Mellin Momentum Amplitudes and onshell boostrap in (A)dS

Focus: boundary correlator or wave function coefficients for n-gluons and gravitons scattering in (A)dS,

Yuyu Mo

Based on

[Jiajie Mei, 2305.13894]

[Jiajie Mei, Y.M., 2402.09111]

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, Y.M. 2407.16052]

Motivation : bootstrap

Principles → **Lagrangian** → **Observables**

Causality, unitarity, locality, symmetries, . . .

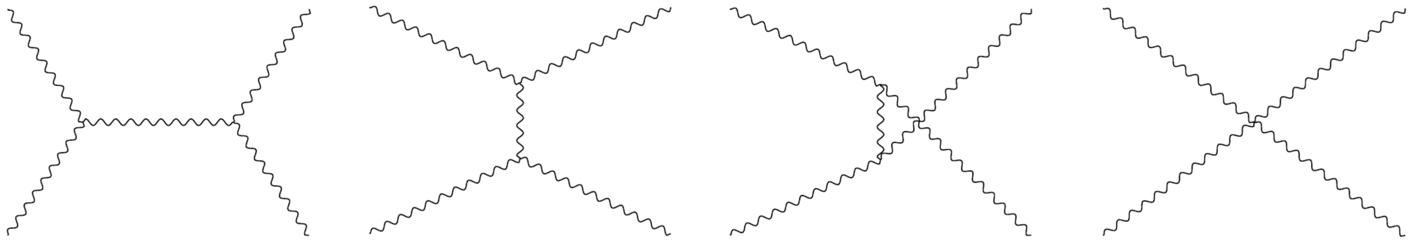
Fields, equations of motion, Feynman diagrams, . . .

Cross sections, decay rates, (Mellin-momentum) amplitudes cosmological correlators, . . .

Bootstrap

Text book approach

$\mathcal{M}^{flat}_{GR,4} \sim$



Amplitude bootstrap

$$\mathcal{M}^{flat}_{GR,4}\left(1^{--}2^{--}3^{++}4^{++}\right) = \frac{\langle 12\rangle^4[34]^4}{stu}$$

[Cheung '17]
[Arkani-Hamed, Huang, Huang '17]

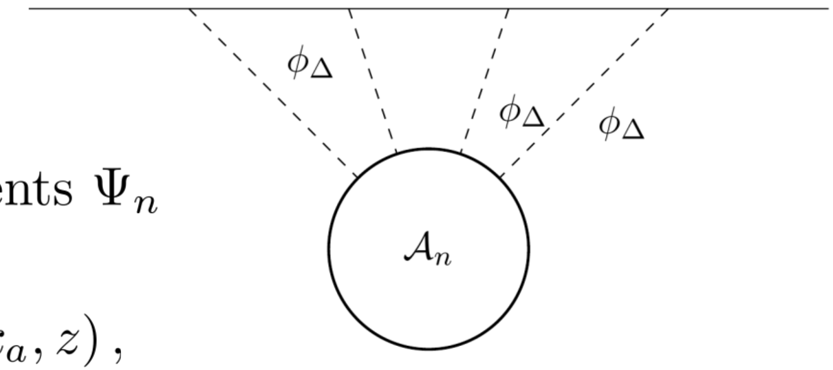
Motivation: why bootstrap and amplitude

Lorentz symmetry, unitarity, and locality ***uniquely*** determined that **two derivative spin-2 particles** with two degrees of freedom can **only** be Einstein Gravity.

Q: What about curved space? Cosmology, Black holes background?

[Arkani-Hamed, Baumann, Lee, Pimentel, 18]

Correlators on the boundary



CFT correlator $\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle$ or Wavefunction coefficients Ψ_n

$$\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle = \int \frac{dz}{z^{d+1}}\mathcal{A}_n\left(z,\partial_z,\vec{k}_a,\vec{\varepsilon}_a\right)\prod_{a=1}^{n}\phi_\Delta\left(k_a,z\right),$$

1. **Not** invariant under field redefinition because of local terms

$$\phi \to \phi + \alpha\phi^3,$$

$$\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \to \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle - \tfrac{1}{3}\alpha\sum_{i=1}^{4}(k_i^3).$$

2. **Not** invariant under symmetry transformation because of local terms

Shift symmetry

Conformal ward identities $\quad \widetilde{K}^i\left\langle J_1^{\pm}O_2\cdots O_n\right\rangle \sim \varepsilon^i_{\pm}\sum_{a=2}^{n}e_a\left\langle O_{\vec{k}_2}\cdots O_{\vec{k}_a+\vec{k}_1}\cdots O_{\vec{k}_n}\right\rangle.$

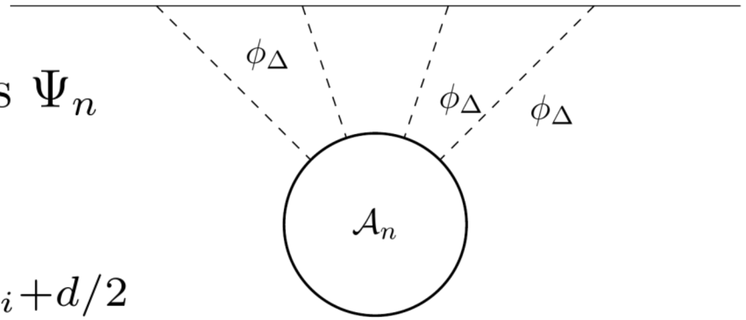
What do we mean by local

We don't have the problem in flat space because of LSZ for amplitudes!

