Entanglement Rényi entropies in celestial holography

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MARIE CURIE ACTIONS

Asymptotic symmetries of flat space

Asymptotic symmetries of flat space: supertranslations and superrotations

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + \frac{4r^2}{(1+|z|^2)^2}\mathrm{d}z\,\mathrm{d}\bar{z} + \dots$$

Superrotations: $z \to w(z) + \dots\ ,$

$$u \to \frac{1+|z|^2}{1+|w|^2}|w'|u + \dots\ , \quad r \to \frac{1+|w|^2}{1+|z|^2}\frac{1}{|w'|}r + \dots\ .$$

$=$ Lorentz transformations when $w(z) = \frac{az+b}{cz+d}$

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4D flat space scattering amplitudes $=$ 2D CFT correlators

Celestial holography

Cosmic strings & superrotations

Superrotation $w(z) = \left(\frac{z-z_1}{z-z_2}\right)^{1/n}$ [Strominger, Zhiboedov, 1610.00639]

Inserts cosmic string: [Nutku, Penrose, 1992]

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + \frac{4r^2}{(1+|z|^2)^2}\mathrm{d}z\,\mathrm{d}\bar{z} \\ &\quad - ur\frac{(n^2-1)}{2n^2}\left[\frac{(z_2-z_1)^2}{(z-z_1)^2(z-z_2)^2}\mathrm{d}z^2 + \frac{(\bar{z}_2-\bar{z}_1)^2}{(\bar{z}-\bar{z}_1)^2(\bar{z}-\bar{z}_2)^2}\mathrm{d}\bar{z}^2\right] \\ &\quad + u^2\,\frac{(n^2-1)^2}{16n^4}\frac{|z_2-z_1|^4}{|z-z_1|^4|z-z_2|^4}(1+|z|^2)^2\mathrm{d}z\,\mathrm{d}\bar{z} \end{aligned}$$

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Coordinate transformation:

$$\mathrm{d}s^2 = -\mathrm{d}u'^2 - 2\mathrm{d}u'\,\mathrm{d}r' + r'^2\left(\mathrm{d}\theta^2 + \frac{\sin^2\theta}{n^2}\,\mathrm{d}\phi^2\right)$$

Entanglement entropy in QFT

Quantum system, $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$.

Density matrix $\rho \quad \Rightarrow \quad$ reduced density matrix $\rho_{\mathcal{A}} = \mathrm{tr}_{\mathcal{B}}\, \rho$.

Entanglement entropy:

$$S = -\,\mathrm{tr}_{\mathcal{A}}\,(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}})$$

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Rényi entropy

$$S_n = \frac{1}{1-n} \ln \mathrm{tr}_{\mathcal{A}}\, \rho_{\mathcal{A}}^n$$

Entanglement entropy

$$S = \lim_{n \to 1} S_n$$

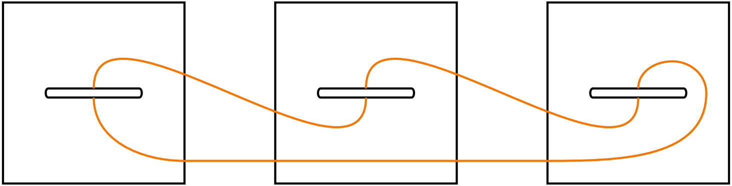
Replica trick

Rényi entropy

$$S_n = \frac{1}{1-n} \ln \operatorname{tr}_{\mathcal{A}} \rho_{\mathcal{A}}^n.$$

Integer n:

$\operatorname{tr} \rho_{\mathcal{A}}^n = Z_n / Z_1^n$ — partition function on n sheets:



Replica trick in 2D CFT

[Calabrese, Cardy, hep-th/0405152]

Entanglement entropy of interval $z \in [z_1, z_2]$

Uniformisation map

$$w(z) = \left(\frac{z - z_1}{z - z_2} \right)^{1/n}$$

n-sheeted replica manifold $\rightarrow$ complex plane

Same as cosmic string superrotation

Replica trick in 2D CFT

Use conformal symmetry:

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln\left(\frac{|z_2 - z_1|}{\varepsilon}\right) + a_n$$

$$S = \frac{c}{3} \ln\left(\frac{|z_2 - z_1|}{\varepsilon}\right) + a_1$$

Weyl transformation $\to$ unit sphere:

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln\left[\frac{2}{\varepsilon} \sin\left(\frac{\ell}{2}\right)\right] + a_n$$

$\ell =$ great circle distance

Entanglement entropy in CCFT?

Working assumption:
CCFT partition function = bulk partition function.

$$Z_{\text{n,CCFT}} = Z_{\text{n,grav.}} = e^{inI_n}$$

I_n = on-shell gravity action in presence of cosmic string, tension

$$\mu = \frac{n-1}{4nG_{\text{N}}}$$

Rényi entropies

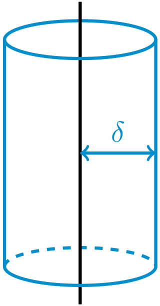
$$S_n = \frac{in}{1-n}\left[I_n - I_1\right]$$

$$\partial_n\left[\left(1-\frac{1}{n}\right)S_n\right] = -i\partial_n I_n$$

c.f. AdS/CFT: [Dong, 1601.06788]

Variation of on-shell action

Conical singularity at location of string
$\rightarrow$ cut out small tube



Einstein gravity (+ minimally coupled matter):

$$\partial_n I_n = -\frac{1}{16\pi G_\mathrm{N}} \int_\mathrm{tube} \mathrm{d}^3 x \sqrt{-\gamma}\, \mathcal{I}$$

$$\mathcal{I} = (\nabla^\mu \partial_n g_{\mu\nu} - g^{\nu\rho} \nabla_m \partial_n g_{\nu\rho})$$

[Lewkowycz, Maldacena, 1304.4926; Dong, 1601.06788]

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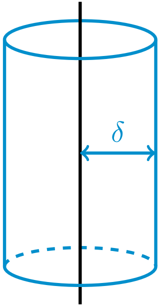

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Key assumption: variation of boundary terms at infinity vanish.

