathrm{Tr}\left(\bar{Z}^k\right)$.

The spherical harmonic is $Y^I = Y^{(k,-k)}$.

At order α^0 boundary condition determines the solution for $\Phi^{(0)}$:

$$\Phi^{(0)} \; = \; K_\Delta(x|\vec{n}) Y^{(k,-k)}(y) \, ,$$

where $K_\Delta(x|\vec{n})$ is a *bulk-to-boundary propagator*.

We use $\Phi^{(0)}$ solution to find $\Phi^{(1)}$ and $\Phi^{(2)}$.

Thus to order α^2 $b^I(\vec{x})$ encodes correlator:

$$b^I(\vec{x}) \; = \; \left\langle H_\alpha | \bar{\mathcal{D}}_k(\vec{n}) \mathcal{D}_k(\vec{x}) | H_\alpha \right\rangle \Big|_{\alpha^2} \; = \; \alpha^2 \left\langle \mathcal{O}_2(0) \bar{\mathcal{O}}_2(\infty) \bar{\mathcal{D}}_k(\vec{n}) \mathcal{D}_k(\vec{x}) \right\rangle$$

We write the *sources* on the RHS of the order α^2 equation

$$\mathcal{J}_1 = \Box^{(2)}\Phi^{(0)}, \qquad \mathcal{J}_2 = \Box^{(1)}\Phi^{(1)}$$

Then the order α^2 equation becomes:

$$\Box^{(0)}\Phi^{(2)} = -(\mathcal{J}_1 + \mathcal{J}_2)$$

To extract $B^{(2)}$, we project the sources $\mathcal{J}_1$, $\mathcal{J}_2$ on the highest-weight spherical harmonic $Y^{(k,k)}$:

$$\langle \mathcal{J}_i \rangle \equiv \frac{1}{||Y_k||^2} \int d\Omega_5 \, \mathcal{J}_i \left(Y^{(k,-k)} \right)^* , \qquad i = 1, 2$$

The 5d equation to be solved:

$$\Box_5^{(0)} B^{(2)} - m^2 B^{(2)} = -\langle \mathcal{J}_1 \rangle - \langle \mathcal{J}_2 \rangle$$

We now transform to Euclidean Poincaré coordinates with line element

$$ds^2_{\mathrm{EAdS}_5} = \frac{1}{w_0^2}\left(dw_0^2 + \sum_{i=1}^{4} dw_i^2\right)$$

The bulk-to-boundary propagator with boundary point at $\vec{x}$:

$$K_\Delta(\mathbf{w}|\vec{x}) \equiv \left(\frac{w_0}{w_0^2 + |\vec{w} - \vec{x}|^2}\right)^\Delta , \quad \mathbf{w} = (w_0, \vec{w}) ,$$

Next we need to rewrite the projected sources in terms of sum of products of three bulk-to-boundary propagators.

The fourth propagator comes from solving 5d equation.

Then the answer can be expressed in terms of D-functions:

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1, x_2, x_3, x_4) = \int d^5\boldsymbol{w}\sqrt{g} \prod_{i=1}^{4} K_{\Delta_i}(\boldsymbol{w}|\vec{x})$$

Mellin space representation

For correlators with pairwise equal dimensions the conformal invariance implies:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \;=\; \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_3}}\mathcal{G}(U,V)\,,$$

$$\begin{aligned}\mathcal{G}(U,V) \;=\; \frac{\pi^2}{2}\int \frac{ds}{4\pi i}\frac{dt}{4\pi i}U^{\frac{s}{2}}V^{\frac{t}{2}-\frac{\Delta_1+\Delta_3}{2}}\Gamma\left[\Delta_1-\frac{s}{2}\right]\Gamma\left[\Delta_3-\frac{s}{2}\right]\\ \times\Gamma^2\left[\frac{\Delta_1+\Delta_3-t}{2}\right]\Gamma^2\left[\frac{\Delta_1+\Delta_3-u}{2}\right]\mathcal{M}(s,t)\,,\end{aligned}$$

where $s+t+u=2\Delta_1+2\Delta_3$.

Result in Mellin space

Up to the overall normalization the result for the LLLL correlator of two CPOs and two descendants $\langle \mathcal{O}_2(0)\bar{\mathcal{O}}_2(\infty)\bar{\mathcal{D}}_k(\vec{n})\mathcal{D}_k(\vec{x})\rangle$:

$$\mathcal{M} \sim \frac{1}{s-2}\Big(\left(k^2+7k+12\right)u^2 - 2\left(k^3+10k^2+41k+60\right)u + k^4+13k^3+78k^2+240k+304\Big) + \frac{8k(k+1)}{u-(k+4)} + (k+3)\left((k+4)u-k^2-2k-16\right),$$

where $u = 2k+12-s-t$.

Superconformal Ward identity

The four-point correlator of CPOs ($\mathcal{O}_p = \mathrm{Tr}\, Z^p,\ Z = \phi_1 + i\phi_2$):

$$\langle \mathcal{O}_2(x_1)\bar{\mathcal{O}}_2(x_2)\bar{\mathcal{O}}_{k+2}(x_3)\mathcal{O}_{k+2}(x_4)\rangle = \frac{1}{(x_{12}^2)^2(x_{34}^2)^{k+2}}\mathcal{G}^{(\mathrm{CPO})}_{2\,,k+2}(U,V)\,,$$

where

$$\mathcal{G}^{(\mathrm{CPO})}_{2\,,k+2}(U,V) \,=\, V^2\mathcal{H}^{(\mathrm{CPO})}_{2\,,k+2}(U,V)\,, \quad \mathcal{H}^{(\mathrm{CPO})}_{2\,,k+2} \,=\, U^{k+2}\,\bar{D}_{k+2,\,k+4,\,2,\,2}$$

The superconformal Ward identity involves the following differential operator

$$\Delta^{(2)} = U\partial_U^2 + V\partial_V^2 + (U+V-1)\,\partial_U\partial_V + 2\,(\partial_U + \partial_V)$$

Drummond, Gallot, Sokatchev 2006

Gonçalves 2014

Comparing the result of WI to supergravity calculation in Mellin space, the coefficient of the pole at $s=2$ determines the overall normalization. We then find precise agreement for $\mathcal{M}$.

Conclusions

Summary

- We computed 4-pt functions of 2 CPOs and 2 descendants in the large N limit by replacing it with 2-pt functions in non-trivial LLM geometry
- LLLL correlators were related by superconformal WI to known 4-pt correlators of CPOs
- New supergravity method bypasses using Witten diagrams

Future directions

- More general correlators with single-traces $\mathcal{O}_2 \to \mathcal{O}_n$
- Correlators with multi-traces $(\mathcal{O}_n)^p$

Thank you for your attention!