$$\partial^\mu\left\langle J_\mu\left(\vec{x}_1\right)O\left(\vec{x}_2\right)\ldots O\left(\vec{x}_n\right)\right\rangle = -\sum_{a=2}^{n}\delta\left(\vec{x}_1-\vec{x}_a\right)\left\langle O\left(\vec{x}_2\right)\ldots\delta O\left(\vec{x}_a\right)\ldots O\left(\vec{x}_n\right)\right\rangle.$$

Mellin momentum amplitudes introduction

CFT correlator $\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle$ or Wavefunction coefficients Ψ_n



[Sleight '19]
[Sleight and Taronna '19 '21]

$$\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle = \int [ds_i]\int \frac{dz}{z^{d+1}} \boxed{\mathcal{A}_n(zk,s)\prod_{i=1}^{n}\phi\left(s_i,k_i\right)}\, z^{-2s_i+d/2},$$

$$\phi_\Delta(k,z) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i} z^{-2s+d/2}\phi_\Delta(s,k),$$

Onshellness

$$\mathcal{D}_k^\Delta \equiv z^2k^2 - z^2\partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d) \quad \Rightarrow \quad \left(z^2k^2 + (d/2-\Delta)^2 - 4s^2\right) = 0$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk,s)$

Definition needs full set of $\prod_{i=1}^{n}\phi\left(s_i,k_i\right)$

- **No** local terms
- **Exactly invariant**
  field redefinition
  symmetry transformation

due to LSZ [Jiajie Mei, 23]

Link to flatspace amplitude

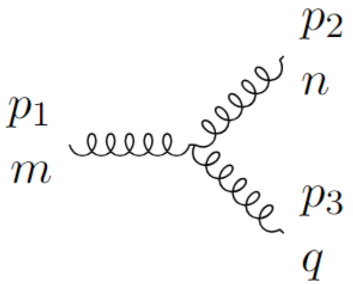
Table 1: Comparison between Minkowski and AdS

Amplitude in Minkowski space	Mellin-Momentum amplitude in AdS
Lorentz-Invariance	Conformal Invariance
Translation symmetry $e^{ik_\mu \cdot x^\mu}$ $(2\pi)^{d+1}\delta^{d+1}(\sum_{i=1}^{n} k_i^\mu)$	Boundary translation + Dilatation $e^{ik_\mu \cdot x^\mu} z^{-2s+d/2}$ $(2\pi)^d\delta^d(\sum_{i=1}^{n} k_i^\mu)$ $\times 2\pi i\delta(d+b+\sum_{i=1}^{n}(2s_i - d/2))$
On-shell condition $k^2+m^2=0$	On-shell condition $z^2k^2+(d/2-\Delta)^2-4s^2=0$
Pole structure $\frac{1}{(k_i+\cdots+k_n)^2}$	Factorization+OPE $\frac{1}{\mathcal{D}_k^\Delta} \quad \frac{1}{(k_I)^2}$

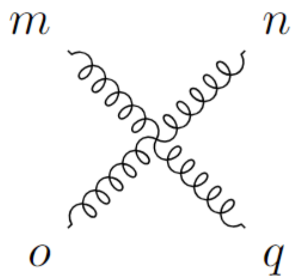
Mellin momentum to flatspace amplitudes $s_i \to \frac{zk_i}{2}, \quad \mathcal{D}_{k_{ij}} \to z^2 S_{ij}$

Example: Feynman rule for gluon in MM amplitude

[J. Mei, YM, 24]



$$: V_{mnq}(p_1,p_2,p_3)=\frac{1}{2}\left(\eta_{mn}(p_1-p_2)_q+\eta_{nq}(p_2-p_3)_m+\eta_{qm}(p_3-p_1)_n\right),$$



$$: V_{mnqo}=\left(\frac{1}{2}\eta_{mq}\eta_{no}-\frac{1}{4}(\eta_{mn}\eta_{qo}+\eta_{mo}\eta_{nq}),\right)$$

$$p^m=(zk^\mu, i(2s-d/2))$$

$$: G_{\mu\nu}=\frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}},$$

$$: G_{zz}=\frac{1}{z^2k^2},$$

$$\Pi_{\mu\nu}=\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}$$

$$\left(\mathcal{D}_k^\Delta(z)\right)^{-1}F(z)=\int\frac{dy}{y^{d+1}}G_\Delta(k,z,y)F(y)$$

$$\mathcal{D}_k^\Delta(z)G_\Delta(k,z,y)=z^{d+1}\delta(z-y)$$

Bootstap higher points (YM and GR)

Starting point $\mathcal{A}_3(1,2,3)$ $\mathcal{M}_3(1,2,3)$

Unitarity: amplitude will factorize into lower point on-shell amplitudes

$$\mathcal{A}_n \to \frac{\sum_h \mathcal{A}_a \mathcal{A}_{n-a+2}}{\mathcal{D}^{\Delta}_{k_I}}$$

