Nearly Critical Superfluids and Holography

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and work in progress

Outline

- Introduction/Generalities
- Field Theory Constructions ($N \leq \infty$)
- Holographic Model ($N \to \infty$)
- Conclusions

Motivation

- At strong coupling often no quasiparticles
- Conserved charges and light Goldstone modes dominate at long wavelengths
- Correlation length ξ diverges close to a transition $\Rightarrow$ Universality?
- Amplitude mode has gap $\sim \xi^{-2} \Rightarrow$ Effective theory?
- Use holography to carry out microscopic computations
- Lessons for EFT?

Field Theory Setup - Microscopics

- Relativistic field theory (with global $U(1)$) at finite temperature T and chemical potential μ
- Charged operator $\mathcal{O}_\psi$ transforms as $\mathcal{O}_\psi \to e^{-iq\alpha}\,\mathcal{O}_\psi$
- Phase transition with $\langle \mathcal{O}_\psi \rangle \neq 0$ at $T < T_c$
- Couple to external gauge field A_μ and scalar source λ

$$\delta S = \int d^n x \left(J^\mu\, \delta A_\mu + \mathcal{O}_\psi^*\, \delta\lambda + \mathcal{O}_\psi\, \delta\lambda^* \right)$$

Field Theory Setup - Microscopics

- Generating function $W[g_{\mu\nu}, A_\mu, \lambda, \lambda^*]$ depends on external gauge field A_μ , background metric $g_{\mu\nu}$ and complex source λ
- Functional differentiation gives the VEVs

$$\langle T^{\mu\nu}\rangle = \frac{i}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}}, \qquad \langle J^\mu\rangle = i\frac{\delta W}{\delta A_\mu}, \qquad \langle \mathcal{O}_\psi\rangle = i\frac{\delta W}{\delta \lambda^*}$$

- Invariance under gauge transformations $\delta A_\mu = -\,\partial_\mu \delta\Lambda, \quad \delta\lambda = i\,q\,\lambda\,\delta\Lambda$

$$\nabla_\alpha \langle J^\alpha\rangle = iq\left(\langle \mathcal{O}_\psi\rangle \lambda^* - \langle \mathcal{O}_\psi^*\rangle \lambda\right)$$

- Invariance under coordinates transformations

$$\nabla_\mu \langle T^{\mu\nu}\rangle = F^{\nu\mu}\langle J_\mu\rangle + \nabla^\nu \lambda\,\langle \mathcal{O}_\psi^*\rangle + \nabla^\nu \lambda^*\,\langle \mathcal{O}_\psi\rangle$$

Hydro away from critical point

Normal phase:

- Express $T_{\mu\nu}$ and J_μ as functions of the fluctuations T, v^μ, μ
- Solve the closed system

$$\nabla_\mu \langle T^{\mu\nu} \rangle = F^{\nu\mu} \langle J_\mu \rangle + \nabla^\nu \lambda \langle \mathcal{O}_\psi^* \rangle + \nabla^\nu \lambda^* \langle \mathcal{O}_\psi \rangle \qquad \nabla_\alpha \langle J^\alpha \rangle = iq \left(\langle \mathcal{O}_\psi \rangle \lambda^* - \langle \mathcal{O}_\psi^* \rangle \lambda \right)$$

Broken phase:

- Include phase ϑ of condensate in the description $\langle \mathcal{O}_\psi \rangle = \langle \mathcal{O}_\psi \rangle_b \, e^{iq\vartheta}$
- Impose Josephson relation $\mu = A_t + \partial_t \vartheta + \cdots$
- Amplitude of VEV has gap $\omega_{\text{gap}} \propto |T - T_c| \rightarrow$ remains frozen

Order Parameter - Current Sector

- Ignore dynamics of stress tensor and conjugate variables T and v^μ
- Focus on coupled sector of charge and order parameter
- Normal phase ($k^2 \gg \omega_{\text{gap}}$)

Charge diffusion

$$\delta J^t = \chi_n \, \delta\mu + \cdots \,, \qquad \delta J^i = -\,\lambda_0^m \, (\partial^i \delta\mu - E^i) + \cdots \,,$$

- Superfluid phase ($k^2 \ll \omega_{\text{gap}}$)

$$\delta J^t = \chi_b \, \delta\mu - \chi_b^2 \, \zeta_3 \, \partial_t^2 \delta\mu + \cdots \,, \qquad \delta J^i = -\,\chi_{JJ} \, (\partial^i \vartheta + A^i) - \lambda_0^m \, (\partial^i \delta\mu - E^i) + \cdots \,,$$

Supercurrent

Hydro away from T_c

- Normal phase ($T \gg k^2 \gg \omega_{\text{gap}}$)

$$\omega = -i\frac{\lambda_0^m}{\chi_b}k^2$$

Charge diffusion

- Superfluid phase ($k^2 \ll T, \omega_{\text{gap}}$)

