

Mellin Momentum Amplitudes and onshell bootstrap in (A)dS

*Focus: boundary correlator or wave function coefficients for
 n -gluons and gravitons scattering in (A)dS,*

Yuyu Mo

Based on

[Jiajie Mei, 2305.13894]

[Jiajie Mei, Y.M., 2402.09111]

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, Y.M. 2407.16052]

Motivation : bootstrap

Principles

Causality, unitarity,
locality, symmetries, ...



Lagrangian

Fields, equations of motion,
Feynman diagrams, ...



Observables

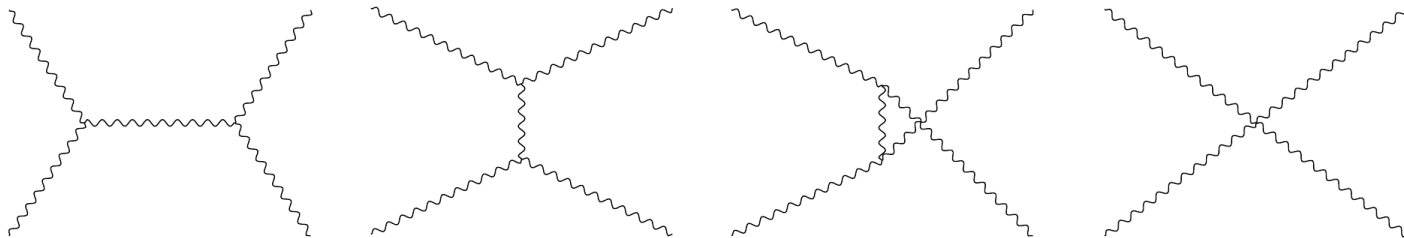
Cross sections, decay rates,
(Mellin-momentum) amplitudes
cosmological correlators, ...



Bootstrap

Text book approach

$$\mathcal{M}_{GR,4}^{flat} \sim$$



Amplitude bootstrap

$$\begin{aligned} & \mathcal{M}_{GR,4}^{flat} (1^{--} 2^{--} 3^{++} 4^{++}) \\ &= \frac{\langle 12 \rangle^4 [34]^4}{stu} \end{aligned}$$

[Cheung '17]

[Arkani-Hamed, Huang, Huang '17]

Motivation: why bootstrap and amplitude

Lorentz symmetry, unitarity, and locality *uniquely*
determined that **two derivative spin-2 particles** with two degrees of freedom can **only** be Einstein Gravity.

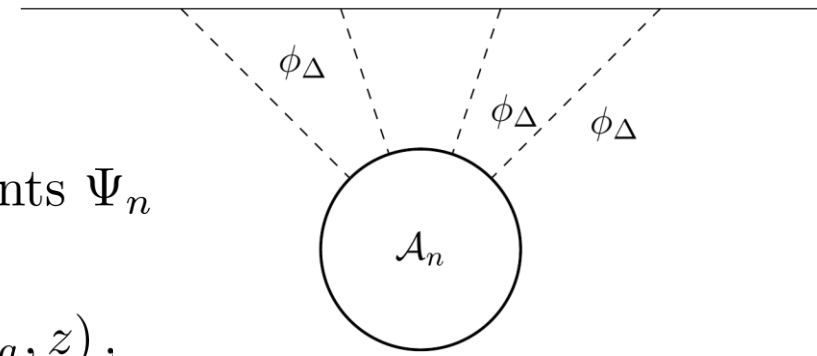
Q: What about curved space? Cosmology, Black holes background?

[Arkani-Hamed, Baumann, Lee, Pimentel, 18]

Correlators on the boundary

CFT correlator $\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle$ or Wavefunction coefficients Ψ_n

$$\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle = \int \frac{dz}{z^{d+1}} \mathcal{A}_n(z, \partial_z, \vec{k}_a, \vec{\varepsilon}_a) \prod_{a=1}^n \phi_\Delta(k_a, z),$$



1. **Not** invariant under field redefinition because of local terms

$$\phi \rightarrow \phi + \alpha \phi^3,$$

$$\langle \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \rangle \rightarrow \langle \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \rangle - \frac{1}{3} \alpha \sum_{i=1}^4 (k_i^3).$$

2. **Not** invariant under symmetry transformation because of local terms

Shift symmetry

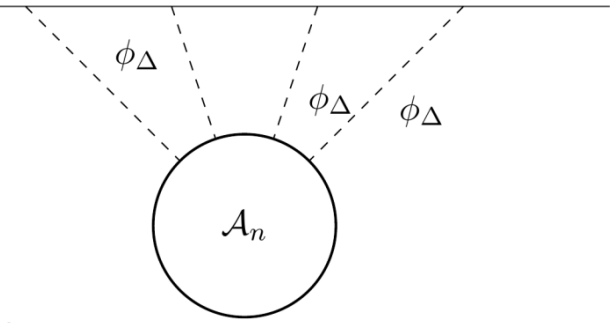
Conformal ward identities
$$\tilde{K}^i \langle J_1^\pm O_2 \dots O_n \rangle \sim \varepsilon_\pm^i \sum_{a=2}^n e_a \langle O_{\vec{k}_2} \dots O_{\vec{k}_a + \vec{k}_1} \dots O_{\vec{k}_n} \rangle.$$

What do we mean by local We don't have the problem in flat space because of LSZ for amplitudes!

$$\partial^\mu \langle J_\mu(\vec{x}_1) O(\vec{x}_2) \dots O(\vec{x}_n) \rangle = - \sum_{a=2}^n \delta(\vec{x}_1 - \vec{x}_a) \langle O(\vec{x}_2) \dots \delta O(\vec{x}_a) \dots O(\vec{x}_n) \rangle.$$

