Non-Invertible Duality Defects in 2+1d QFTs from Half Spacetime Gauging

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Based on 2406.09261 *with W. Cui and B. Haghighat*

Non-Invertible Symmetries

In a relativistic QFT

ordinary (invertible) **global symmetry** $\Rightarrow$ **topological defect**

Is the converse true? Already in 1+1d topological line defects satisfy

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \mathcal{L}_c , \quad N_{ab}^c \in \mathbb{N}_{\geq 0}$$

For the converse to be true we need non-invertible symmetries

- Symmetries do not follow a group-like composition
- Not every symmetry generator has an inverse $\mathcal{L} \times \mathcal{L}^{-1} = 1$

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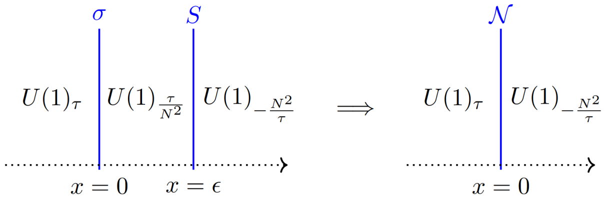
Consider **duality defects** [Choi,Cordova,Hsin,Lao,Shao '21; Kaidi,Ohmori, Zheng '21]: QFT $\mathcal{Q}$ with finite and non-anomalous symmetry $G^{(p)}$ has a non-invertible duality symmetry when

$$\mathcal{Q} \simeq \mathcal{Q}/G^{(p)}$$

Duality Defects: Half Spacetime Gauging

Example: 4d Maxwell theory at $\tau = iN$ [Choi,Cordova,Hsin,Lao,Shao '21]

- **Electric symmetry**: symmetry operator $\eta = \exp \oint \star F$ acting on Wilson lines. Gauging $\mathbb{Z}_N^{(1)} \subset U(1)_e^{(1)}$ rescales $A \to A/N$ or $\tau \to \tau/N^2$
- **S-duality**: isomorphism between the theories at $\tau \simeq -1/\tau$



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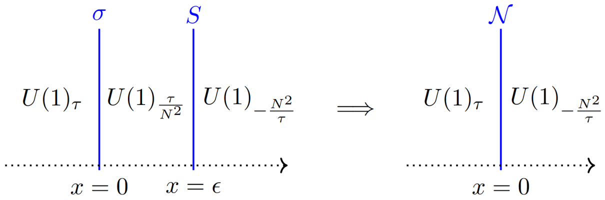


The worldvolume of the $\mathcal{N} = \sigma \circ S$ is

$$\frac{\mathrm{i}N}{2\pi} \int_{x=0} A_L \wedge dA_R$$

Fusion rule gives a **condensation defect**, gauging $\mathbb{Z}_N^{(1)}$ on a codimension one submanifold (one-gauging) [Roumpedakis,Seifnashri,Shao '22]

$$\mathcal{N} \times \overline{\mathcal{N}} = \sum_{S \in H_2(M_4, \mathbb{Z}_N)} \eta(S) = \mathcal{C}_{\mathbb{Z}_N^{(1)}}$$

Duality Defects: 't Hooft Anomaly

4d QFT $\mathcal{Q}'$ with $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ symmetries and a mixed 't Hooft anomaly

$$\pi \int_{M_5} A^{(1)} \cup \frac{1}{2}\mathcal{P}(B^{(2)}), \quad \partial M_5 = M_4,\ M_4 \text{ spin}$$

In $\mathcal{Q} = \mathcal{Q}'/\mathbb{Z}_2^{(1)}$ the defect D generating $\mathbb{Z}_2^{(0)}$ is not gauge invariant, however it gives rise to a non-invertible symmetry **coupled to a TQFT such that the anomaly vanishes** [Kaidi,Ohmori,Zheng '21]

$$\mathcal{N} = D \circ \mathcal{A}^{2,1}, \quad \mathcal{A}^{2,1} = U(1)_2 \text{ Chern-Simons theory}$$

$$\mathcal{N} \times \overline{\mathcal{N}} = \mathcal{C}_{\mathbb{Z}_N^{(1)}}, \quad \text{as } \mathcal{A}^{2,1} \times \overline{\mathcal{A}}^{2,1} = \mathbb{Z}_2 \text{ gauge theory}$$

