Noise and Spatial Resolution

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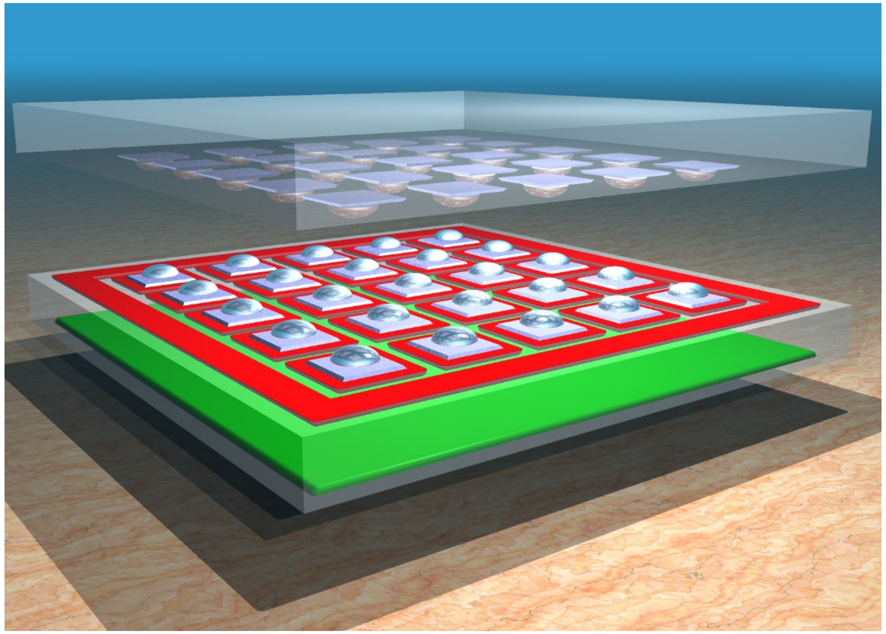
Why this talk?

- Spatial resolution pops up everywhere…
- Good to know same basic mechanisms
- I love the topic since my PhD.
  Always wanted to write a paper…
  Some parts are now written up in N. Wermes's book..
- Mathematics can be fun.
- There will be some 'take home messages…'
- Sorry for the old-fashioned style file. This used to be the 'corporate design' of Uni Heidelberg…

What is it about

- In Strips / Pixels / …, ‘Hits’ (particles going through, X-rays, Photon) produce signals
- These are measured on one or more channels
- The data is used to reconstruct the position.



- Questions:
  - What is the spatial resolution?
  - How does it depend on the reconstruction algorithm?
  - How does it depend on noise?
  - How does it depend on the ‘charge sharing’ mechanism?
  - …

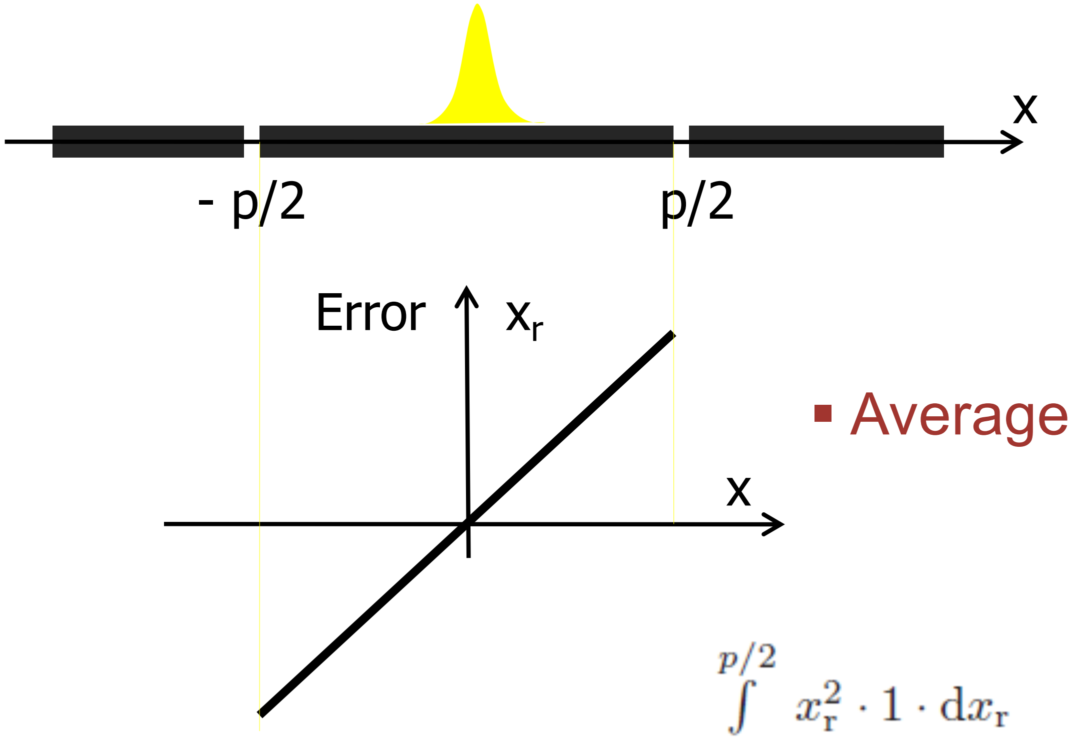
Overview

- Warmup:
  Resolution with binary readout, optimal signal width
- Error of Center-of-Gravity: When do we need a fit?
- Influence of noise on spatial resolution
  - Higher Moments
  - Correlated Noise
  - 2D structures
  - Wide signals
- Error when doing ‘Eta-reconstruction’
  - Search for ‘best’ response function

BINARY READOUT OF BOX SIGNALS

Spatial Resolution of Narrow Signals

- Consider very narrow signal
- → Only **one** strip is hit → Binary ‘yes/no’ - readout
- Reconstructed position = strip center. Error = offset in strip.



- Average Error is 0! (No bias)

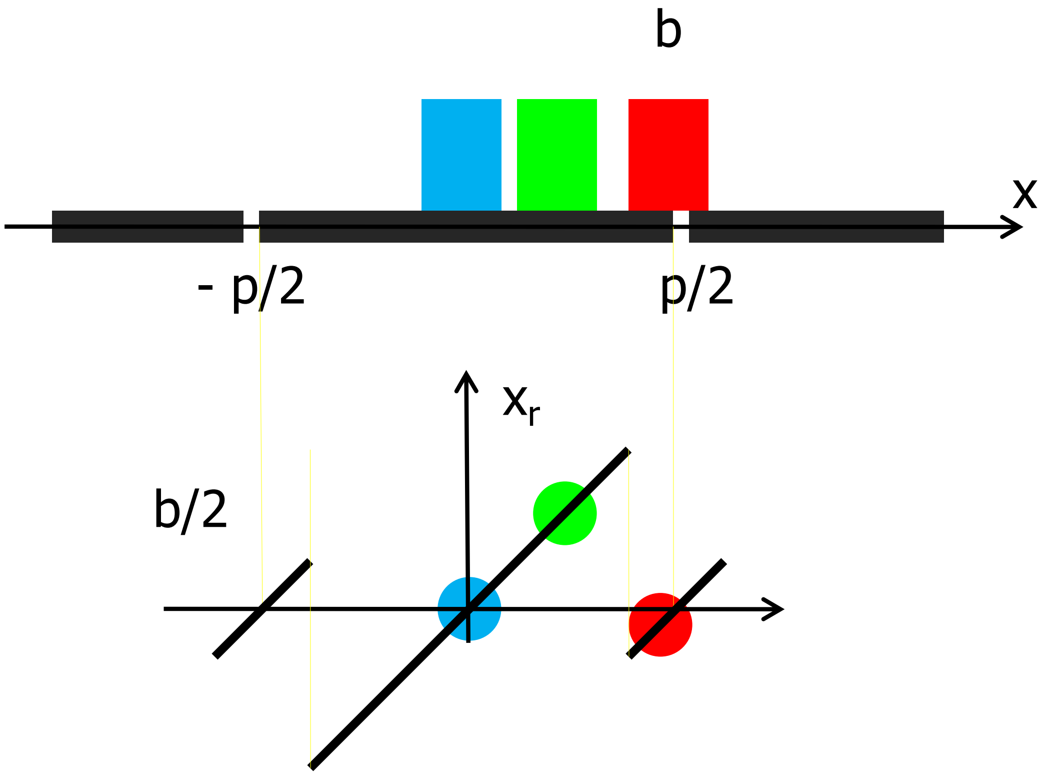
- Sigma of Error $\sigma^2_{\text{position}} = \frac{\int_{-p/2}^{p/2} x_r^2 \cdot 1 \cdot dx_r}{\int_{-p/2}^{p/2} 1 \cdot dx_r} = \frac{p^2}{12}$