Mellin momentum amplitudes introduction

CFT correlator $\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle$ or Wavefunction coefficients Ψ_n



$$\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i+d/2},$$

$$\phi_\Delta(k, z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s, k),$$

[Sleight '19]

[Sleight and Taronna '19 '21]

Onshellness

$$\mathcal{D}_k^\Delta \equiv z^2 k^2 - z^2 \partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d)$$



$$(z^2 k^2 + (d/2 - \Delta)^2 - 4s^2) = 0$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk, s)$

- No local terms
- **Exactly invariant**
field redefinition
symmetry transformation

Definition needs full set of $\prod_{i=1}^n \phi(s_i, k_i)$

due to LSZ [Jiajie Mei, 23]

Link to flatspace amplitude

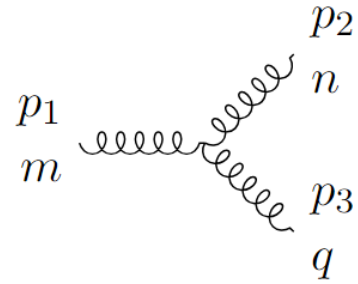
Table 1: Comparison between Minkowski and AdS

Amplitude in Minkowski space	Mellin-Momentum amplitude in AdS
Lorentz-Invariance	Conformal Invariance
Translation symmetry $e^{ik_\mu \cdot x^\mu}$ $(2\pi)^{d+1} \delta^{d+1} \left(\sum_{i=1}^n k_i^\mu \right)$	Boundary translation + Dilatation $e^{ik_\mu \cdot x^\mu} z^{-2s+d/2}$ $(2\pi)^d \delta^d \left(\sum_{i=1}^n k_i^\mu \right)$ $\times 2\pi i \delta \left(d + b + \sum_{i=1}^n (2s_i - d/2) \right)$
On-shell condition $k^2 + m^2 = 0$	On-shell condition $z^2 k^2 + (d/2 - \Delta)^2 - 4s^2 = 0$
Pole structure $\frac{1}{(k_i + \dots + k_n)^2}$	Factorization+OPE $\frac{1}{\mathcal{D}_k^\Delta} \quad \frac{1}{(k_I)^2}$

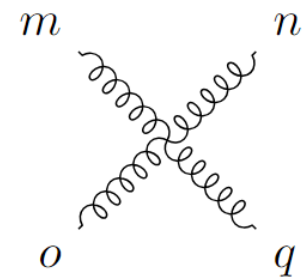
Mellin momentum to flatspace amplitudes $s_i \rightarrow \frac{zk_i}{2}$, $\mathcal{D}_{k_{ij}} \rightarrow z^2 S_{ij}$

Example: Feynman rule for gluon in MM amplitude

[J. Mei, YM, 24]

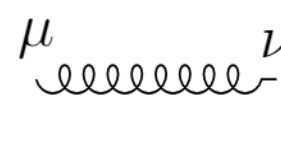


$$: V_{mnq}(p_1, p_2, p_3) = \frac{1}{2} (\eta_{mn}(p_1 - p_2)_q + \eta_{nq}(p_2 - p_3)_m + \eta_{qm}(p_3 - p_1)_n),$$



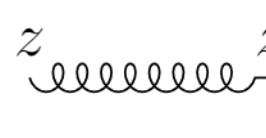
$$: V_{mnqo} = \left(\frac{1}{2} \eta_{mq} \eta_{no} - \frac{1}{4} (\eta_{mn} \eta_{qo} + \eta_{mo} \eta_{nq}), \right)$$

$$p^m = (z k^\mu, i(2s - d/2))$$



$$: G_{\mu\nu} = \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}},$$

$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$



$$: G_{zz} = \frac{1}{z^2 k^2},$$

$$(\mathcal{D}_k^\Delta(z))^{-1} F(z) = \int \frac{dy}{y^{d+1}} G_\Delta(k, z, y) F(y)$$

$$\mathcal{D}_k^\Delta(z) G_\Delta(k, z, y) = z^{d+1} \delta(z - y)$$

Bootstrap higher points (YM and GR)

Starting point $\mathcal{A}_3(1, 2, 3)$ $\mathcal{M}_3(1, 2, 3)$

Unitarity: amplitude will factorize into lower point on-shell amplitudes

$$\mathcal{A}_n \rightarrow \frac{\sum_h \mathcal{A}_a \mathcal{A}_{n-a+2}}{\mathcal{D}_{k_I}^\Delta}$$

Soft (OPE) limit: amplitude should have vanishing residue in OPE limit

$$\text{Res}_{k_I^2 \rightarrow 0} \mathcal{A}_n = 0$$

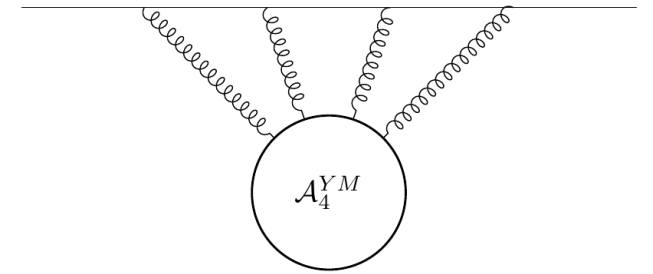
Flatspace limit: the particles are ignoring the curvature correction and the Lorentz symmetry will emerge

$$\lim_{R \rightarrow \infty} \mathcal{A}_n \rightarrow A_n$$

MM amplitudes reduce to amplitudes in flat space

Four point YM from bootstrap

[Jiajie Mei, 23]



