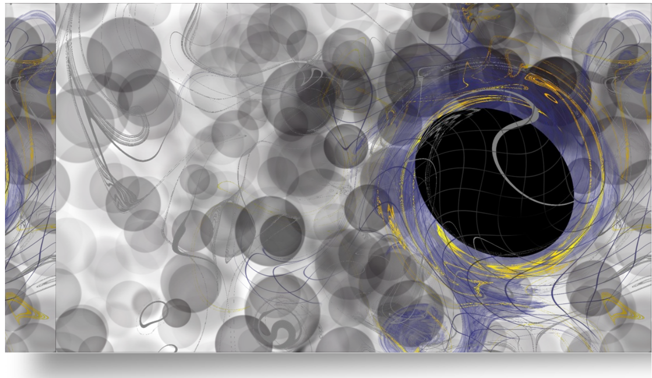
The light we can see



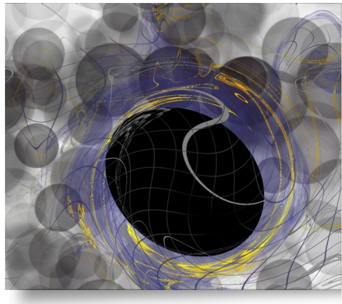
Extracting Black Holes from Weak Jacobi Forms

Alejandra Castro, DAMTP

EuroStrings 2024, Southampton, UK

2407.06260 [hep-th] with Luis Apolo, Suzanne Bintanja, Diego Liska

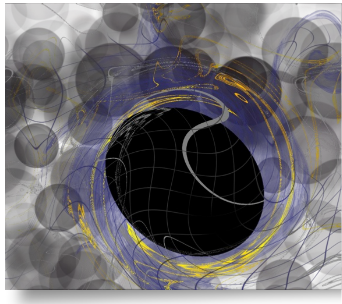
Black Hole Entropy & Modular Forms



Modular forms play a pivotal role in the counting of black hole microstates.

Our goal is to revisit the connection between modular forms and black hole entropy, and tie it with other consistency conditions of AdS/CFT.

Black Hole Entropy & Modular Forms



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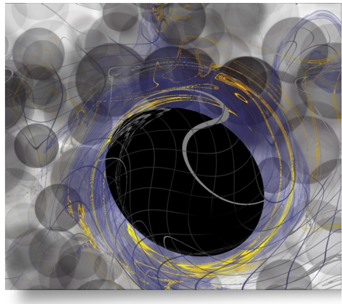
Context: BPS+extremal black holes in 4D and 5D ungauged supergravity

$$S_{BH} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

[Banerjee, Gupta, Mandal, Sen; Sen 2011]

Perfect agreement between gravitational computation and the counting formula.

Black Hole Entropy & Modular Forms



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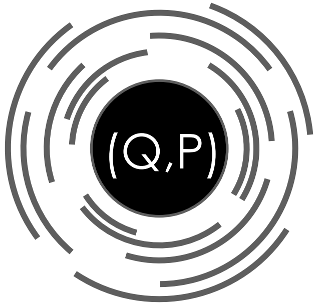
Perfect agreement between gravitational computation and the counting formula.

By matching the leading term and logarithmic correction, what did we learn about the microscopic theory?

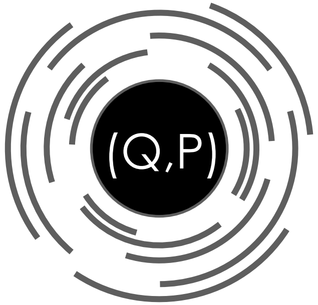
Gravitational Derivation

Microscopic Derivation



$$\mathcal{Z}(Q,P) = \int_{\mathcal{M}} \mathcal{D}g\, \mathcal{D}\phi_i\, e^{-I(g,\phi_i)}$$

Gravitational Derivation



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$$= e^{(\underbrace{S^{(0)}}_{\text{Tree-level}}+\underbrace{S^{(1)}}_{\text{One-loop}}+\cdots)} + \cdots$$

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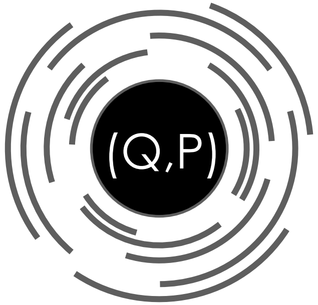

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Tree-level One-loop

$$\boxed{\begin{gathered} Q \to \Lambda^{\mathrm{D}-3}\, Q\,, \qquad P \to \Lambda^{D-3} P \\ A_H \to \Lambda^{\mathrm{D}-2} A_{\mathrm{H}} \\ \Lambda \gg 1 \end{gathered}} \quad (1)$$

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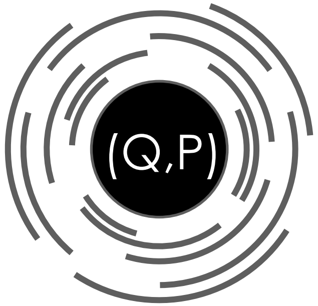
Microscopic Derivation



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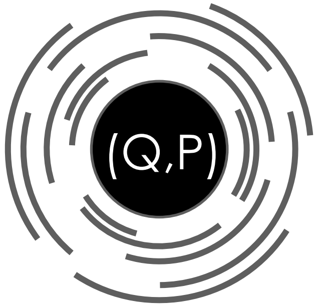
Local contribution,
from massless fields

Zero modes,
from isometries

Gravitational Derivation

Microscopic Derivation



AdS$_2$x S^n

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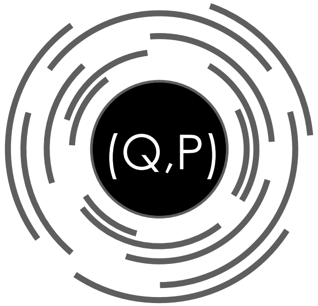