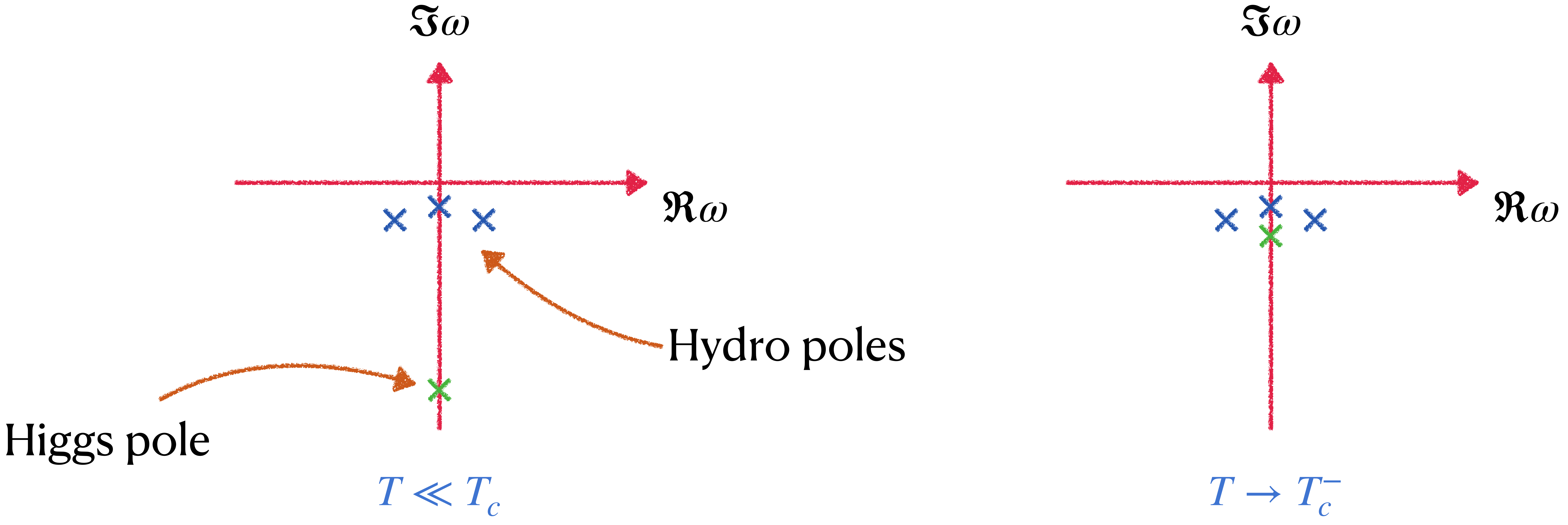
$$\omega_{\pm} = \pm\sqrt{\frac{\chi_{JJ}}{\chi_b}}k - \frac{i}{2\chi_b}\left(\lambda_0^m + \chi_b\,\chi_{JJ}\,\zeta_3\right)\,k^2$$

Second sound

- Current susceptibility χ_{JJ} goes to zero close to the transition
- Third bulk viscosity ζ_3 blows up
- $\chi_{JJ}\,\zeta_3$ stays finite

Donos, Kailidis, Pantelidou

Hydrodynamics at $T < T_c$



- At $T \ll T_c$ linear response is dominated by hydro poles
  - Higgs mode is integrated out
- As $T \to T_c^-$ Higgs pole becomes gapless → Transport coefficients blow up
  - Include Higgs mode in hydro description