Starting point

$$\mathcal{A}_3(1, 2, 3)$$

Unitarity

$$a(1, 2, 3, 4) = \sum_{h=\pm} \mathcal{A}_3(1, 2, -k_s^h) \mathcal{A}_3(k_s^{-h}, 3, 4).$$

$$\sum_r \varepsilon_i(k, r) \varepsilon_j(k, r)^* = \eta_{ij} - \frac{k_i k_j}{k^2} \equiv \Pi_{ij}$$

Vanishing residue
in OPE limit

$$b(1, 2, 3, 4) = -\text{Res}_{k_s^2 \rightarrow 0} \left(\frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^{d-1}} \right) = \frac{(s_1 - s_2)(s_3 - s_4) \varepsilon_{12,34}}{z^2 k_s^2}.$$

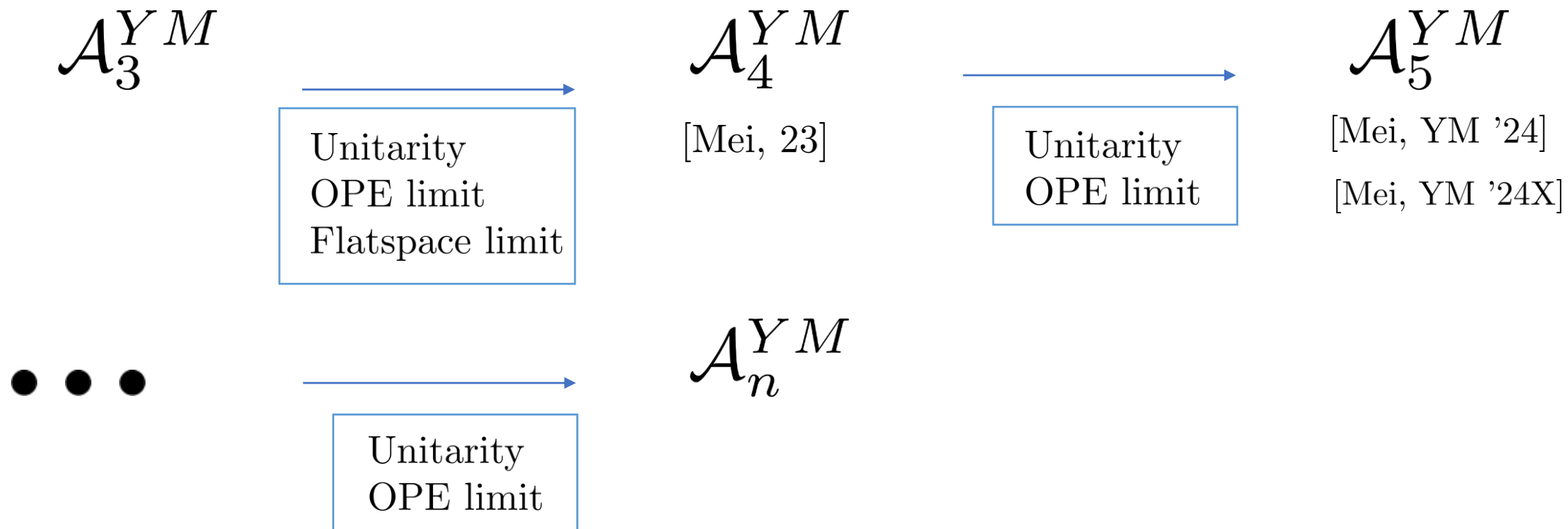
Consistency check:
Reproducing the Feynman rule computation

$$\mathcal{A}_4^s = \left(\frac{V^{12\mu} \Pi_{\mu\nu} V^{\nu 34}}{\mathcal{D}_{k_s}^{d-1}} + \frac{V^{12z} V^{z 34}}{z^2 k_s^2} \right)$$

Flatspace limit

$$\mathcal{A}_4^c = \left(\frac{1}{2} \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \frac{1}{4} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 \right)$$

N point YM from bootstrap



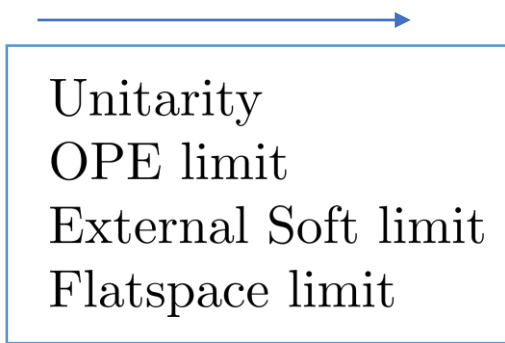
Final scalar integrals give

$$\langle JJ \dots J \rangle$$

The main result: 1, 5 point graviton amplitude

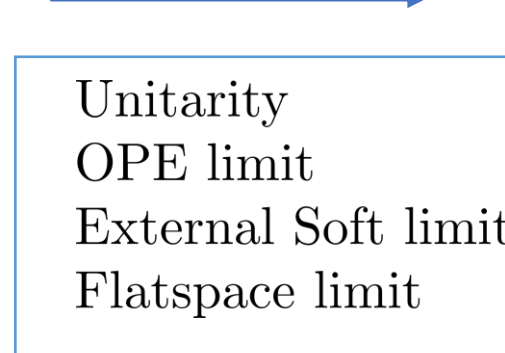
Following similar procedure

$$\mathcal{M}_3^{GR}$$



[Jiajie Mei, 23]

$$\mathcal{M}_4^{GR}$$



[Jiajie Mei, YM, 24]

[Jiajie Mei, YM, 24X]