AdS$_2$x S^n $\xrightarrow[\text{Uplift}]{}$ AdS$_3$x S^m

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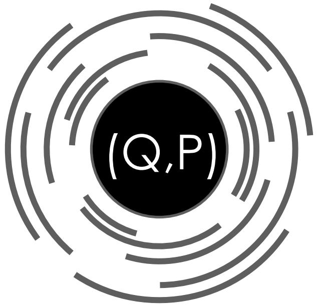
$$\varphi(\tau,z) = \sum_{n,\ell} d(n,\ell) e^{2\pi i \tau n} e^{2\pi i z \ell}$$

↓

Modular form, counts BPS states Tr $(-1)^F$ …

Gravitational Derivation



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$\Delta = n - \frac{\ell^2}{4t}$: discriminant, t index

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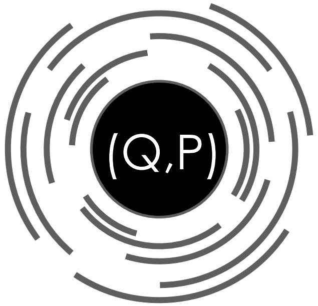

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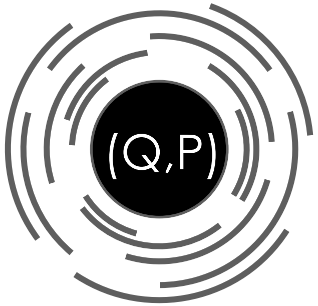

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→ $$\text{“}\log d(n,\ell) = \mathrm{S}_{\mathrm{BH}}\text{”}$$

What is the microscopic information that controls a_Δ and a_{Δ_0}?

Does the agreement with a_{grav} follow from symmetries or dynamics?

$$S_{BH} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

Outline

- Asymptotic expansion of $d(n,\ell)$
  - Implement holographic conditions (HKS)
  - Symmetric Product Orbifolds & Exponential Lifts
- Revisit black hole entropy
  - ¼-BPS BH $\mathcal{N}$=4, D=4
  - ¼ -BPS BH $\mathcal{N}$ =4, D=5 (BMPV)
  - BPS BH N=2, D=4 (MSW)

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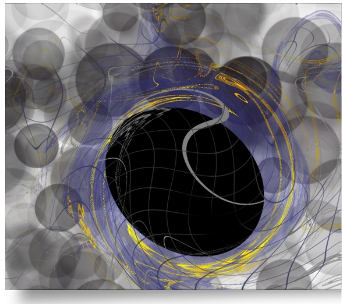
Asymptotic expansion of $d(n,\ell)$



Weak Jacobi form

$$\varphi(\tau,z) = \sum_{\substack{n\geq 0\\ \ell\in\mathbb{Z}}} d(n,\ell)e^{2\pi i\tau n}e^{2\pi i z\ell}$$

Modular properties

$$\varphi\left(\frac{a\tau\ +b}{c\tau\ +d},\frac{z}{c\tau\ +d}\right) \ =\ (c\tau+d)^{\underset{\text{weight}}{k}}e^{\frac{2\pi i\ \overset{\text{index}}{t}\ c\ z^2}{c\tau+d}}\ \varphi(\tau,z)$$

Elliptic properties

$$\varphi(\tau,z+\lambda\tau+\mu)\ = e^{-2\pi i t(\lambda^2\tau+2\lambda z+\mu)}\varphi(\tau,z)$$

$$\lambda,\mu\in\mathbb{Z}$$

Convenient to organize states via the discriminant

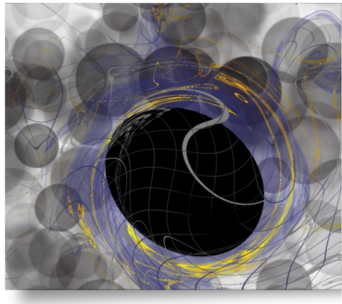
$$(n,\ell)\ \rightarrow\ \Delta = n-\frac{\ell^2}{4t}\ ,\qquad \ell\ (mod\ 2t)$$

$\Delta < 0$: polar states (light)

$\Delta \geq 0$: non-polar states (heavy)

$\Delta_0 = -\frac{b^2}{4t}\ ,\qquad b\leq t$: most polar state

Asymptotic expansion of $d(n,\ell)$

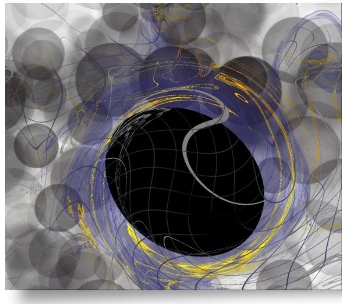


Rademacher Expansion

$$d(n,\ell) = \sum_{\Delta'<0} \sum_{\ell'=-t}^{t-1} d(n',\ell') \sum_{c=1}^{\infty} \frac{2\pi}{c} \left(-\frac{\Delta'}{\Delta}\right)^{\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{4\pi}{c}\sqrt{-\Delta\Delta'}\right) Kl(\Delta,\ell,\Delta',\ell';c)$$

For simplicity, weight $k = 0$

Asymptotic expansion of $d(n,\ell)$



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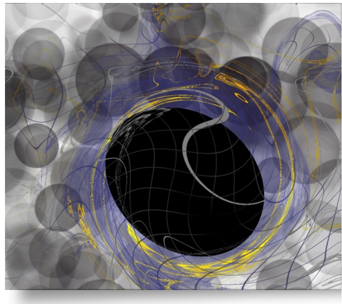

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Input: polar/light states

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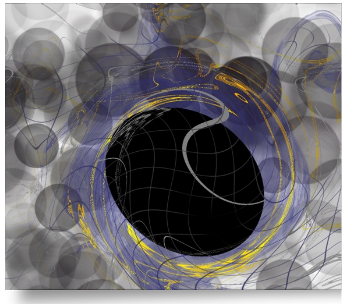
Asymptotic expansion of $d(n,\ell)$



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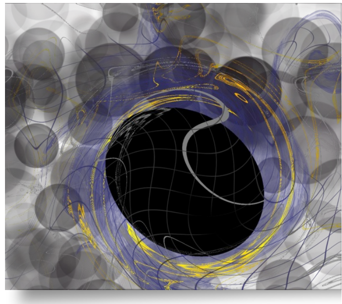
$$I_{\frac{3}{2}}\left(\frac{4\pi}{c}\sqrt{-\Delta\Delta'}\right) \sim e^{\frac{4\pi}{c}\sqrt{-\Delta\Delta'}} + \cdots \longrightarrow d(n,\ell) = e^{4\pi\sqrt{\Delta|\Delta_0|}+\cdots} + \cdots$$

Asymptotic expansion of $d(n,\ell)$



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We will not use this. Why?

