Celestial chiral algebras in higher dimensions

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to appear, with Iustin Surubaru

First things first...

Welcome to the Flat Holography parallel sessions!

Three(!) sessions this week:

- Today, Wednesday and Friday

Lots of exciting things to hear about from lots of exciting people!

This talk:

Focus on a particular aspect of a particular approach to flat holography, namely:

Chiral algebras in the context of *celestial holography*

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What are these things?

Celestial holography

Pipe dream: $\exists$ CFT living on the celestial sphere (CCFT) s.t. correlators $=$ massless S-matrix in asymp. flat space-times

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OK...

...but in real money, typically less ambitious and much more concrete

Celestial chiral algebras

One of the key outputs of celestial holography is the following:

Observation: the positive helicity soft sector of perturbative GR in Minkowski space forms an algebra (under collinear limit)

$$\left[g^p_{m,r},\, g^q_{n,s}\right] = 2\left(m(q-1) - n(p-1)\right)\, g^{p+q-2}_{m+n,r+s}$$

Generators $g^p_{m,r}$ a re-writing of positive helicity soft gravitons:

- $2p-2 \in \mathbb{Z}_{\geq 0} \leftrightarrow \omega^{2p-4}$-order soft expansion [Guevara]
- $|m| \leq p-1 \leftrightarrow \overline{\mathrm{SL}(2,\mathbb{R})}$ weight in momentum
- $r \in \mathbb{Z} \leftrightarrow$ loop parameter weight $\leftrightarrow$ Laurent expansion on celestial sphere

[Guevara-Himwich-Pate-Strominger, Strominger]

Upshot

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u, v holomorphic coords on $\mathbb{C}^2$, with natural Poisson bracket

$$\{f, g\} := \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}$$

Basis of Hamiltonian functions

$$w^p_m = u^{p+m-1}\, v^{p-m-1}\,, \qquad 2p-2 \geq 0\,, \quad |m| \leq p-1$$

with $g^p_{m,r} := w^p_m / z^r$, $z \in \mathbb{C}$ complex loop parameter.

Other facts about celestial chiral algebras:

- similar story for self-dual Yang-Mills $\to \mathcal{L}\mathfrak{g}[\mathbb{C}^2]$
- $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$ closely related to $\mathcal{L}w^{\wedge}_{1+\infty}$ [Hoppe, Bakas]
- can also be derived from asymptotic symmetries, and extended to action on hard gravitons [Himwich-Pate-Singh, Jiang]
- natural derivation using *twistor theory* [TA-Mason-Sharma]

Who cares?

- ∞-dim. algebras $\Rightarrow$ many constraints on scattering
- enable bootstrap of some amplitudes in flat space and beyond [Costello-Paquette, Bittleston, Costello, Zeng]
- all-orders collinear expansions of some helicity sectors [TA-Bu-Casali-Sharma, Ren-Schreiber-Sharma-Wang]
- generalizations/deformations in curved SD spaces [Bu-Heuveline-Skinner, Costello-Paquette-Sharma, Bittleston-Heuveline-Skinner, Garner-Paquette, TA-Bu-Zhu, Zhu-Taylor, Bogna-et al.]
- underpin only known top-down constructions [Costello-Paquette-Sharma, Bittleston-Costello-Zeng]

Higher dimensions

Very little work (certainly nothing comparable to 4d)...why?

- conformal group on S^{d-2} finite-dimensional
- asymp. symmetry algebras finite-dimensional

  [Hollands-Ishibashi, Hollands-Ishibashi-Wald]
- soft symmetries in $d > 4$ finite-dimensional [Pano-Puhm-Trevisani]

Today

Celestial chiral algebras actually exist for gauge theory and gravity whenever $d = 4k$!