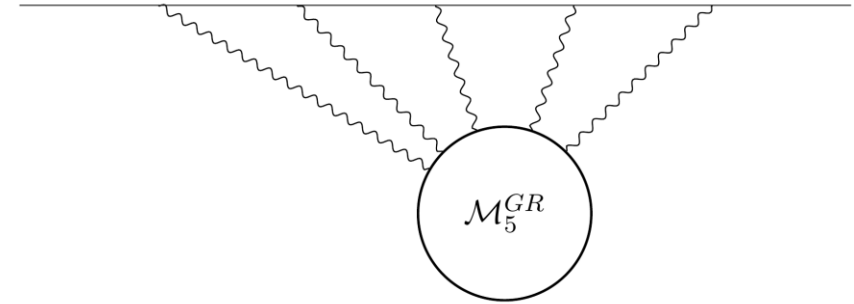
$$\mathcal{M}_5^{GR}$$

$$\mathcal{M}_n^{GR}$$



Final scalar integrals give

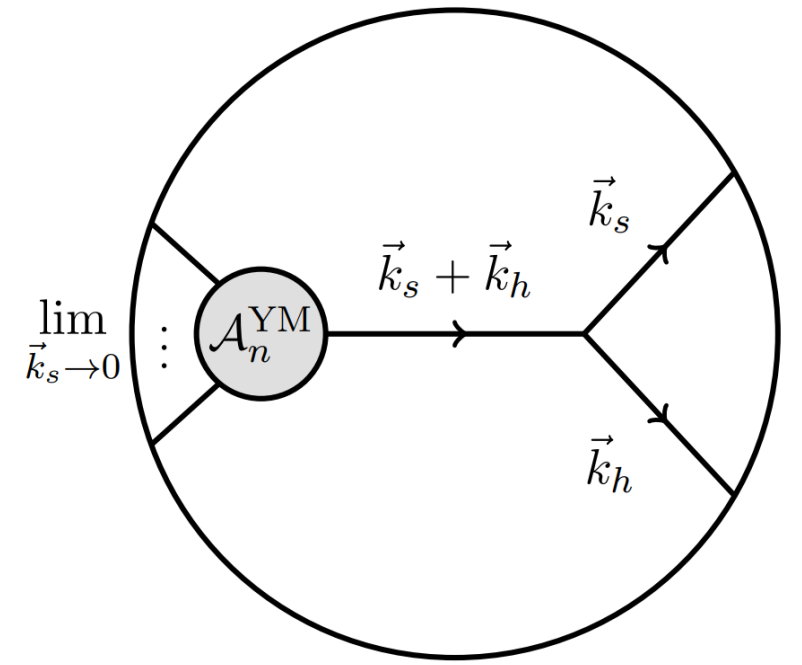
$$\langle hh\dots h \rangle$$



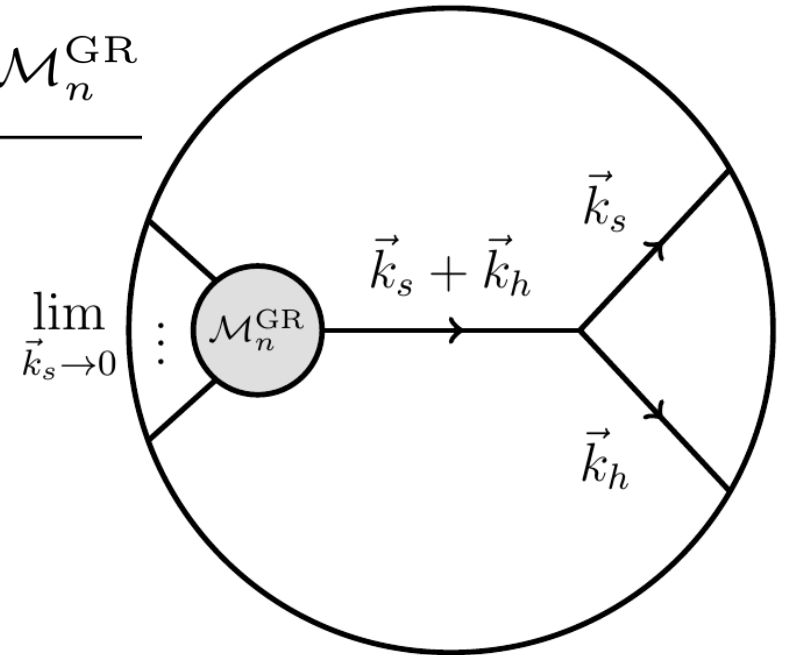
The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

$$\lim_{\vec{k}_s \rightarrow 0} \mathcal{A}_{n+1}^{\text{YM1}} = -\frac{z \varepsilon_s \cdot k_h \mathcal{A}_n^{\text{YM}}}{\mathcal{D}_{k_h}^{d-1}} - \varepsilon_s \cdot \varepsilon_h \frac{k_h \cdot \partial_{\varepsilon_h} \mathcal{A}_n^{\text{YM}}}{2z k_h^2}$$



$$\lim_{\vec{k}_s \rightarrow 0} \mathcal{M}_{n+1}^{\text{GR1}} = \frac{z^2 (\varepsilon_s \cdot k_h)^2 \mathcal{M}_n^{\text{GR}}}{\mathcal{D}_{k_h}^d} + \frac{\varepsilon_s \cdot \varepsilon_h \varepsilon_s \cdot k_h \varepsilon_h^i k_h^j \partial_{\varepsilon_h^{ij}} \mathcal{M}_n^{\text{GR}}}{k_h^2}$$



Mellin momentum to flatspace amplitudes $s_i \rightarrow \frac{z k_i}{2}$, $\mathcal{D}_{k_{ij}} \rightarrow z^2 S_{ij}$