Effective Field Theory

- Identify order parameter ψ and integrate out gapped degrees of freedom

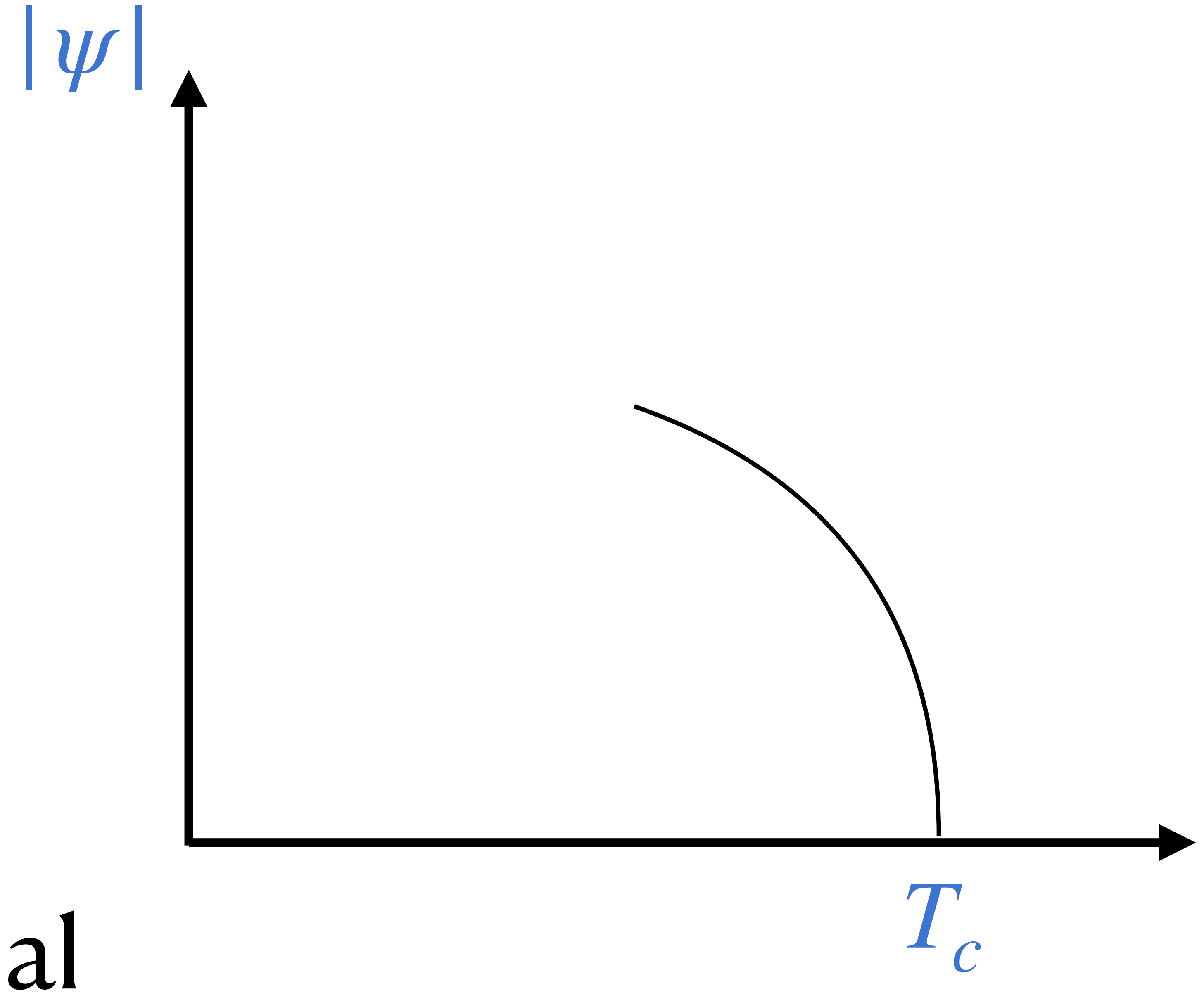
- Introduce UV cutoff Λ

- Long wavelength Boltzmann distribution $P[\psi, \psi^\star] = Z^{-1}\, e^{-\beta\, W_0[\psi,\psi^\star]}$

- Partition function $Z = \int D\psi\, D\psi^\star\, e^{-\beta\, W_0[\psi,\psi^\star]}$

- Free energy $w = -\, k_B T \ln Z$

Ginzburg - Landau Mean Field Theory



$$W_0 = \int d^D x \left(\frac{\hbar^2}{2m(T)} |\vec{\nabla}\psi|^2 + a'(T)|\psi|^2 + \frac{b'(T)}{2}|\psi|^4 \right) + W_n(T, \ldots)$$

$$m(T) = m + \cdots \qquad b'(T) = b' + \cdots \qquad a'(T) = T_c\,\alpha\,(1 - T/T_c) + \cdots$$

- Most probable configuration extremises G-L energy potential

$$|\psi_0|^2 = \begin{cases} 0 & , T > T_c \\ -\frac{T_c\alpha}{b'}(1 - T/T_c) & , T < T_c \end{cases}$$

- Second order transition $\Delta F/\text{vol} = -\dfrac{\alpha^2}{2b'}(T - T_c)^2$
- Thermal fluctuations significant for $D < 4$ or finite N

Critical Dynamics - Model A

- Scalar order parameter ϕ_I

- Postulate stochastic kinetic equation

External Force

$$\partial_t \phi_I = -\Gamma_0 \left(\frac{\delta W_0}{\delta \phi_I} - h_I \right) + \theta_I$$

Kawasaki
Halperin, Hohenberg

Damping

Thermal noise

- Introduce finite kinetic coefficient Γ_0

- Thermal fluctuations of modes beyond cutoff Λ captured by Gaussian noise θ_I

$$\langle \theta_I(t, \mathbf{x}) \rangle = 0 \qquad \langle \theta_I(t, \mathbf{x})\, \theta_J(t', \mathbf{x}') \rangle = 2\, \delta_{IJ}\, D\, \delta(t - t')\, \delta(\mathbf{x} - \mathbf{x}')$$

Critical Dynamics - Model F

$$\partial_t \psi = -2\Gamma_0 \frac{\delta W_0}{\delta \psi} - i g_0 \psi \frac{\delta W_0}{\delta m} + \theta$$

Mode Coupling

$$\partial_t m = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta m} + 2 g_0 \, \mathrm{Im}\left(\psi^\star \frac{\delta W_0}{\delta \psi^\star}\right) + \zeta$$

Halperin, Hohenberg, Ma

- Include conserved charge density $m \Rightarrow$ Affects long wavelength physics
- Energy functional at fixed charge density $\Rightarrow \mu = \dfrac{\delta W_0}{\delta m}$
- Non-dissipative mode coupling fixed by Josephson relation $\mu = -g_0^{-1} \partial_t \arg \psi$
- Γ_0 can be complex. What is it?