$$\boxed{\sigma_{\text{position}} = \frac{p}{\sqrt{12}}}$$

50 µm pitch -> σ=12 µm, FWHM = 28 µm

Resolution with **wider** Signals (‘Binary’ Readout!)

- Consider ‘Box’ Signals for simplicity. Still binary readout.
- When 2 strips are hit → reconstruct at edge → small error



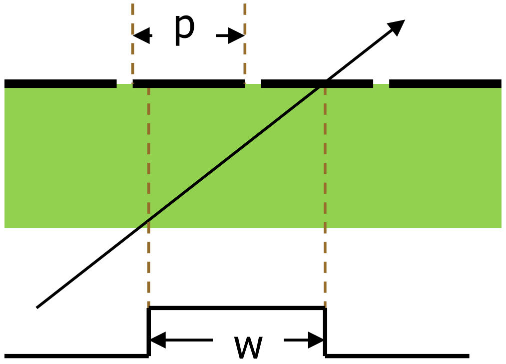
- Minimum Error for b = p/2. Error becomes **half**: $\sigma = \frac{1}{2}\, p/\sqrt{12}$
- Note that we have **50% single and 50% double hits**
- Note: Signals wider than p add no information!

- Mathematica !

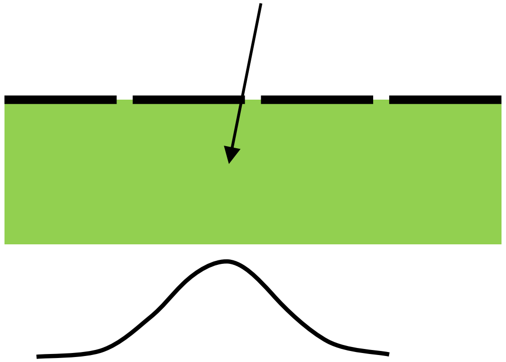
ANALOG READOUT

Realistic wide Signals

- When we know the AMPLITUDE in each strip/pixel, ce can do better.
- Simplest approach: x_{rek} = Center of Gravity of the signals



- Thick detector, Inclined track -> Box
  - -> Tilt detector!
  - Lorentz Angle helps
- **Perfect** reconstruction of w = p

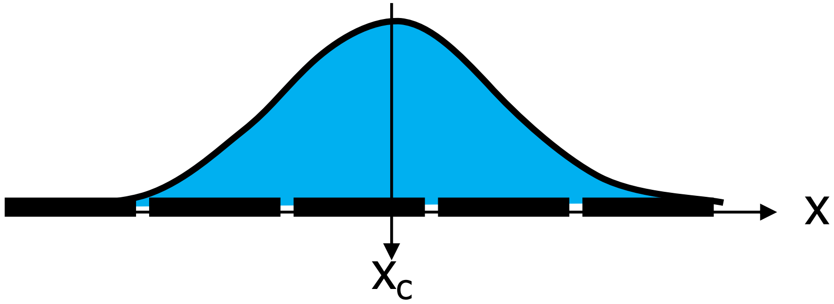


- Diffusion -> Gauss
  - Reconstruction ?
  - Fit !
  - Can we do simpler ? How good is CoG?

CENTER-OF-GRAVITY RECONSTRUCTION:
WHEN IS IT SUFFICIENT - OR -
WHEN DO WE NEED A FIT?

The question:

- A 1D signal with (spatial) shape f(x) falls onto a strip structure with pitch *a*
  - We assume **$\int f(x)dx = 1$** and **f(x) symmetric**.
- This generates (analogue) signals on several strips.
- We assume for now that **noise = 0**.
- Question:
  What is the reconstruction error for CoG reconstruction?
  - More precisely: Error for a single event? Average error? Sigma?



- We expect the answer to depend on
  - signal shape
  - Strip pitch a
  - signal position (for single events)

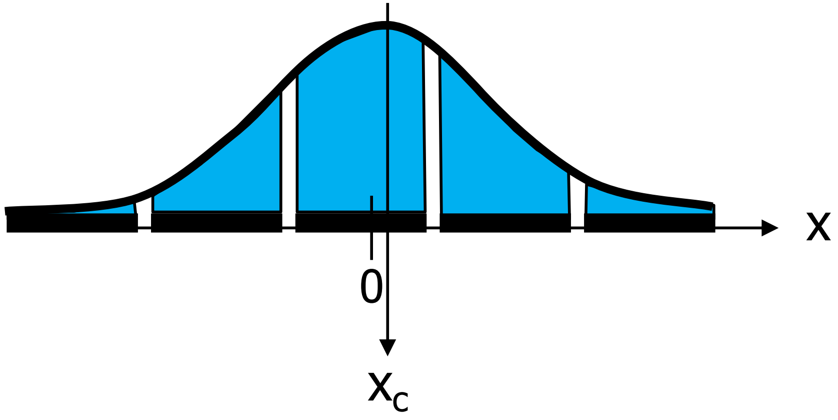
Remark

- The following calculation involves partial integrals over arbitrary function.
- Normally we must give up soon analytically (consider Gaussians..)
- But it turns out that we can go quite a way…

'never give up'

- Maybe showing the derivation would not really be necessary, but I like the fact that so many 'simple' aspects of basic analysis show up…

1. Signal on Strips



We **assume** the signal on a strip m is the integral of f(x)

Strip centers are at **m·a**

$$S_m(x_c) = \int_{(m-1/2)a}^{(m+1/2)a} f(x - x_c)\, dx$$

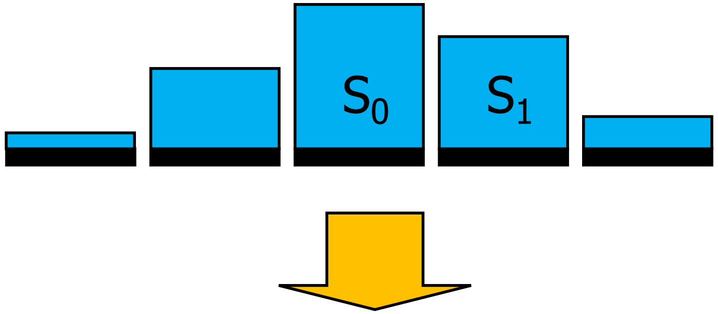
This is the signal in strip m when the charge cloud is centered around x_c

'Signal' function centered at x_c

$$= \int_{-\infty}^{\infty} f(x) \cdot \mathrm{Rect_a}(x - (ma - x_c))\, dx$$

Box of width **a** centered at **m·a-x$_c$**

2. Reconstructed Position with Center of Gravity



x_{rek} ?