The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

Back to momentum space

$$\begin{aligned}\lim_{\vec{k}_{n+1} \rightarrow 0} \langle\langle J \dots J \rangle\rangle_{n+1} &= \mathcal{N}_{d-1} \left\{ \frac{\varepsilon_{n+1} \cdot k_n}{2k_n} \partial_{k_n} \langle\langle J \dots J \rangle\rangle_n - \frac{\varepsilon_{n+1} \cdot \varepsilon_n}{2k_n^2} k_n \cdot \partial_{\varepsilon_n} \langle\langle J \dots J \rangle\rangle_n \right\} - \left(\vec{k}_n \rightarrow \vec{k}_1, \vec{\varepsilon}_n \rightarrow \vec{\varepsilon}_1 \right) + \dots \\ &= \frac{\mathcal{N}_{d-1}}{2} \left\{ \varepsilon_{n+1}^i \partial_{k_{ni}} \langle\langle J \dots J \rangle\rangle_n - \varepsilon_{n+1}^i \partial_{k_{1i}} \langle\langle J \dots J \rangle\rangle_n \right\} + \dots\end{aligned}$$

$$\begin{aligned}\lim_{\vec{k}_{n+1} \rightarrow 0} \langle\langle T \dots T \rangle\rangle_{n+1} &= \mathcal{N}_d \left\{ \sum_{a=1}^n -\frac{(\varepsilon_{n+1} \cdot k_a)^2}{2k_a} \partial_{k_a} \langle\langle T \dots T \rangle\rangle_n + \frac{\varepsilon_{n+1} \cdot \varepsilon_a \varepsilon_{n+1} \cdot k_a}{2k_a^2} \varepsilon_a^{(i} k_a^{j)} \partial_{\varepsilon_a^{ij}} \langle\langle T \dots T \rangle\rangle_n \right\} + \dots, \\ &= -\frac{\mathcal{N}_d}{2} \sum_{a=1}^n \varepsilon_{n+1}^{ij} k_{ai} \partial_{k_{aj}} \langle\langle T \dots T \rangle\rangle_n + \dots,\end{aligned}$$

$$\partial_{k^i} = \frac{k_i}{k} \partial_k, \quad \frac{\partial \varepsilon_i}{\partial k^j} = -\frac{\varepsilon_j k_i}{k^2}.$$

For spin 1/2 see [Chowdhury, Chowdhury, Moga, Singh '24]

Conclusion

- Mellin momentum amplitude make use of LSZ and onshell condition which allow us to bootstrap higher points boundary correlators
- We bootstrap 5 point graviton amplitude in AdS and also Class I soft theorem for YM and GR, without relying on spacetime symmetry

Future step

- Parke taylor amplitude for (A)dS
- Extrapolate to other spacetime FRW, black hole, etc
- Understand soft theorem from Lambda-BMS

Thank you!

Back up slides

Onshell

CFT correlator

or

Wavefunction coefficients

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i+d/2},$$

$$\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

[C. Sleight 19]

[C. Sleight and M. Taronna, 19, 21]

[Jiajie Mei, 23]

Onshellness

$$\mathcal{D}_k^\Delta \phi_\Delta(k, z) = 0$$

$$\mathcal{D}_k^\Delta \equiv z^2 k^2 - z^2 \partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d)$$



$$(z^2 k^2 + (d/2 - \Delta)^2 - 4s^2) \phi_\Delta(s, k) = 0$$

$$\text{Flat space analogy } k^2 + m^2 = 0$$

Mellin Momentum Amplitude

$$\mathcal{A}_n(zk, s)$$

No $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_{n-1}) \rangle \delta^d(x_i - x_j)$

Exactly invariant under
field redefinition

and symmetry transformation

due to LSZ

[Jiajie Mei, 23]

Definition needs full set of

$$\prod_{i=1}^n \phi(s_i, k_i)$$

Back up onshell

CFT correlator $\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle$ or Wavefunction coefficients Ψ_n

$$\Psi_n = \int \frac{dz}{z^{d+1}} \mathcal{A}_n(z, \partial_z, \vec{k}_a, \vec{\varepsilon}_a) \prod_{a=1}^n \phi_\Delta(k_a, z),$$

[C. Sleight 19]
 [C. Sleight and M. Taronna, 19, 21]
 [Jiajie Mei, 23]