Soft (OPE) limit: amplitude should have vanishing residue in OPE limit

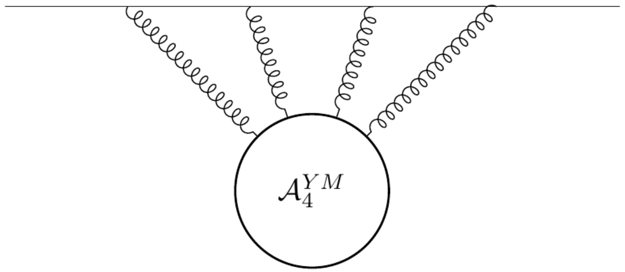
$$\mathrm{Res}_{k_I^2 \to 0} \mathcal{A}_n = 0$$

Flatspace limit: the particles are ignoring the curvature correction and the Lorentz symmetry will emerge

$$\lim_{R\to\infty} \mathcal{A}_n \to A_n$$

MM amplitudes reduce to amplitudes in flat space

Four point YM from bootstrap



[Jiajie Mei, 23]

Starting point $\mathcal{A}_3(1,2,3)$

Unitarity $a(1,2,3,4) = \sum_{h=\pm} \mathcal{A}_3\left(1,2,-k_s^h\right)\mathcal{A}_3\left(k_s^{-h},3,4\right).$

$\sum_r \varepsilon_i(k,r)\varepsilon_j(k,r)^* = \eta_{ij} - \frac{k_i k_j}{k^2} \equiv \Pi_{ij}$

Vanishing residue in OPE limit $b(1,2,3,4) = -\underset{k_s^2\to 0}{\mathrm{Res}}\left(\frac{a(1,2,3,4)}{\mathcal{D}_{k_s}^{d-1}}\right) = \frac{(s_1-s_2)(s_3-s_4)\varepsilon_{12,34}}{z^2 k_s^2}.$

Consistency check:
Reproducing the Feynman rule computation

$$\mathcal{A}_4^s = \left(\frac{V^{12\mu}\Pi_{\mu\nu}V^{\nu 34}}{\mathcal{D}_{k_s}^{d-1}} + \frac{V^{12z}V^{z34}}{z^2k_s^2}\right)$$

Flatspace limit

$$\mathcal{A}_4^c = \left(\frac{1}{2}\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 - \frac{1}{4}\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 - \frac{1}{2}\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3\right)$$

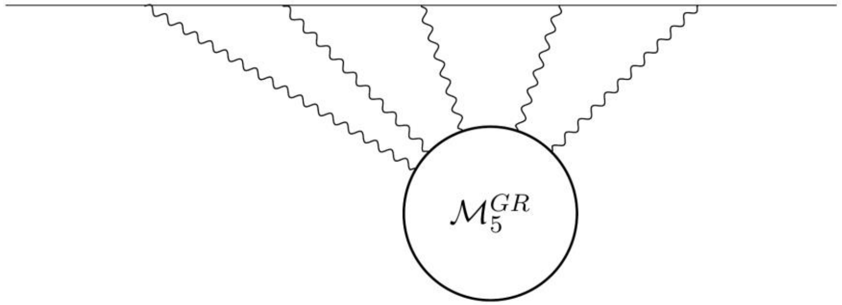
N point YM from bootstrap

$\mathcal{A}_3^{YM}$ → (Unitarity, OPE limit, Flatspace limit) → $\mathcal{A}_4^{YM}$ [Mei, 23] → (Unitarity, OPE limit) → $\mathcal{A}_5^{YM}$ [Mei, YM ’24] [Mei, YM ’24X]

$\bullet\bullet\bullet$ → (Unitarity, OPE limit) → $\mathcal{A}_n^{YM}$

Final scalar integrals give $\langle JJ...J \rangle$

The main result: I, 5 point graviton amplitude



Following similar procedure

[Jiajie Mei, 23]

$\mathcal{M}_3^{GR}$ → $\mathcal{M}_4^{GR}$ → $\mathcal{M}_5^{GR}$

Unitarity
OPE limit
External Soft limit
Flatspace limit

Unitarity
OPE limit
External Soft limit
Flatspace limit

[Jiajie Mei, YM, 24]

[Jiajie Mei, YM, 24X]

