W-symmetries, anomalies and heterotic backgrounds with SU holonomy

Loukas Grimanellis

King's College London

Eurostrings 2024
University of Southampton
3 September 2024

Based on arXiv:2305.19793 with G. Papadopoulos and E. Perez-Bolaños

Review

- Sigma models play a central role in string theory as they describe the string propagation in a spacetime M^n.
- 2-dimensional non-linear sigma models with target space M^n.

Interest:

- Algebra of symmetries
- Anomalies
- Anomaly cancellation

Review

- $\hat{\nabla}$-covariantly constant forms generate symmetries in 2-dim supersymmetric sigma models [Odake; Howe, Papadopoulos]

$$\hat{\nabla} = \nabla + H/2, \text{ a metric connection with torsion } H = db.$$

- These symmetries arise from the reduction of the holonomy of $\hat{\nabla}$ to a subgroup of $O(n)$
- Known as *holonomy symmetries* and satisfy a W-algebra. [Howe, Papadopoulos]
- Some of these backgrounds have been considered as target spaces of heterotic sigma models. [de La Ossa, Fiset]

Review

- Symmetries of heterotic sigma models are anomalous in the quantum theory due to the presence of the chiral worldsheet fermions.
- E.g. gauge invariance is necessary for the global definition of the theory.
- To preserve the geometric interpretation, the anomalies of those symmetries must cancel.
- The anomaly cancellation of the spacetime frame rotations and the gauge sector transformations has been investigated. [Hull, Witten]
- The classification of heterotic backgrounds has revealed a general class of spacetimes admitting $\hat{\nabla}$-covariantly constant forms constructed as Killing spinor bilinears. [Gran, Gutowski, Papadopoulos]

Heterotic Backgrounds with SU Holonomy

Let the (1,0)-supersymmetric 2-dim sigma model with target space heterotic backgrounds

$$S = -i \int d^2\sigma\, d\theta^+ \left((g+b)_{\mu\nu} D_+ X^\mu \partial_= X^\nu + i h_{ab} \psi_-^a \mathcal{D}_+ \psi_-^b \right) \tag{1}$$

- *Sigma model symmetries:* Diffeomorphisms as well as spacetime frame rotations, and gauge transformations with connections ω and Ω resp.

- The transformations $\delta_L X^\mu = a_L\, L^\mu{}_{\nu_1 \dots \nu_\ell}\, D_+ X^{\nu_1 \dots \nu_\ell}$, and $\Delta_L \psi_-^a = 0$ leave the action invariant when

$$\hat{\nabla}_\nu L_{\lambda 1} \dots \lambda_{\ell+1} = 0 \ , \quad F_{\nu[\lambda_1} L^\nu{}_{\lambda_2 \dots \lambda_{\ell+1}]} = 0 \tag{2}$$

with F the curvature of Ω.

Heterotic Backgrounds with SU Holonomy

We investigated two classes:

- $hol(\hat{\nabla}) \subset SU(2)$: Six 1-forms e^a, and three 2-forms I_r.

$$\delta_K X^\mu = a_K^a e_a^\mu \ , \quad \delta_I X^\mu = a_I^r (I_r)^\mu{}_\nu D_+ X^\nu \tag{3}$$

- $hol(\hat{\nabla}) \subset SU(3)$: Four 1-forms e^a, one 2-form I, and two 3-forms L_s

$$\delta_L X^\mu = a_L^s (L_s)^\mu{}_{\nu\rho} D_+ X^{\nu\rho} \tag{4}$$

Algebra

We calculated explicitly the algebra of holonomy symmetries which can be rearranged as [Papadopoulos, Pérez-Bolaños, LG]

$$[\delta_L, \delta_M] = \delta_N + \delta_S + \delta_{JP} \tag{5}$$

$$\begin{aligned}\delta_S X_\mu &= \alpha_S \hat{\nabla}_+ D_+ X^\nu S_{\nu,\mu Q}\, D_+ X^Q + \frac{(-1)^q}{q+1} \hat{\nabla}_+ (\alpha_S S_{\mu,\nu Q}\, D_+ X^{\nu Q}) \\ &\quad - \frac{q+3}{3(q+1)} \alpha_S H_{[\mu\nu\rho} Q_{Q]} D_+ X^{\nu\rho Q}\ , \quad S^\mu{}_{\nu Q} = \delta^\mu{}_{[\nu} Q_{Q]} \\ \delta_{JP} X_\mu &= (m'+1) c_{L'} J_{L'}\, \delta_{M'} X_\mu + (\ell'+1) c_{M'} J_{M'}\, \delta_{L'} X_\mu\end{aligned}$$

- Each variation in the RHS is a symmetry of the action.
- Generated by $\hat{\nabla}$-covariantly constant forms N, S, P.
- a_N, a_S, a_{JP} depend on a_L, a_M, but a_{JP} also on conserved currents J.