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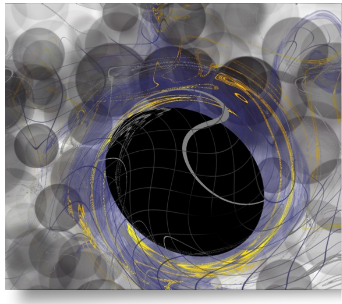

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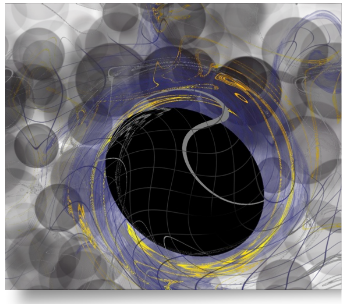
We will not use this. Why?

Gravity path integral (holography or black holes) is coarse: $t \rightarrow \Lambda^{\#}\, t$ with $\Lambda \gg 1$,
And this enters in the logarithmic corrections.

Asymptotic expansion of $\mathrm{d}(n,\ell)$

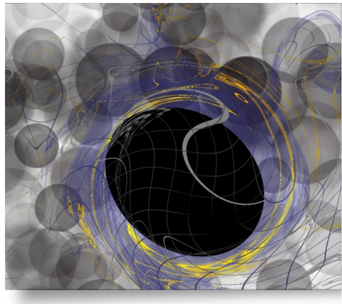


Crossing Kernels

$$\varphi(\tau,z) = \int_{0}^{\infty} dE \int_{-\infty}^{\infty} dj \, \underbrace{\rho(E,j)} \, e^{2\pi i\tau\, E + 2\pi i z\, j}$$

Delta functions supported on $(E,j) = (n,\ell)$

Asymptotic expansion of $\mathrm{d}(n,\ell)$



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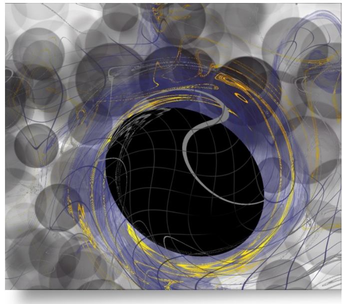
Delta functions supported on $(E,j)=(n,\ell)$

$$\varphi\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \tau^{k} e^{\frac{2\pi i\, t\, z^2}{\tau}}\, \varphi(\tau,z) \longrightarrow$$

$$\rho_H(E,j) := \int_{0}^{\infty} dE' \int_{-\infty}^{\infty} dj'\, \rho_L(E',j')\, \mathbb{P}_{\{E,j\};\{E',j'\}} + \ldots$$

$$\frac{d(n,\ell)}{\rho_H(n,\ell)} = 1 + O\left(e^{-2\pi\sqrt{-\Delta\Delta'}}\right)$$

Asymptotic expansion of $\mathrm{d}(n, \ell)$



Incorporate Gravity

$$d(n,\ell) \sim \rho_H(n,\ell) = e^{4\pi\sqrt{\Delta|\Delta_0|}+\cdots} + \cdots$$

Universally valid if $\Delta \gg |\Delta_0|$ (high temperature regime) and usually Δ_0 and t are fixed (Cardy regime)

In the limit $t, \Delta_0 \to \infty$, the validity of the Cardy regime will be extended if

[Hartman, Keller, Stoica 2014; Benjamin, Cheng, Kachru, Moore, Paquette 2015]

$$\rho_L(n,\ell) \lesssim e^{2\pi(\Delta-\Delta_0)}, \qquad \Delta_0 < \Delta < 0$$

Sparseness condition on light states.

$$d(n,\ell) \sim \rho_H(n,\ell) = e^{4\pi\sqrt{\Delta|\Delta_0|}+\cdots} + \cdots, \qquad \Delta \gtrsim |\Delta_0| \gg 1$$

Crossing Kernels

$$\rho_H(E,j) := \int_0^\infty dE' \int_{-\infty}^\infty dj' \, \rho_L(E',j') \, \mathbb{P}_{\{E,j\};\{E',j'\}} + \ \dots$$

+

Sparseness criteria (HKS)

$$\rho_L(n,\ell) \lesssim e^{2\pi(\Delta-\Delta_0)} \,, \qquad \Delta_0 < \Delta < 0$$

Crossing Kernels

$$\rho_H(E,j) := \int_0^\infty dE' \int_{-\infty}^\infty dj'\, \rho_L(E',j')\, \mathbb{P}_{\{E,j\};\{E',j'\}} + \ \dots$$

+

Sparseness criteria (HKS)

$$\rho_L(n,\ell) \lesssim e^{2\pi(\Delta-\Delta_0)}, \qquad \Delta_0 < \Delta < 0$$

Subleading corrections fall into two categories ($k = 0$, for simplicity):

Universal Regime $\Delta \gtrsim |\Delta_0| \gg 1$

$$d(n,\ell) \approx \frac{\rho_0(\Delta_0)}{\Delta}\sqrt{\frac{|\Delta_0|}{t}}\, e^{4\pi\sqrt{\Delta|\Delta_0|}} + \cdots$$

Only data needed from the light spectrum is

$\rho_0(\Delta_0)$: ground state degeneracy.

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Only data needed from the light spectrum is

$\rho_0(\Delta_0)$: ground state degeneracy.