Critical Dynamics - Model F

$$\partial_t m = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta m} + 2\, g_0\, \mathrm{Im}\left(\psi^\star \frac{\delta W_0}{\delta \psi^\star}\right) + \zeta$$

Define density $W_0 = \int d^d x\, F$

$$\partial_t m = -\nabla_i \overbrace{\left(-\lambda_0^m \nabla^i \mu + 2\, g_0\, \mathrm{Im}\left(\psi^\star \frac{\partial F}{\partial \nabla_i \psi^\star}\right)\right)}^{J^i} + 2\, g_0\, \mathrm{Im}\left(\psi^\star \frac{\partial F}{\partial \psi^\star}\right) + \zeta$$

Diffusion — Supercurrent — Scalar source

- Model F captures:
  - Current conservation
  - Josephson relation
  - Amplitude mode dynamics

Critical Dynamics & Renormalisation

- Statics (Thermodynamics):
  - Renormalise by integrating out unwanted fast modes in probability distribution
- Real Time Dynamics :
  - Package all fast modes in Gaussian noise to write kinetic equations

➡ Assumes large separation of relaxation rates

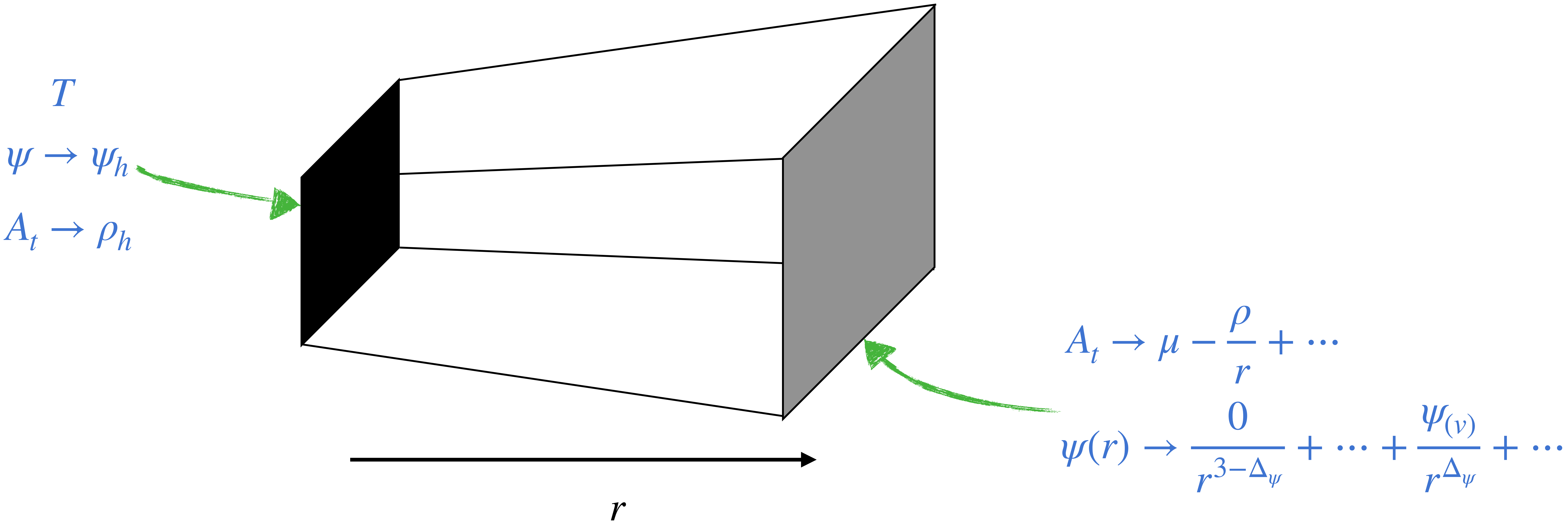
➡ Rethink thermal noise in Keldysh-Schwinger formalism

Chen-Lin, Delacretaz, Hartnoll

Jain, Kovtun

Donos, Kailidis

Thermal States in AdS/CFT



- Introduce planar event horizon at Hawking temperature T
- Fix μ on the boundary
- Study source free quasinormal modes

Setup

Minimal bulk action includes a complex scalar ψ and neutral scalar ϕ

$$\mathscr{L} = -V(\phi, |\psi|^2) - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}(D_\mu\psi)(D^\mu\psi)^* - \frac{1}{4}\tau(\phi, |\psi|^2)\,F^{\mu\nu}F_{\mu\nu}$$

$$D_\mu\psi = \nabla_\mu\psi + i\,q\,A_\mu\,\psi$$

- Invariant under $\psi \to e^{-iq\Lambda}\psi,\ A_\mu \to A_\mu + \partial_\mu\Lambda$
- Decouple stress tensor ➜ Fixed background black hole
- Scenario were we deform by neutral scalar ϕ while ψ breaks $U(1)$ spontaneously
- Suppress story about ϕ from now on

Holographic Setup

- Boundary theory gets deformed to

$$S[\phi_s, a_\mu, \delta g_{\mu\nu}] = S_{CFT} + \int d^{d+1}x \left(\phi_s(x)\, \mathcal{O}(x) + a_\mu(x)\, J^\mu(x) + \frac{1}{2}\delta g_{\mu\nu}(x) T^{\mu\nu}(x) \right)$$

- Holographic conjecture relates partition functions

$$Z_{CFT}[\phi_s, a_\mu, \delta g_{\mu\nu}] = Z_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}] \approx e^{iS_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}]}$$