- Examples include time reversal in 4d YM and the axial symmetry in 4d QED and QCD
- Duality defects constructed in this way can be **obtained via half spacetime gauging**, viceversa is not true

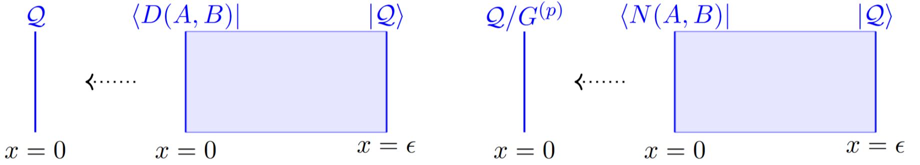
SymTFT

Non-invertible duality defects can be conveniently described by the Symmetry TFT [Kaidi,Ohmori,Zheng '22], a TFT on a slab in d+1 which decouples the dynamics of $\mathcal{Q}$ from its symmetries $G^{(p)}$, and is **independent on topological manipulations** of $\mathcal{Q}$

On the two ends of the interval one imposes boundary conditions

$$\text{top: } \langle D(A)| = \sum_a \delta(a - A)\langle a|, \quad \langle N(A)| = \sum_a \exp\left(-\mathrm{i}\int a \cup A\right)\langle a|$$

$$\text{phys: } |\mathcal{Q}\rangle = \sum_a Z_{\mathcal{Q}}[a]|a\rangle, \qquad a \text{ set of flat connections of } G^{(p)}$$



Group Theoretical vs. Intrinsically Non-Invertible

A non-invertible duality defect $\mathcal{N}$ in $\mathcal{Q}$ is said group theoretical if it can be **mapped via a topological manipulation (e.g. gauging) to an invertible defect** in $\mathcal{Q}'$. It is denoted intrinsic otherwise

> $\Rightarrow$ All duality defects constructed via a 't Hooft anomaly are group theoretical. Intrinsic non-invertible ones only via half spacetime gauging

Given a generic duality defect, the SymTFT, being invariant under topological manipulations, can be used to determine whether $\mathcal{N}$ is group theoretical [Sun,Zheng '23]

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- The SymTFT of a theory with invertible symmetries is a gauged anomaly theory, that is a **Dijkgraaf-Witten theory**, and viceversa
- Example: in 4d gauging in half of spacetime $\mathbb{Z}_N^{(1)}$ gives rise to group theoretical defects only for $N = L^2 M$ where -1 is a quadratic residue of M

Non-Invertible Duality Defects in 3d

- Group theoretical duality defects starting from $\mathcal{Q}'$ with a **mixed 't Hooft anomaly** between $\mathbb{Z}_{2,1}^{(0)} \times \mathbb{Z}_{2,2}^{(0)} \times \mathbb{Z}_2^{(1)}$ [Kaidi,Ohmori,Zheng '21]

  $\Rightarrow$ In $\mathcal{Q} = \mathcal{Q}'/(\mathbb{Z}_{2,2}^{(0)} \times \mathbb{Z}_2^{(1)})$ the anomalous symmetry $\mathbb{Z}_{2,1}^{(0)}$ gives rise to a non-invertible symmetry

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- In [Cui,Haghighat,LR '24] the goal is to construct duality defects via **half spacetime gauging**: $\mathcal{Q}$ with $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)}$ symmetries and vanishing 't Hooft anomaly, and

$$\mathcal{Q} \simeq \mathcal{Q}/(\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)})$$

  The fusion rules are: $\mathcal{N}_2 \times \overline{\mathcal{N}}_2 = \mathcal{C}_{\mathbb{Z}_N^{(1)}} \times \mathcal{C}_{\mathbb{Z}_N^{(0)}}$ where the rhs corresponds to gauging on a codimension one submanifold (one-gauging) $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)}$ [Roumpedakis,Seifnashri,Shao '22]

4d SymTFT

Not assuming $\mathcal{Q} \simeq \mathcal{Q}/(\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)})$ the SymTFT is a **4d BF theory**

$$S_{4d} = \frac{4\pi}{N} \int_{M_4} (\delta a_1 \cup b_1 + \delta a_2 \cup b_2)\,,$$