Key idea: In $4k$-d, there are integrable sectors of GR/YM which have all the features of the self-dual sector in 4d

Hyperkähler sector

$(\mathscr{M}_{4k}, g)$ hyperkähler if $\mathrm{Hol}(g) \subset \mathrm{Sp}(k, \mathbb{C})$

HK $\Rightarrow$ Ricci-flat

- $k = 1 \Leftrightarrow$ vacuum + self-duality

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For HK manifolds, $T\mathscr{M} \cong \mathbb{S} \otimes \tilde{\mathbb{S}}$

- $\mathbb{S}$ rank 2 with flat $\mathrm{SL}(2, \mathbb{C})$ connection; indices $\alpha, \beta, \ldots = 1, 2$
- $\tilde{\mathbb{S}}$ rank $2k$ with $\mathrm{Sp}(k, \mathbb{C})$ connection; indices $\dot{\alpha}, \dot{\beta}, \ldots = 1, \ldots, 2k$.

$$\mathrm{d}s^2 = \epsilon_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}}\, e^{\alpha\dot{\alpha}}\, e^{\beta\dot{\beta}}$$

Flat model

Let x^a be holomorphic coords on $\mathbb{C}^{4k}$, encode in matrix

$$x^{\alpha\dot\alpha} = \begin{pmatrix} z^{\dot\alpha} \\ \tilde{z}^{\dot\alpha} \end{pmatrix}$$
$$= \begin{pmatrix} x^1 + \mathrm{i}x^{4k} & \mathrm{i}x^2 + x^{4k-1} & \cdots & \mathrm{i}x^{2k} + x^{2k+1} \\ -x^{2k+1} + \mathrm{i}x^{2k} & \mathrm{i}x^{2k+2} - x^{2k-1} & \cdots & -\mathrm{i}x^{4k} + x^1 \end{pmatrix}$$

Flat metric $\mathrm{d}s^2 = \epsilon_{\dot\alpha\dot\beta}\,\mathrm{d}z^{\dot\alpha} \odot \mathrm{d}\tilde{z}^{\dot\beta} = \sum_{a=1}^{4k}(\mathrm{d}x^a)^2$

HK structure encoded in triplet of 2-forms

$$\Sigma^{11} = \mathrm{d}z^{\dot\alpha} \wedge \mathrm{d}z_{\dot\alpha}\,, \quad \Sigma^{12} = \mathrm{d}z^{\dot\alpha} \wedge \mathrm{d}\tilde{z}_{\dot\alpha}\,, \quad \Sigma^{22} = \mathrm{d}\tilde{z}^{\dot\alpha} \wedge \mathrm{d}\tilde{z}_{\dot\alpha}$$

Twistor space

Twistor space of $\mathbb{C}^{4k}$:

$$\mathbb{PT} = \left\{ Z^A = (\mu^{\dot\alpha}, \lambda_\alpha) \in \mathbb{CP}^{2k+1} \,|\, \lambda_\alpha \neq 0 \right\}$$

Points $x \in \mathbb{C}^{4k} \leftrightarrow$ holomorphic curves $\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$ in $\mathbb{PT}$

Some important structures:

- holomorphic fibration $p : \mathbb{PT} \to \mathbb{CP}^1$, $[Z^A] \mapsto [\lambda_\alpha]$
- holomorphic, weighted Poisson structure on fibres:

$$\{\cdot,\, \cdot\} = \epsilon^{\dot\alpha\dot\beta} \frac{\partial}{\partial \mu^{\dot\alpha}} \wedge \frac{\partial}{\partial \mu^{\dot\beta}}$$

Key result

Theorem (Penrose, Salamon)

There is a 1:1 correspondence between:

- *HK metrics*
- *complex deformations* $\mathbb{P}\mathscr{T}$ *of* $\mathbb{PT}$ *which preserve* $p : \mathbb{P}\mathscr{T} \to \mathbb{CP}^1$ *and the Poisson structure on the fibres (+ some technical stuff)*

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Upshot: symmetries of HK sector $\leftrightarrow$ deformations of $\mathbb{P}\mathscr{T}$ preserving fibration over $\mathbb{CP}^1$ and Poisson structure

Deformations of $\mathbb{PT}$ determined by Hamiltonians $g(Z)$ which are

- homogeneous of degree $+2$
- polynomial in $\mu^{\dot{\alpha}}$
- Laurent in λ_{α}

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In real money: can expand $g(Z)$ in modes

$$g[\mathbf{m}; r] := \frac{(\mu^{\dot 1})^{m_1}\,(\mu^{\dot 2})^{m_2}\cdots(\mu^{\dot{2k}})^{m_{2k}}}{\lambda_0^{M-2-r}\,\lambda_1^r}$$

where

- $\mathbf{m} = (m_1, \ldots, m_{2k}) \in \mathbb{N}_0^{2k}$
- $M := \sum_{i=1}^{2k} m_i \in \mathbb{N}_0$
- $r \in \mathbb{Z}$