- Powerful tool to extract VEVs of operators

$$\langle \mathcal{O}(x) \rangle = \frac{1}{i} \frac{\delta}{\delta \phi_s(x)} \ln Z_{CFT}[\phi_s, a_\mu, \delta g_{\mu\nu}] \approx \frac{\delta}{\delta \phi_s(x)} S_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}]$$

Symplectic Current

- Cast the bulk action in terms of first derivatives

$$S_{bulk} = \int_M d^{d+1}x\, \mathscr{L}(\partial\phi, \phi) + counterterms$$

- Vary with respect to bulk field to find

$$\langle \mathcal{O}_\phi \rangle\, \delta\phi_s = \int_{\partial M} d^d x\, \delta\phi\, \frac{\delta \mathscr{L}}{\delta \partial_r \phi} + \cdots$$

- Non-trivial information from knowing on shell value of $\dfrac{\delta \mathscr{L}}{\delta \partial_r \phi}$ close to the boundary
- Useful to think of it as momentum density

Papadimitriou

Symplectic Current

- Need to know variation $\delta\left(\frac{\delta\mathscr{L}}{\delta\partial_r\phi}\right)$ against specific bulk perturbations
- For any two perturbations $\delta_1\phi_A$ and $\delta_2\phi_A$ define the symplectic current density

$$P^\mu = \delta_1\phi_A\,\delta_2\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_A}\right) - \delta_2\phi_A\,\delta_1\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_A}\right)$$

- Divergence free when evaluated on-shell

$$\partial_\mu P^\mu = 0$$

- More general way of thinking about the membrane paradigm

Policastro, Son, Starinets
Iqbal, Liu
Donos, Gauntlett

Hydro Modes

- Useful in a hydro/derivative expansion
- Consider set of static solutions e.g. thermodynamic/zero mode perturbations $\delta_1\phi_A^{(s)}$
- Construct hydro perturbation in derivative expansion

$$\delta_2\phi_A = e^{-i\epsilon\,\omega\,t+i\epsilon\,k\,x}(\delta_1\phi_A^{(s)} + \epsilon\,\delta\phi_A^{(1)} + O(\epsilon^2))$$

First dissipative correction

- Construct P^μ out of $\delta_1\phi_A^{(s)}$ and $\delta_2\phi_A$
- Expand conservation of P^μ in ϵ, integrate along radial direction to study dynamics of $\delta\phi_A^{(1)}$

Symplectic Current

- Component P^r interesting in holography
- At leading order in boundary derivatives

$$P^r_{\delta_1,\delta_2} = \frac{1}{r^3}\left(\delta_1\varphi^I_{(s)}\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_I\rangle\right) - \delta_2\varphi^I_{(s)}\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_I\rangle\right)\right)$$
$$+\frac{1}{r^3}\frac{1}{2}\left(\delta_1 A_a\,\delta_2\left(\sqrt{-\gamma}\,\langle J^a\rangle\right) - \delta_2 A_a\,\delta_1\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)\right) + \cdots$$
$$+\frac{1}{r^3}\frac{1}{2}\left(\delta_1\gamma_{ab}\,\delta_2\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right) - \delta_2\gamma_{ab}\,\delta_1\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right)\right) + \cdots$$

- Use known static solutions to read off VEVs of hydro perturbation
- Use to derive effective theories of hydrodynamics

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Holographic Model F

- Use symplectic current techniques to deduce long wavelength dynamics

Donos, Kailidis

$$\partial_t \psi = -2\Gamma_0 \frac{\delta W_0}{\delta \psi^\star} - i g_0 \psi \frac{\delta W_0}{\delta m}\,, \qquad \partial_t m = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta m} + 2\, g_0\, \mathrm{Im}(\psi^\star \frac{\delta W_0}{\delta \psi^\star})\,,$$

$$W_0[\psi, m] = \int d^2x \left(\frac{w_0}{2} |\nabla \psi|^2 + \frac{\tilde{r}_0}{2} |\psi|^2 + \tilde{u}_0 |\psi|^4 + \frac{1}{2\, C_0} m^2 + \gamma_0\, m\, |\psi|^2 \right)$$

- Holography reproduces the classical part of Model F
- All constants and transport coefficients fixed by thermodynamics and horizon data

Holographic Model F

$$C_0 = \chi_n\,, \qquad \gamma_0 = -\frac{\Delta\varrho}{\chi_n \langle \mathcal{O}_\psi \rangle^2}\,, \qquad w_0 = \frac{\chi_{JJ}}{q_e^2 \langle \mathcal{O}_\psi \rangle^2}\,, \qquad \tilde{u}_0 = -\frac{1}{\langle \mathcal{O}_\psi \rangle^4}\Delta E\,, \qquad \tilde{r}_0 = 4\frac{\Delta w_{FE}}{\langle \mathcal{O}_\psi \rangle^2}\,,$$

$$\Gamma_0 = \frac{\langle \mathcal{O}_\psi \rangle^2}{\varpi^\star}\,, \qquad \lambda_0^m = \tau_h\,,$$