With line and surface operators generating $\mathbb{Z}_N^{(2)} \times \mathbb{Z}_N^{(2)} \times \mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$

$$L_{(l_1,l_2)}(\gamma) = \exp\left(\frac{2\pi i}{N} \oint_\gamma l_1 a_1\right) \exp\left(\frac{2\pi i}{N} \oint_\gamma l_2 a_2\right), \quad (l_1, l_2) \in \mathbb{Z}_N \times \mathbb{Z}_N,$$

$$S_{(s_1,s_2)}(\sigma) = \exp\left(\frac{2\pi i}{N} \oint_\sigma s_1 b_1\right) \exp\left(\frac{2\pi i}{N} \oint_\sigma s_2 b_2\right), \quad (s_1, s_2) \in \mathbb{Z}_N \times \mathbb{Z}_N.$$

There is also a $\mathbb{Z}_4^{\text{EM}}$ **electromagnetic exchange symmetry**

$$(a_1, a_2) \to (-a_2, a_1), \qquad (b_1, b_2) \to (-b_2, b_1)$$

generated by a condensation defect $\mathcal{C}_{\mathbb{Z}_N^{(2)}} \times \mathcal{C}_{\mathbb{Z}_N^{(1)}}$, 1-gauging $\mathbb{Z}_N^{(2)} \times \mathbb{Z}_N^{(1)}$

4d SymTFT for Duality Defect

Twist defect generating one-gauging on $\partial M_3 = M_2$, imposing Dirichlet boundary conditions on M_2

$$V_{(0,0)}(M_3, M_2) = \mathcal{C}_{\mathbb{Z}_N^{(1)}}(M_3, M_2) \times \mathcal{C}_{\mathbb{Z}_N^{(0)}}(M_3, M_2)$$

Imposing that $\mathcal{Q} \simeq \mathcal{Q}/(\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)})$ is equivalent to gauge $\mathbb{Z}_4^{\mathrm{EM}}$, **the bulk of $V_{(0,0)}$ becomes transparent**

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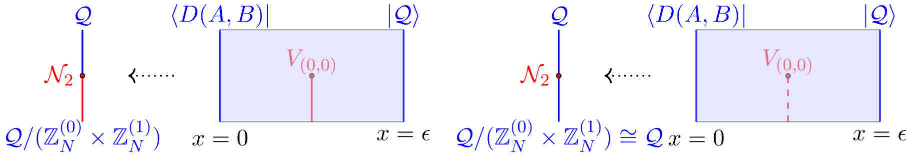
Imposing that $\mathcal{Q} \simeq \mathcal{Q}/(\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)})$ is equivalent to gauge $\mathbb{Z}_4^{\text{EM}}$, **the bulk of $V_{(0,0)}$ becomes transparent**. Upon shrinking the slab:



- The SymTFT becomes a $\mathbb{Z}_4^{\text{EM}}$ gauged version of the 4d BF theory
- Full symmetry is **higher categorical analog of the Tambara-Yamagami category**, $\text{TY}(\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)})$

$U(1) \times U(1)$ Gauge Theory

Combining gauging a subgroup of $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)} \subset U(1)_{M,1}^{(0)} \times U(1)_{E,2}^{(1)}$ and S-transformation

$$\tau = \mathrm{i}e_1/e_2, \quad \tau \to -1/\tau$$

As in 4d Maxwell theory, the effect of gauging σ is

- $\mathbb{Z}_N^{(0)}$: $A_1 \to A_1 N$ or $e_1 \to e_1 N$
- $\mathbb{Z}_N^{(1)}$: $A_2 \to A_1/N$ or $e_2 \to e_2/N$

At $e_1 = e_2 N$ ($\tau = \mathrm{i}N$) the combined action gives rise to duality defect

$$\tau \xrightarrow{\quad\sigma\quad} \tau N^2 \xrightarrow{\quad S\quad} -N^2\tau^{-1}$$

The worldvolume of the duality defect is

$$S = \frac{\mathrm{i}N}{2\pi}\int_{x=0}(d\phi_1^L \wedge A_2^R + \phi_1^R \wedge dA_2^L)$$

Product Theories

From a 3d QFT $\mathcal{T}$ with anomaly free symmetry $G^{(0)}$, consider

$$\mathcal{Q} = \mathcal{T} \times (\mathcal{T}/G^{(0)})$$

(for lattice theories [Choi,Sanghavi,Shao,Zheng '24)]). It admits a non-invertible duality defect as

$$\mathcal{Q} \simeq \mathcal{Q}/(G^{(0)} \times \widehat{G}^{(1)}), \quad \widehat{G}^{(1)} \text{ “quantum symmetry” of } \mathcal{T}/G^{(0)}$$