position reconstructed by CoG
(we assumed a normalized signal)

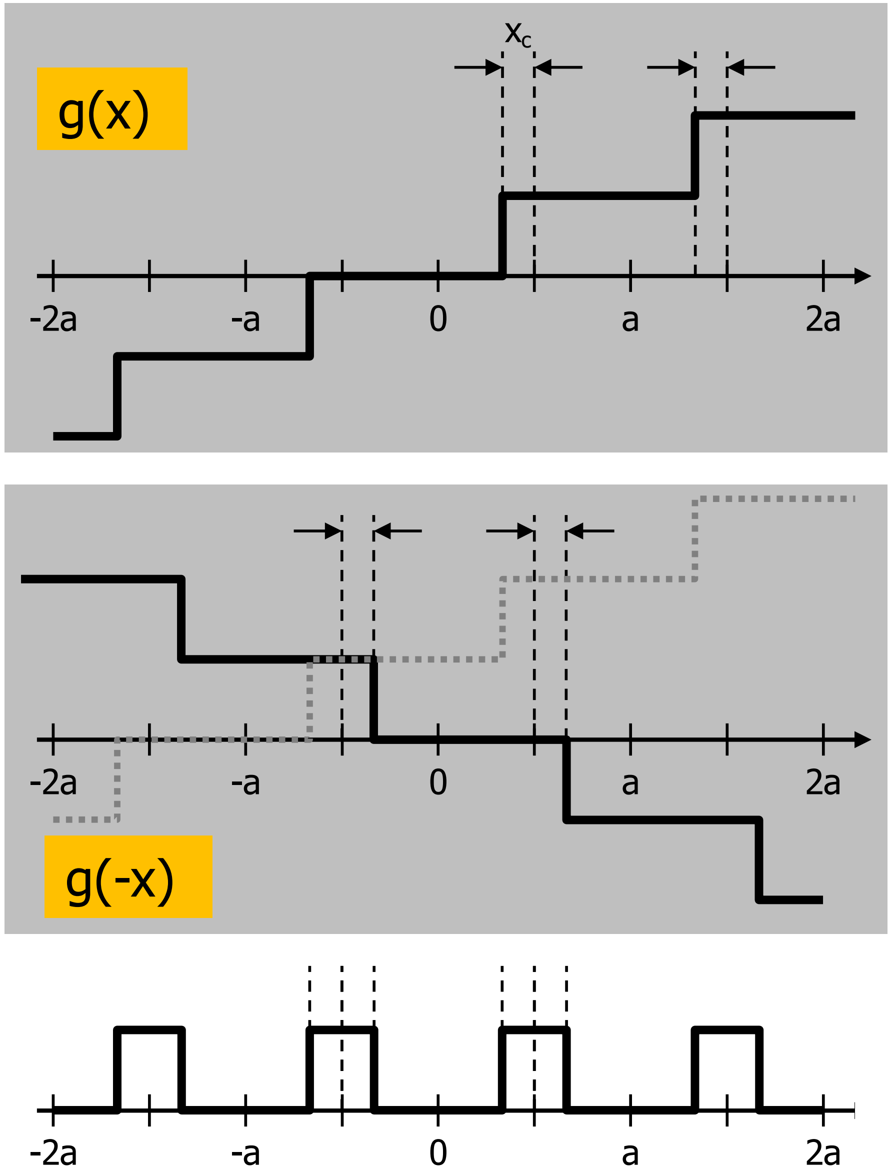
$$x_{rek}(x_c) = \sum_m ma \cdot S_m(x_c)$$

$$= a \int f(x) \underbrace{\sum_m m \cdot \mathrm{Box_a}(x + x_c - ma)\, dx}$$

$$= a \int f(x)\, g(x)\, dx$$

Staircase = g(x)

3. Divide Staircase in sym. / antisym. parts



$$g(x) = g_{\mathrm{sym}}(x) + g_{\mathrm{antisym}}(x)$$

$$g_{\mathrm{sym}}(x) = \frac{g(x) + g(-x)}{2}$$

NB: this is only valid for $a-2x_c > 0$, i.e. $x_c < a/2$. This will be sufficient.

$$g_{\mathrm{sym}}(x) = \frac{1}{2} - \frac{1}{2} \sum_m \mathrm{Box}_{\mathrm{a-2x_c}}(x - ma)$$

4. Simplify the integrals. Move to Fourier Space

- Integral of $g_{\mathrm{antisym}}(x)$ is zero because f is assumed symmetric
- We are left with

Write Sum of Boxes as convolution of a single Box with Dirac Comb

$$x_{rek}(x_c) = a \int f(x)\, g_{\mathrm{sym}}(x)\, dx$$

$$= \frac{a}{2} - \frac{a}{2} \int f(x) \sum_{m} \mathrm{Box}_{\mathrm{a}-2\mathrm{x_c}}(x - ma)\, dx$$

$$= \frac{a}{2} - \frac{a}{2} \int f(x) \left[\mathrm{Box}_{\mathrm{a}-2\mathrm{x_c}}(x) \star \mathrm{Comb}_{\mathrm{a}}(x) \right] dx.$$

- To solve this, move to Fourier Space with

$$\tilde{f}(k) := \int f(x)\, e^{-2\pi i k x} dx$$

- We can use $\int a(x)\, b(x)\, \mathrm{d}x = \int \tilde{a}(k)\, \tilde{b}(k)\, \mathrm{d}k$ and $\widetilde{a \star b} = \tilde{a} \cdot \tilde{b}$ (for symmetrical a, b)

5. Get rid of the Integral

$$x_{rek}(x_c) = \frac{a}{2} - \frac{a}{2}\int f(x)\Big[\mathrm{Box}_{a-2x_c}(x) \star \mathrm{Comb}_a(x)\Big]\,dx$$

$$= \frac{a}{2} - \frac{a}{2}\int \widetilde{f}(k)\cdot\widetilde{\mathrm{Box}}_{a-2x_c}(k)\cdot\widetilde{\mathrm{Comb}}_a(k)\,dk$$

This is again a Dirac Comb, i.e. a **sum** of peaks at distances 1/a

sinc

integral can be carried out.
Sum is left

$$= \frac{a}{2} - \frac{a}{2}\sum_{m=-\infty}^{\infty}\widetilde{f}\left(\frac{m}{a}\right)\frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m\pi}$$

6. Use Symmetry, Simplify the Sin() function

$$x_{rek} = \frac{a}{2} - \frac{a}{2} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m\pi}$$

Treat m=0:
~f(0) = 1,
Sin(m k)/m → k
-> 1-2x$_c$/a

$$= x_c - \frac{a}{\pi} \sum_{m=1}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m}$$

Use symmetry
m ⇔ -m.

Center position
x$_c$ shows up !

$$\begin{aligned} \sin(m\pi - x) &= \sin(m\pi)\cos(x) - \cos(m\pi)\sin(x) \\ &= -(-1)^m \sin(x) \end{aligned}$$

$$\boxed{x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_c}{a}\right)}$$

PUH…..

