Index for (0,2) superconformal quantum mechanics

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$|\text{Black hole}> =?$

- Success of AdS/CFT: ‘microstate counting’ $\rightarrow$ BH entropy
- $|\mathit{\Gamma}_{BPS}> \longrightarrow$ Many options! (black holes, single and multi-centered, black rings, fuzzballs..)
- which string theory vacua should we trust in?
- further complication: wall-crossing $\rightarrow$ moduli-dependent ‘index’

“D-brane quantum mechanics”

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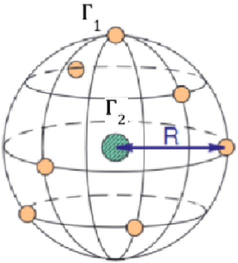
BPS bound states in $d = 4$ and $d = 1$

- BPS solutions in $\mathcal{N} = 2$ string theories, obtained by compactifying Type II on CY.
- In the $d = 4, \mathcal{N} = 2$ sugra description, they can appear as bound states of multi-centered BPS molecules.

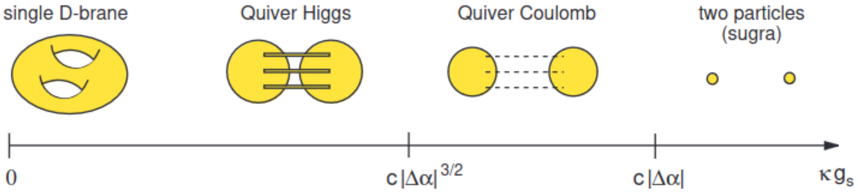


$$R = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\mathrm{Im}\,(e^{-i\alpha} Z_1)_\infty}$$

(Denef,2000)

BPS bound states in $d = 4$ and $d = 1$

* In the $g_S \to 0$ limit, the BPS states appear as wrapped D-branes localized at a single position in the non-compact space. These wrapped D-branes are described by $\mathcal{N} = 4$ susy gauge theory in 0+1d with $\prod_{i=1}^{k} U(N_i)$ gauge group, called quiver mechanics.



(Denef,2002)

* Coulomb branch bound states $\longleftrightarrow$ SUGRA bound states

Susy mechanics of D-particles

* $(d=1, \mathcal{N}=4)$ Effective dynamics: $(x_i, \lambda_\alpha, \bar{\lambda}^\alpha, D)$, (Smilga,1986), (Denef,2002).

$$L_B = \frac{1}{2}\left(\mu + \frac{\kappa}{4|\vec{x}|^3}\right)\left(\dot{\vec{x}}^2 + D^2\right) - \left(f + \frac{\kappa}{2|\vec{x}|}\right) D - \kappa \vec{A}^d \cdot \vec{x}$$

* Susy minima: $\frac{\delta L}{\delta D} = 0$, $D = 0 \Rightarrow r = -\frac{\kappa}{2f}$

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* Susy minima: $\frac{\delta L}{\delta D} = 0$, $D = 0 \Rightarrow r = -\frac{\kappa}{2f}$

$$\Rightarrow \boxed{r = r_{\text{Denef}}}, \text{ as } \kappa = \langle Q, q \rangle,\ f = \mu \sin(\alpha_q - \alpha)|_{r=\infty}.$$

* Straightforwardly generalizes to N-centers $\rightarrow$ Effective Coulomb branch.

Scaling Solutions

- There exists a particular regime called the deep scaling regime of the effective Coulomb quiver mechanics where a full superconformal invariance appears (Anninos, Anous, de Lange, Konstantinidis, 2013). This is described by a $\mathcal{N} = 4$ gauged superconformal mechanics, with susy ground states given by a certain twisted Dolbeault cohomology (Mirfendereski, Raeymaekers, C.S., V.d.Bleeken, 2022).

- This allows for a BPS state counting via a superconformal index

$$\Omega = \mathrm{tr}\left[(-1)^F e^{-\beta(L_0 - J)} \ldots\right].$$

- A useful information for the supergravity description.

