Non-Lorentzian Geometry and String Theory

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Introduction

- Is there a NR limit of the AdS/CFT correspondence?
- What are the main actors in such a correspondence? Is there a NR gravity regime?
- What is NR quantum gravity? How to backreact a quantum system onto a Newton–Cartan geometry?
- Can NR techniques be used to learn more about ordinary string theory?
- What is the landscape of UV complete non-Lorentzian string theories?
- NL geometry has many applications outside the areas of quantum gravity/string theory.

Outline

- Non-Lorentzian geometry
- Non-relativistic fluid/gravity
- Non-relativistic closed and open strings

Non-Lorentzian geometries in a nutshell

- Nowhere vanishing 1-forms (vielbeine) τ^A, e^a that transform linearly under some group $G \supset SO(1,p) \times SO(d-p)$. ($A = 0, 1, \ldots, p$ and $a = p+1, \ldots, d$)
- G is typically a subgroup or a contraction of the Lorentz group $SO(1,d)$.
- Typically the geometry comes equipped with additional gauge fields that also transform under G.
- Newton–Cartan geometry: τ, e^a ($a = 1, \ldots, d$), m with $G = \mathbb{R}^d \rtimes SO(d)$ acting as

$$\tau' = \tau\,, \qquad e'^a = R^a{}_b e^b + \Lambda^a \tau\,, \qquad m' = m + \delta_{ab}\Lambda^a e^b + \frac{1}{2}\delta_{ab}\Lambda^a\Lambda^b\tau$$

- m is also a gauge field transforming as $m' = m + d\sigma$.

- Typical examples are:

Newton–Cartan geometry	$\delta\tau = 0$	$\delta e^a = \lambda^a{}_b e^b + \lambda^a \tau$
Carroll geometry	$\delta\tau = \lambda^a e^a$	$\delta e^a = \lambda^a{}_b e^b$
Aristotelian geometry	$\delta\tau = 0$	$\delta e^a = \lambda^a{}_b e^b$

- There also exist stringy versions of these geometries.
- Using vielbein postulates one can define connections and curvatures.
- Typical form of connection ($T^\rho_{\mu\nu}$ is some tensor):

$$\Gamma^\rho_{\mu\nu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + T^\rho_{\mu\nu}$$

- $(\tau^\mu\,, e^\mu_a)$ inverse to $(\tau_\mu\,, e^a_\mu)$ and $h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu$, similarly for $h^{\mu\nu}$.

- Non-Lorentzian geometries appear e.g. in:
  - approximations of GR (e.g. PN corrections use Newton–Cartan geometry)
  - boundaries of spacetimes (e.g. asymptotically Lifshitz, Schroedinger and flat spacetimes)
  - null hypersurfaces (black hole horizons are Carrollian)
  - BKL singularities (can be formulated using Carroll geometry, see talk by Gerben Oling on Tuesday)
  - low energy field theories and fluid dynamics (Newton–Cartan and Aristotelian geometry)
  - limits of string theory
  - cosmology on super-Hubble scales (Carrollian)
  - HL gravity (diffeo invariance: Aristotelian geometry)

Non-relativistic Gravity

- Write $g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$ and expand

$$T_\mu = \tau_\mu + c^{-2} m_\mu + \mathcal{O}(c^{-4})\,, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + c^{-2}\Phi_{\mu\nu} + \mathcal{O}(c^{-4})$$

- Use a new connection $C^\rho_{\mu\nu}$ defined as

$$C^\rho_{\mu\nu} = -T^\rho \partial_\mu T_\nu + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_\mu \Pi_{\nu\sigma} + \partial_\nu \Pi_{\mu\sigma} - \partial_\sigma \Pi_{\mu\nu}\right)$$