Non-Universal Regime $\Delta_0 \gtrsim |\Delta| \gg 1$

Sensitive to light spectrum. In a **democracy**:

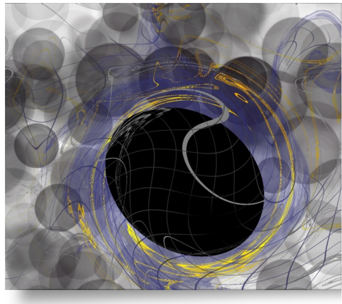
$$d(n,\ell) \approx \frac{\rho_0(\Delta_0)}{\Delta}\sqrt{\frac{|\Delta_0|}{t}}\left(\frac{|\Delta_0|}{\Delta}\right)^{\frac{w+2-\alpha}{2(1-\alpha)}} e^{4\pi\sqrt{\Delta|\Delta_0|}} + \cdots$$

$$\Delta_0 \gtrsim |\Delta| \gg |\Delta_0|^{2\alpha-1} \gg 1$$

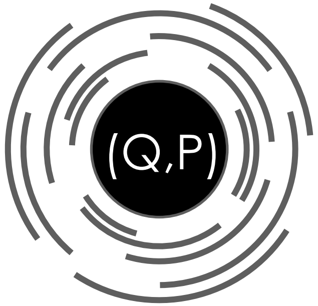
$$\rho_L(\Delta) \approx \rho_0(\Delta_0)\,(\Delta - \Delta_0)^w e^{2\pi\gamma(\Delta-\Delta_0)^\alpha}$$

$$0 < \alpha < 1\,, \qquad \gamma > 0$$

Revisit Black Hole Entropy



Gravitational Derivation



$$\mathcal{Z}(Q,P) = \int_{\mathcal{M}} \mathcal{D}g\, \mathcal{D}\phi_i\, e^{-I(g,\phi_i)}$$

$$= e^{(S^{(0)}+S^{(1)}+\cdots)} + \cdots$$

Tree-level One-loop

$$Q \to \Lambda^{\mathrm{D}-3}\, Q\,, \qquad P \to \Lambda^{D-3} P$$
$$A_H \to \Lambda^{\mathrm{D}-2} A_{\mathrm{H}} \qquad (1)$$
$$\Lambda \gg 1$$

$$S_{BH} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

Local contribution, from massless fields

Zero modes, from isometries

Microscopic Derivation

$$\varphi(\tau,z) = \sum_{n,\ell} d(n,\ell) e^{2\pi i\tau n} e^{2\pi i z\ell}$$

Counting BPS states Tr $(-1)^F$ …

HKS: $\rho_L(n,\ell) \lesssim e^{2\pi(\Delta-\Delta_0)}$, $\Delta_0 < \Delta < 0$

$$\log d(n,\ell) = 4\pi\sqrt{\Delta|\Delta_0|} + a_\Delta \log\Delta + a_{\Delta_0}\log|\Delta_0| + \cdots$$

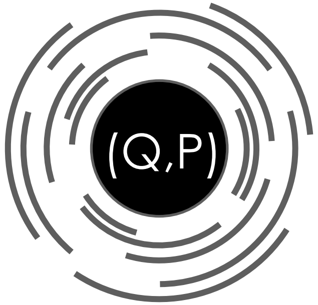
Universal Regime $\Delta \gtrsim |\Delta_0| \gg 1$

Only ground state data is needed

Non-Universal Regime $\Delta_0 \gtrsim |\Delta| \gg 1$

$$\rho_L(\Delta) \approx \rho_0(\Delta_0)\, (\Delta - \Delta_0)^w e^{2\pi\gamma\Delta^\alpha}$$

¼-BPS $\mathcal{N}$=4 D=4



$$\boxed{\begin{array}{c} Q \to \Lambda\, Q\,, \qquad P \to \Lambda P \\ \Lambda \gg 1 \end{array}}$$

$$\frac{A_H}{4G} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}$$

$$S_{BH} = \frac{A_H}{4G} + a_{grav}\log\left(\frac{A_H}{4G}\right) + \cdots$$

Local contribution, from massless fields

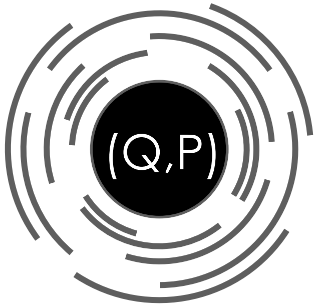
$$\frac{1}{12}(11(3-\mathcal{N}) - n_V + n_H) = -1$$

$$\mathcal{N} = 4, \qquad n_V = n_H + 1$$

Zero modes, from isometries

$$\frac{1}{2}(-6+8) = 1$$

¼-BPS $\mathcal{N}$=4 D=4



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$$\mathcal{N} = 4, \qquad n_V = n_H + 1$$

Microscopic Derivation

$$\frac{1}{\Phi_{10}} = Exp - Lift(-2\varphi_{0,1})(\tau, z, \sigma)$$

$$t = b, \qquad \Delta_0 = -\frac{t}{4} = -\frac{Q^2}{8},$$

$$n = \frac{P^2}{2}, \qquad \ell = Q\cdot P,$$

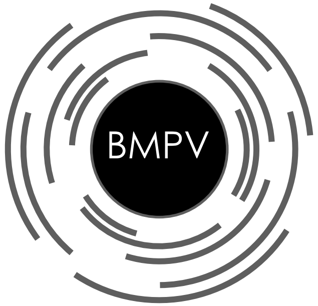
$$4\pi\sqrt{\Delta|\Delta_0|} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}$$

Universal Regime $\Delta \sim |\Delta_0| \gg 1$

$$d(n,\ell;t) \approx \frac{\Delta_0^{d(0,-1)-1}}{\Delta} e^{4\pi\sqrt{\Delta|\Delta_0|}} + \cdots$$

$d(0,-1) = 2$: Ramond grounds states.
Compatible with AdS$_3$xS^{3}xK3 and dual CFT$_2$