(Scaling) Quiver Mechanics

$$
\begin{aligned}
L &= (-f_a D^a) + \left(-U_a D^a + A_{ia}\dot{x}^{ia} + \partial_{ib}U_a\bar{\lambda}^a\sigma_i\lambda^b\right) \\
&+ \left(\frac{1}{2}G_{ab}(\dot{x}^{ia}\dot{x}^{ib} + D^aD^b + i(\bar{\lambda}^a\dot{\lambda}^b - \dot{\lambda}^b\lambda^a)\right. \\
&- \left.\frac{1}{2}\partial_{ic}G_{ab}(\bar{\lambda}^a\sigma_i\lambda^bD^c + \epsilon_{ijk}\bar{\lambda}^a\sigma_j\lambda^B\dot{x}^{kc}) - \frac{1}{8}\partial_{jc}\partial_{jd}G_{ab}\lambda^a\lambda^b\bar{\lambda}^c\bar{\lambda}^d\right) \\
U_a &= \sum_{b,b\neq a}\frac{\kappa_{ab}}{2r_{ab}} \\
A_{ia} &= -\sum_{b,b\neq a}\kappa_{ab}\frac{\epsilon_{ijk}n^jx^k_{ab}}{2r(x^l_{ab}n^l - r)}
\end{aligned}
$$

* D-brane physics determines: κ_{ab} (DSZ), μ_{ab} (mass), f_a (FI).

* Scaling limit has the net effect of putting mass and FI couplings to zero: $f_a = 0$, $\mu_{ab} = 0$, while the action remains finite and develops a $D(2,1;0)$ superconformal symmetry.

Scaling quivers

* 3-centered: $x^A := x^{ia}$, $A = 1, \cdots, 9$. **(scaling limit $\rightarrow \mu_a = f_a = 0$)** (Anninos, Anous, de Lange, Konstantinidis, 2013).

$$L_B = \frac{1}{2} G_{AB} \dot{x}^A \dot{x}^B + A_A \dot{x}^A - V(x)$$

* $SL(2,\mathbb{R})$ invariant: $L_\xi G_{AB} = -G_{AB}$, $L_\xi V = V$, (dilatations)

$\xi = -\frac{1}{2} dK$, $i_\xi F = 0$ (special conformal).

Puzzles

- “pure-Higgs” $\longrightarrow$ exponential growth (Bena,Berkooz, de Boer, El-Showk, V.d. Bleeken, 2012). What is the fate of ‘pure-Higgs’ states at strong gravitational coupling?
- What becomes of the Higgs-Coulomb equivalence (Berkooz,Verlinde,1999)?
- Are ‘single-centered’ black holes distinguished from scaling solutions ? Relevant for BPS index of (Manschot, Pioline, Sen, 2011-2014).
- An explicit example for AdS_2/CFT_1? (cfr. D0-D4-D2 system of (Gaiotto,Strominger,Yin,2006))

General $\mathcal{N} = 2B$ sigma models

* geometry of $\mathcal{N} = 2B$ superconformal mechanics
(Michelson,Strominger,2000), (Papadopoulos,2000)

$$\begin{aligned} L &= L^{(1)} + L^{(2)} \\ L^{(1)} &= A_A \dot{x}^A - \frac{i}{2} F_{AB} \chi^A \chi^B, \qquad (A = 1, \cdots, 2N) \\ L^{(2)} &= \frac{1}{2} G_{AB} \dot{x}^A \dot{x}^B + \frac{i}{2} G_{AB} \chi^A \hat{\nabla}_t \chi^B - \frac{1}{12} \partial_{[A} C_{BCD]} \chi^A \chi^B \chi^C \chi^D \\ \hat{\nabla}_t \chi^A &:= \dot{\chi}^A + \left(\Gamma^A_{BC} + \frac{1}{2} C^A{}_{BC} \right) \dot{x}^B \chi^C \end{aligned}$$

* Lagrangian invariance:

$$F_{AC}(J)^C{}_B + F_{CB}(J)^C{}_A = 0, \qquad G_{AC}(J)^C{}_B + G_{CB}(J)^C{}_A = 0$$

$$\check{\nabla}_A \Omega_{BC} = 0; \qquad \Omega_{AB} = J_{AB}$$

$$L_\omega G = L_\omega J = L_\omega C = 0, \qquad i_\omega F = 0$$

$\mathcal{N}=2$ superconformal index

* $\mathcal{N}=2$ superconformal algebra:

$$\begin{aligned}
[D,Q^\alpha] &= -\frac{i}{2}Q^\alpha \quad [D,S^\alpha]=\frac{i}{2}S^\alpha \quad [H,S^\alpha]=-i\,Q^\alpha \quad [K,Q^\alpha]=iS^\alpha \\
[R,Q^\alpha] &= i\epsilon^{\alpha\beta}Q^\beta \quad [R,S^\alpha]=i\epsilon^{\alpha\beta}S^\beta \\
\{Q^\alpha,S^\beta\} &= -2\delta^{\alpha\beta}D+\epsilon^{\alpha\beta}R
\end{aligned}$$