$$\phi_\Delta(k, z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s, k),$$

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i+d/2},$$

$$\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk, s)$

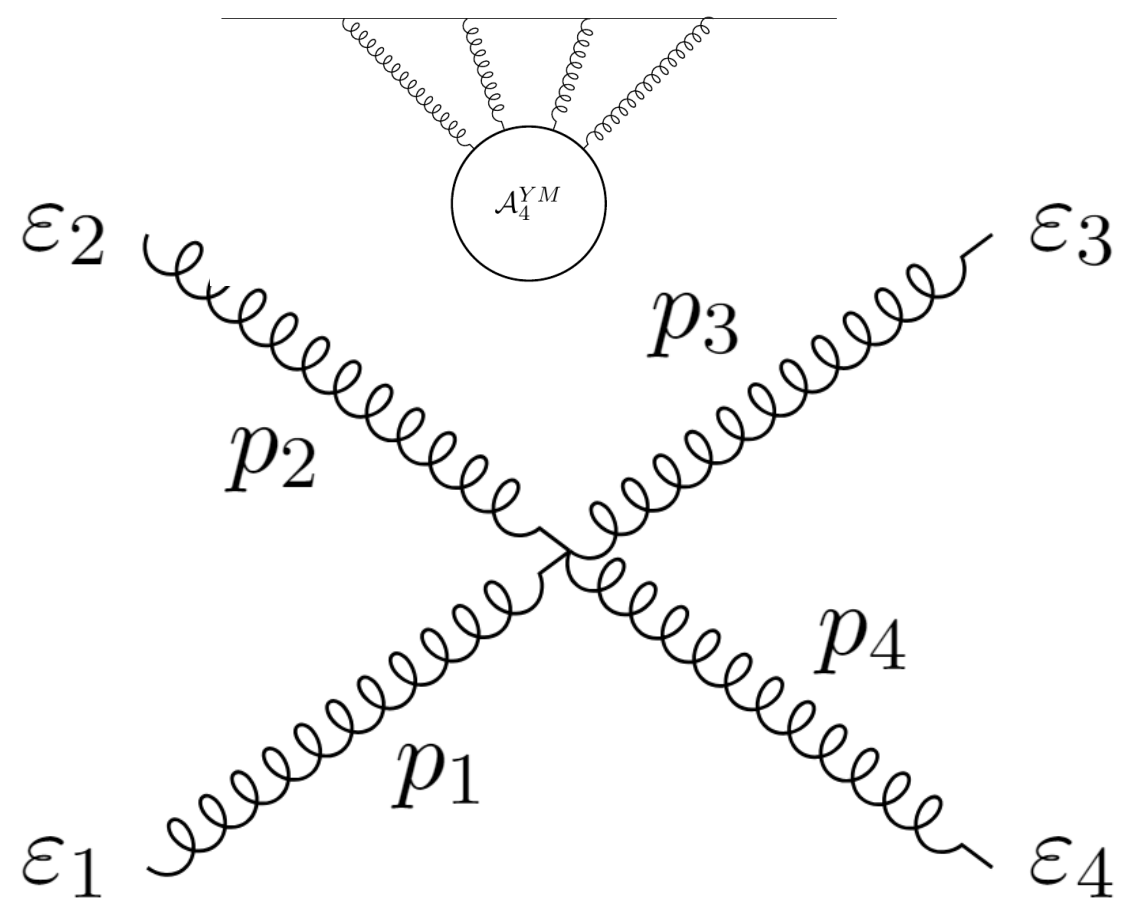
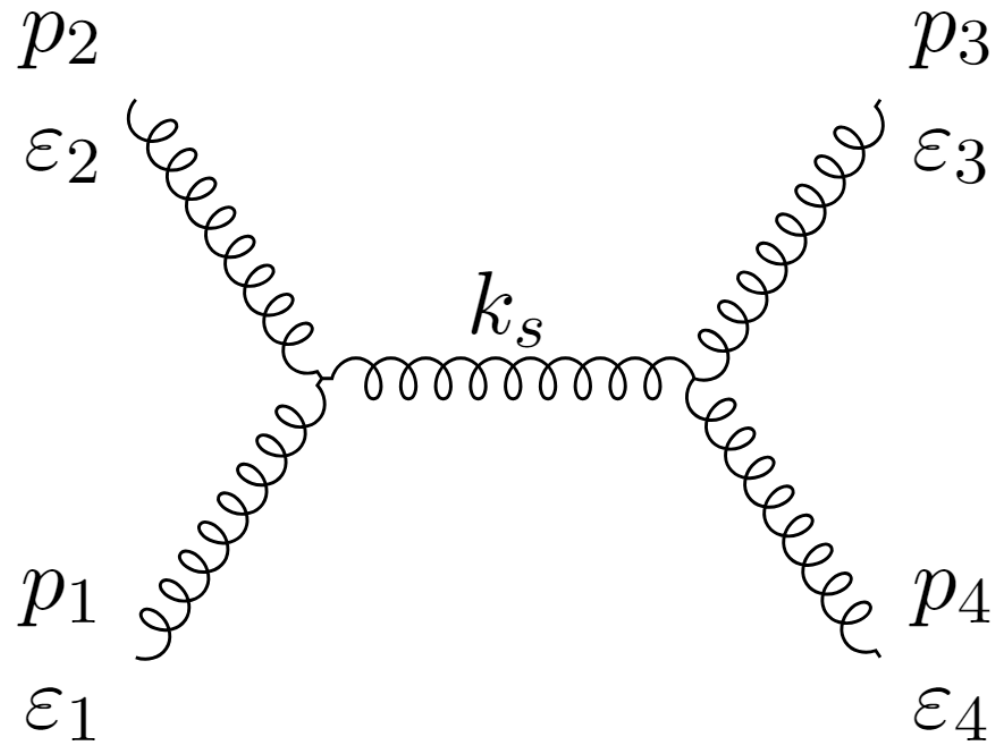
No $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_{n-1}) \rangle \delta^d(x_i - x_j)$

Exactly invariant
 field redefinition
 symmetry transformation
 due to LSZ

Definition needs full set of $\prod_{i=1}^n \phi(s_i, k_i)$

[Jiajie Mei, 23]

Four point exchange



$$\mathcal{A}_4^s = \left(\frac{V^{12\mu} \Pi_{\mu\nu} V^{\nu 34}}{\mathcal{D}_{k_s}^{d-1}} + \frac{V^{12z} V^{z 34}}{z^2 k_s^2} \right)$$

$$\mathcal{A}_4^c = \left(\frac{1}{2} \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \frac{1}{4} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 \right)$$

$$s_i \rightarrow \frac{z k_i}{2}, \quad \mathcal{D}_{k_{ij}} \rightarrow z^2 S_{ij}$$

Flatspace limit recovers usual 4 gluon amplitude

How to bootstrap higher points

$$\sum_r \varepsilon_i(k, r) \varepsilon_j(k, r)^* = \eta_{ij} - \frac{k_i k_j}{k^2} \equiv \Pi_{ij}$$

$$\sum_r \varepsilon_{ij}(k, r) \varepsilon_{kl}(k, r)^* = \frac{1}{2} \Pi_{ik} \Pi_{jl} + \frac{1}{2} \Pi_{il} \Pi_{kj} - \frac{1}{d-1} \Pi_{ij} \Pi_{kl}$$