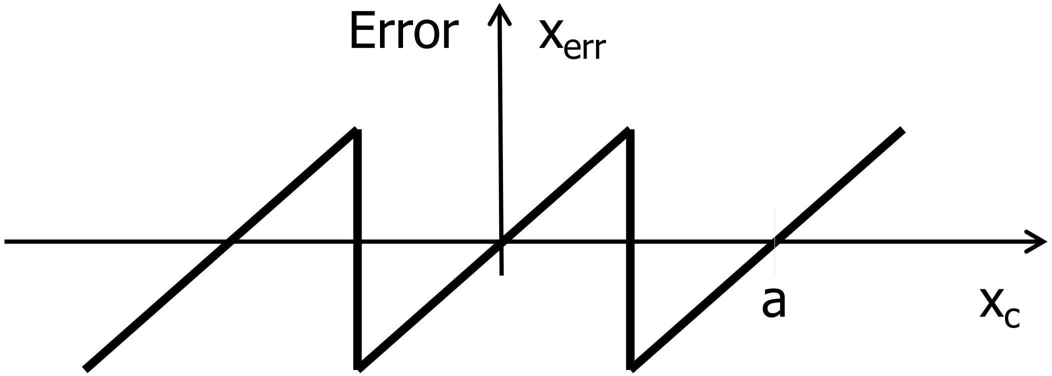
A First Check

$$x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_c}{a}\right)$$

- For very narrow f(x), f(x) → Dirac(x) and therefore $\tilde{f}(k) \to 1$ so that

$$x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin\left(m\pi \frac{2x_c}{a}\right)$$

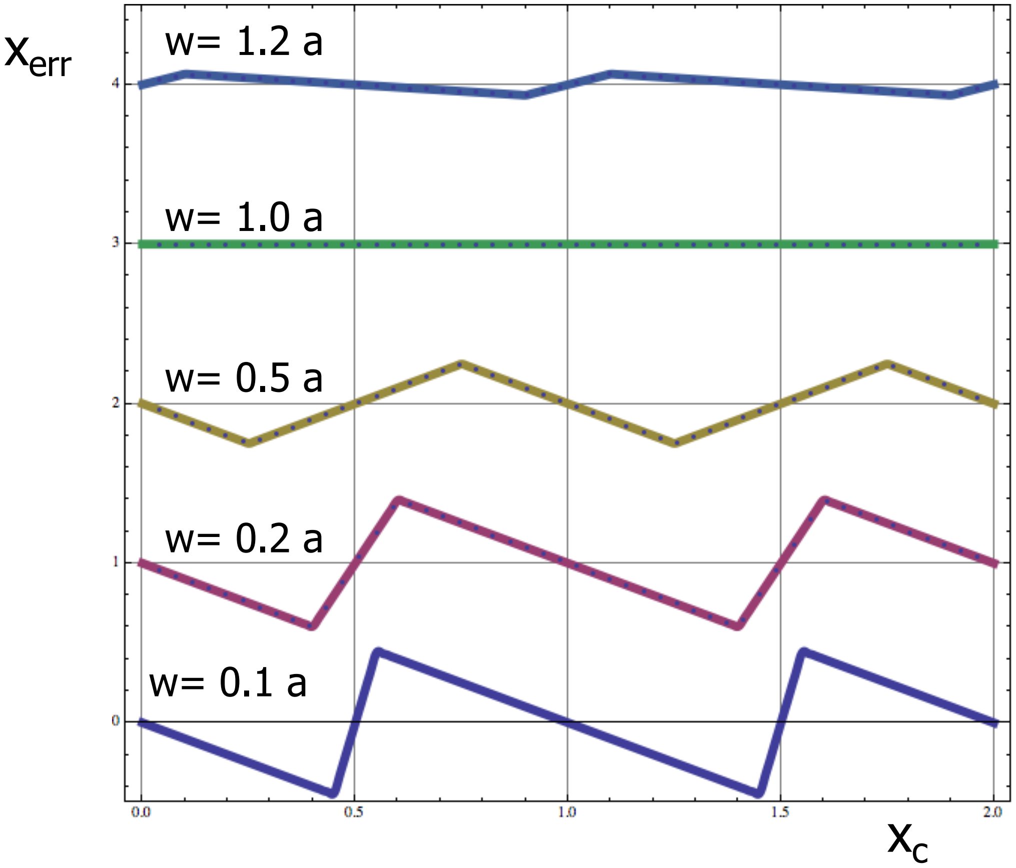
- This is the Fourier Series of a Saw-Tooth, as expected!