* Go to a different basis:

$$\begin{aligned}
\mathcal{G}_{\pm\frac{1}{2}} &= \frac{1}{2}(\omega^{-\frac{1}{2}}(Q^1+iQ^2)\mp i\omega^{\frac{1}{2}}(S^1+iS^2)) \\
L_0 &= \frac{1}{2}(\omega^{-1}H+\omega K) \\
L_{\pm1} &= \frac{1}{2}(\omega^{-1}H-\omega K)\pm iD.
\end{aligned}$$

$\mathcal{N} = 2$ superconformal index

* $\mathcal{N} = 2$ superconformal algebra:

$$[L_m, L_n] = (m-n)L_{m+n}$$

$$[L_0, \mathcal{G}_{\pm\frac{1}{2}}] = \mp\frac{1}{2}\mathcal{G}_{\pm\frac{1}{2}}, \qquad [L_{\mp 1}, \mathcal{G}_{\pm\frac{1}{2}}] = \mp\mathcal{G}_{\mp\frac{1}{2}}, \quad [R, \mathcal{G}_{\pm\frac{1}{2}}] = \mathcal{G}_{\pm\frac{1}{2}}$$

$$\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^{\dagger}_{\pm\frac{1}{2}}\} = 2L_0 \pm R, \qquad \{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^{\dagger}_{\mp\frac{1}{2}}\} = 2L_{\pm 1}, \quad \{\mathcal{G}_\alpha, \mathcal{G}_\beta\} = 0$$

* lowest weight state: $|h, r\rangle$

$$\mathcal{G}_{+1/2}\,|h, r\rangle = \mathcal{G}^{\dagger}_{-1/2}\,|h, r\rangle = 0$$

* lowest weight chiral primary:

$$\mathcal{G}_{+1/2}\,|h, r\rangle = \mathcal{G}^{\dagger}_{-1/2}\,|h, r\rangle = \mathcal{G}_{-1/2}\,|h, r\rangle = 0$$

$\mathcal{N} = 2$ superconformal index

* (unitary, lowest-weight, infinite-dimensional) Irreps:

$$\text{chiral}: \qquad (h)_r + \left(h + \frac{1}{2}\right)_{r-1}, \qquad r = 2h$$

$$\text{anti}-\text{chiral}: \qquad (h)_r + \left(h + \frac{1}{2}\right)_{r+1}, \qquad r = -2h$$

$$\text{long}: \quad (h)_r + \left(h + \frac{1}{2}\right)_{r-1} + \left(h + \frac{1}{2}\right)_{r+1} + (h+1)_r, \quad |r| < 2h$$

$\mathcal{N} = 2$ superconformal index

* Well-defined '(refined-)superconformal-index':

$$\Omega_\pm(\zeta) = \mathrm{tr}(-1)^F e^{-\beta\mathcal{H}_\pm}\zeta^{\pm J} \ , \qquad \mathcal{H}_\pm = \omega\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\pm\frac{1}{2}}\} = \omega(2L_0 \pm R)$$

which counts

$$\begin{aligned}\Omega_+(\zeta) &= \sum_h \left(N^+_{\mathrm{anti-chiral}}(h) - N^-_{\mathrm{anti-chiral}}(h)\right)\zeta^J, \\ \Omega_-(\zeta) &= \sum_h \left(N^+_{\mathrm{chiral}}(h) - N^-_{\mathrm{chiral}}(h)\right)\zeta^{-J}\end{aligned}$$

* Crucial observation:

$$U(1)_J \neq U(1)_R \qquad J = -R + \frac{i}{2}J_{AB}\chi^A\chi^B \qquad [J, su(1,1|1)] = 0 \ !$$

A Simple Example

* 2-dimensional free-particle:

$$
\begin{aligned}
L &= \frac{1}{2}\dot{x}_A\dot{x}^A + \frac{i}{2}\chi_A\dot{\chi}^A \ , \qquad A = 1,2 \\
&= \dot{z}\dot{\bar{z}} + i\bar{\psi}\dot{\psi}
\end{aligned}
$$