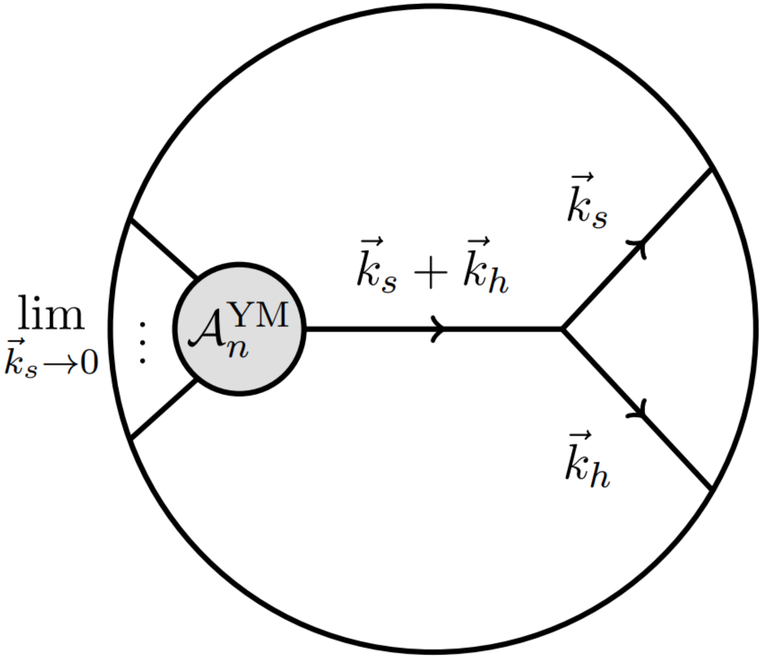
● ● ● → $\mathcal{M}_n^{GR}$

Final scalar integrals give $\langle hh...h\rangle$

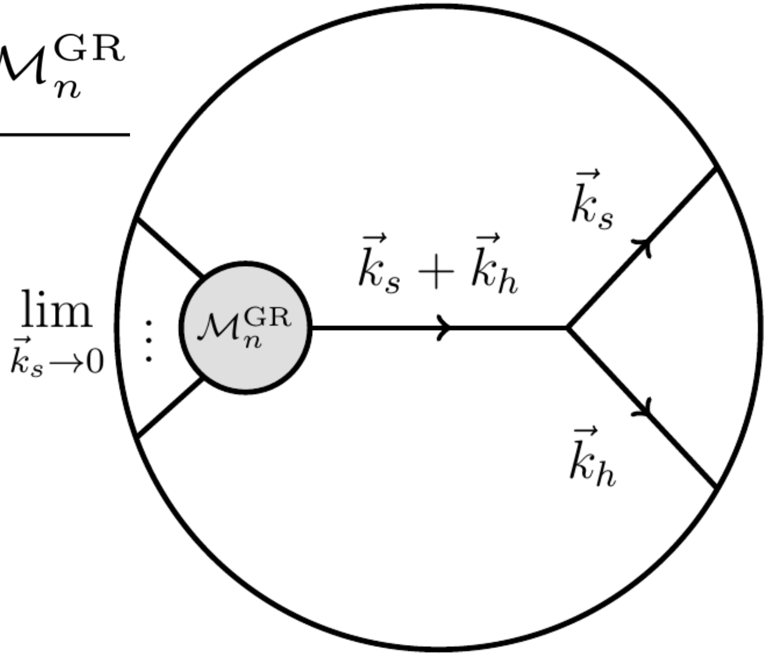
The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

$$\lim_{\vec{k}_s\to 0}\mathcal{A}_{n+1}^{\text{YM1}} = -\frac{z\varepsilon_s\cdot k_h\mathcal{A}_n^{\text{YM}}}{\mathcal{D}_{k_h}^{d-1}} - \varepsilon_s\cdot\varepsilon_h\frac{k_h\cdot\partial_{\varepsilon_h}\mathcal{A}_n^{\text{YM}}}{2zk_h^2}$$



$$\lim_{\vec{k}_s\to 0}\mathcal{M}_{n+1}^{\text{GR1}} = \frac{z^2(\varepsilon_s\cdot k_h)^2\mathcal{M}_n^{\text{GR}}}{\mathcal{D}_{k_h}^{d}} + \frac{\varepsilon_s\cdot\varepsilon_h\varepsilon_s\cdot k_h\varepsilon_h^i k_h^j\partial_{\varepsilon_h^{ij}}\mathcal{M}_n^{\text{GR}}}{k_h^2}$$



Mellin momentum to flatspace amplitudes $s_i \to \frac{zk_i}{2}, \quad \mathcal{D}_{k_{ij}} \to z^2 S_{ij}$

The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

Back to momentum space

$$\lim_{\vec{k}_{n+1}\to 0}\langle\langle J\ldots J\rangle\rangle_{n+1} = \mathcal{N}_{d-1}\left\{\frac{\varepsilon_{n+1}\cdot k_n}{2k_n}\partial_{k_n}\langle\langle J\ldots J\rangle\rangle_n - \frac{\varepsilon_{n+1}\cdot\varepsilon_n}{2k_n^2}k_n\cdot\partial_{\varepsilon_n}\langle\langle J\ldots J\rangle\rangle_n\right\} - \left(\vec{k}_n\to\vec{k}_1, \vec{\varepsilon}_n\to\vec{\varepsilon}_1\right) + \ldots$$

$$= \frac{\mathcal{N}_{d-1}}{2}\left\{\varepsilon^i_{n+1}\partial_{k_{ni}}\langle\langle J\ldots J\rangle\rangle_n - \varepsilon^i_{n+1}\partial_{k_{1i}}\langle\langle J\ldots J\rangle\rangle_n\right\} + \ldots$$

$$\lim_{\vec{k}_{n+1}\to 0}\langle\langle T\ldots T\rangle\rangle_{n+1} = \mathcal{N}_d\left\{\sum_{a=1}^{n} -\frac{(\varepsilon_{n+1}\cdot k_a)^2}{2k_a}\partial_{k_a}\langle\langle T\ldots T\rangle\rangle_n + \frac{\varepsilon_{n+1}\cdot\varepsilon_a\varepsilon_{n+1}\cdot k_a}{2k_a^2}\varepsilon_a^{(i}k_a^{j)}\partial_{\varepsilon_a^{ij}}\langle\langle T\ldots T\rangle\rangle_n\right\} + \ldots,$$

$$= -\frac{\mathcal{N}_d}{2}\sum_{a=1}^{n}\varepsilon^{ij}_{n+1}k_{ai}\partial_{k_{aj}}\langle\langle T\ldots T\rangle\rangle_n + \ldots,$$

$$\partial_{k^i} = \frac{k_i}{k}\partial_k, \quad \frac{\partial\varepsilon_i}{\partial k^j} = -\frac{\varepsilon_j k_i}{k^2}.$$

For spin 1/2 see [Chowdhury, Chowdhury, Moga, Singh '24]

Conclusion

- Mellin momentum amplitude make use of LSZ and onshell condition which allow us to bootstrap higher points boundary correlators
- We bootstrap 5 point graviton amplitude in AdS and also Class I soft theorem for YM and GR, without relying on spacetime symmetry

Future step

- Parke taylor amplitude for (A)dS
- Extrapolate to other spacetime FRW, black hole, etc
- Understand soft theorem from Lambda-BMS

Thank you!