Algebra

- W-algebra as its structure constants depend on conserved currents.
- Closes under the addition of the stress-energy tensor T and Casimir operator C

$$\delta_T X^\mu = 2i\alpha_T \partial_{\ddagger} X^\mu + D_+ \alpha_T D_+ X^\mu$$
$$\delta_C X^a = \alpha_C \hat{\nabla}_+ D_+ X^a + \hat{\nabla}_+(\alpha_C D_+ X^a)$$

- E.g. in $SU(3)$

$$[\delta_I, \delta_I'] = \delta_T + \delta_C + \delta_K \tag{6}$$

with $\alpha_T = a_I' a_I$, $\alpha_C = -a_I' a_I$, and $a_K^a = a_I' a_I H^a{}_{bc} J_K^b J_K^c$, where J_K the current of e^a.

Anomalies

Due to the worldsheet fermions, the symmetries of heterotic sigma models are anomalous.

- The anomalies of spacetime frame rotations and gauge transformations are known. [Zumino; Hull, Townsend]
- About the former, let the curvature R of the frame connection ω. Using the Poincaré lemma and the descent equations, locally

$$P_4 = TrR^2 = dQ_3^0 \tag{7}$$

with $\delta_\ell Q_3^0 = dQ_2^1$, where Q_3^0 the Chern-Simons form.

Anomalies

- The frame rotation anomaly is

$$\Delta(\ell) = \frac{i\hbar}{4\pi} \int d^2\sigma \, d\theta^+ \, Q_2^1(\omega, \ell)_{\mu\nu} \, D_+ X^\mu \, \partial_= X^\nu \tag{8}$$

  and similarly for $\Delta(u)$.

- For a theory invariant under the algebra of symmetries with

$$[\delta_A, \delta_B] = \delta_{[A,B]} \tag{9}$$

- The effective action Γ will satisfy the Wess-Zumino consistency condition

$$\delta_A \Delta(a_B) - \delta_B \Delta(a_A) = \Delta(a_{[A,B]}) \tag{10}$$

  with $\delta_A \Gamma = \Delta(a_A)$ the anomaly of δ_A.

Anomalies

- As $[\ell, L] = [u, L] = 0$, then $\Delta(a_{[\ell,L]}) = \Delta(a_{[u,L]}) = 0$

$$\delta_\ell \Delta(a_L) - \delta_L \Delta(\ell) = 0 \tag{11}$$
$$\delta_u \Delta(a_L) - \delta_L \Delta(u) = 0 \tag{12}$$

- A solution of these gives the anomaly of the holonomy symmetries

$$\Delta(a_L) = \frac{i\hbar}{4\pi} \int d^2\sigma \, d\theta^+ \, Q_3^0(\omega, \Omega)_{\mu\nu\rho} \, \delta_L X^\mu D_+ X^\nu \, \partial_= X^\rho \tag{13}$$

where $Q_3^0(\omega, \Omega) = Q_3^0(\omega) - Q_3^0(\Omega)$.

Anomalies

$$\begin{aligned}
&\delta_L \Delta(a_M) - \delta_M \Delta(a_L) \\
&\qquad = \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \, P_4(R,F)_{\mu\nu\rho\sigma} \delta_L X^\mu \delta_M X^\nu D_+ X^\rho \partial_= X^\sigma \\
&\qquad + \frac{i\hbar}{4\pi} \int d^2\sigma d\theta^+ \, Q_3^0(\omega,\Omega)_{\mu\nu\rho} [\delta_L, \delta_M] X^\mu D_+ X^\nu \partial_= X^\rho
\end{aligned}$$

- P_4 term may be a potential inconsistency, cf. WZ
- Dependence on the symmetries in the RHS of $[\delta_L, \delta_M]$
- If L, M are not anomalous, the whole RHS must vanish.
- All anomalies are consistent at 1-loop if the CS forms are in terms of the frame connection associated to the connection with torsion $-H$.