* Noether Charges:

$$
\begin{aligned}
H &= p_z p_{\bar{z}}, \qquad K = z\bar{z}, \qquad D = -\frac{1}{2}(zp_z + \bar{z}p_{\bar{z}}) \\
R &= -i(zp_z - \bar{z}p_{\bar{z}}) - [\psi, \bar{\psi}] \\
Q^1 + iQ^2 &= 2ip_{\bar{z}}\bar{\psi}, \qquad S^1 + iS^2 = 2iz\bar{\psi}
\end{aligned}
$$

A Simple Example

* Recall: the 'superconformal-index'

$$\Omega_\pm = \mathrm{tr}(-1)^F e^{-\beta\mathcal{H}_\pm}\ , \qquad \mathcal{H}_\pm = \omega\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\pm\frac{1}{2}}\} = \omega(2L_0 \pm R)$$

$\longrightarrow \equiv$ Witten index for the auxiliary model.

* $N = 2$ auxiliary model:

$$\begin{aligned} \mathcal{G}_{\pm\frac{1}{2}} &= \frac{i}{\sqrt{\omega}}\left(p_{\bar z} - \mathcal{A}^\pm_{\bar z}\right)\bar\psi \\ \mathcal{H}_\pm &= \left(p_z - \mathcal{A}^\pm_z\right)\left(p_{\bar z} - \mathcal{A}^\pm_{\bar z}\right) + i\mathcal{F}^\pm_{z\bar z}\psi\bar\psi \\ \mathcal{A}^\pm_z &= \mp i\omega\bar z, \qquad \mathcal{A}^\pm_{\bar z} = \pm i\omega z \end{aligned}$$

* Witten parity:

$$F = \hat{\bar\psi}\hat\psi, \qquad W = (-1)^F = \sigma_3$$

A Simple Example

* Hilbert space: $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$.

$$\hat{p}_z = -i\partial_z, \qquad \hat{p}_{\bar{z}} = -i\partial_{\bar{z}}$$
$$\hat{\psi} = \sigma_+, \qquad \hat{\bar{\psi}} = \sigma_-$$

* Superconformal generators:

$$\begin{aligned} \hat{\mathcal{G}}_{\pm\frac{1}{2}} &= (\omega)^{-\frac{1}{2}}(\partial_{\bar{z}} \pm \omega z)\sigma_- \\ \hat{L}_0 &= (2\omega)^{-1}(-\partial_z\partial_{\bar{z}} + \omega^2 z\bar{z}) \\ \hat{R} &= -(z\partial_z - \bar{z}\partial_{\bar{z}} + \sigma_3) \end{aligned}$$

* (anti-)chiral primaries:

$$\chi_{\frac{n+1}{2}} = \bar{z}^n e^{-\omega z\bar{z}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{\chi}_{\frac{n+1}{2}} = z^n e^{-\omega z\bar{z}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Refined Index

* $J = z\partial_z - \bar{z}\partial_{\bar{z}} + \frac{\sigma_3}{2}$, $[J, \mathcal{G}_{\pm 1/2}] = [J, W] = 0$

* acting on primaries:

$$J\chi_{\frac{n+1}{2}} = -\left(n + \frac{1}{2}\right)\chi_{\frac{n+1}{2}}, \qquad J\tilde{\chi}_{\frac{n+1}{2}} = \left(n + \frac{1}{2}\right)\chi_{\frac{n+1}{2}},$$

* refines the index:

$$\Omega_{\pm}(\zeta) = \mathrm{tr}(-1)^F e^{-\beta\mathcal{H}_{\pm}}\zeta^{\pm J}$$

* which then becomes:

$$\Omega_{\pm}(\zeta) = \pm\frac{\zeta^{1/2}}{1-\zeta}$$

Index $\equiv$ Euclidean Path Integral

* Refined Index (with Twisted Boundary Conditions):

$$\begin{aligned}\zeta = e^{i\theta} \quad &\Rightarrow \quad \Omega_\pm(e^{i\theta}) = \mathrm{tr}(-1)^F e^{\pm i\theta J} e^{-\beta \mathcal{H}_\pm} \\ &\Rightarrow \quad z(\tau+\beta) = e^{\pm i\theta} z(\tau)\ ,\ \psi(\tau+\beta) = e^{\pm i\theta}\psi(\tau) \\ &\Rightarrow \quad \Omega_\pm[e^{i\theta}] = \int [DzD\bar{z}D\psi D\bar{\psi}]_{\pm\theta} e^{-S_E^\pm}\end{aligned}$$