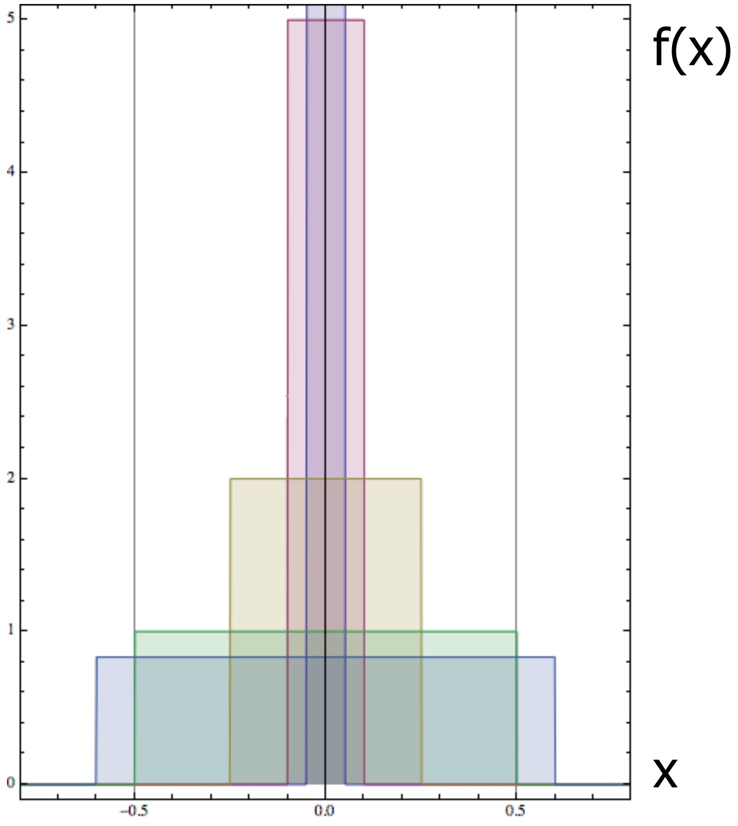


Check with f(x) = Box

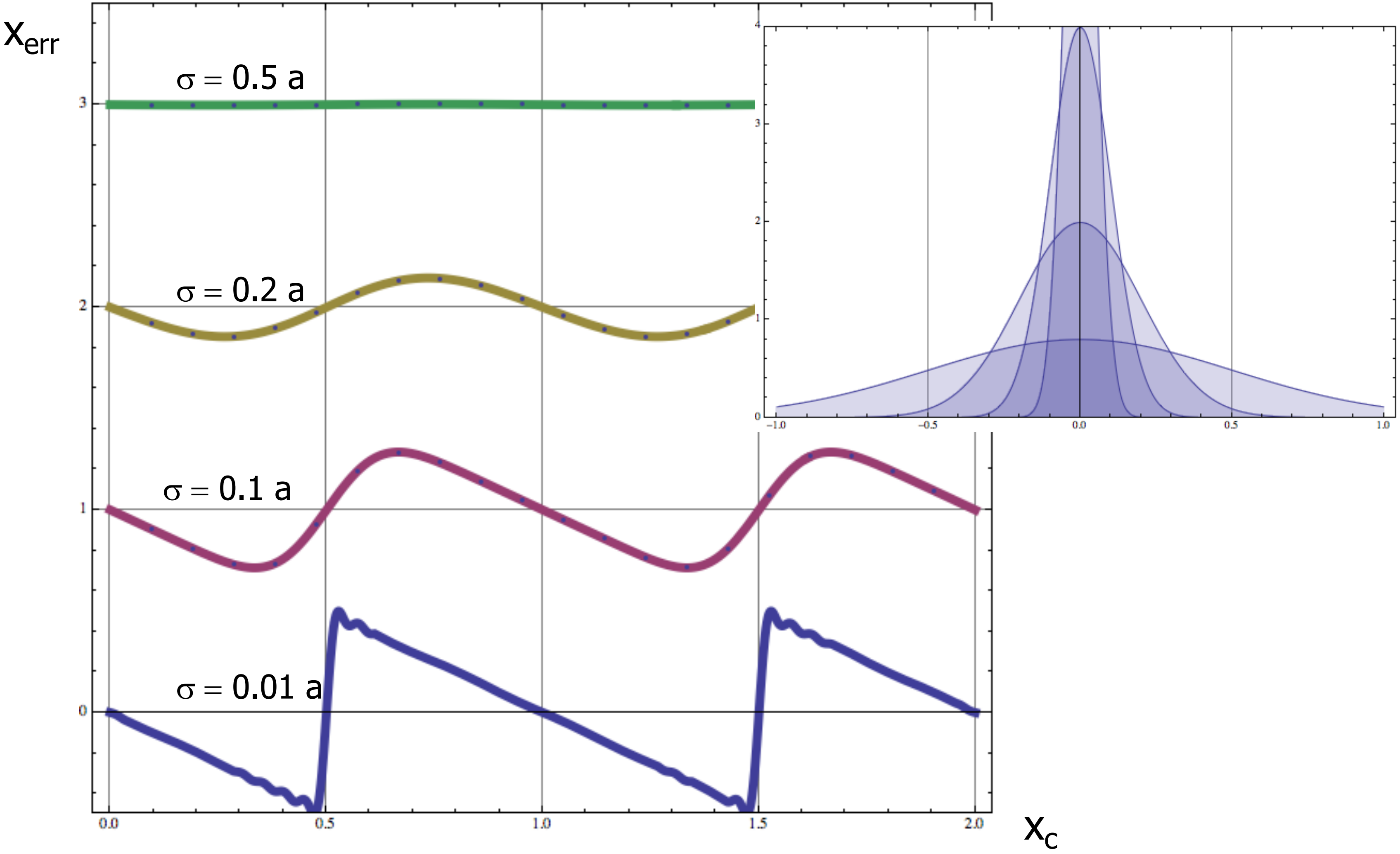
$$x_{err}(x_c) = \frac{a}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\tilde{f}\left(\frac{m}{a}\right)\sin\left(\frac{2\pi m x_c}{a}\right)$$

- For a box of width a, $\tilde{f}\left(\frac{m}{a}\right) = \frac{\sin m\pi}{m\pi}$ is zero for m∈N.
  → reconstruction is perfect. Same for width = multiple of a.





Check with Gaussians



- Error already very small for $\sigma = 0.5$a

Going Further: **Sigma** of $x_{\rm err}$?

$$x_{err}(x_c) = \frac{a}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\tilde{f}\left(\frac{m}{a}\right)\sin\left(\frac{2\pi m x_c}{a}\right)$$

$$\begin{aligned}
\sigma^2_{\rm rec} &= \frac{1}{a}\int_{-a/2}^{a/2} x^2_{\rm err}(x_{\rm c})\,dx_{\rm c} \\
&= \frac{a}{\pi^2}\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\tilde{f}\left(\frac{n}{a}\right)\tilde{f}\left(\frac{m}{a}\right)\underbrace{\int_{-a/2}^{a/2}\sin\frac{2\pi n x_{\rm c}}{a}\sin\frac{2\pi m x_{\rm c}}{a}\,dx_{\rm c}} \\
&= \frac{a}{\pi^2}\sum_{n,m=1}^{\infty}\frac{\cancel{(-1)^{n+m}}}{nm}\tilde{f}\left(\frac{n}{a}\right)\tilde{f}\left(\frac{m}{a}\right)\frac{a}{2}\boxed{\delta_{n,m}}
\end{aligned}$$

sin() are orthorgonal!

$$\boxed{\sigma^2_{\rm rec} = \frac{a^2}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\tilde{f}^2\left(\frac{m}{a}\right).}$$

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Check This for Narrow Signal

$$\boxed{\sigma^2_{\rm rec} \;=\; \frac{a^2}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\,\tilde{f}^2\left(\frac{m}{a}\right).}$$

- For very narrow signals, we have again $\tilde{f}(k) \to 1$ so that

$$\left(\frac{\sigma_{\rm err}}{a}\right)^2 = \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{1}{12}$$

as expected….

$\pi^2/6$

- This is probably the most complicated way to get the 1/12…

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f(x) = Gaussian(x) or Box(x)

- For a Gaussian signal with width σ

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{x^2}{2\sigma_s^2}\right) \quad \text{with} \quad \tilde{G}(k) = \exp\left(-2\pi^2 k^2 \sigma_s^2\right)$$

we get

$$\boxed{\left(\frac{\sigma_{\mathrm{err}}}{a}\right)^2 = \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\exp\left(-\frac{4m^2\pi^2\sigma_s^2}{a^2}\right)}{m^2}}$$

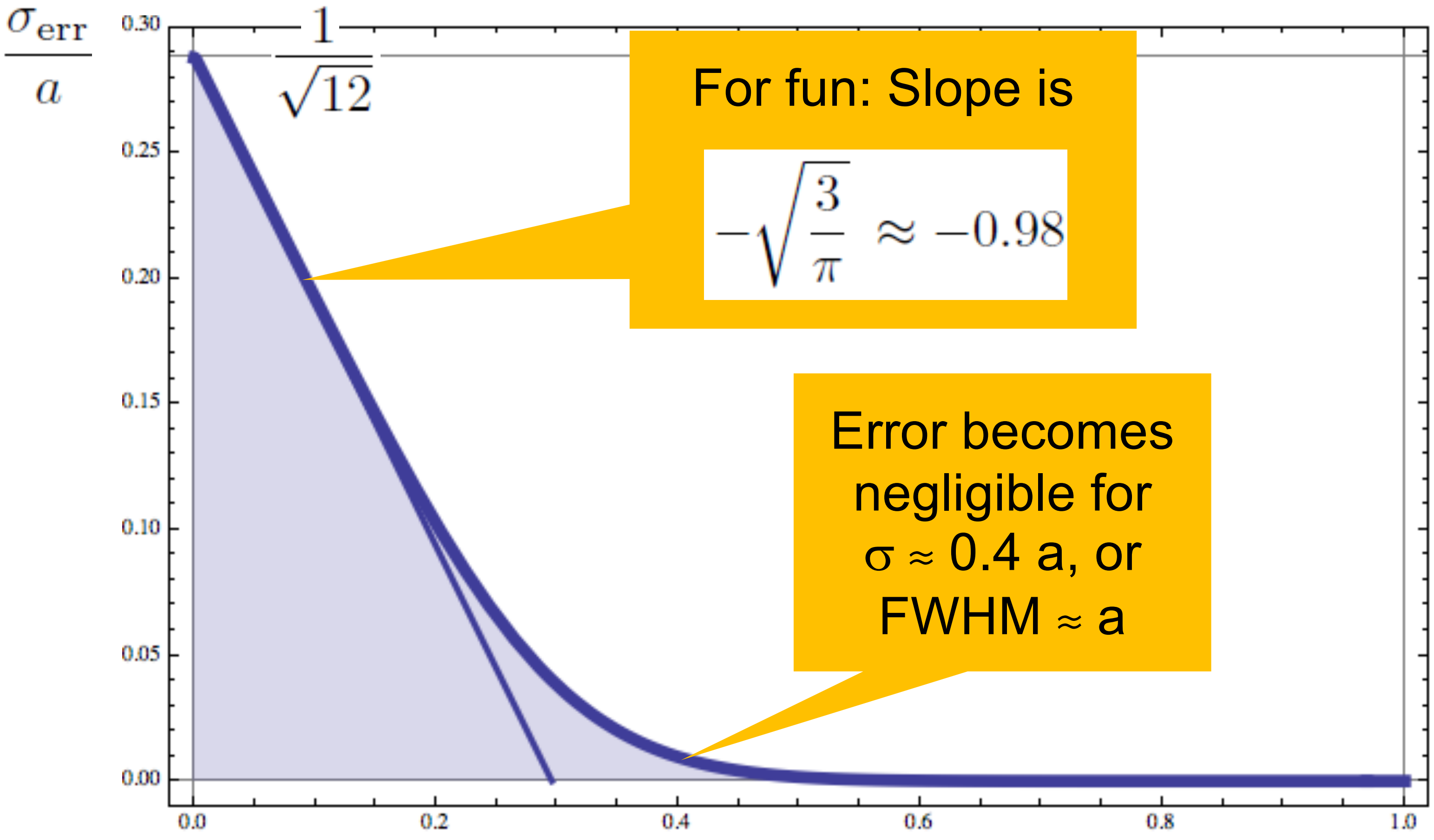
- For a Box of width s·a:

$$\left(\frac{\sigma_{\mathrm{err}}}{a}\right)^2 = \frac{1}{360 s^2} - \frac{1}{4\pi^4 s^2} \sum_{m=1}^{\infty} \frac{\cos(2\pi m s)}{m^4}$$

=1/48 for s=0.5
=0 for s integer

- For integer width s, cos(..)=1, so the sum is not 0…
- But σ becomes zero thanks to $\sum_m (1/m^4) = \pi^4/90$..

Plot this for f(x) = Gauss(x)



For fun: Slope is

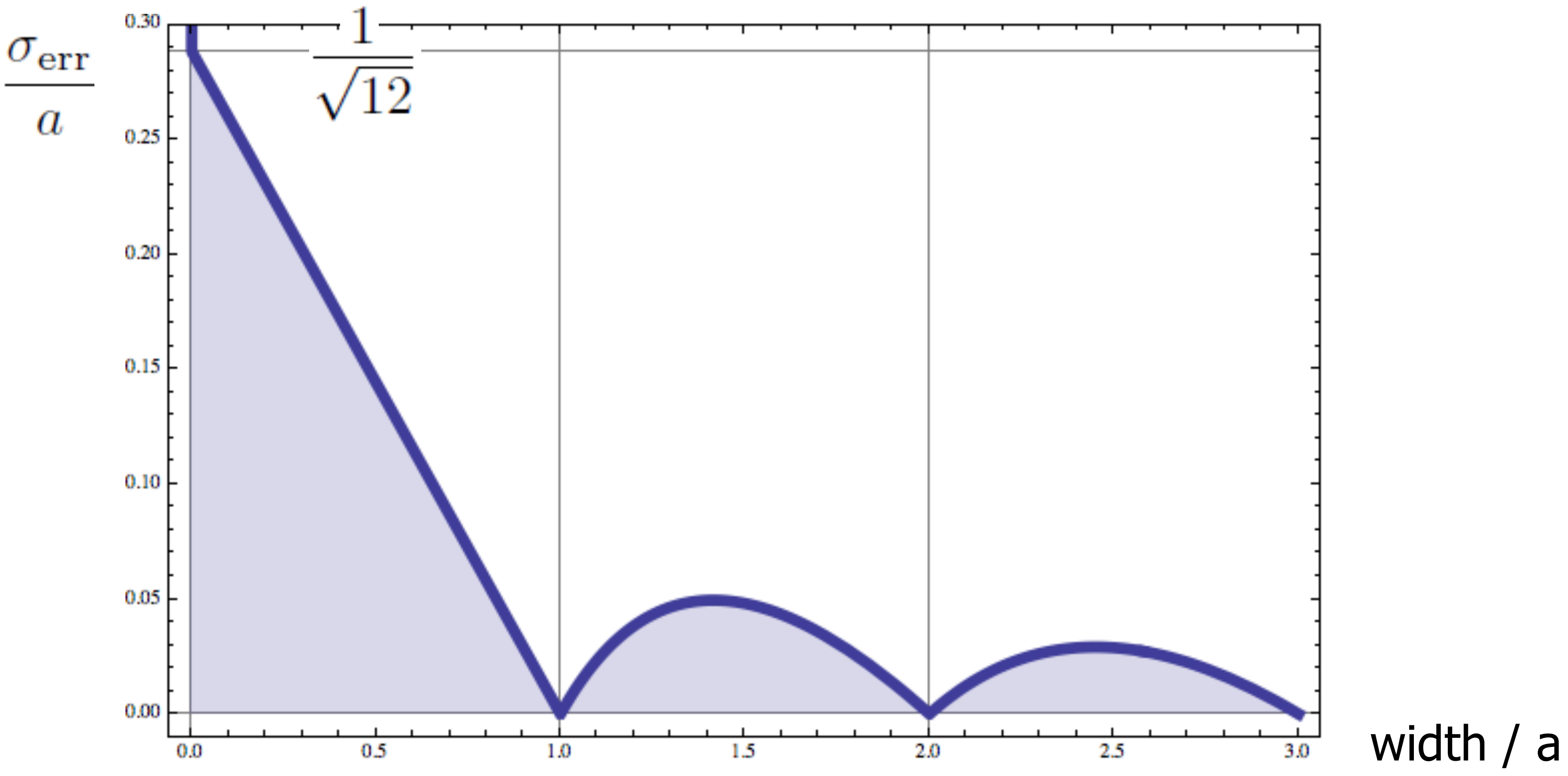
$$-\sqrt{\frac{3}{\pi}} \approx -0.98$$

Error becomes negligible for σ ≈ 0.4 a, or FWHM ≈ a

The result ‘Error ≈ 0 for FWHM ≈ a’ can be found for many pulse shapes. We knew this… but now we know *for sure*…

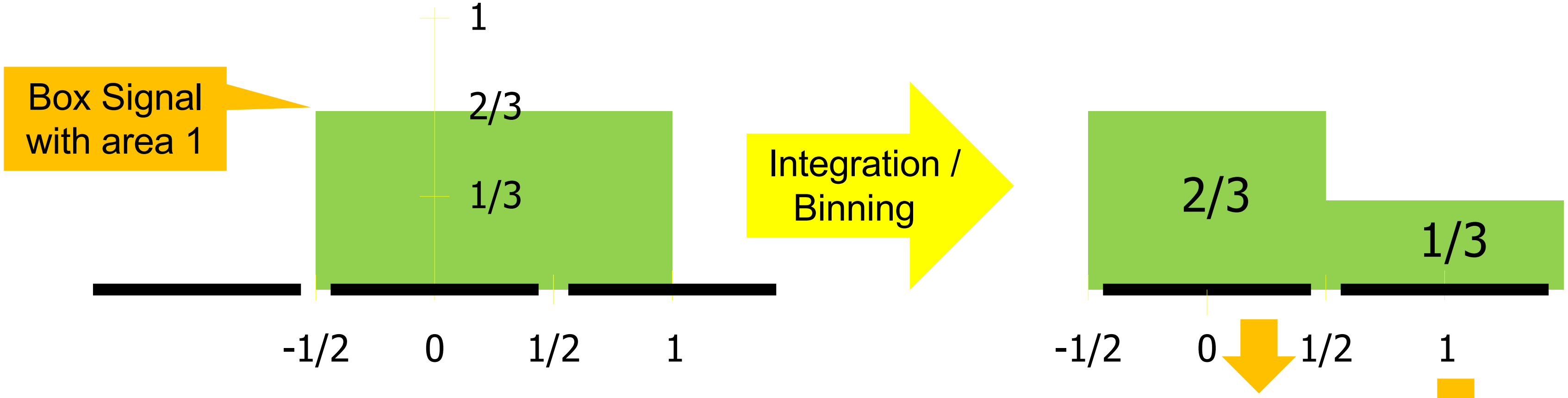
Plot this for f(x) = Box(x)