Anomaly cancellation

The anomalies of frame rotations and gauge transformations cancel after assigning an anomalous variation to b [Hull, Witten]

$$\delta_\ell b = \frac{\hbar}{4\pi} Q^1_2(\ell, \omega) \ , \quad \delta_u b = -\frac{\hbar}{4\pi} Q^1_2(u, \Omega) \tag{14}$$

There are two ways to cancel the holonomy anomalies.

- First, by adding finite local counterterms in the effective action. As $P_4 = dQ^0_3$, then Q^0_3 is specified up to an exact form, $Q^0_3 \to Q^0_3 + dW$.

  E.g. in $SU(2)$, there is a $\tilde{P}_4$ on N^4 such that $P_4 = \pi^* \tilde{P}_4$

  As $d\tilde{P}_4 = 0$, there is $\tilde{Q}^0_3$ such that $\tilde{P}_4 = d\tilde{Q}^0_3$, thus

  $$Q^0_3 = \pi^* \tilde{Q}^0_3 + dW \tag{15}$$

  with W a 2-form on M^{10}.

Anomaly cancellation

Adding in Γ the finite local counterterm

$$\Gamma^{fl} = -\frac{i\hbar}{4\pi}\int d^2\sigma d\theta^+ \, W_{\mu\nu}\, D_+ X^\mu \,\partial_= X^\nu \tag{16}$$

$$\Delta(a_L) + \delta_L \Gamma^{fl} = \frac{i\hbar}{4\pi}\int d^2\sigma d\theta^+ \,(\pi^* \tilde{Q}^0_3)_{\mu\nu\rho}\, \delta_L X^\mu \, D_+ X^\nu \,\partial_= X^\rho = 0 \tag{17}$$

when $i_L \pi^* \tilde{Q}^0_3 = 0$ which holds for $L = K$ and C.

Re-expressing the anomalies by replacing Q^0_3 with $\pi^* \tilde{Q}^0_3$ results their cancellation. Analogous treatment holds for I.

Anomaly cancellation

- Second, when L receives quantum corrections, $L^\hbar$.

$$\delta_L^\hbar\Gamma = \delta_L^\hbar(\Gamma^{(0)} + \hbar\Gamma^{(1)}) = \Delta(a_L) \implies$$

$$- i\int d^2\sigma d\theta^+ (a_L \frac{2(-1)^\ell}{\ell+1}\hat{\nabla}_\mu L^\hbar_{L+1}\partial_= X^\mu D_+ X^{L+1}$$

$$- i a_L L_L^{\hbar\mu} F^\hbar_{\mu\nu ab}\psi_-^a\psi_-^b D_+ X^{L\nu} + 2i\Delta_L^\hbar\psi_-^a\mathcal{D}_+^\hbar\psi_{-a}) = 0 + \mathcal{O}(\hbar^2)$$

where $\hat{\nabla}^\hbar$ the quantum corrected connection with

$$H^\hbar = H - \frac{\hbar}{4\pi}Q_3^0(\omega,\Omega) + \mathcal{O}(\hbar^2) \qquad (18)$$

Provided that $\hat{\nabla}^\hbar L^\hbar = 0$ and $i_{L^\hbar}F^\hbar = 0$, the anomaly cancels.

Consistent with the fact that the KSEs of heterotic supergravity retain their form up to and including two loops [Bergshoeff, de Roo]

All the holonomy anomalies cancel for both SU(2) and SU(3).

Conclusions

- We showed that the Killing spinor bilinears of heterotic backgrounds satisfy a W-algebra
- We calculated the anomaly of holonomy symmetries using the Wess-Zumino consistency condition
- We argued that these anomalies can be cancelled either adding finite local counterterms or with an appropriate quantum correction of the bilinears
- Heterotic backgrounds with compact holonomy G_2. Can we approach them similarly?
- Anomalies of holonomy symmetries can possibly include δ_ℓ- and δ_u-invariant terms $\Delta_{inv}(a_L)$. Further investigation required.