- In terms of T_μ, $\Pi_{\mu\nu}$ the EH Lagrangian is [Hansen, JH, Obers, 2019]

$$\begin{aligned}\mathcal{L}_{\text{EH}} &= \frac{E}{16\pi G}\Bigg[\frac{c^6}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}\left(\partial_\mu T_\rho - \partial_\rho T_\mu\right)\left(\partial_\nu T_\sigma - \partial_\sigma T_\nu\right) + c^4\left(\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - 2\Lambda\right) \\ &\quad + \frac{c^2}{4}\left(\Pi^{\mu\nu}\Pi^{\rho\sigma} - \Pi^{\mu\rho}\Pi^{\nu\sigma}\right)\mathcal{L}_T \Pi_{\mu\rho}\mathcal{L}_T\Pi_{\nu\sigma}\Bigg] \\ &= c^6\mathcal{L}_{\text{LO}} + c^4\mathcal{L}_{\text{NLO}} + c^2\mathcal{L}_{\text{NNLO}} + \mathcal{O}(c^0)\,.\end{aligned}$$

NR fluid/gravity correspondence [JH, Mehra, Musaeus, 2024]

- The LO theory only tells us that $\tau \wedge d\tau = 0$.
- EOM are repeated.
- $\mathcal{L}_{\text{NNLO}}$ is a theory for τ_μ, $h_{\mu\nu}$, m_μ and $\Phi_{\mu\nu}$.

NR AdS: $$\tau = \frac{dt}{r}\,, \qquad h = \frac{1}{r^2}\left(dr^2 + dx^i dx^i\right)\,, \qquad m = 0 = \Phi$$

- Killing vectors form the Galilean conformal algebra
- A Penrose-like boundary at $r = 0$ which is a NC geometry.
- Planar excitation: $m = A r^{D-2} dt\,, \quad \Phi = -2 A r^{D-3} dr^2$
- Perform a Galilean boost (asymptotic symmetry). Bdry EMT:

$$T^t{}_t = -\rho\,, \qquad T^i{}_t = -\frac{D-1}{D-2}\rho v^i\,, \qquad T^i{}_t = 0\,, \qquad T^i{}_j = \frac{\rho}{D-2}\delta^i_j$$

- Massless Galilean. Different vacuum: massive Galilean EMT?

Strings in non-Lorentzian target spaces

- Is there a string whose beta functions give NRG?
- To define a NR expansion the string needs to wrap something: rest mass with respect to which we can define a NR spectrum.
- Write $g_{MN} = c^2\left(-T^0_M T^0_N + \tilde{c}^{-2} T^1_M T^1_N\right) + \Pi_{MN}$
- Target space is a NC geometry plus a circle: also called string NC geometry: τ^A_M with $A = 0, 1$ and e^a_M
- Pullback $\tau_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N \left(-\tau^0_M \tau^0_N + \tilde{c}^{-2}\tau^1_M \tau^1_N\right)$ is 2D Lorentzian metric [Bidussi, Harmark, JH, Obers, Oling, 2021]

$$\mathcal{L} = -\frac{\tilde{c}T_{\text{NR}}}{2}\left(\sqrt{-\tau}\tau^{\alpha\beta}h_{\alpha\beta} + \varepsilon^{\alpha\beta}m_{\alpha\beta}\right)$$

- $m_{\alpha\beta}$ includes the pullback of a Kalb–Ramond 2-form

- For a flat target space $\mathbb{R}^{1,24} \times S^1$ with radius $c\tilde{c}^{-1}R_{\text{NR}}$ the energy of the string with tension $T_{\text{NR}} = c\tilde{c}^{-1}T$ and nonzero winding w is [JH, Have, 2022]

$$E = \frac{\alpha'_{\text{NR}}}{2wR_{\text{NR}}}p^2 + \frac{\tilde{c}}{wR_{\text{NR}}}\left(N + \bar{N} - 2\hbar\right)$$

- For a flat target space the symmetries form the string Bargmann algebra (with an infinite lift due to the 2D Lorentzian submanifold). [Bergshoeff, Gomis, Rosseel, Simsek, Yan, 2019]

- Canonical quantisation and finding irreps of the string Bargmann algebra in the string spectrum has not been done yet.

- Physical states all have nonzero winding.

- In string scattering we have to include zero winding states as off shell fields (these lead to instantaneous forces mediated by string Newton–Cartan background fields) [Danielsson, Güijosa, Kruczenski, 2001].