- Error is zero for integer box width.
- Behavior in-between is not trivial (see next slide)…



Understanding the BOX-Behavior

- Why does the error → 0 for wider Gauss while it is ≠ 0 also for wide boxes?
  - This very reasonable question has been asked after the talk.
- We consider an example case:
  $a = 1$, $b = 1.5$, $x_c = 0.25$



  - The central part has weight 2/3 and position 0
  - The right part has weight 1/3 and position 1
    → reconstructed position is 1/3 and NOT 0.25
- This shows: the ‘central parts’ of the box carry no information, the edges are badly assigned to bins

Noise

- This was for an ideal, noise-free case.
- The ‘reconstruction error’ was systematical, or from insufficient knowledge (small box)
- But even for wide signals with ‘good’ shape, NOISE will degrade the reconstruction

LIMIT OF SPATIAL RESOLUTION FROM NOISE

The Question

- How is spatial resolution degraded by noise?
- We all ‘know’ $\sigma_{\mathrm{err}} = \kappa \cdot \sigma_{\mathrm{n}} = \dfrac{\kappa}{SNR}$.

  This states, that the resolution degrades with noise ‘linearly to first order’.
- The proportionality κ is empirical. We want to *calculate* it
- We also want to check what happens with *correlated noise*
- We want to see what happens to *higher order*
  - What is this here? It is the distribution of the noise…
- We assume we can reconstruct with CoG (more later…)
- We restrict on a 1D treatment, but 2D is straight forward

1. Write down x_{rek} with noise

- A Signal at $\vec{x}$ is distributed over N strips at positions $\vec{x}_{\mathrm{i}}$
- Signal on i-th strip is $S_{\mathrm{i}}(\vec{x})$
- The sum of all signals shall be normalized to 1 (‘trivial’):

$$\sum S_{\mathrm{i}} = 1$$

①

- Assume we can perfectly reconstruct the position as center of gravity:

$$\vec{x} = \frac{\sum S_{\mathrm{i}}\vec{x}_{\mathrm{i}}}{\sum S_{\mathrm{i}}} = \sum S_{\mathrm{i}}\vec{x}_{\mathrm{i}}$$

②

- ***Now*** assume noise n_{i} on all strips, signals are then S_i+n_i
- the reconstructed position is:

$$\vec{x}_{\mathrm{rek}}(\vec{x}) = \frac{\sum(S_{\mathrm{i}}+n_{\mathrm{i}})\vec{x}_{\mathrm{i}}}{\sum(S_{\mathrm{i}}+n_{\mathrm{i}})} = \frac{\vec{x}+\sum n_{\mathrm{i}}\vec{x}_{\mathrm{i}}}{1+\sum n_{\mathrm{i}}}$$

2. Assume noise is small. Get the standard dev.

- This becomes (Trick: Taylor Expansion of Denominator):

$$\vec{x}_{\mathrm{rek}}(\vec{x}) = \frac{\vec{x} + \sum n_{\mathrm{i}}\vec{x}_{\mathrm{i}}}{1 + \sum n_{\mathrm{i}}} = \left(\vec{x} + \sum n_{\mathrm{i}}\vec{x}_{\mathrm{i}}\right)\left(1 - \sum n_{\mathrm{i}} + \mathcal{O}(n^2)\right)$$

- The reconstruction *error* (x_{err}=x_{rek}-x) is:

$$\vec{x}_{\mathrm{err}}(\vec{x}) = \sum_i n_{\mathrm{i}}(\vec{x}_{\mathrm{i}} - \vec{x}) + \mathcal{O}(n^2).$$

- We need the standard deviation:

$$\sigma_{\mathrm{err}}^2 = \langle \vec{x}_{\mathrm{err}}^2 \rangle - \langle \vec{x}_{\mathrm{err}} \rangle^2$$

Average error is zero!

We need to average over
- ALL possible positions x
- ALL noise values

3. Do the averaging

$$\sigma_{\mathrm{err}}^2 = \langle \vec{x}_{\mathrm{err}}^2 \rangle$$

$\vec{x}_{\mathrm{err}}(\vec{x}) = \sum_i n_{\mathrm{i}}(\vec{x}_{\mathrm{i}} - \vec{x}) + \mathcal{O}(n^2).$

$$= \sum_{i,j} \langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle \langle (\vec{x}_{\mathrm{i}} - \vec{x})(\vec{x}_{\mathrm{j}} - \vec{x}) \rangle + \langle \mathcal{O}(n^3) \rangle$$

For **un**correlated noise

$\langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle = \delta_{ij} \cdot \sigma_{\mathrm{n}}^2$

$$= \sigma_{\mathrm{n}}^2 \cdot \sum_i \langle (\vec{x}_{\mathrm{i}} - \vec{x})^2 \rangle + \mathcal{O}(\sigma_{\mathrm{n}}^3)$$

- If we chose the origin such that

$$\sum_i \vec{x}_{\mathrm{i}} = \vec{0}$$

3

this simplifies to:

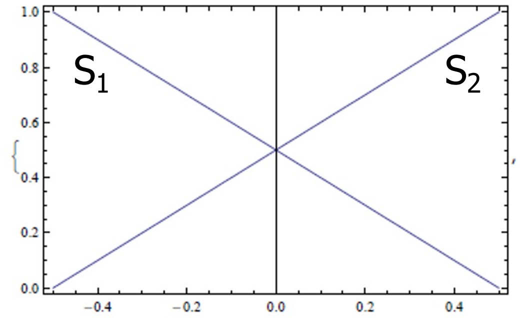
This is κ² !

$$\boxed{\sigma_{\mathrm{err}}^2 = \sigma_{\mathrm{n}}^2 \left(\sum_{i=1}^{N} \vec{x}_{\mathrm{i}}^2 + N \langle \vec{x}^2 \rangle \right) + \mathcal{O}(\sigma_{\mathrm{n}}^3).}$$

Example: Two Strips with linear signal sharing

- Consider two strips at $x_1 = -a/2$ and $x_2 = +a/2$ (N = 2)
- Signals for a hit at x shall depend linearly on x:

$$S_1(x) = (x_2 - x)/a \text{ and } S_2(x) = (x + x_2)/a$$



- ①, ② and ③ are fulfilled:

$S_1 + S_2 = 1$; $x_1 S_1 + x_2 S_2 = x$; $x_1 + x_2 = 0$

- We get $$\left(\frac{\sigma_{\mathrm{err}}}{\sigma_{\mathrm{n}}}\right)^2 \approx x_1^2 + x_2^2 + \frac{2}{a}\int_{x_1}^{x_2} x^2 dx = \frac{2}{3}a^2$$

- Or $$\boxed{\sigma_{\mathrm{err}} = 0.816 \cdot a \cdot \sigma_{\mathrm{n}}}$$

$$\sigma_{\mathrm{err}}^2 = \sigma_{\mathrm{n}}^2 \left(\sum_{i=1}^{N} \vec{x}_{\mathrm{i}}^{\,2} + N\langle \vec{x}^{\,2}\rangle\right) + \mathcal{O}(\sigma_{\mathrm{n}}^3).$$

50 µm pitch
S/N =10
-> σ=4 µm
(FWHM = 10 µm)

- For σ_n **= 0.1** (Signal/Noise = 10), **resolution = 8% · a**
- Resolution is better than optimal binary readout for **S/N>5.6**

Correlated Noise ?

