Logarithmic soft graviton theorems from superrotation Ward identities

based on 2309.11220 w/ Laura Donnay, Kevin Nguyen and Romain Ruzziconi

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Motivation

Motivation

- Building a holographic description of quantum gravity in asymptotically flat spacetimes
- Understanding infrared structure of gauge and gravity theories
- Finding the symmetries of asymptotically flat spacetimes
- Constructing the celestial stress tensor for the proposed CCFT dual

Soft Theorem

Leading soft theorem

- In his seminal paper, Weinberg showed that in a scattering process, the dependence on soft momentum can be factored out:

$$\lim_{\omega \to 0} \mathcal{M}_{n+1}(p_i, \omega \hat{q}) = \frac{1}{\omega} S^{(0)}(p_i, \omega \hat{q}) \mathcal{M}_n(p_i) \tag{1}$$

$$S^{(0)}(p_i, \hat{q}) = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} p_i^{\mu} p_i^{\nu}}{p_i \cdot \hat{q}} \quad \text{[Weinberg, 1965]} \tag{2}$$

- This term is exact to all orders in perturbation theory in the sense that:

$$\lim_{\omega \to 0} \omega \mathcal{M}_{n+1}^{\ell-loop} = S^{(0)} \mathcal{M}_n^{\ell-loop} \tag{3}$$

- $S^{(0)}$ is also universal in the sense that it does not depend on the matter content of the theory

Subleading soft theorem

- Cachazo and Strominger computed the subleading term in ω expansion,

$$\lim_{\omega \to 0} \mathcal{M}_{n+1} = \left(\frac{1}{\omega} S^{(0)} + S^{(1)tree} \right) \mathcal{M}_n \tag{4}$$

$$S^{(1)tree}(p_i, \hat{q}) = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} p_i^{\mu} \hat{q}_{\rho}}{p_i \cdot \hat{q}} J_i^{\nu\rho} \quad \text{[Cachazo, Strominger, '14]} \tag{5}$$

- Tree level computations can be checked explicitly to give

$$\lim_{\omega \to 0} (1 + \omega \partial_{\omega}) \mathcal{M}_{n+1}^{tree} = S^{(1)tree} \mathcal{M}_n^{tree} \tag{6}$$

Logarithmic soft theorem

- Sahoo and Sen performed an explicit computation of the two amplitudes at one loop **without** assuming any expansion in ω
- Then they were able to factor out the soft momentum dependent terms as,

$$S(p_i, \omega\hat{q}) = \lim_{\omega \to 0} \frac{\mathcal{M}_{n+1}^{tree} + \mathcal{M}_{n+1}^{1-loop}}{\mathcal{M}_{n}^{tree} + \mathcal{M}_{n}^{1-loop}} \tag{7}$$

- They found the expansion in ω to be,

$$S = \frac{1}{\omega} S^{(0)} + \ln \omega \, S^{(\ln)} + S^{(1)tree} + \mathcal{O}(\omega \ln \omega) \quad \text{[Sahoo, Sen, '18]} \tag{8}$$

- Thus we can write the subleading soft theorem as,

$$\lim_{\omega \to 0} (1 + \omega \partial_\omega) \mathcal{M}_{n+1} = [\ln \omega S^{(\ln)} + S^{(1)tree}] \mathcal{M}_n \tag{9}$$

Logarithmic soft theorem

$$
\begin{aligned}
S_n^{(\ln)} &= \frac{i\kappa}{8\pi}\sum_i \frac{\varepsilon_{\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot q}\sum_j \delta_{\eta,\eta_j}\, q\cdot p_j \\
&+ \frac{i\kappa}{16\pi}\sum_i \frac{\varepsilon_{\mu\nu}p_i^\nu q_\rho}{p_i\cdot q}\sum_j \delta_{\eta_i,\eta_j}(p_i\cdot p_j)\left(p_i^\mu p_j^\rho - p_j^\mu p_i^\rho\right)\frac{2(p_i\cdot p_j)^2 - 3p_i^2p_j^2}{\left[(p_i\cdot p_j)^2 - p_i^2p_j^2\right]^{3/2}} \\
&- \frac{\kappa}{8\pi^2}\sum_i \frac{\varepsilon_{\mu\nu}p_i^\nu p_i^\nu}{p_i\cdot q}\sum_j q\cdot p_j \ln|\hat{q}\cdot\hat{p}_j| \\
&- \frac{\kappa}{32\pi^2}\sum_i \frac{p_i^\mu\varepsilon_{\mu\nu}q_\lambda}{p_i\cdot q}\left(p_i^\lambda\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\lambda}}\right) \\
&\sum_j \frac{2(p_i\cdot p_j)^2 - p_i^2p_j^2}{[(p_i\cdot p_j)^2 - p_i^2p_j^2]^{1/2}}\ \ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\right)\ .
\end{aligned}
$$

Logarithmic soft theorem

- We can write down the log soft factor compactly as,

$$S^{(\ln)} = \sigma'_{n+1}(p_i, \hat{q})S^{(0)} - S^{(1)tree}\sigma_n(p_i) \tag{9}$$

[Bern, Davies, Nohle, ‘14; Sahoo, Sen, ‘18]

- The first term comes due to the interaction of external soft particle with the loop
- The second term comes due to the interaction of external hard particles with the loop

Central Question

Can we understand the logarithmic corrections to soft theorem as coming from superrotation Ward Identities?