Back up slides

Onshell

CFT correlator or Wavefunction coefficients

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \boxed{\mathcal{A}_n(zk,s) \prod_{i=1}^n \phi(s_i,k_i)}\, z^{-2s_i+d/2},$$
$$\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

[C. Sleight 19]
[C. Sleight and M. Taronna, 19, 21]
[Jiajie Mei, 23]

Onshellness

$$\mathcal{D}_k^\Delta \phi_\Delta(k,z) = 0$$
$$\mathcal{D}_k^\Delta \equiv z^2k^2 - z^2\partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d)$$

$$\left(z^2k^2 + (d/2-\Delta)^2 - 4s^2\right)\phi_\Delta(s,k) = 0$$

Flat space analogy $k^2 + m^2 = 0$

Mellin Momentum Amplitude $\mathcal{A}_n(zk,s)$

Definition needs full set of $\prod_{i=1}^n \phi(s_i,k_i)$

No $\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_{n-1})\rangle\, \delta^d(x_i - x_j)$

Exactly invariant under
field redefinition
and symmetry transformation

due to LSZ

[Jiajie Mei, 23]

Back up onshell

CFT correlator $\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle$ or Wavefunction coefficients Ψ_n

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$$\Psi_n = \int \frac{dz}{z^{d+1}} \mathcal{A}_n\left(z, \partial_z, \vec{k}_a, \vec{\varepsilon}_a\right) \prod_{a=1}^n \phi_\Delta\left(k_a, z\right),$$

$$\phi_\Delta(k,z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s,k),$$

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi\left(s_i, k_i\right) z^{-2s_i+d/2},$$

$$\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk, s)$

Definition needs full set of
$\prod_{i=1}^n \phi\left(s_i, k_i\right)$

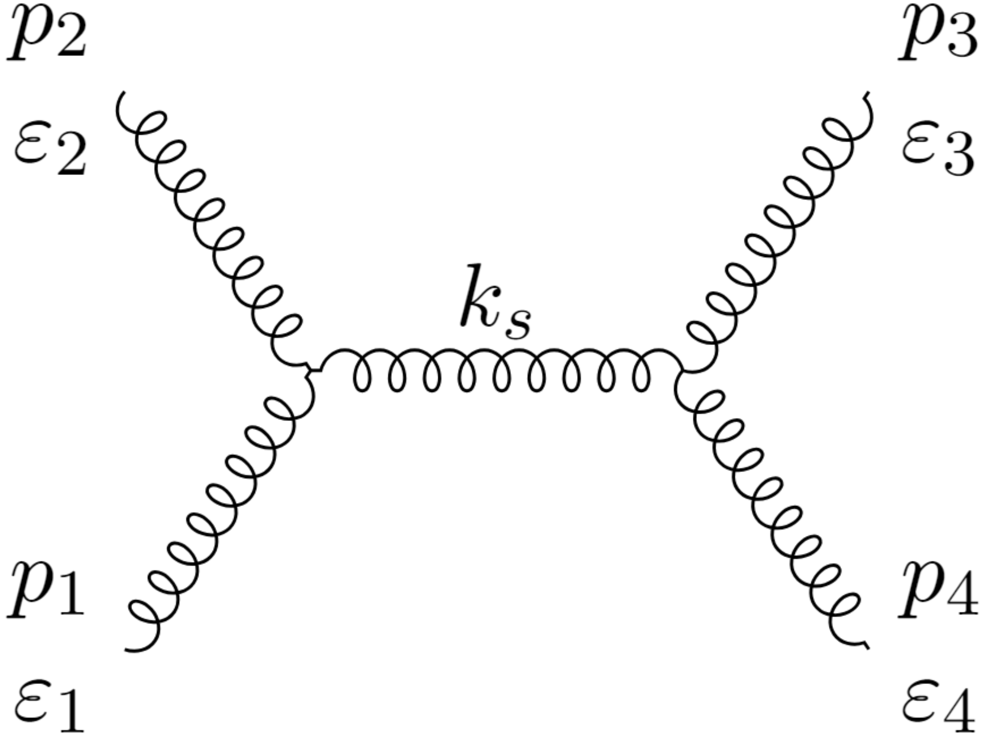
No $\left\langle \mathcal{O}\left(x_1\right)\ldots\mathcal{O}\left(x_{n-1}\right)\right\rangle \delta^d\left(x_i - x_j\right)$