¼-BPS $\mathcal{N}$ =4 D=5



$$Q_1 \to \Lambda^2\, Q_1\,,\;\; Q_5 \to \Lambda^2\, Q_5\,,$$
$$P \to \Lambda^2 P, \qquad J \to \Lambda^3 J$$
$$\Lambda \gg 1$$

$$\frac{A_H}{4G} = 2\pi\sqrt{Q_1 Q_5 P - J^2}$$

$$S_{BH} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

Local contribution,
from massless fields

Zero modes,
from isometries

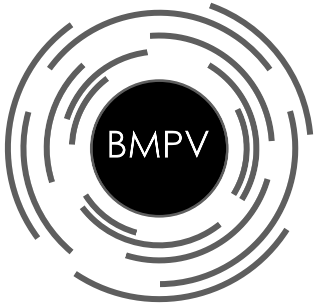
0

$$-\frac{1}{6}(n_V + 21 - 24) = -4$$

$$n_V = 20 + 2x3 + 1 = 27$$

for Type IIB on K3xS¹

¼-BPS $\mathcal{N}$ =4 D=5



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Local contribution, from massless fields

Zero modes, from isometries

0

$$-\frac{1}{6}(n_V + 21 - 24) = -4$$

$$n_V = 20 + 2x3 + 1 = 27$$

for Type IIB on K3xS^1

Microscopic Derivation

$$\mathcal{Z}(\sigma,\tau,z) = \sum_N \varphi(\tau,z;Sym^N(K3))e^{2\pi i\sigma N}$$

$$t = b, \qquad \Delta_0 = -\frac{t}{4} = -\frac{Q_1 Q_5}{4},$$
$$n = P, \qquad \ell = J,$$
$$4\pi\sqrt{\Delta|\Delta_0|} = 2\pi\sqrt{Q_1 Q_5 P - J^2}$$

Non-Universal Regime $\Delta_0 \gg |\Delta| \gg 1 : \Delta_0 \sim \Lambda^4, \Delta \sim \Lambda^2$

$$d(n,\ell;t) \approx \frac{\Delta_0^{d(0,-1)-1}}{\Delta}\left(\frac{|\Delta_0|}{\Delta}\right)^{w+\frac{3}{2}} e^{4\pi\sqrt{\Delta|\Delta_0|}} + \cdots$$

$$\rho_L(\Delta) \approx \rho_0(\Delta_0)\,(\Delta - \Delta_0)^w e^{2\pi\gamma\Delta^{1/2}}$$

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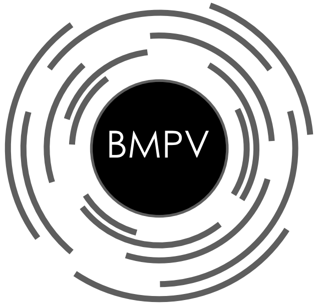

$$\boxed{\begin{array}{c} Q_1 \to \Lambda^2\, Q_1\,,\;\; Q_5 \to \Lambda^2\, Q_5\,, \\ P \to \Lambda^2 P, \qquad J \to \Lambda^3 J \\ \Lambda \gg 1 \end{array}}$$

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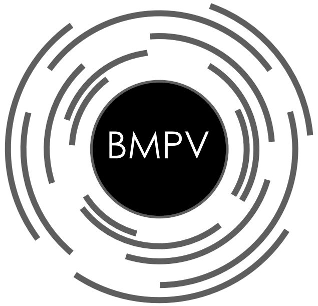
$$d(n,\ell;t) \approx \frac{\Delta_0^{d(0,-1)-1}}{\Delta}\left(\frac{|\Delta_0|}{\Delta}\right)^{w+\frac{3}{2}} e^{4\pi\sqrt{\Delta|\Delta_0|}} + \ldots$$

$d(0,-1) = 2$: Ramond grounds states.

$$w + \frac{3}{2} = d(0,-1) - \frac{d(0,0)}{4} = -3$$

$d(0,0) = h^{1,1} = 20$ for K3

½ -BPS $\mathcal{N}$ =2 D=4



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$$\Lambda \gg 1$$

$$\frac{A_H}{4G} = 2\pi\sqrt{\widehat{q_0} c_{IJK} p^I p^J p^k}$$

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from massless fields

$$\frac{1}{12}(11(3-\mathcal{N}) - n_V + n_H)$$

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$$\frac{1}{2}(-6+8) = 1$$

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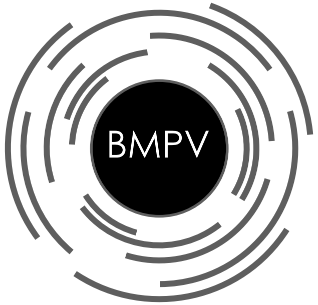

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Microscopic Derivation: MSW / OSV

$$\mathcal{Z}(\tau,\bar\tau,z^I) = \mathrm{Tr}(F^2(-1)^F e^{2\pi i\,\tau\, L_0}\, e^{2\pi i\bar\tau \bar L_0} y^{\ell_I})$$

$$t_{IJ} = c_{IJK}p^k, \qquad \Delta_0 = -\frac{t}{4} = -\frac{c_{IJK}p^I p^J p^k}{4},$$

$$q_0 = L_0 - \bar L_0, \qquad \ell_I = q_I\,, \qquad \Delta = \widehat{q_0}$$

$$4\pi\sqrt{\Delta|\Delta_0|} = 2\pi\sqrt{\widehat{q_0} c_{IJK} p^I p^J p^k}$$

Non-Universal Regime $\Delta_0 \gg |\Delta| \gg 1$: $\Delta_0 \sim \Lambda^3, \Delta \sim \Lambda$

$$d(n,\ell)$$

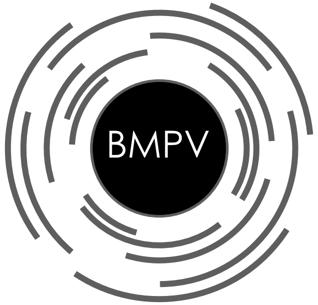
$$\approx\; |\Delta_0|\sqrt{\det t_{IJ}}\left(\frac{|\Delta_0|}{\Delta}\right)^{1+\frac{n_V}{4}} \frac{|\Delta_0|^{\omega+\frac{2}{3}}}{(\Delta|\Delta_0|)^{\frac{1}{4}}} e^{4\pi\zeta|\Delta_0|^{\frac{2}{3}}} + \cdots$$

$$w = -\frac{\chi+24}{36}$$

$\chi = 2(n_V - n_H + 1)$: Euler number CY$_3$

Extremely sensitive to the light spectrum!