HK chiral algebra

These modes form a chiral algebra $\mathcal{L}\mathfrak{ham}(\mathbb{C}^{2k})$ under the Poisson bracket on twistor space:

$$\{g[\mathbf{m}; r], g[\mathbf{n}; s]\} = \sum_{i=0}^{k-1} (m_{i+1}\, n_{2k-i} - n_{i+1}\, m_{2k-i})\, g[\mathbf{m} + \mathbf{n} - \mathbf{1}_{i+1} - \mathbf{1}_{2k-i}; r + s]$$

for $(\mathbf{1}_i)_j = \delta_{ij}$

Gauge theory

Similar story for gauge theory!

in $4k$-dim., gauge field strength valued in $\mathfrak{g}$ decomposes

$$F_{ab} = \epsilon_{\alpha\beta}\,\tilde{F}_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\,F_{\alpha\beta} + \check{F}_{\alpha\beta\,\dot\alpha\dot\beta}$$

where $\tilde{F}_{(\dot\alpha\dot\beta)}$, $F_{(\alpha\beta)}$ and $\check{F}_{(\alpha\beta)[\dot\alpha\dot\beta]}$, $\epsilon^{\dot\alpha\dot\beta}\check{F}_{\alpha\beta\,\dot\alpha\dot\beta} = 0$

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Gauge field is *hyperholomorphic* if $F_{\alpha\beta} = 0 = \check{F}_{\alpha\beta\,\dot\alpha\dot\beta}$ [Ward, Corrigan-Goddard-Kent, Verbitsky]

$$\text{HH} \Rightarrow \text{Yang-Mills}$$

Twistor construction

Theorem (Ward)

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Upshot: symmetries of HH sector $\leftrightarrow$ deformations of partial connection on holomorphic bundles $E \to \mathbb{PT}$

HH chiral algebra

Deformations given by $a(Z)$

- valued in $\mathfrak{g}$, homogeneous of weight zero
- polynomial in $\mu^{\dot\alpha}$
- Laurent in λ_α

$$\text{Modes:} \quad S^a[\mathbf{m}; r] = T^a \frac{(\mu^{\dot 1})^{m_1} (\mu^{\dot 2})^{m_2} \cdots (\mu^{\dot{2k}})^{m_{2k}}}{\lambda_0^{M-r} \lambda_1^r}$$

$$\text{Algebra:} \quad \left[S^a[\mathbf{m}; r], S^b[\mathbf{n}; s] \right] = f^{abc} S^c[\mathbf{m} + \mathbf{n}; r + s]$$

This is the chiral algebra $\mathcal{L}\mathfrak{g}[\mathbb{C}^{2k}]$

What else can we do?

HK/HH sectors classically described by a 2d CFT [TA-Mason-Sharma, TA-Bu-Casali-Sharma]

- Action for holomorphic curves/frames in twistor space
- On-shell, gives potentials for HK metric [Plebanski] / HH connection [Sparling]
- HK/HH perturbations $\leftrightarrow$ vertex operators in 2d CFT

Chiral algebras encoded by OPEs between VOs, e.g.:

$$S^{\mathsf{a}}[\mathbf{m}](z)\, S^{\mathsf{b}}[\mathbf{n}](z') \sim \frac{f^{\mathsf{abc}}}{z - z'}\, S^{\mathsf{c}}[\mathbf{m}+\mathbf{n}](z')$$

$\Rightarrow$ these really are algebras living in 2-dimensions

Who cares?

Not terribly surprising that these sectors have infinite-dim. symmetry algebras (classically integrable),
...but:

- linked with $4k$-dim. versions of MHV scattering [TA-Surubaru]
- contrast with general expectations
- underlying structure still 2d CFT...
- ...challenges expectations for celestial/twisted holography

Lots to do!

- Connection – if any – with conformal primary representations
- Soft and collinear limits?
- Link with asymptotic symmetries and radiative data
- Top-down manifestations – a HK version of twisted holography?

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Thanks, and stay tuned for an exciting week of talks!