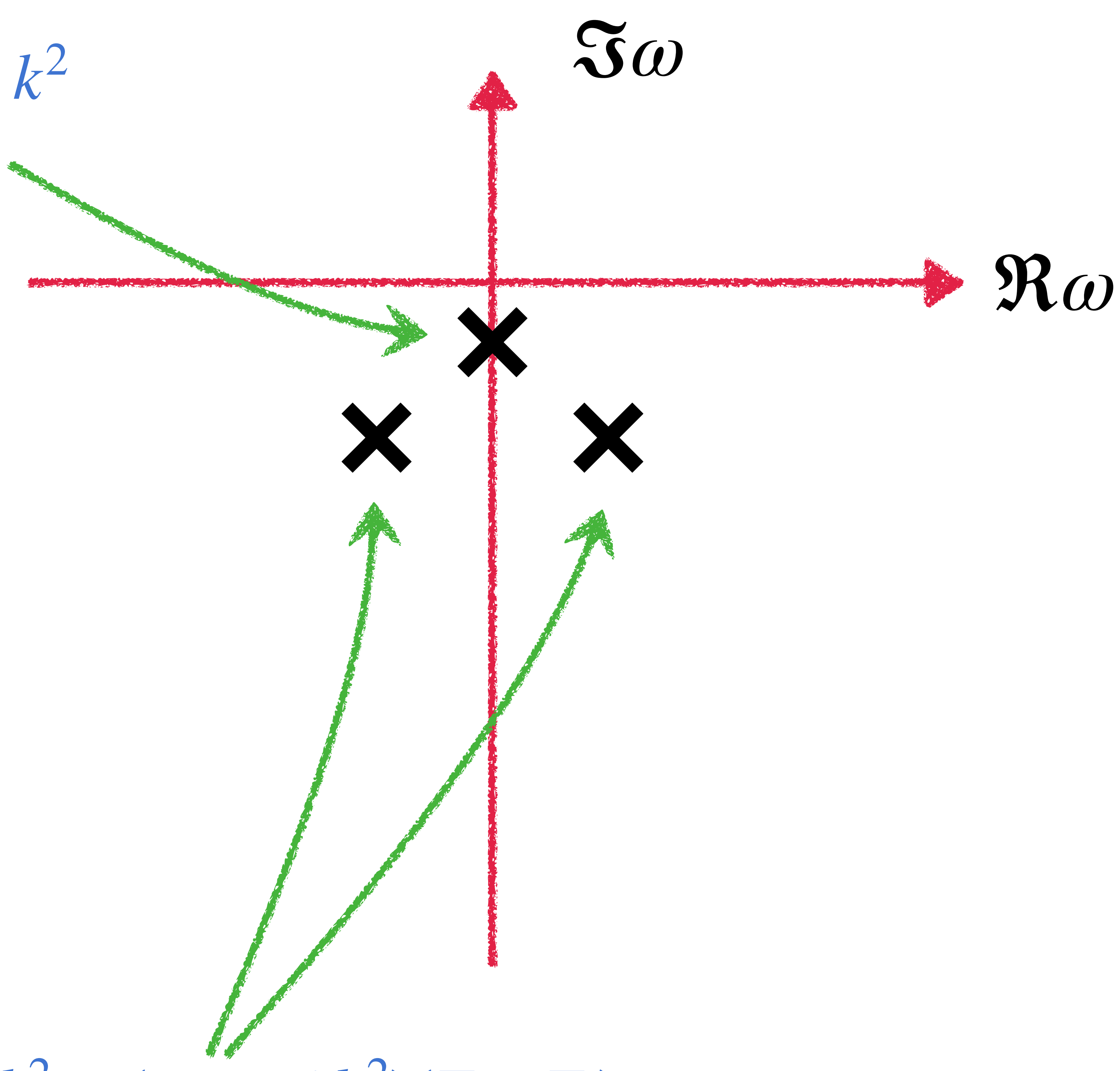
$$\varpi = \frac{s_c}{4\pi}|\psi_h|^2 + i\,\frac{2}{q_e}\left(\varrho_* - \varrho_{h*}\right)$$

- Defined $\Delta R = R(\text{broken phase}) - R(\text{normal phase})$ at fixed T and μ
- The kinetic coefficient Γ_0 is finite at the transition
- Integrate out amplitude to find $\zeta_3 \propto \langle \mathcal{O}_\psi \rangle^{-2}$