As an example take $\mathcal{T} = SO(N)_K$ with N_f adjoint scalars

- $N, K, N_f = 0$ mod 4 with symmetries $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^C \times \mathbb{Z}_2^M$ and a mixed 't Hooft anomaly. $\mathcal{T}/(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^C)$ has group theoretical duality defects [Kaidi,Ohmori,Zheng '21]
- Generic N, K, N_f, the magnetic $\mathbb{Z}_2^M$ is always present. Then $\mathcal{T}/\mathbb{Z}_2^M = Spin(N)_K$ with N_f adjoint scalars and $\mathcal{Q} = \mathcal{T} \times (\mathcal{T}/\mathbb{Z}_2^M)$ has a non-invertible duality defect

6d $\mathcal{N} = (2,0)$ SCFTs

Start from 6d $\mathcal{N} = (2,0)$ SCFTs of type A_{N-1}, a **relative theory** on the boundary of a 7d TFT, and compactify on M_3 ([Bashmakov,Del Zotto,Hasan '22; Chen,Chen,Cui,Haghighat '22] for M_2, M_4)

$$S_{7d} = \frac{N}{4\pi}\int_{M_7} c \wedge dc \quad \longrightarrow \quad S_{4d} = \frac{N}{4\pi}\sum_{ij} Q^{ij}\int_{M_4} a_i \wedge db_j$$

$$a_i = \int_{\zeta_i} c, \quad b_i = \int_{\eta_i} c, \quad \zeta_i, \eta_i \text{ basis of } H_2(M_3, \mathbb{Z}_N), H_1(M_3, \mathbb{Z}_N)$$

- A choice of **polarization** $\Lambda_i \subset H_1(M_3, \mathbb{Z}_N) \oplus H_2(M_3, \mathbb{Z}_N)$ (maximal isotropic lattice), determines the symmetries of absolute theories in 3d
- On top of gauging, also $\mathrm{MCG}(M_3)$ (group of diffeomorphisms of M_3 connected to the identity) transforms between absolute theories

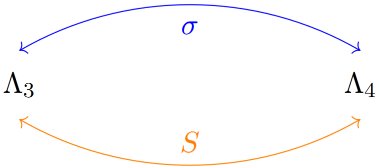
Connected Sums of $S^2 \times S^1$

$\mathrm{MCG}((S^2 \times S^1)\#(S^2 \times S^1)) = (\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2) \times \mathbb{Z}_2$ and the overall $\mathbb{Z}_2$ factor switches the two $S^2 \times S^1$ components

$$S : (S^2 \times S^1)_1 \leftrightarrow (S^2 \times S^1)_2.$$

Absolute theories $N = p$, $l_i \in H_1((S^2 \times S^1)_i, \mathbb{Z}_N)$, $s_i \in H_2((S^2 \times S^1)_i, \mathbb{Z}_N)$

$$\Lambda_3 = \langle l_1, s_2 \rangle \;\rightarrow\; \mathbb{Z}^{(1)}_{p,\mathrm{E}} \times \mathbb{Z}^{(0)}_{p,\mathrm{M}}, \quad \Lambda_4 = \langle l_2, s_1 \rangle \;\rightarrow\; \mathbb{Z}^{(1)}_{p,\mathrm{M}} \times \mathbb{Z}^{(0)}_{p,\mathrm{E}}$$



Combining action of gauging σ and S gives back Λ_3 but a **different value of the coupling constant**

$$\tau = \frac{\mathrm{vol}(S^2 \times S^1)_1}{\mathrm{vol}(S^2 \times S^1)_2}$$

At $\mathrm{vol}(S^2 \times S^1)_1 = \mathrm{vol}(S^2 \times S^1)_2$ we have a **non-invertible defect**

Conclusions

We have found:

- Non-invertible duality defects in 2+1d via **half spacetime gauging** and studied the SymTFT describing them
- Various theories admitting such duality defects: $U(1) \times U(1)$ gauge theories, product theories and via compactification

Future directions:

- Study conditions under which the duality defects are **group theoretical**
- Find 3d gauge theories admitting duality defects and which are not product theories, using the large web of dualities in 3d
- Consider deformation triggering RG flow and preserving the non-invertible symmetry to study IR phases of 3d theories