Orbifold Averages

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based on work with
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Outline

- Introduction
- Orbifold CFT's
- Averaging
- Bulk Theory
- Conclusions

- Maldacena 1997: AdS/CFT duality, e.g. type IIB on $AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ super Yang Mills
- more recently AdS/Ensembles of CFT's duality, e.g. JT gravity in 2d dual to random matrix model (ensemble of Quantum Mechanics) (Saad, Shenker, Stanford 2019)
- Here we are interested in ensembles of 2d CFT's

Introduction: Free Bosons on T^D's

(Afkhami-Jeddi, Cohn, Hartman, Tajdini 2020; Maloney, Witten 2020)

$$Z_{T^D} = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(p_L,p_R)\in\Gamma_{D,D}} q^{\frac{1}{2}p_L^2}\, \overline{q}^{\frac{1}{2}p_R^2} = \frac{\Theta_h(0,0,\tau)}{|\eta(\tau)|^{2D}}$$

with $\Theta_h(0,0,\tau)$ Siegel-Narain Theta function (h encodes moduli dependence)
average over $\Gamma_{D,D} \in O(D,D,\mathbb{Z}) \backslash O(D,D) / O(D) \times O(D)$
(Siegel-Weil formula)

$$\langle Z_{T^D} \rangle = \frac{E_{D/2}(\tau)}{\tau_2^D\, |\eta(\tau)|^{2D}}$$

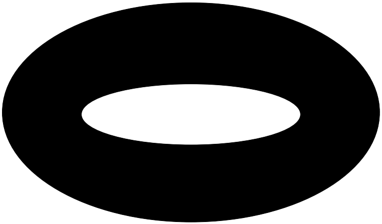
with real analytic Eisenstein series

$$E_s = \sum_{\gamma\in P\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s\,, \quad P : \tau \mapsto \tau + \text{Integer}$$

Introduction: Free Bosons on T^D's, Bulk Dual

$$\langle Z_{T^D} \rangle = \sum_{\text{solid tori}} Z_{CS}$$

where $Z_{CS} =$ Chern-Simons partition function for $U(1)^{2D}$, i.e. bulk dual is Chern-Simons coupled to topological gravity



boundary $= T^2$ with complex structure $\equiv \tau$

Introduction: Free Bosons on $T^D/\mathbb{Z}_2$'s

(Benjamin, Keller, Ooguri, Zadeh 2021)

$$T^D = \mathbb{R}^D/\Lambda_D \text{ and } \mathbb{Z}_2 : \vec{x} \in \mathbb{R}^D \mapsto -\vec{x}$$

- lattice Λ_D is always invariant $\to$ moduli space same as before
- partition function: insert projector $(1+\theta)/2$ into trace and trace also over twisted sector
- contributions with θ insertion and/or over twisted sector do not depend on moduli

$$\langle Z_{T^D/\mathbb{Z}_2}\rangle = \frac{1}{2}\langle Z_{T^d}\rangle + \text{contributions from } \theta \text{ insertion and twisted sector}$$

- in 3D: Chern-Simons $U(1)^{2D} \times \mathbb{Z}_2$
- partition functions match if projector is inserted into trace and vortex contributions are added

Example: $T^2/\mathbb{Z}_2'$

(SF, Jockers, Kames-King, Kanargias, Zadeh 2024)

- consider $\mathbb{Z}_2$ acting only on a subspace of $\mathbb{R}^2$, e.g.

$$\mathbb{Z}_2' : (x, y) \in \mathbb{R}^2 \mapsto (-x, y)$$

- moduli space is reduced:

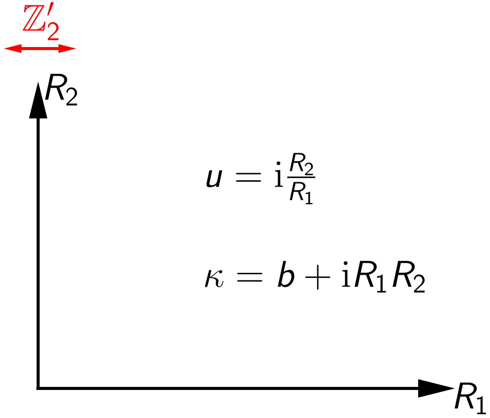
$$b_{xy} = b \overset{!}{=} -b + \text{Integer} \to b \in \left\{0, \frac{1}{2}\right\},$$

i.e. real part of Kähler structure is zero or 1/2

$$T^2 = \mathbb{C}/2\pi\Lambda$$

Λ has to be invariant under $\mathbb{Z}_2'$

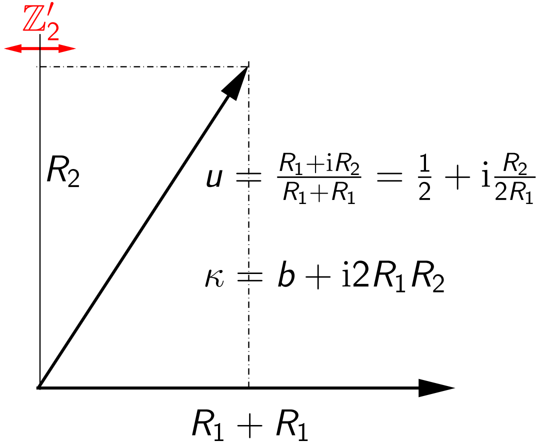
factorisable T^2: real part of complex structure is zero



$$u = \mathrm{i}\frac{R_2}{R_1}$$

$$\kappa = b + \mathrm{i}R_1 R_2$$

non factorisable T^2: real part of complex structure is 1/2



we will consider factorisable T^2 with $b = 0$, non-factorisable T^2 with $b = 0$ (T-dual to factorisable T^2 with $b = 1/2$)

factorisable T^2:

$$Z_{T^2/\mathbb{Z}_2}(\tau, R_1, R_2) = Z_{S^1/\mathbb{Z}_2}(\tau, R_1)\, Z_{S^1}(\tau, R_2)$$

with

$$Z_{S^1}(\tau, R) = \frac{1}{|\eta(\tau)|^2} \sum_{w,m\in\mathbb{Z}} \exp\left[-2\pi \mathrm{i} \tau_1 w m - \pi \tau_2 \left(\frac{w^2}{R^2} + m^2 R^2\right)\right]$$

$$Z_{S^1/\mathbb{Z}_2}(\tau, R) = \frac{1}{2} Z_{S^1}(\tau, R) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|$$

non factorisable T^2 (see e.g. Erler, Klemm 92; Wendland 00):
here, partition function can be expressed in terms of various circle partition functions:

$$
\begin{aligned}
Z_{T^2/\mathbb{Z}_2'} = & \frac{1}{4}\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}\left(2\tau;\sqrt{2}R_1\right) Z_{S^1}\left(2\tau;\sqrt{2}R_2\right) \\
& + \frac{1}{4}\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}\left(\frac{\tau}{2};\sqrt{2}R_1\right) Z_{S^1}\left(\frac{\tau}{2};\sqrt{2}R_2\right) \\
& + \frac{1}{4}\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}\left(\frac{\tau+1}{2};\sqrt{2}R_1\right) Z_{S^1}\left(\frac{\tau+1}{2};\sqrt{2}R_2\right) \\
& - \frac{1}{2} Z_{S^1}\left(\tau;R_1\right) Z_{S^1}\left(\tau;2R_2\right) - \frac{1}{2} Z_{S^1}\left(\tau;2R_1\right) Z_{S^1}\left(\tau;R_2\right) \\
& + \frac{1}{2} Z_{S^1}\left(2\tau;\sqrt{2}R_2\right) + \frac{1}{2} Z_{S^1}\left(\frac{\tau}{2};\sqrt{2}R_2\right) + \frac{1}{2} Z_{S^1}\left(\frac{\tau+1}{2};\sqrt{2}R_2\right)
\end{aligned}
$$