½ -BPS $\mathcal{N}$ =2 D=4



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Match!

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$$\mathcal{Z}(\tau,\bar{\tau},z^I) = \mathrm{Tr}(F^2(-1)^F e^{2\pi i\,\tau\, L_0}\, e^{2\pi i \bar{\tau}\bar{L}_0} y^{\ell_I})$$

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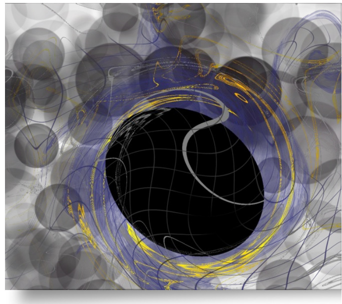
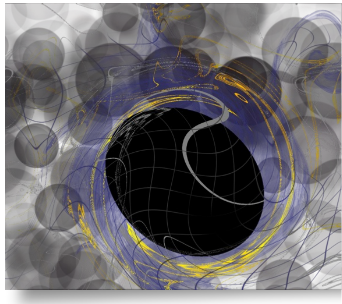
Conclusion



Conclusion



$$\mathcal{Z}(Q,P) = \int_{\mathcal{M}} \mathcal{D}g\, \mathcal{D}\phi_i\, e^{-I(g,\phi_i)}$$

$$\log \mathcal{Z} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

- Local contribution, from massless fields.
- Zero modes, from isometries.

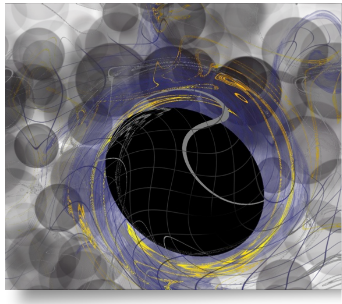
$$\varphi(\tau,z) = \sum_{n,\ell} d(n,\ell) e^{2\pi i \tau n} e^{2\pi i z \ell}$$

$$\log d(n,\ell) = 4\pi\sqrt{\Delta|\Delta_0|} + a_\Delta \log \Delta + a_{\Delta_0} \log|\Delta_0| + \cdots$$

- Universal Regime $\Delta \gtrsim |\Delta_0| \gg 1$
- Non-Universal Regime $\Delta_0 \gtrsim |\Delta| \gg 1$

Quantified the imprint of light states on a_Δ and a_{Δ_0}

Conclusion



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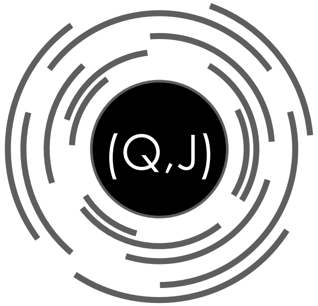
- Universal Regime $\Delta \gtrsim |\Delta_0| \gg 1$
- Non-Universal Regime $\Delta_0 \gtrsim |\Delta| \gg 1$

Quantified the imprint of light states on a_Δ and a_{Δ_0}

Universality/Simplicity that might happen on each side is not preserved.
Important to develop tools that allow to quantify CFT observables for continuous N.

Gravitational Derivation

Microscopic Derivation



Supersymmetric BHs in AdS$_4$

$$S_{BH} = \frac{A_H}{4G} + a_{grav} \log\left(\frac{A_H}{4G}\right) + \cdots$$

Local contribution,
from massless fields

Zero modes,
from isometries

If the EFT is AdS$_4$xS^7 (infinite number of fields):

a_{grav} : A rational number independent of charges.

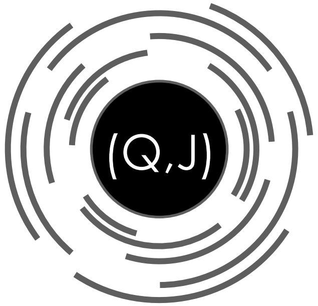
S. Bhattacharyya, A. Grassi, M. Mariño, A. Sen 2012
J. T. Liu, L. Pando Zayas, V. Rathee, W. Zhao 2017

If the EFT in AdS$_4$ has a finite number of fields:

$a_{grav}\left(\frac{J}{Q^2}\right)$: A non trivial function of the charges!

M. David, V. Godet, Z. Liu, L. Pando Zayas 2023
N. Bobev, M. David, J. Hong, V. Reys, X. Zhang 2024

Gravitational Derivation



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$a_{grav}\left(\frac{J}{Q^2}\right)$: A non trivial function of the charges!

Microscopic Derivation

$$Z_{index}(\tau, z) = \sum_{n,\ell} d(n,\ell) e^{2\pi i \tau n} e^{2\pi i z \ell}$$