Asymptotic Symmetries

Asymptotically flat spacetime

- The metric for an asymptotically flat spacetime in Bondi coordinates (u, r, x^A) is,

$$
\begin{aligned}
ds^2 = &\left(\frac{2M}{r} + \mathcal{O}(r^{(-2)})\right) du^2 - 2(1 + \mathcal{O}(r^{-2}))dudr \\
&+ (r^2 q_{AB} + rC_{AB} + \mathcal{O}(r^0))dx^A dx^B \\
&+ \left(\frac{1}{2} D_B C^B_A + \frac{2}{3r}(N_A + \frac{1}{4} C^B_A D_C C^C_B) + \mathcal{O}(r^{-2})\right) dudx^A
\end{aligned} \tag{10}
$$

- The asymptotic fall off of this metric is preserved by the diffeomorphism given by,

$$
\begin{aligned}
&\xi^u = \mathcal{T} + \frac{u}{2}(\partial \mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}}),\ \xi^r = -\frac{r}{2}(\partial \mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}}) + \mathcal{O}(r^0) \\
&\xi^z = \mathcal{Y} + \mathcal{O}(r^{-1}),\ \xi^{\bar{z}} = \bar{\mathcal{Y}} + \mathcal{O}(r^{-1})
\end{aligned} \tag{11}
$$

[Bondi, Van der Burg, Metzner, ‘62; Sachs, ‘62; Barnich, Troessaert, ‘10]

- $\mathcal{T}(z, \bar{z})$ parametrized **supertranslations** while **superrotations** are parametrized by $\mathcal{Y}(z)$ and $\bar{\mathcal{Y}}(\bar{z})$ called the **BMS symmetries**

Phase space at null infinity

- The free data at null infinity constitutes the radiative phase space Γ
- The radiative phase space can be separated into hard (defined on full null infinity) and soft (defined on the celestial sphere) pieces,

$$\begin{aligned}\Gamma^{\text{hard}} &= \{C_{zz}, N_{zz} = \partial_u C_{zz}, C_{\bar{z}\bar{z}}, N_{\bar{z}\bar{z}}, \text{matter}\} \\ \Gamma^{\text{soft}} &= \{C^{(0)} = \frac{1}{2}(C_+ + C_-), \mathcal{N}^{(0)}_{zz} = \int du\, N_{zz}, \mathcal{N}^{(0)}_{\bar{z}\bar{z}}, \\ &\quad \mathcal{N}^{(1)}_{zz} = \int du\, u\, N_{zz}, \mathcal{N}^{(1)}_{\bar{z}\bar{z}}, \varphi, \bar{\varphi}\} \end{aligned} \tag{12}$$

[Donnay, Nguyen, Ruzziconi, '22]

- $\partial^2 C_\pm$ is the boundary value of C_{zz} and $C^{(0)}$ is the Goldstone mode of supertranslation. We also define $C^{(0)}_{zz} = \partial^2 C^{(0)}$.

BMS Fluxes

- The BMS fluxes that act as canonical transformation on this phase space were computed to be,

$$
\begin{aligned}
F_{\mathcal{T}}^{\text{hard}} &= -\frac{1}{16\pi G}\int du d^2z\, \mathcal{T}\left[N_{zz}N_{zz} + 16\pi G T_{uu}^{(2)}\right] \\
F_{\mathcal{T}}^{\text{soft}} &= \frac{1}{8\pi G}\int d^2z\, \mathcal{T}\left[\partial^2 \mathcal{N}_{\bar{z}\bar{z}}^{(0)}\right] \\
F_{\mathcal{Y}}^{\text{hard}} &= \frac{1}{16\pi G}\int du d^2z\, \mathcal{Y}\Big[\frac{3}{2}C_{zz}\partial N_{\bar{z}\bar{z}} + \frac{1}{2}N_{\bar{z}\bar{z}}\partial C_{zz} + \frac{u}{2}\partial(N_{zz}N_{\bar{z}\bar{z}}) \\
&\quad + 16\pi G\left(\frac{u}{2}\partial T_{uu}^{(2)} - T_{uz}^{(2)}\right)\Big] \\
F_{\mathcal{Y}}^{\text{soft}} &= \frac{1}{16\pi G}\int d^2z\, \mathcal{Y}\left[-\partial^3\mathcal{N}_{\bar{z}\bar{z}}^{(1)} + \frac{3}{2}C_{zz}^{(0)}\partial\mathcal{N}_{\bar{z}\bar{z}}^{(0)} + \frac{1}{2}\mathcal{N}_{\bar{z}\bar{z}}^{(0)}\partial C_{zz}^{(0)}\right] \\
&\equiv F_{\mathcal{Y}}^{\text{soft}(0)} + F_{\mathcal{Y}}^{\text{soft(new)}}
\end{aligned}
\tag{13}
$$