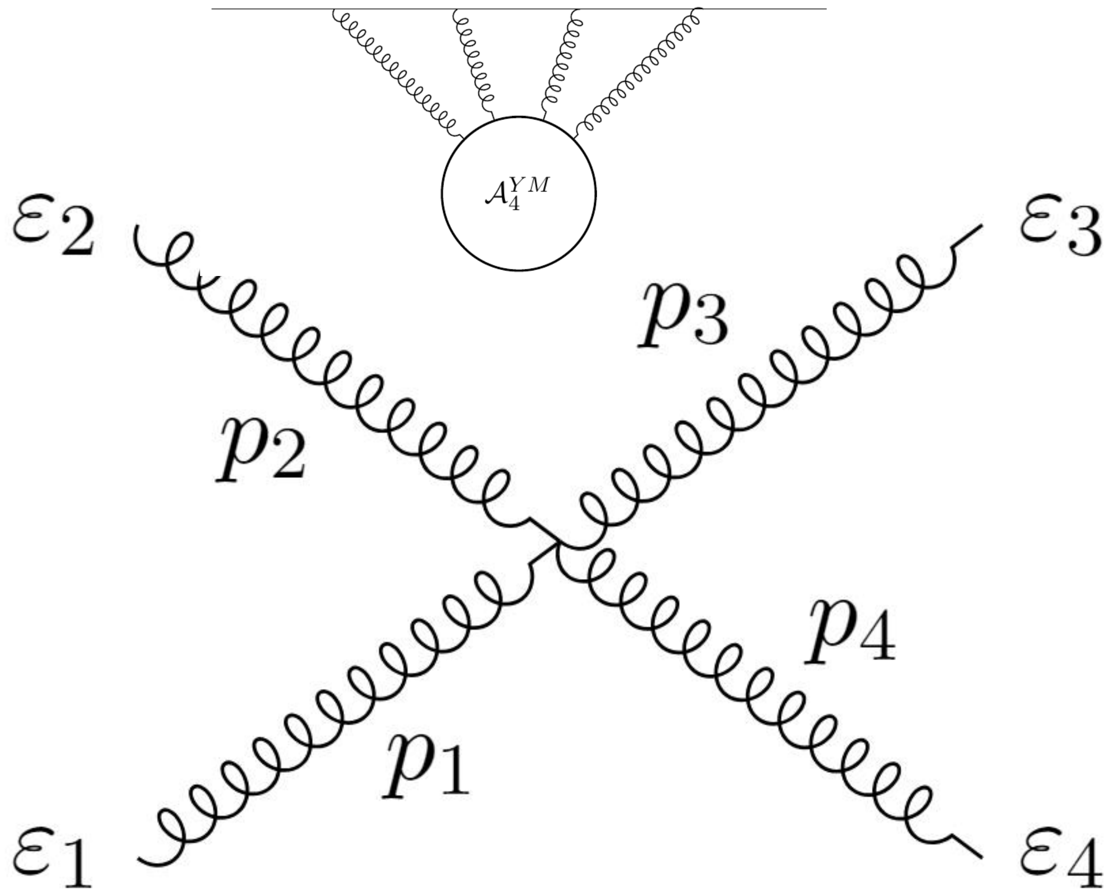
Exactly invariant
field redefinition
symmetry transformation

due to LSZ

[Jiajie Mei, 23]

Four point exchange





$$\mathcal{A}_4^c = \left(\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 - \tfrac{1}{4}\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3\right)$$

$$\mathcal{A}_4^s = \left(\frac{V^{12\mu}\Pi_{\mu\nu}V^{\nu 34}}{\mathcal{D}_{k_s}^{d-1}} + \frac{V^{12z}V^{z34}}{z^2k_s^2}\right)$$

$$s_i \to \frac{zk_i}{2}, \quad \mathcal{D}_{k_{ij}} \to z^2 S_{ij}$$

Flatspace limit recovers usual 4 gluon amplitude

How to bootstap higher points

$$\sum_r \varepsilon_i(k,r)\varepsilon_j(k,r)^* = \eta_{ij} - \frac{k_i k_j}{k^2} \equiv \Pi_{ij}$$

$$\sum_r \varepsilon_{ij}(k,r)\varepsilon_{kl}(k,r)^* = \frac{1}{2}\Pi_{ik}\Pi_{jl} + \frac{1}{2}\Pi_{il}\Pi_{kj} - \frac{1}{d-1}\Pi_{ij}\Pi_{kl}.$$

Starting point $\mathcal{A}_3\,(1,2,3)$ $\mathcal{M}_3\,(1,2,3)$

Unitarity

YM: $\mathcal{A}_n \to \mathcal{A}_a^{\mu} \frac{\sum_h \varepsilon_\mu^h \varepsilon_\nu^{*h}}{\mathcal{D}_{k_I}^{d-1}} \mathcal{A}_{n-a+2}^{\nu}$

- GR: $\mathcal{M}_n \to \mathcal{M}_a^{\mu_1\mu_2} \frac{\sum_h \varepsilon_{\mu_1\mu_2}^h \varepsilon_{\nu_1\nu_2}^{*h}}{\mathcal{D}_{k_I}^{d}} \mathcal{M}_{n-a+2}^{\nu_1\nu_2}$

OPE limit

- $\mathrm{Res}_{k_I^2\to 0}\,\mathcal{A}_n = 0$

Vanishing residue

- Lorentz invariance at short distance

Gluon propagation fixed

External soft limit

- $\langle h_i h_j hhh\ldots\rangle \xrightarrow{\varepsilon_i\cdot\varepsilon_j=1,\varepsilon_i\cdot k_a=0,\varepsilon_j\cdot k_a=0} \langle \phi_i\phi_j hhh\ldots\rangle$