$$\log d(n,\ell) = 4\pi N\, f(n,\ell) + a_n \log \mathrm{N} + \cdots$$

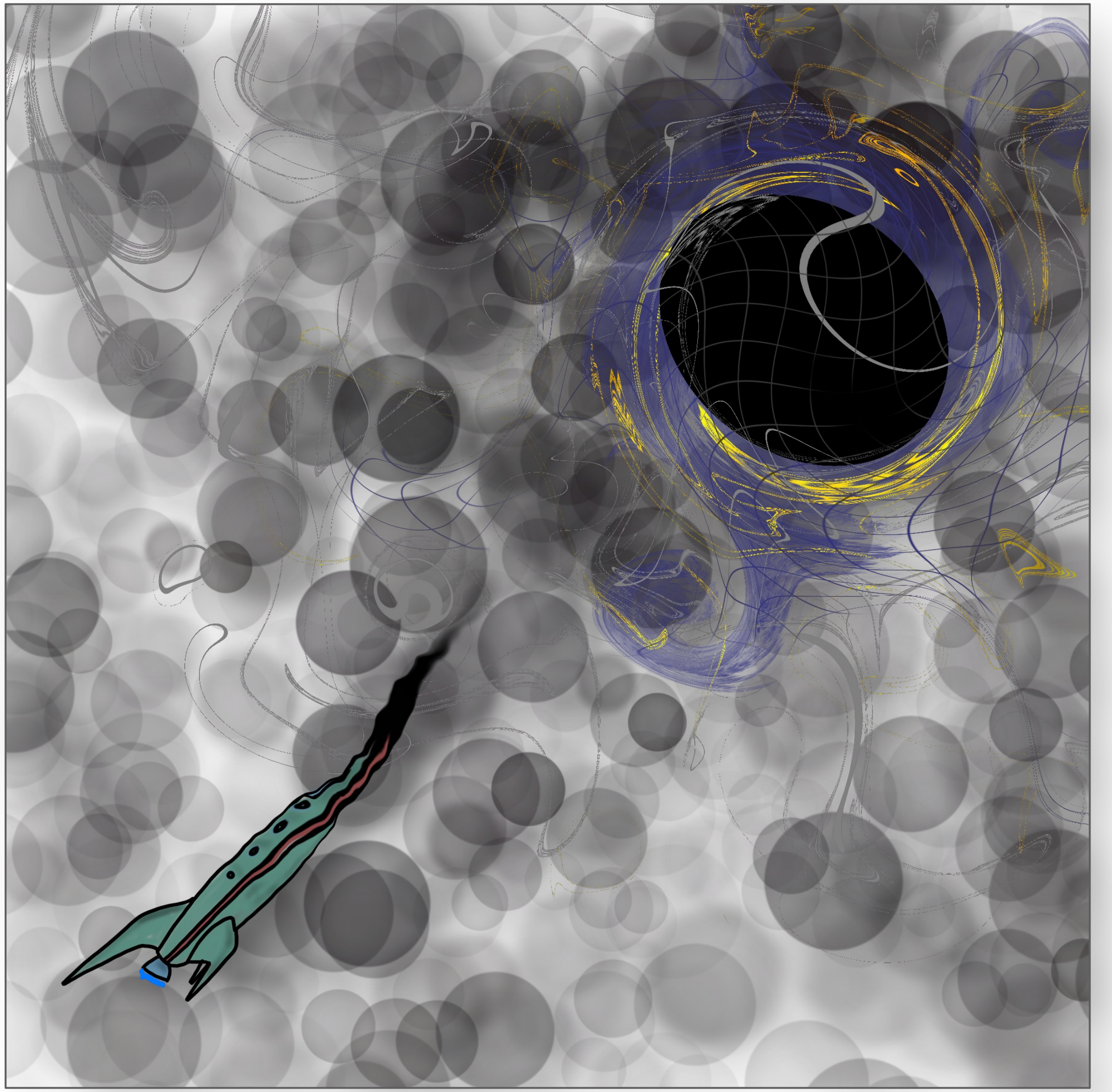
To date: not a single example of $a_n\left(\frac{J}{Q^2}\right)$.

Always independent of charges!

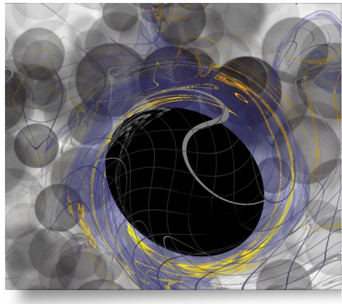
Lessons:

- GPI responds differently at the quantum level depending on the matter content.
- Could provide non-trivial insight about scale separation in AdS/CFT.
- We need a better understanding of what controls a_n in the CFT!

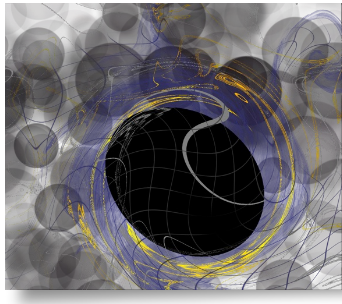
Thank you!



Extra material



Symmetric Product Orbifolds and Exponential Lifts



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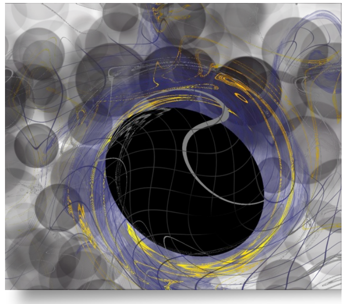


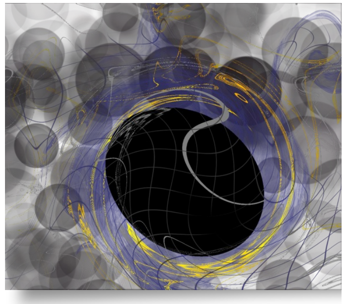
Rademacher Expansion

$$d(n,\ell) = \sum_{\Delta'<0} \sum_{\ell'=-t}^{t-1} d(n',\ell') \sum_{c=1}^{\infty} \frac{2\pi}{c} \left(-\frac{\Delta'}{\Delta}\right)^{\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{4\pi}{c}\sqrt{-\Delta\Delta'}\right) Kl(\Delta,\ell,\Delta',\ell';c)$$

We will not use this. Why?

Gravity path integral (holography or black holes) is coarse: $t \to \Lambda^{\#}\, t$ with $\Lambda \gg 1$,
And this enters in the logarithmic corrections.