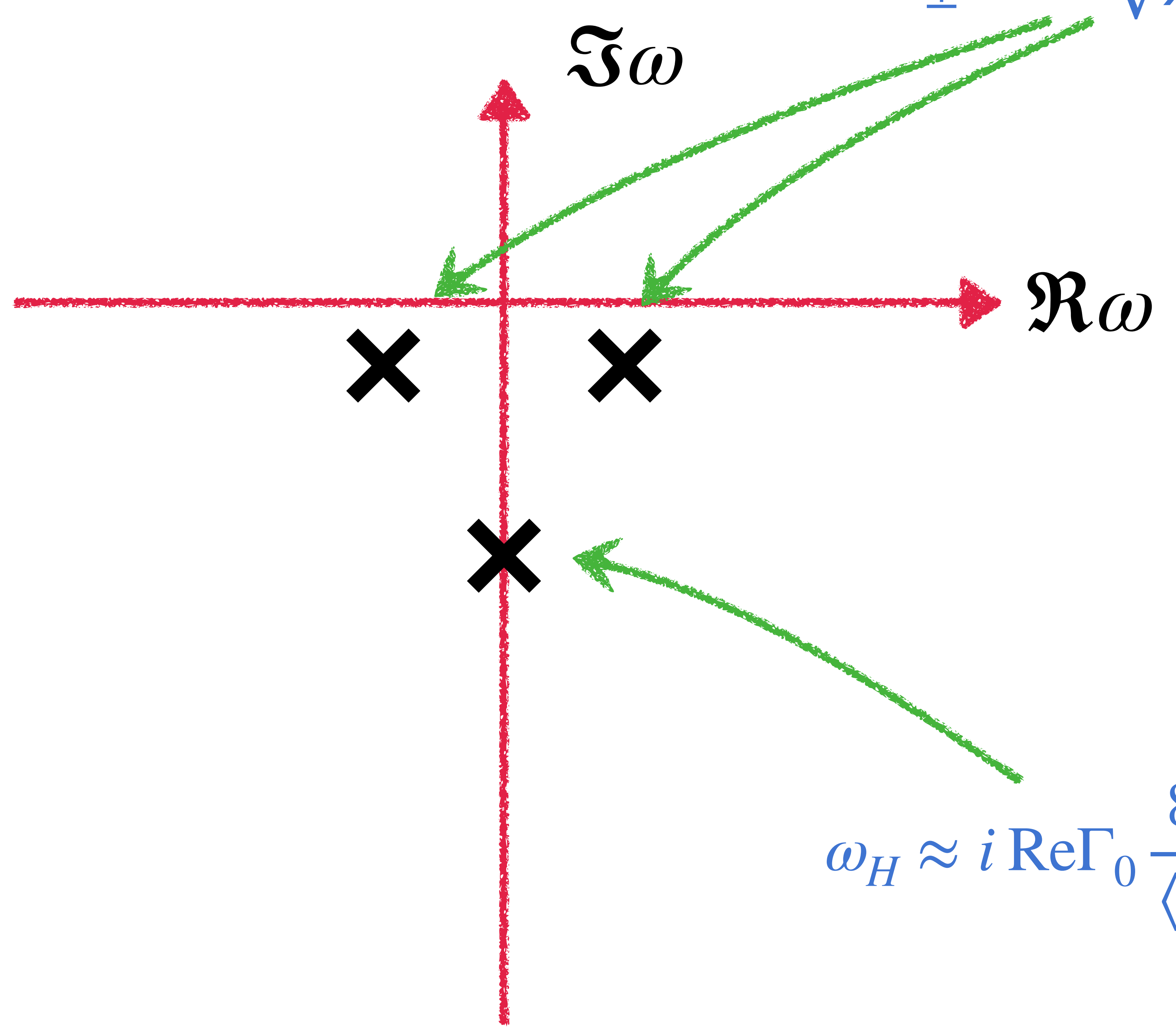
Modes away from critical point

$$k^2 \ll |T - T_c|$$

$T > T_c$



$T < T_c$



Modes of Neutral Superfluid

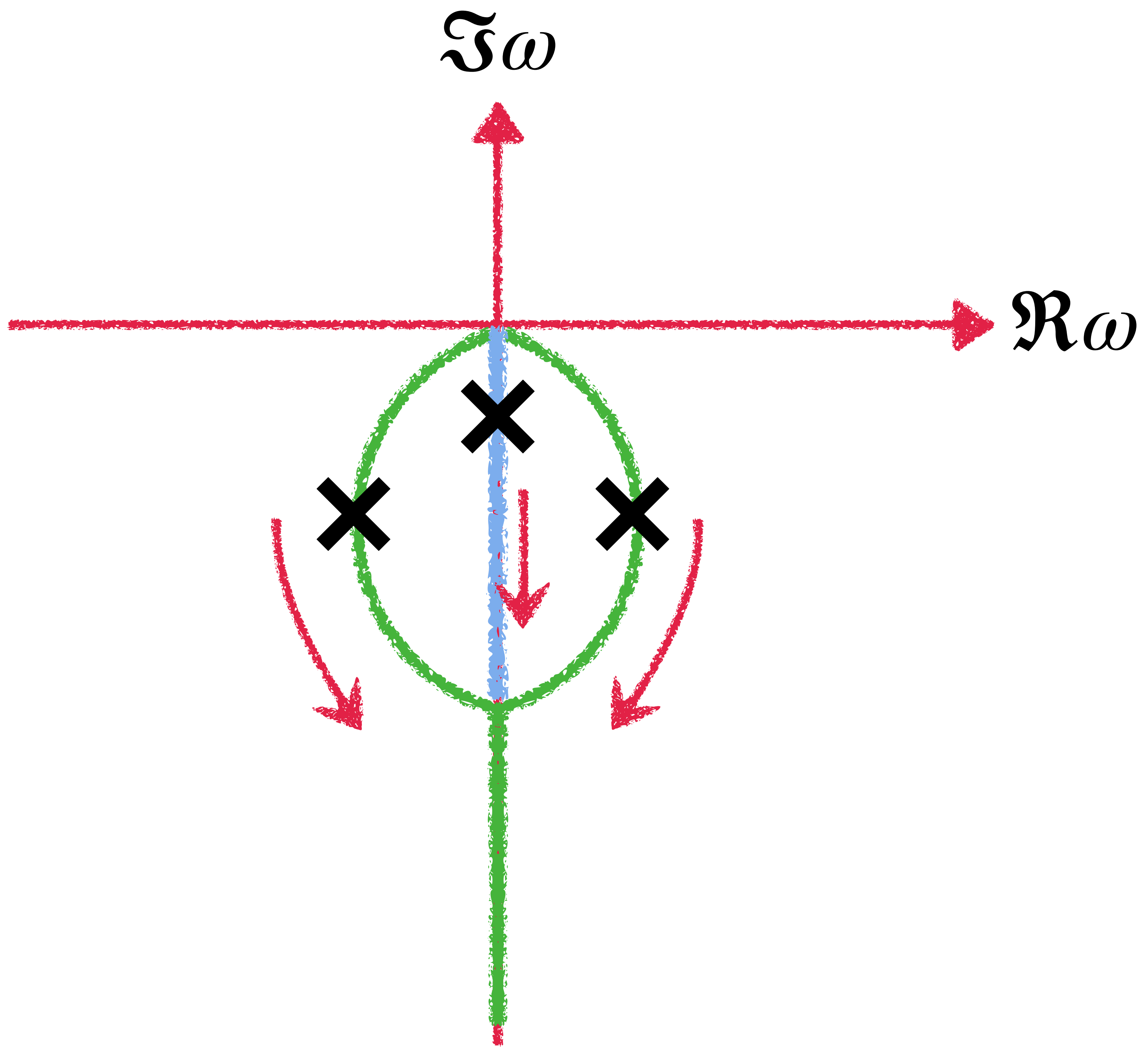
- Gapped mode ω_H and two sound modes $\omega_\pm$
- Collision of sound modes at $k_c^2 = \dfrac{4\,\chi_{JJ}\,\chi_n}{\left(\lambda_0^m - w_0\,\mathrm{Re}\Gamma_0\,\chi_n\right)^2}$
- At $k \gg k_c$