Open strings

- Endpoints of open string ending on D25 move at the speed of light but not for a D24 brane: massive particle.
- Winding circle in transverse v-direction to D24 brane.
- Target space symmetries that are respected by Dirichlet bcs: endpoint is a Bargmann particle.
- $1/c$ exp of Nambu–Goto action in flat spacetime but with D24 brane with world-volume fields turned on [JH, Have, 2024]:

$$S = -\frac{\tilde{c}T_{\text{NR}}}{2}\int_\Sigma d^2\sigma\,\sqrt{-\tau}\tau^{\alpha\beta}\partial_\alpha x^i\partial_\beta x^i + \int_{\partial\Sigma} d\sigma^0\, a_\mu \dot{x}^\mu\Big|_{\sigma^1=0}^{\sigma^1=\pi}$$

used a critical B field and $\tau^{\alpha\beta}$ is the inverse of

$$\tau_{\alpha\beta} = -\partial_\alpha x^t\partial_\beta x^t + \tilde{c}^{-2}\partial_\alpha x^v\partial_\beta x^v\,, \qquad x^v = x_0^v(\sigma) + \frac{1}{T_{\text{NR}}}\phi(x^\mu)$$

NR limit of DBI action

- DBI in static gauge with a critical B field ($B = -\tilde{c}^{-1}c^2 dt \wedge dv$)

$$S_{\mathsf{D24}} = -cT_{\mathsf{D24}} \int d^{25}\sigma \sqrt{-\det M_{\mu\nu}}$$

$$M_{\mu\nu} = -c^2 T_\mu^+ T_\nu^- + \delta_\mu^i \delta_\nu^i + \frac{1}{\tilde{c}T_{\mathsf{NR}}} F_{\mu\nu}\,, \qquad T_\mu^\pm = \delta_\mu^t \pm \frac{1}{\tilde{c}T_{\mathsf{NR}}} \partial_\mu \Phi$$

- NR expansion: $A_\mu = a_\mu + \mathcal{O}(c^{-2})$ and $\Phi = \phi + \mathcal{O}(c^{-2})$.
- NR worldvolume gauge field a_μ (with field strength $f_{\mu\nu}$) and transverse fluctuations ϕ.

- At leading order in $1/c$ we get the NR DBI action

$$S_{\text{nrD24}} = -T_{24}c^2 \int d^{25}\sigma \sqrt{\det\, M_{ij}} \Big[\tau_t^{+}\tau_t^{-} + \frac{1}{\tilde{c}^2 T_{\text{NR}}^2}\left(\tau_t^{-} M^{ij} + \tau_t^{+} M^{ji}\right) f_{it}\partial_j\phi$$
$$+ \frac{1}{\tilde{c}^4 T_{\text{NR}}^4}\left(M^{ij}M^{kl} - M^{jk}M^{il}\right) f_{it} f_{lt} \partial_j\phi\partial_k\phi\Big]^{1/2}$$

where

$$M_{ij} = \delta_{ij} + \frac{1}{\tilde{c}T_{\text{NR}}} f_{ij}\,, \qquad \tau_t^{\pm} = 1 \pm \frac{1}{\tilde{c}T_{\text{NR}}}\partial_t\phi$$

- This is the action for nonlinear Galilean electrodynamics (GED) [Gomis, Yan, Yu, 2020] & [JH, Have, 2024].

- Expanding in $\frac{1}{T_{\text{NR}}^2}$ leads to (standard) GED [Festuccia, Hansen, JH, Obers, 2016]:

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 + \partial_i\phi f_{ti} - \frac{1}{4} f_{ij} f_{ij}$$

NR holography?

- The $\tilde{c} \to \infty$ limit leads to spin matrix strings [Bidussi, Harmark, JH, Obers, Oling 2023].
- Spin matrix theory is a limit of $\mathcal{N} = 4$ SYM whose dual description (for large N) is the $\tilde{c} \to \infty$ limit [Harmark, Orselli, 2014].
- Carroll strings & evidence of a duality web of various NL strings and limits of ordinary strings theory [Bagchi, Banerjee, JH, Have, Kolekar, 2024] & [Blair, Lahnsteiner, Obers, Yan, 2023].
- Hints for NR holography:
  - NR fluid/gravity
  - Spin matrix limits (see talk by Stefano Baiguera)
  - Existence of open and closed NR strings
- Stay tuned for next talks!