[Kapec, Lysov, Pasterski, Strominger, '14][Donnay, Nguyen, Ruzziconi, '22]

[Donnay, Ruzziconi, '21; SA, Donnay, Nguyen, Ruzziconi, '23]

Ward Identities

Supertranslation = Leading soft theorem

- The fields in the soft flux can be mode expanded using,

$$C_{zz}(u,z,\bar{z}) = \lim_{r\to\infty} \frac{1}{r} h^{\text{out}}_{zz}(r,u,z,\bar{z}) \tag{14}$$

- Now the invariance of $\mathcal{S}$-matrix under supertranslation can be written as,

$$\langle \text{out}| F^{\text{soft}}_{\mathcal{T}} \mathcal{S} - \mathcal{S} F^{\text{soft}}_{\mathcal{T}} |\text{in}\rangle = -\langle \text{out}| F^{\text{hard}}_{\mathcal{T}} \mathcal{S} - \mathcal{S} F^{\text{hard}}_{\mathcal{T}} |\text{in}\rangle \tag{15}$$

- Using the mode expansion and the form of supertranslation transformation on the external hard particles,

$$\begin{aligned} &\lim_{\omega\to 0} \omega \langle \text{out}| a(\omega,\hat{q}) \mathcal{S} - \mathcal{S} a^{\dagger}(\omega,\hat{q}) |\text{in}\rangle = S^{(0)} \langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &\lim_{\omega\to 0} \omega \mathcal{M}_{n+1} = S^{(0)} \mathcal{M}_n \quad \text{[He, Lysov, Mitra, Strominger, '14]} \end{aligned} \tag{16}$$

Tree level subleading soft theorem as Superrotation Ward Identity

- Similar to supertranslation case, the superrotation invariance was found to be,

$$\langle \mathrm{out}|F_{\mathcal{Y}}^{\mathrm{soft}(0)}\mathcal{S} - \mathcal{S}F_{\mathcal{Y}}^{\mathrm{soft}(0)}|\mathrm{in}\rangle = -\langle \mathrm{out}|F_{\mathcal{Y}}^{\mathrm{hard}}\mathcal{S} - \mathcal{S}F_{\mathcal{Y}}^{\mathrm{hard}}|\mathrm{in}\rangle \tag{17}$$

- And again replacing with the mode expansions and symmetry action,

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle \mathrm{out}|a(\omega,\hat{q})\mathcal{S} - \mathcal{S}a^\dagger(\omega,\hat{q})|\mathrm{in}\rangle = S^{(1)tree}\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle \tag{18}$$

[Kapec, Lysov, Pasterski, Strominger, '14]

- Recall however the full sub-leading soft theorem was given by

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\mathcal{M}_{n+1} = [\ln\omega(\sigma'_{n+1}S^{(0)} - S^{(1)tree}\sigma_n) + S^{(1)tree}]\mathcal{M}_n \tag{19}$$

Loop correction from soft Flux

- To include the correction from $F_{\mathcal{Y}}^{\text{soft(new)}}$ we also need to know the insertion of $C^{(0)}$ in the $\mathcal{S}$-matrix,

$$\langle \text{out}|C^{(0)}(\hat{q})\mathcal{S}|\text{in}\rangle = -i \ln \omega\, \sigma'_{n+1}(p_i, \hat{q})\langle \text{out}|\mathcal{S}|\text{in}\rangle \tag{20}$$

- Then immediately the correction from $F_{\mathcal{Y}}^{\text{soft(new)}}$ can be computed,

$$\langle \text{out}|F_{\mathcal{Y}}^{\text{soft(new)}}\mathcal{S} - \mathcal{S}F_{\mathcal{Y}}^{\text{soft(new)}}|\text{in}\rangle = \ln \omega \sigma'_{n+1} S^{(0)} \langle \text{out}|\mathcal{S}|\text{in}\rangle \tag{21}$$

[Donnay, Nguyen, Ruzziconi, '22; SA, Donnay, Nguyen, Ruzziconi, '23]

- So the correction to the soft flux accounts for the interaction of external soft particle with the loop
- We are now left with a term coming from hard interaction