Starting point

$\mathcal{A}_3(1, 2, 3)$

$\mathcal{M}_3(1, 2, 3)$

Unitarity

YM: $\mathcal{A}_n \rightarrow \mathcal{A}_a^\mu \frac{\sum_h \varepsilon_\mu^h \varepsilon_\nu^{*h}}{\mathcal{D}_{k_I}^{d-1}} \mathcal{A}_{n-a+2}^\nu$

● GR: $\mathcal{M}_n \rightarrow \mathcal{M}_a^{\mu_1 \mu_2} \frac{\sum_h \varepsilon_{\mu_1 \mu_2}^h \varepsilon_{\nu_1 \nu_2}^{*h}}{\mathcal{D}_{k_I}^d} \mathcal{M}_{n-a+2}^{\nu_1 \nu_2}$

OPE limit

● $\text{Res}_{k_I^2 \rightarrow 0} \mathcal{A}_n = 0$

Vanishing residue

● Lorentz invariance at short distance

Glueon propagation fixed

External soft limit

● $\langle h_i h_j h h h \dots \rangle \xrightarrow{\varepsilon_i \cdot \varepsilon_j = 1, \varepsilon_i \cdot k_a = 0, \varepsilon_j \cdot k_a = 0} \langle \phi_i \phi_j h h h \dots \rangle$

$\mathcal{A}_{\langle \phi_i \phi_j h h h \dots \rangle} |_{k_i \rightarrow 0} = 0$ due to $\nabla^m \phi \nabla^n \phi h_{mn}$

Graviton propagation fixed

Flatspace limit

● $s_i \rightarrow \frac{z k_i}{2}, \quad \mathcal{D}_{k_{ij}} \rightarrow z^2 S_{ij}$
Fixes remaining contact interaction

The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

$$\lim_{\vec{k}_s \rightarrow 0} \mathcal{M}_{n+1}^{\text{GR1}} = \frac{z^2 (\varepsilon_s \cdot k_h)^2 \mathcal{M}_n^{\text{GR}}}{\mathcal{D}_{k_h}^d} + \frac{\varepsilon_s \cdot \varepsilon_h \varepsilon_s \cdot k_h \varepsilon_h^i k_h^j \partial_{\varepsilon_h^{ij}} \mathcal{M}_n^{\text{GR}}}{k_h^2}$$

Derivation

Ansatz
$$\mathcal{M}_{n+1}^{\text{GR1}} = \frac{a_{s+h}^{\text{GR}}}{\mathcal{D}_{k_{s+h}}^d} + \frac{b_{s+h}^{(1)}}{k_{s+h}^2} + \frac{b_{s+h}^{(2)}}{k_{s+h}^4} + c^{\text{AZ}}$$

Unitarity

General Kinematics

$$a_{s+h}^{\text{GR}} = \sum_r \mathcal{M}_3^{\text{GR}, i_1 i_2} \left(\frac{1}{2} \Pi_{i_1 j_1} \Pi_{i_2 j_2} + \frac{1}{2} \Pi_{i_1 j_2} \Pi_{j_1 i_2} - \frac{1}{d-1} \Pi_{i_1 i_2} \Pi_{j_1 j_2} \right) \mathcal{M}_n^{\text{GR}, j_1 j_2}$$

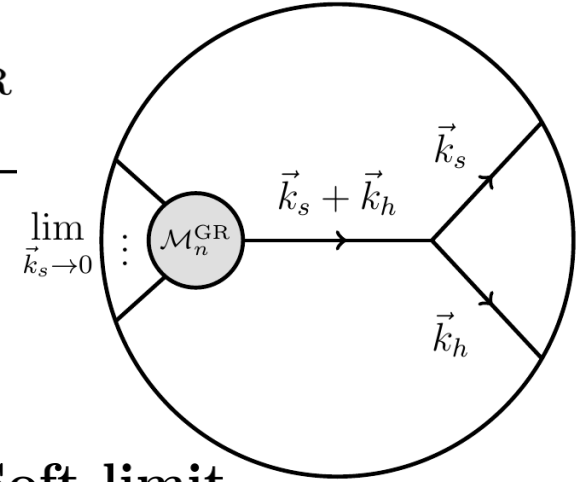
Zero residue
in OPE limit

$$\frac{b_{s+h}^{(1)}}{k_{s+h}^2} + \frac{b_{s+h}^{(2)}}{k_{s+h}^4} = -\frac{1}{k_{h+s}^2} \left(\text{Res}_{k_{s+h}^2 \rightarrow 0} \frac{a_{s+h}}{\mathcal{D}_{k_{s+h}}^d} \right) - \frac{1}{k_{h+s}^4} \left(\text{Res}_{k_{s+h}^4 \rightarrow 0} \frac{a_{s+h}}{\mathcal{D}_{k_{s+h}}^d} \right)$$

Adler zeros condition

$$\varepsilon_s \cdot \varepsilon_h = 1, \varepsilon_s \cdot k_a = 0, \varepsilon_h \cdot k_a = 0 \quad \mathcal{M}_{\langle hhhhh... \rangle} \rightarrow \mathcal{M}_{\langle \phi \phi hhh... \rangle}$$

$$\mathcal{M}_{\langle \phi_s \phi_h hhh... \rangle} |_{k_s \rightarrow 0} = 0 \text{ due to } \nabla^m \phi \nabla^n \phi h_{mn}$$