(for $R_1 = R_2$ same as permutation S_2 orbifold (Kames-King, Kanargias, Knighton, Usatyuk 23))

Ensemble Averages

- lower dimension: moduli spaces have infinity volume, need to regularise
- take R_1 and R_2 as moduli, Zamalodchikov measure in factorisable and non-fractorisable case $\sim \frac{dR_1}{R_1}\frac{dR_2}{R_2}$ (factorises!)
- average: just integrate over naive range

$$R_1, R_2 \in (0, \infty) = \lim_{\epsilon \to 0} (\epsilon, 1/\epsilon)$$

- normalise with volume computed in the same way (multiplicities due to discrete symmetries cancel)
- one obtains for factorisable

$$\langle Z_{T^2/\mathbb{Z}_2} \rangle_{\text{reg}} = \frac{1}{2} \langle Z_{S^1} \rangle^2_{\text{reg}} + \left(\left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| \right) \langle Z_{S^1} \rangle_{\text{reg}}$$

- regularised non-factorisable average can also be written as sum of products of regularised averages of circle partition functions
- circle partition function: limit $\epsilon \to 0$ diverges (Maloney, Witten 20)

- consider $T^{2D} = T^D \times T^D = \mathbb{R}^D/\Lambda \times \mathbb{R}^D/\Lambda$
- take Λ canonical, i.e. generated by basis $\{(1,0,\dots,0),(0,1,0,\dots,0),\dots(0,\dots,0,1)\}$ and encode everything in metric (lattice vectors = coordinate basis)
- $\mathbb{Z}_2 : (\vec{x},\vec{y}) \to (\vec{x},-\vec{y})$
- moduli

$$G^{(2D)} = \begin{pmatrix} G^{(D)} & 0 \\ 0 & \tilde{G}^{(D)} \end{pmatrix} \;,\quad B^{(2D)} = \begin{pmatrix} B^{(D)} & 0 \\ 0 & \tilde{B}^{(D)} \end{pmatrix}$$

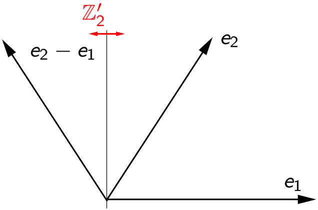
- moduli spaces factorise

$$\langle Z_{T^{2D}/\mathbb{Z}_2'} \rangle = \langle Z_{T^D} \rangle \langle Z_{T^D/\mathbb{Z}_2} \rangle$$

- use results of Afkhami-Jeddi, Cohn, Hartman, Tajdini; Maloney and Witten, respectively of Benjamin, Keller, Ooguri, Zadeh

$$\langle Z_{T^{2D}/\mathbb{Z}_2'} \rangle = \frac{E_{D/2}(\tau)^2}{2\tau_2^D |\eta(\tau)|^{4D}} + 2^{D-1} \left(\left| \frac{\eta(\tau)}{\theta_2(\tau)} \right|^D + \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right|^D + \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right|^D \right) \frac{E_{D/2}(\tau)}{\tau_2^{D/2} |\eta(\tau)|^D}$$

notice: by choice of equivalent basis in T^2 lattice we can have $\mathbb{Z}_2'$ permuting the two basis vectors



generalise for T^{2D}:

$$\mathbb{Z}_2' : (\vec{x}, \vec{y}) \to (\vec{y}, \vec{x})$$

moduli:

$$G^{(2D)} = \frac{1}{2}\begin{pmatrix} g+\tilde{g} & g-\tilde{g} \\ g-\tilde{g} & g+\tilde{g} \end{pmatrix}, \quad B^{(2D)} = \frac{1}{2}\begin{pmatrix} b+\tilde{b} & b-\tilde{b} \\ b-\tilde{b} & b+\tilde{b} \end{pmatrix}$$