$\mathcal{A}_{\langle\phi_i\phi_j hhh\ldots\rangle}|_{k_i\to 0} = 0$ due to $\nabla^m\phi\nabla^n\phi h_{mn}$

Graviton propagation fixed

Flatspace limit

- $s_i \to \frac{zk_i}{2},\quad \mathcal{D}_{k_{ij}} \to z^2 S_{ij}$

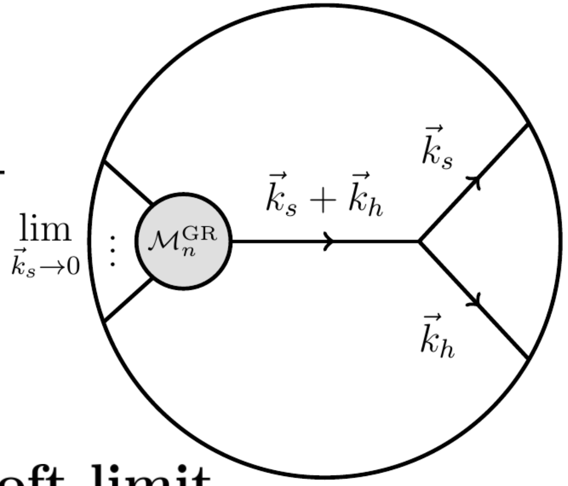
Fixes remaining contact interaction

The main result: II, soft limit

[Chandramouli Chowdhury, Arthur Lipstein, Jiajie Mei, YM, 24]

$$\lim_{\vec{k}_s\to 0}\mathcal{M}_{n+1}^{\mathrm{GR1}}=\frac{z^2(\varepsilon_s\cdot k_h)^2\mathcal{M}_n^{\mathrm{GR}}}{\mathcal{D}_{k_h}^d}+\frac{\varepsilon_s\cdot\varepsilon_h\,\varepsilon_s\cdot k_h\,\varepsilon_h^i k_h^j\partial_{\varepsilon_h^{ij}}\mathcal{M}_n^{\mathrm{GR}}}{k_h^2}$$

Class I diagram
Plot by Chandramouli Chowdhury



Derivation

Ansatz $$\mathcal{M}_{n+1}^{\mathrm{GR1}}=\frac{a_{s+h}^{\mathrm{GR}}}{\mathcal{D}_{k_{s+h}}^d}+\frac{b_{s+h}^{(1)}}{k_{s+h}^2}+\frac{b_{s+h}^{(2)}}{k_{s+h}^4}+c^{AZ}$$

General Kinematics

Unitarity

$$a_{s+h}^{\mathrm{GR}}=\sum_r\mathcal{M}_3^{\mathrm{GR},i_1i_2}\left(\tfrac{1}{2}\Pi_{i_1j_1}\Pi_{i_2j_2}+\tfrac{1}{2}\Pi_{i_1j_2}\Pi_{j_1i_2}-\tfrac{1}{d-1}\Pi_{i_1i_2}\Pi_{j_1j_2}\right)\mathcal{M}_n^{\mathrm{GR},j_1j_2}$$

Zero residue
in OPE limit

$$\frac{b_{s+h}^{(1)}}{k_{s+h}^2}+\frac{b_{s+h}^{(2)}}{k_{s+h}^4}=-\frac{1}{k_{h+s}^2}\left(\mathrm{Res}_{k_{s+h}^2\to 0}\frac{a_{s+h}}{\mathcal{D}_{k_{s+h}}^d}\right)-\frac{1}{k_{h+s}^4}\left(\mathop{\mathrm{Res}}_{k_{s+h}^4\to 0}\frac{a_{s+h}}{\mathcal{D}_{k_{s+h}}^d}\right)$$

Adler zeros condition

$$\varepsilon_s\cdot\varepsilon_h=1,\varepsilon_s\cdot k_a=0,\varepsilon_h\cdot k_a=0 \qquad \mathcal{M}_{\langle hhhhh\ldots\rangle}\to\mathcal{M}_{\langle\phi\phi hhh\ldots\rangle}$$

$$\mathcal{M}_{\langle\phi_s\phi_h hhh\ldots\rangle}|_{k_s\to 0}=0 \text{ due to } \nabla^m\phi\nabla^n\phi h_{mn}$$

Soft limit

$$\lim_{\vec{k}_s\to 0}\frac{a_{s+h}^{\mathrm{GR}}}{\mathcal{D}_{k_{s+h}}^d}=\frac{z^2(\varepsilon_s\cdot k_h)^2\mathcal{M}_n^{\mathrm{GR}}}{\mathcal{D}_{k_h}^d}$$

$$\frac{\varepsilon_s\cdot\varepsilon_h\,\varepsilon_s\cdot k_h\,\varepsilon_h^i k_h^j\partial_{\varepsilon_h^{ij}}\mathcal{M}_n^{\mathrm{GR}}}{k_h^2}-\frac{(\varepsilon_s\cdot\varepsilon_h)^2\delta^{ij}\partial_{\varepsilon_h^{ij}}\mathcal{M}_n^{\mathrm{GR}}}{4(1-d)}$$

$$\lim_{\vec{k}_s\to 0}c^{AZ}=\frac{(\varepsilon_s\cdot\varepsilon_h)^2\delta^{ij}\partial_{\varepsilon_h^{ij}}\mathcal{M}_n^{\mathrm{GR}}}{4(1-d)}$$