* Refined Index (with Periodic Boundary Conditions):

$$\begin{aligned}\zeta = e^{-\beta\lambda} \quad &\Rightarrow \quad \Omega_\pm[e^{-\beta\lambda}] = \mathrm{tr}(-1)^F e^{-\beta(\mathcal{H}_\pm \pm \lambda J)} \\ &\Rightarrow \quad \Omega_\pm[e^{-\beta\lambda}] = \int [DzD\bar{z}D\psi D\bar{\psi}]_{\mathrm{PBC}} e^{-S_{E,\lambda}^\pm}\end{aligned}$$

General $\mathcal{N} = 2B$ superconformal sigma models

* Superconformal charges:

$$\mathcal{G}_{\pm\frac{1}{2}} = \frac{1}{2\sqrt{\omega}}\chi^{\bar{m}}\left(i(\tilde{p}_{\bar{m}} - A_{\bar{m}} - \mathcal{A}^{\pm}_{\bar{m}}) + \omega_{\bar{m}\underline{\bar{p}}\underline{n}}\chi^{\underline{\bar{p}}}\chi^{\underline{n}} + \frac{1}{8}\partial_{\bar{m}}\ln G\right)$$

* with the effective magnetic field:

$$\mathcal{A}^{\pm} = \pm 2\omega J\cdot\xi = \mp 2\omega\omega^{\flat}, \qquad \mathcal{F}^{\pm} = \pm 2\omega(\Omega - i_{\omega}C) = \pm 2i\partial\bar{\partial}K$$

* Superconformal index:

$$\Omega_{\pm} = \prod_{n=1}^{d_{\mathbb{C}}}\frac{i}{2\sin\left(\pm\frac{\mu\omega_n}{2}\right)}, \qquad \nabla^{A}\rho_B(r=0) = \begin{pmatrix} 0 & \omega_1 & \dots \\ -\omega_1 & 0 & \\ & & \ddots \end{pmatrix}$$

Open Questions

1. Index-Decoding (more information about the spectrum)
2. Index for (gauged) $\mathcal{N} = 4B$ mechanics
3. Scaling quiver ground states $\leftrightarrow$ black hole microstates in sugra
4. Generalizations: higher order terms in superpotential, non-primitive nodes, comparison with Higgs branch results
5. Type-A / Type-B analogy? (gauging, relation to higher d SCFTs, SCQM/AdS holography...(Dorey, Singleton, Barns-Graham, Zhang, Crew, Zhao, Mouland)).
6. Explicit checks for $\mathcal{I}$-extremization/R-extremization and relation to asymptotically AdS black holes (Gauntlett, Martelli, Sparks,Yau),(Benini,Hristov,Zaffaroni).

Appendices

General $\mathcal{N} = 2B$ superconformal sigma models

* Superconformal invariance:

$$\mathcal{L}_\xi G_{AB} = -G_{AB}, \qquad \mathcal{L}_\xi J^A{}_B = 0$$
$$\mathcal{L}_\xi C = -C, \qquad i_\xi C = 0.$$

* also fixes

$$\omega^A = -J^A{}_B \xi^B \qquad K = \xi^2, \qquad \xi^\flat = -\frac{1}{2} dK$$

* and, the Kahler form:

$$\Omega = -d\omega^\flat + i_\omega C$$

* ‘Free-particle’:

$$F_{AB} = 0, \qquad G_{AB} = \delta_{AB},$$
$$J^A{}_B = \epsilon^A{}_B, \qquad \xi^A = -\frac{1}{2} x^A, \quad \omega^A = \frac{1}{2} \epsilon^A{}_B x^B$$

Always a Cone

* Radial coordinate: $r^2 = 2K \quad G = dr^2 + r^2\tilde{G}\ ,\ \xi = -\frac{1}{2}r\partial_r.$
* Constraints then imply

$$i_\xi d\Omega = -i_\rho C \ \rightarrow\ \Omega = -d\rho^\flat + i_\rho C$$

so

$$\text{G is Kahler} \iff i_\rho C = 0$$

* In this case, then: the base of the cone is a Sasakian manifold, with ρ being the Reeb vector field, and the Kahler form becomes:

$$\Omega = i\partial\bar{\partial}K$$

* So K is the Kahler potential.