Variation of on-shell action

Cosmic string spacetime:

$$
\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + \frac{4r^2}{(1+|z|^2)^2}\mathrm{d}z\,\mathrm{d}\bar{z} \\
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Variation of on-shell action

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Coordinate transformation:

$$\mathrm{d}s^2 = -\mathrm{d}U\mathrm{d}V + V^2\left(\mathrm{d}W^2 + \frac{W^2}{n^2}\mathrm{d}\phi^2\right)$$

Integral

Area integral:

$$\partial_n I_n = -\frac{1}{n^2}\frac{1}{8G_{\mathrm{N}}}\int \mathrm{d}U\,\mathrm{d}V$$

Regulators:

$$|uv| \leq L^2 \qquad\qquad r \leq r_c$$

$$\updownarrow \qquad\qquad \updownarrow$$

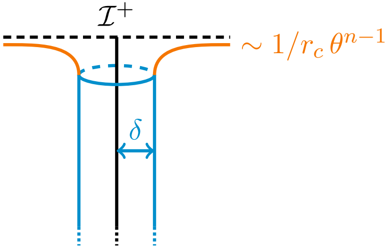
$$|UV| \leq L^2 \qquad\qquad V \leq \frac{2n|z_2 - z_1|W^{n-1}r_c}{|W^n e^{in\phi} - 1|^2 + |z_2 W^n e^{in\phi} - z_1|^2}$$

Similar cutoffs: [Cheung et al., 1609.00732; Ogawa et al., 2207.06735]

Single interval Rényi entropy

$$\lim_{r_c \to \infty} \lim_{\delta \to 0} \neq \lim_{\delta \to 0} \lim_{r_c \to \infty}$$

Finite near string cutoff until end.



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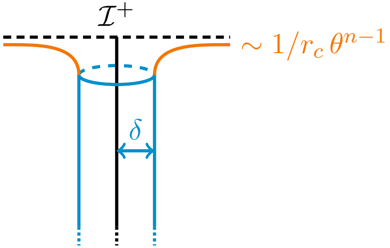


$$S_n = \frac{iL^2}{2G_{\rm N}}\left(1+\frac{1}{n}\right)\ln\left[\frac{2}{\varepsilon}\sin\left(\frac{\ell}{2}\right)\right] \qquad \varepsilon = \frac{\delta}{nr_c}$$

$$= \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{2}{\varepsilon}\sin\left(\frac{\ell}{2}\right)\right] \quad \checkmark$$

$$c = \frac{3iL^2}{G_{\rm N}} \quad \checkmark$$

[Cheung et al., 1609.00732]
[Pasterski, Verlinde, 2201.01630]
[Ogawa et al., 2207.06735]

Why not $c = 0$?

Higher-dimensions

Asymptotically flat spacetime with conical singularity:

(i) Hyperbolic slicing

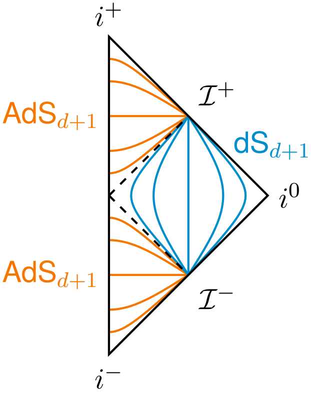
[de Boer, Solodukhin, hep-th/0303006]

(ii) Generalise AdS/CFT CHM method

[Casini, Huerta, Myers, 1102.0440]

$\rightarrow$ Rényi entropy for sphere in CFT$_d$

- No issue with cutoff limits
- Consistent with previous result for $d = 2$



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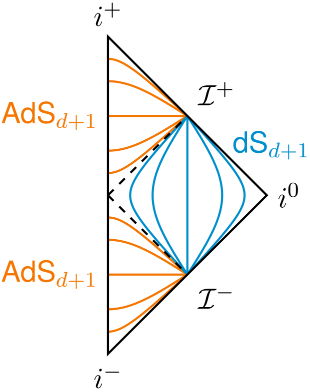
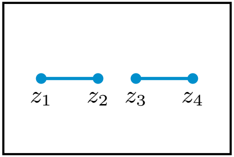


No type A Weyl anomaly when $d = 4\mathbb{Z}$?

Outlook

- Why $c \neq 0$?
- Multiple intervals? Entanglement inequalities?
- Parallel strings
- Meaning of central charge $\propto$ (IR cutoff)^d?
- Relation to Carollian approach?



$$S = \frac{c}{3} \ln \left[\frac{|z_{12} z_{34} z_{14} z_{23}|}{\varepsilon^2 |z_{13} z_{24}|} \mathcal{F}\left(\frac{z_{12} z_{34}}{z_{13} z_{24}} \right) \right]$$

[Calabrese, Cardy, Tonni, 0905.2069]

Thanks!