Back up onshell

CFT correlator $\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle$ or Wavefunction coefficients Ψ_n

[C. Sleight 19]
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[Jiajie Mei, 23]

$$\Psi_n = \int \frac{dz}{z^{d+1}} \mathcal{A}_n\left(z, \partial_z, \vec{k}_a, \vec{\varepsilon}_a\right) \prod_{a=1}^{n} \phi_\Delta\left(k_a, z\right),$$

$$\phi_\Delta(k,z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s,k),$$

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk,s) \prod_{i=1}^{n} \phi\left(s_i,k_i\right) z^{-2s_i+d/2},$$

$$\int [ds_i] = \prod_{i=1}^{n} \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}.$$

Mellin Momentum Amplitude $\mathcal{A}_n(zk,s)$

Definition needs full set of $\prod_{i=1}^{n} \phi\left(s_i,k_i\right)$

No $\left\langle \mathcal{O}\left(x_1\right)\ldots\mathcal{O}\left(x_{n-1}\right)\right\rangle \delta^d\left(x_i - x_j\right)$

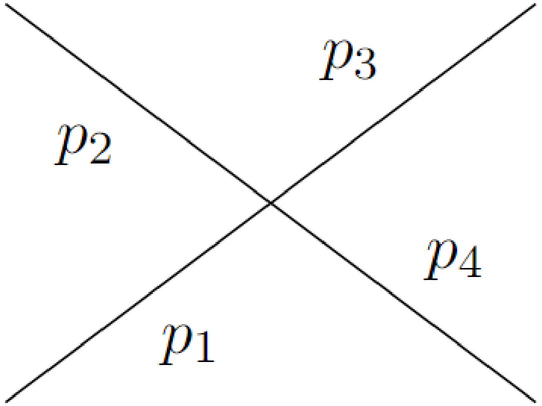
Exactly invariant
field redefinition
symmetry transformation

due to LSZ

[Jiajie Mei, 23]

Example

Four-point contact diagram for $\lambda\phi^4$ theory:

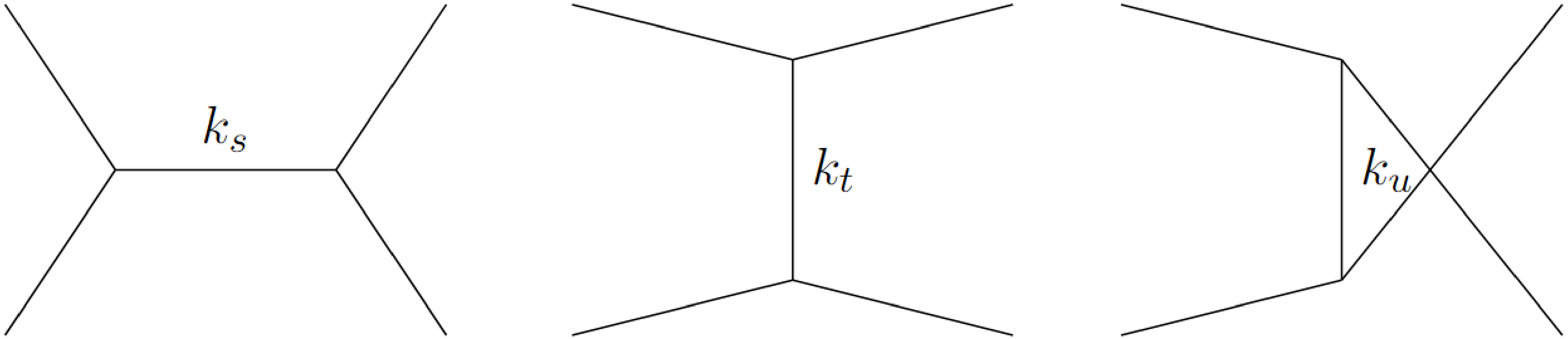


$$\mathcal{A}_{4,c}^{\phi^4} = \lambda \times (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu\right) \times 2\pi i \delta \left(d + \sum_{i=1}^4 (2s_i - d/2)\right)$$

$$\left\langle\left\langle \mathcal{O}\left(\vec{k}_1\right) \ldots \mathcal{O}\left(\vec{k}_n\right)\right\rangle\right\rangle = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi\left(s_i, k_i\right) z^{-2s_i+d/2},$$

Example

Four-point exchange diagram for $\lambda\phi^3$ theory:



$$\mathcal{A}_4^{\phi^3} = \lambda^2 (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu\right) \left(\frac{1}{\mathcal{D}^\Delta_{k_s}} + \frac{1}{\mathcal{D}^\Delta_{k_t}} + \frac{1}{\mathcal{D}^\Delta_{k_u}}\right)$$

$$(\mathcal{D}(z))^{-1}\mathcal{O}(z) = \int \frac{dy}{y^{d+1}} G(z,y)\mathcal{O}(y)$$

$$\mathcal{D}_k^\Delta G(z,y) = z^{d+1}\delta(z-y)$$

$$\int_0^\infty \frac{dz}{z^{d+1}} \phi_\Delta(z_1,k_1)\,\phi_\Delta(z_1,k_2)\,G(z_1,z_2,k_s)\phi_\Delta(z_2,k_3)\,\phi_\Delta(z_2,k_4)$$