Symmetric Product Orbifolds and Exponential Lifts

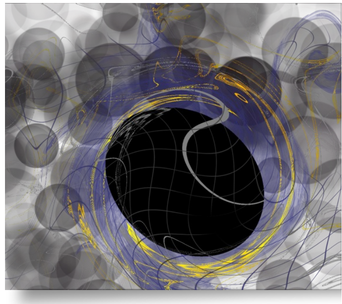


Gravity path integral (holography or black holes) is coarse: $t \to \Lambda^{\#}\, t$ with $\Lambda \gg 1$,
And this enters in the logarithmic corrections.

$$\varphi_{k,t}(\tau, z) = \sum_{\substack{n \geq 0 \\ \ell \in \mathbb{Z}}} d(n, \ell) e^{2\pi i \tau n} e^{2\pi i z \ell} \longrightarrow \mathcal{Z}(\sigma, \tau, z) = \sum_{t} \varphi_{k,t}(\tau, z) e^{2\pi i \sigma t}$$

Sum over theories!

Symmetric Product Orbifolds and Exponential Lifts

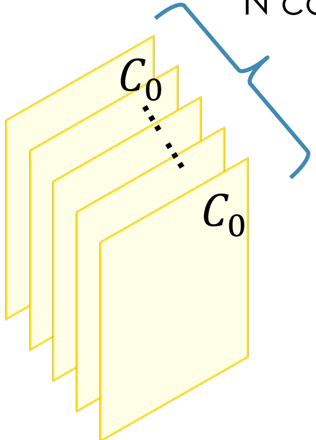


A class of modular forms where it is natural to sum over the index

$$\varphi_{0,t_o}(\tau,z) = \sum_{\substack{n\geq 0\\ \ell\in\mathbb{Z}}} d(n,\ell)e^{2\pi i\tau n}e^{2\pi iz\ell} \xrightarrow[\text{Symm Prod Orb}]{} \mathcal{Z}(\sigma,\tau,z) = \sum_{N} \varphi_{0,t_oN}(\tau,z;Sym^N(C_0))e^{2\pi i\sigma t_o N}$$

$$= \prod_{m,n,\ell} \frac{1}{(1-e^{2\pi i\tau n}e^{2\pi iz\ell}e^{2\pi i\sigma m})^{d(nm,\ell)}}$$

[DMVV]

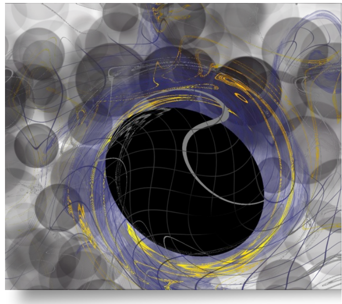


N copies of a unitary and compact CFT$_2$.
C_0 = seed theory

Orbifold by the permutation group S_N

$$Sym^N(C_0) = \frac{C_0^{\otimes N}}{S_N}$$

Symmetric Product Orbifolds and Exponential Lifts



Coefficients of symmetric product orbifolds

$$\mathcal{Z}(\sigma,\tau,z) = \sum_N \varphi_{0,t_oN}(\tau,z;Sym^N(C_0))e^{2\pi i\sigma t_oN} = \prod_{m,n,\ell} \frac{1}{(1-e^{2\pi i\tau n}e^{2\pi i z\ell}e^{2\pi i\sigma m})^{d(nm,\ell)}}$$

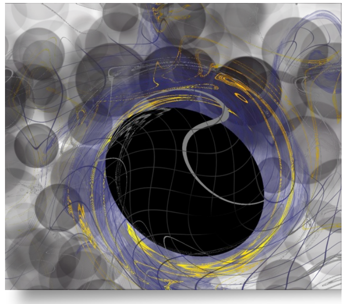
$t = t_0N,$
$b = b_0N$

Use poles in $\mathcal{Z}(\sigma,\tau,z)$ and countour deformations. Main assumption is $\Delta|\Delta_0| \gg 1$

[David, Sen; Sen; Belin, AC, Gomes, Keller]

$$d(n,\ell;t) = \int_C d\tau dz \frac{N^{d(0,-b_0)-1}}{(d(0,-b_0)-1)!}\,\varphi_\infty(\tau,z)\,e^{-2\pi i\left(n\tau+\frac{tz^2}{\tau}-\frac{bz}{\tau}+\ell\,z\right)} + \cdots$$

Symmetric Product Orbifolds and Exponential Lifts



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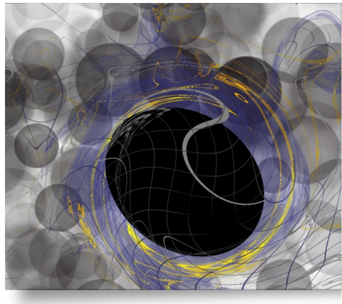
[Belin, AC, Muehlmann, Keller 2019]

Symmetric Product Orbifolds and Exponential Lifts



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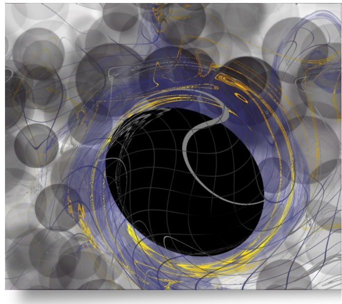
$\varphi_\infty(\tau,z)$: for K3 generating of perturbative ¼-BPS 6D SUGRA states on AdS_3xS^3

Symmetric Product Orbifolds and Exponential Lifts



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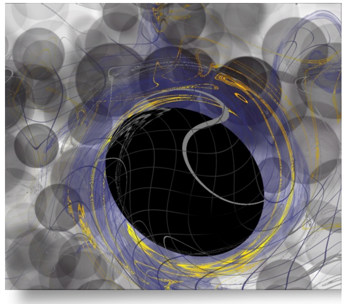
Complete classification,
Controlled by seed C_0

[Belin, AC, Muehlmann, Keller 2019]

Symmetric Product Orbifolds and Exponential Lifts



Symm Prod Orb

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In perfect agreement with crossing kernels!

Universal Regime $\Delta \gtrsim |\Delta_0| \gg 1$

Non-Universal Regime $\Delta_0 \gtrsim |\Delta| \gg 1$