Correction to hard flux

- The hard flux can be written as a charge constructed from the asymptotic value of the bulk stress-tensor for bulk fields,

$$F_{\mathcal{Y}}^{\text{hard}} = \int d\Sigma_\mu T^\mu_\nu \xi^\nu_{\mathcal{Y}} \quad \text{[Campiglia, Laddha, '16]} \tag{22}$$

- For free massless fields this simply gives,

$$F_{\mathcal{Y}}^{\text{hard}} = \int du d^2 z \, \mathcal{Y} \left(\frac{u}{2} \partial T^{(2)}_{uu} - T^{(2)}_{uz} \right) \tag{23}$$

- Fields interacting with gravity in the bulk do not asymptote to free fields but **dressed** fields which can be accounted for by,

$$b^\dagger(\omega, \hat{p}) \to e^{i\omega C^{(0)}(\hat{p})} b^\dagger(\omega, \hat{p}) \tag{24}$$

[Himwich, Narayana, Pate, Paul, Strominger, '20]

- This corrects the hard flux by a factor

$$\Delta F_{\mathcal{Y}}^{\text{hard}} = \int du d^2 z \left(\mathcal{Y} \partial - \frac{1}{2} \partial \mathcal{Y} \right) C^{(0)} T^{(2)}_{uu} \tag{25}$$

[SA, Donnay, Nguyen, Ruzziconi, '23]

Loop correction from hard flux

- The correction to Ward Identity from $\Delta F_{\mathcal{Y}}^{\text{hard}}$ can again be computed,

$$\langle \text{out}|\Delta F_{\mathcal{Y}}^{\text{hard}}\mathcal{S} - \mathcal{S}\Delta F_{\mathcal{Y}}^{\text{hard}}|\text{in}\rangle = \ln\omega S^{(1)tree}\sigma_n\langle \text{out}|\mathcal{S}|\text{in}\rangle \tag{26}$$

[SA, Donnay, Nguyen, Ruzziconi, '23]

- Thus from the Ward Identity of the full flux,

$$\langle \text{out}|F_{\mathcal{Y}}^{\text{soft}}\mathcal{S} - \mathcal{S}F_{\mathcal{Y}}^{\text{soft}}|\text{in}\rangle = -\langle \text{out}|F_{\mathcal{Y}}^{\text{hard}}\mathcal{S} - \mathcal{S}F_{\mathcal{Y}}^{\text{hard}}|\text{in}\rangle \tag{27}$$

- We obtain the full sub-leading soft theorem,

$$\begin{aligned} &\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle \text{out}|a(\omega,\hat{q})\mathcal{S} - \mathcal{S}a^\dagger(\omega,\hat{q})|\text{in}\rangle \\ &\qquad = \left[\ln\omega(\sigma'_{n+1}S^{(0)} - S^{(1)tree}\sigma_n) + S^{(1)tree}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle \end{aligned} \tag{28}$$

Conclusion

Results and Conclusions

- At **one loop** the **infrared divergent** loop correction to sub-leading soft theorem and the **logarithmic** soft theorem are equivalent.
- The infrared factorization of amplitudes contains a real and an imaginary part (previously disregarded). To account for the imaginary part as well we need to modify the two-point function for the Goldstone mode,

$$\langle C^{(0)}(z,\bar{z})C^{(0)}(w,\bar{w})\rangle = \frac{1}{\epsilon}\frac{\kappa^2}{(4\pi)^2}\eta_z\eta_w|z-w|^2\left(\ln|z-w|^2 - i\pi\delta_{\eta_z,\eta_w}\right) \tag{29}$$

- We accounted for the loop corrections to sub-leading soft theorem with appropriate corrections to superrotation flux.

Future Directions

A pyramid of soft theorems

Soft theorem	Tree level	1-loop	2-loop	n-loop
Leading	ω^{-1}			
Sub-leading	ω^{0}	$\ln\omega$		
Subsub-leading	ω	$\omega\ln\omega$	$\omega(\ln\omega)^2$	
$(\text{Sub})^n$-leading	ω^{n-1}	$\omega^{n-1}\ln\omega$	$\omega^{n-1}(\ln\omega)^2$	$\omega^{n-1}(\ln\omega)^n$

[Ghosh, Sahoo, '22]

- Can higher loop corrections be recovered from appropriately corrected higher charges?
- Is there a generalization to towers at **higher loops** of the $w_{1+\infty}$ algebra found at tree level?

  [Guevara, Himwich, Pate, Strominger, '21; Krishna, '23]
- How does the one loop correction to the sub-leading charge affect the celestial stress tensor? [Donnay, Nguyen, Ruzziconi, '22]
- What are the bulk diffeomorphisms related to the various soft factors? [Choi, Laddha, Puhm, '24]