$\mathcal{N} = (0,4)$ (gauged) superconformal mechanics

* geometry of $\mathcal{N} = 4B$ (gauged) superconformal mechanics (Michelson,Strominger,2000), (Papadopoulos,2000),(Mirfendereski, Raeymaekers, C.S., V.d.Bleeken, 2022):

$$\begin{aligned} L &= L_B + L_F \\ L_B &= A_A \dot{x}^A + a^I v_I + \frac{1}{2} G_{AB} D_t x^A D_t x^B \\ L_F &= -\frac{i}{2} F_{AB} \chi^A \chi^B + \frac{i}{2} G_{AB} \chi^A \check{D}_t \chi^B - \frac{1}{12} \partial_{[A} C_{BCD]} \chi^A \chi^B \chi^C \chi^D \end{aligned}$$

* a weakly HyperKahler with Torsion target:

$$\begin{aligned} F_{AC}(J^i)^C{}_B + F_{CB}(J^i)^C{}_A &= 0 \\ G_{AC}(J^i)^C{}_B + G_{CB}(J^i)^C{}_A &= 0 \\ \check{\nabla}_A (J^i)^B{}_C &= 0 \end{aligned}$$

$\mathcal{N} = (0,4)$ (gauged) superconformal mechanics

* Standard formulation of Coulomb Quiver Mechanics: n-(3,4,1) multiplets. Recasted into a description by n-(4,4,0) multiplets:

$$x^A := x^{\mu a} = (x^{ia}, x^{4a}) \;, \qquad \dot{x}^{4a} \leftrightarrow D^a \qquad a = 1, \ldots, n$$

$$\begin{aligned}
G_{\mu a\,\nu b} &= \delta_{\mu\nu} G_{ab} \;\;,\;\; G_{ab} = \frac{1}{2}\partial_{ia}\partial_{ib}\mathcal{H} \\
\mathcal{H} &= \sum_{a,b,a\neq b}\left(\frac{\mu_{ab}|\Phi^{ab}|^2}{6} - \frac{|\kappa_{ab}|}{4|\Phi^{ab}|}\log|\Phi^{ab}|\right) \\
\partial_{ia}G_{bc} &= \partial_{i(a}G_{bc)} \\
C_{\mu a\,\nu b\,\rho c} &= \partial_{\lambda a}G_{bc}\,\epsilon_{\lambda\mu\nu\rho} \\
(J^i)^{\mu a}{}_{\nu b} &= (j^i_+)_{\mu\nu}\delta^a_b \\
\partial_{4a}A_{\mu b} &= \partial_{4a}G_{bc} = 0, \quad F_{\mu a\,\nu b} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho a\,\sigma b} \\
k_a &= \partial_{4a} \;,\; \mathfrak{g} = u(1)^n \\
v_a &= -A_{4a} \;\;,\;\; (A_A) = (A_{\mu a}) = (A_{ia}, -f_a - U_a)
\end{aligned}$$

Upshot

1. Scaling quiver theory ⊂ gauged superconformal mechanical models with $D(2,1;0)$ superconformal symmetry.

2. Use $D(2,1,0)$ Superconformal index (Gaiotto, Simons, Strominger, Yin, 2006):

$$\mathrm{tr}[(-1)^F y^{L_0+J_L} z^{L_0+J_R}]$$

Upshot

1. Scaling quiver theory ⊂ gauged superconformal mechanical models with $D(2,1;0)$ superconformal symmetry.

2. Use $D(2,1,0)$ Superconformal index (Gaiotto, Simons, Strominger, Yin, 2006):

$$\mathrm{tr}[(-1)^F y^{L_0+J_L} z^{L_0+J_R}]|_{\text{scaling quivers}} = ?$$

3. formulated recently for the $(\mathcal{N}=2)$ case (J. Raeymaekers, C.S., D.V.d.Bleeken,2024).