Zamalodchikov measure factorises

$$d\,(\text{moduli}) = \det\left(h^{-1}dh\right)\det\left(\tilde{h}^{-1}d\tilde{h}\right)$$

with

$$h = \begin{pmatrix} \frac{1}{2}g^{-1} & \frac{1}{2}g^{-1}b \\ -\frac{1}{2}bg^{-1} & \frac{1}{2}\left(g-bg^{-1}b\right) \end{pmatrix}, \tilde{h} = \begin{pmatrix} \frac{1}{2}\tilde{g}^{-1} & \frac{1}{2}\tilde{g}^{-1}\tilde{b} \\ -\frac{1}{2}\tilde{b}\tilde{g}^{-1} & \frac{1}{2}\left(\tilde{g}-\tilde{b}\tilde{g}^{-1}\tilde{b}\right) \end{pmatrix}$$

$$Z_{T^{2D}/\mathbb{Z}_2'} = \frac{1}{2\left|\eta(\tau)\right|^{4D}} \sum_{\Delta\in\{0,1\}^{2D}} \Theta_h\left(0,\frac{\Delta}{2},2\tau\right)\Theta_{\tilde{h}}\left(0,\frac{\Delta}{2},2\tau\right)$$
$$+\frac{1}{2}\left(Z_{T^D}\left(2\tau;g,b\right)+Z_{T^D}\left(\frac{\tau}{2};g,b\right)+Z_{T^D}\left(\frac{\tau+1}{2};g,b\right)\right).$$

where $\Theta_h(a,b,c)$ Siegel-Narain Θ function (Siegel 44; see also Dong, Hartman, Jiang 2021)

It is known how to average it over moduli space $\{g,b\}$. (Siegel-Weil formula)

$$\langle Z_{T^{2D}/\mathbb{Z}_2'}\rangle = \frac{E_{D/2}\left(2\tau\right)^2+E_{D/2}\left(\frac{\tau}{2}\right)^2+E_{D/2}\left(\frac{\tau+1}{2}\right)^2}{2\left|\eta(\tau)\right|^{4D}\operatorname{Im}\left(\tau\right)^D\left(2^D-1\right)}$$
$$+\frac{1}{2}\left(\frac{E_{D/2}\left(2\tau\right)}{\operatorname{Im}\left(2\tau\right)^{\frac{D}{2}}\left|\eta\left(2\tau\right)\right|^{2D}}+\frac{E_{D/2}\left(\frac{\tau}{2}\right)}{\operatorname{Im}\left(\frac{\tau}{2}\right)^{\frac{D}{2}}\left|\eta\left(\frac{\tau}{2}\right)\right|^{2D}}+\frac{E_{D/2}\left(\frac{\tau+1}{2}\right)}{\operatorname{Im}\left(\frac{\tau+1}{2}\right)^{\frac{D}{2}}\left|\eta\left(\frac{\tau+1}{2}\right)\right|^{2D}}\right)$$

Bulk Dual?

factorisable T^{2D}

$$\langle Z_{T^{2D}/\mathbb{Z}_2'} \rangle = \langle Z_{T^D} \rangle \langle Z_{T^D/\mathbb{Z}_2} \rangle$$

- Each factor can be obtained as partition function from known 3d theory.
- Take two solid tori, one with $U(1)^{2D}$ Chern-Simons theory (for first factor), one with $U(1)^{2D} \times \mathbb{Z}_2$ (for second factor) and sum over all pairs of solid tori with boundary T^2's complex structure $\equiv \tau$

non-factorisable T^{2D}: bulk dual is not known

Conclusion

- Considered $\mathbb{Z}_2'$ orbifolds of T^{2D} in which only half the directions where reflected.
- There are several discrete choices for $\mathbb{Z}_2'$ invariant backgrounds.
- Computed averages of partition function for two examples.
  - factorisable: average= product of known averages
  - non-factorisable: average = sum of products of known averages
- Bulk dual?
- Extend to other orbifolds, e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$: choice of discrete torsion.

Thanks for listening!