$$\omega_\pm \to -i\,\frac{\lambda_0^m}{\chi_n}\,k^2 + \cdots \quad \text{Charge Diffusion}$$

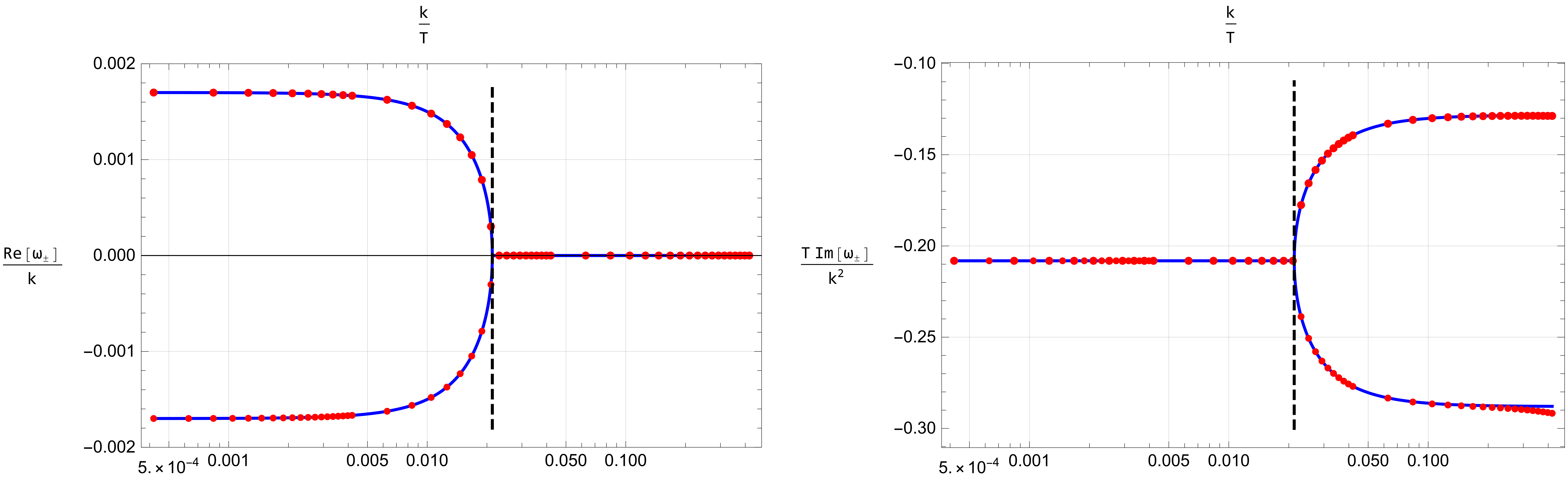
$$\omega_\pm \to -i\,w_0\,\Gamma_0\,k^2 + \cdots$$

$$\omega_H \to -i\,w_0\,\Gamma_0\,k^2 + \cdots$$

Agree with double pole of order parameter fluctuations as $T \to T_c^+$



Numerical Checks



- Check for sound pole dispersion relations

Conclusions & Outlook

- Holographic (probe) nearly critical superfluids captured by (mean field) Model F
- Third bulk viscosity diverges close to the transition. Thermal fluctuations?
- Include stress tensor in the description (KS, holography…)
- New fixed points?