Soft limit

$$\lim_{\vec{k}_s \rightarrow 0} \frac{a_{s+h}^{\text{GR}}}{\mathcal{D}_{k_{s+h}}^d} = \frac{z^2 (\varepsilon_s \cdot k_h)^2 \mathcal{M}_n^{\text{GR}}}{\mathcal{D}_{k_h}^d}$$

$$\frac{\varepsilon_s \cdot \varepsilon_h \varepsilon_s \cdot k_h \varepsilon_h^i k_h^j \partial_{\varepsilon_h^{ij}} \mathcal{M}_n^{\text{GR}}}{k_h^2} - \frac{(\varepsilon_s \cdot \varepsilon_h)^2 \delta^{ij} \partial_{\varepsilon_h^{ij}} \mathcal{M}_n^{\text{GR}}}{4(1-d)}$$

$$\lim_{\vec{k}_s \rightarrow 0} c^{\text{AZ}} = \frac{(\varepsilon_s \cdot \varepsilon_h)^2 \delta^{ij} \partial_{\varepsilon_h^{ij}} \mathcal{M}_n^{\text{GR}}}{4(1-d)}$$

Back up onshell

CFT correlator $\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle$ or Wavefunction coefficients Ψ_n

$$\Psi_n = \int \frac{dz}{z^{d+1}} \mathcal{A}_n(z, \partial_z, \vec{k}_a, \vec{\varepsilon}_a) \prod_{a=1}^n \phi_\Delta(k_a, z),$$

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 [Jiajie Mei, 23]

$$\phi_\Delta(k, z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s, k),$$

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i+d/2},$$

$$\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk, s)$

No $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_{n-1}) \rangle \delta^d(x_i - x_j)$

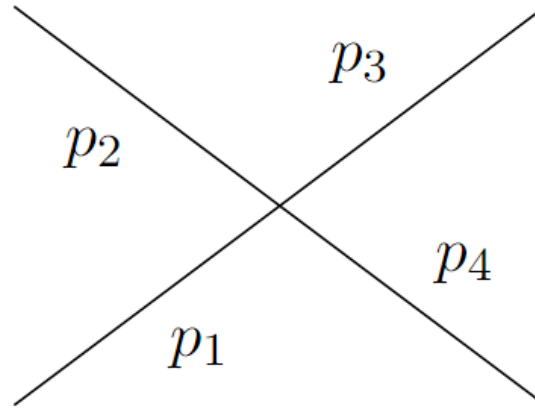
Exactly invariant
 field redefinition
 symmetry transformation
 due to LSZ

Definition needs full set of $\prod_{i=1}^n \phi(s_i, k_i)$

[Jiajie Mei, 23]

Example

Four-point contact diagram for $\lambda\phi^4$ theory:

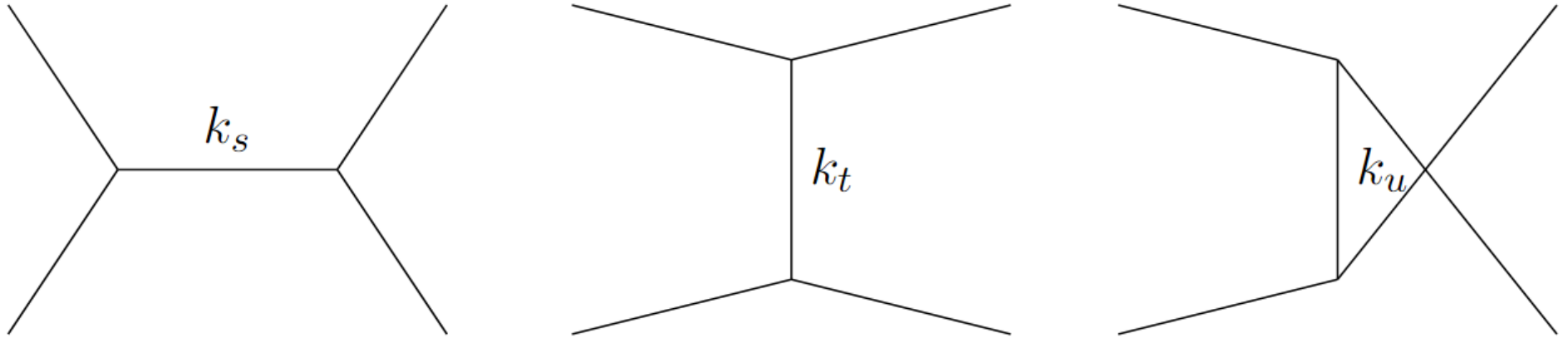


$$\mathcal{A}_{4,c}^{\phi^4} = \lambda \times (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu \right) \times 2\pi i \delta \left(d + \sum_{i=1}^4 (2s_i - d/2) \right)$$

$$\langle\langle \mathcal{O}(\vec{k}_1) \dots \mathcal{O}(\vec{k}_n) \rangle\rangle = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i + d/2},$$

Example

Four-point exchange diagram for $\lambda\phi^3$ theory:



$$\mathcal{A}_4^{\phi^3} = \lambda^2 (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu \right) \left(\frac{1}{\mathcal{D}_{k_s}^\Delta} + \frac{1}{\mathcal{D}_{k_t}^\Delta} + \frac{1}{\mathcal{D}_{k_u}^\Delta} \right)$$

$$(\mathcal{D}(z))^{-1} \mathcal{O}(z) = \int \frac{dy}{y^{d+1}} G(z, y) \mathcal{O}(y)$$

$$\mathcal{D}_k^\Delta G(z, y) = z^{d+1} \delta(z - y)$$

$$\int_0^\infty \frac{dz}{z^{d+1}} \phi_\Delta(z_1, k_1) \phi_\Delta(z_1, k_2) G(z_1, z_2, k_s) \phi_\Delta(z_2, k_3) \phi_\Delta(z_2, k_4)$$