Index via Path Integral

* Auxiliary Lagrangian

$$\mathcal{L}_\pm = \dot{z}\dot{\bar{z}} + \mathcal{A}_z^\pm \dot{z} + \mathcal{A}_{\bar{z}}^\pm \dot{\bar{z}} + i\bar{\psi}\dot{\psi} + i\mathcal{F}_{z\bar{z}}^\pm \bar{\psi}\psi$$

* Euclidean action

$$S_E^\pm = \int_0^\beta d\tau \left(\dot{z}\dot{\bar{z}} - i(\mathcal{A}_z^\pm \dot{z} + \mathcal{A}_{\bar{z}}^\pm \dot{\bar{z}}) + \bar{\psi}\dot{\psi} - i\mathcal{F}_{z\bar{z}}^\pm \bar{\psi}\psi \right)$$

* With susies $\mathcal{G}_{\pm 1/2}$ generating:

$$\delta z = 0, \qquad \delta \bar{z} = \frac{i}{\sqrt{\omega}}\bar{\psi}$$

$$\delta \psi = \frac{i}{\sqrt{\omega}}\dot{z}, \qquad \delta \bar{\psi} = 0$$

* BPS saddles: $z = z_0$, $\bar{z} = \bar{z}_0$, $\psi = \bar{\psi} = 0$.

Index via Path Integral

* Expanding to quadratic order in $z = z_0 + \zeta$, $\psi = 0 + \chi$:

$$\begin{aligned} \Omega_\pm &= \mathcal{M} \int dz_0 d\bar{z}_0 \frac{\det_{\mathrm{PBC}}(\partial_\tau - i\mathcal{F}^\pm_{z\bar{z}}(z_0, \bar{z}_0))}{\det'_{\mathrm{PBC}}(-\partial_\tau^2 + i\mathcal{F}^\pm_{z\bar{z}}(z_0, \bar{z}_0)\partial_\tau)} \\ &= \frac{1}{2\pi} \int \mathcal{F}^\pm \end{aligned}$$

A Simple Example

* saturate the bound:

$$
\begin{aligned}
2\hat{L}_0 \chi_{\frac{n+1}{2}} &= \hat{R}\chi_{\frac{n+1}{2}} = (n+1)\chi_{\frac{n+1}{2}}\,, \\
2\hat{L}_0 \tilde{\chi}_{\frac{n+1}{2}} &= -\hat{R}\tilde{\chi}_{\frac{n+1}{2}} = (n+1)\tilde{\chi}_{\frac{n+1}{2}}\,.
\end{aligned}
$$

* Unrefined index counts Lowest Landau Levels:

$$\Omega_{\pm} = \pm\infty$$

Refined Index via Path Integral

1. Same $S_E^{\pm}$, with Twisted Boundary Conditions:
   - BPS saddles: $z = \psi = 0$.
   - $\Omega_{\pm}[e^{i\theta}] = \frac{\det_{\pm\theta}(\partial_\tau + i\mathcal{F}^{\pm}_{z\bar{z}})}{\det_{\pm\theta}(-\partial_\tau^2 - i\mathcal{F}^{\pm}_{z\bar{z}}\partial_\tau)} = (\det_{\pm\theta}(-\partial_\tau))^{-1}$
     $= \pm\left(e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}\right)^{-1}$

2. $S_E^{\pm} \to S_{E,\lambda}^{\pm}$, with Periodic Boundary Conditions:

$$S_{E,\lambda}^{\pm} = \int_0^\beta d\tau \left[\dot{z}\dot{\bar{z}} \pm 2\lambda\bar{z}\dot{z} - \lambda^2|z|^2 + \bar{\psi}\dot{\psi} \mp \lambda\bar{\psi}\psi \right.$$
$$\left. -i(\mathcal{A}_z^{\pm}\dot{z} + \mathcal{A}_{\bar{z}}^{\pm}\dot{\bar{z}}) + i\mathcal{F}^{\pm}_{z\bar{z}}\bar{\psi}\psi + 2\omega\lambda|z|^2\right].$$

$\to$ localizes to $|z| = 0$, as $\lambda \to \infty$.

$\to \Omega_{\pm}[e^{-\beta\lambda}] = \frac{\det_{\mathrm{PBC}}(\partial_\tau \mp 1)}{\det_{\mathrm{PBC}} - (\partial_\tau \mp 1)^2} = \pm\left(e^{\frac{\lambda\beta}{2}} - e^{-\frac{\lambda\beta}{2}}\right)^{-1}$