- For FULLY correlated noise, $n_i = n_j$ and $\langle n_i n_j \rangle = \sigma_n^2$
- We get $\sigma_{err}^2 \approx \sigma_n^2 N^2 \langle \vec{x}^2 \rangle$
- For the strip example

$$\sigma_{err} = a\ \sigma_n / \sqrt{3} = 0.57\ a\ \sigma_n \quad \text{(instead of 0.816..)}$$

- **Correlated noise is less harmful** than 'normal' noise
- Note: For mixed noise, superimpose both components
- Note: If the Amplitude of the signal is *KNOWN* (X-ray), **noise becomes correlated** and resolution improves!

Higher Orders (in noise)

- Noise can have different distributions.
- They have different higher moments:

$$\begin{aligned} \langle n_i^2 \rangle &= \sigma_\mathrm{n}^2, \\ \langle n_i^4 \rangle &= \beta \cdot \sigma_\mathrm{n}^4. \end{aligned}$$

- They are $\beta := \frac{\int n^4\, p(n)\, dn}{\left(\int n^2\, p(n)\, dn\right)^2} = \begin{cases} 3 : \mathrm{Gauss} \\ 9/5 : \mathrm{Box} \\ 1 : \mathrm{Peaks} \end{cases}$

- We need then higher order correlations (not trivial..):

$$\begin{aligned} \langle n_\mathrm{i} \rangle &= 0 \\ \langle n_\mathrm{i} n_\mathrm{j} \rangle &= \delta_{ij} \sigma_\mathrm{n}^2 \\ \langle n_\mathrm{i} n_\mathrm{j} n_\mathrm{k} \rangle &= 0 \\ \langle n_\mathrm{i} n_\mathrm{j} n_\mathrm{k} n_\mathrm{l} \rangle &= \delta_{ij}\delta_{jk}\delta_{kl}\,(\beta - 3)\,\sigma_\mathrm{n}^4 \quad + (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\sigma_\mathrm{n}^4 \end{aligned}$$

Higher Orders

- Repeating the derivation yields

$$\sigma_{\mathrm{err}}^2 = \sigma_{\mathrm{n}}^2 \cdot (A + N\,B)$$
$$\cdot \left(1 + 3\left[\beta - 3 + N\frac{A + 3\,N\,B}{A + N\,B}\right]\sigma_{\mathrm{n}}^2\right)$$

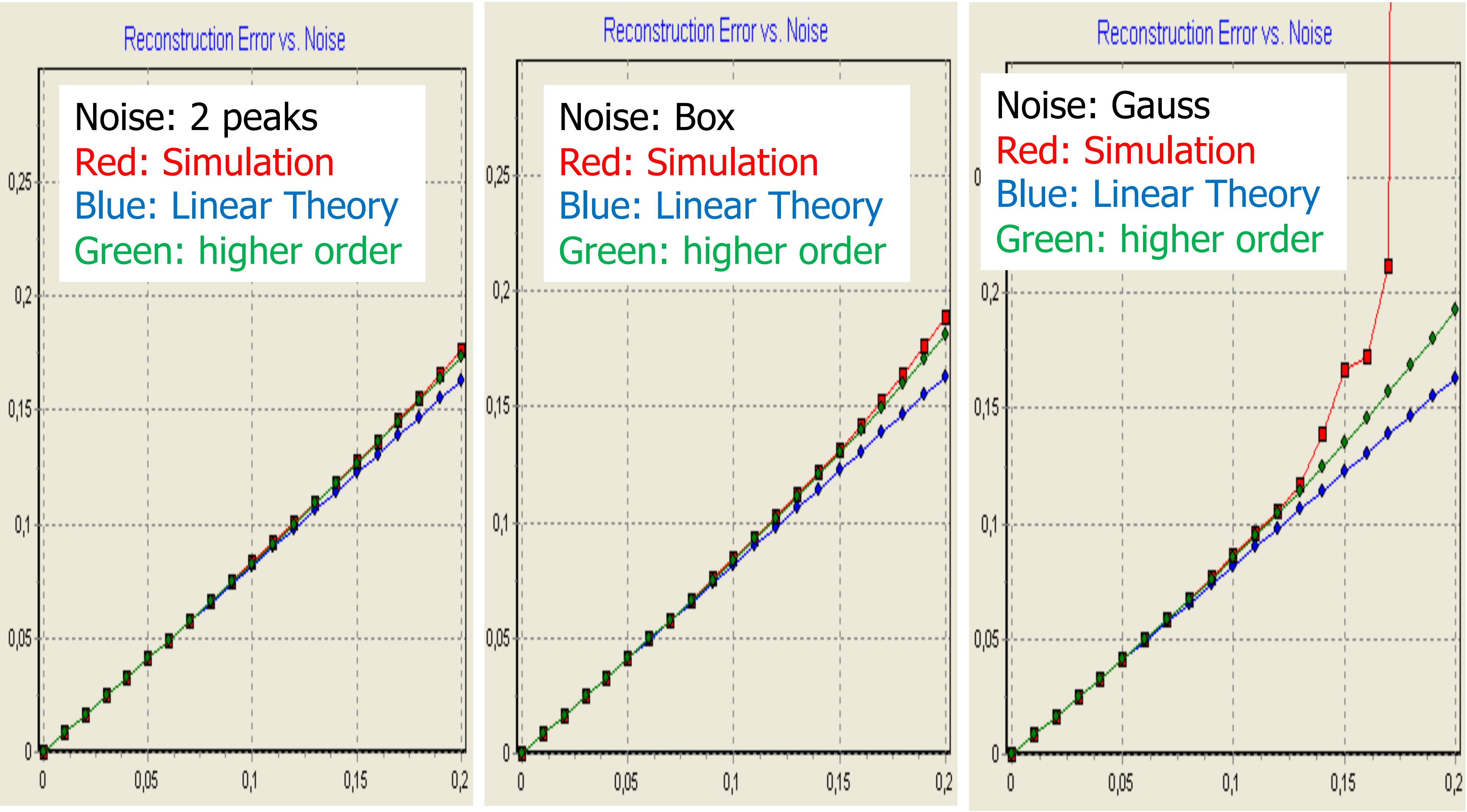
Previous result

Correction

- Only the *correction* depends on the ‘type’ (shape) of noise.

- Remember:
  - For small noise, **there is *no need* to simulate Gaussian noise**
  - Randomly adding or subtracting ± σ_n has the **same** effect!

Is this true? → Small Monte Carlo: Error vs. Noise



- Reconstruction for Gauss noise fails completely in few cases due to very high noise values

2D Structures

- Can be treated similarly
- Observations:
  - Small number of electrodes is good
  - Well confined acceptance is good (‘circle’)

$$\sigma_{\mathrm{err}}^2 = \sigma_{\mathrm{n}}^2 \left(\sum_{i=1}^{N} \vec{x}_{\mathrm{i}}^{\,2} + N \langle \vec{x}^{\,2} \rangle \right) + \mathcal{O}(\sigma_{\mathrm{n}}^3).$$

Geometry	σ_{1D} theory linear	 correction	Value (A=p=1)
strips	$\sqrt{\frac{2}{3}} \cdot p$	$\sqrt{1 + 3\beta\sigma_{\mathrm{n}}^2}$	0.8165
square	$\frac{2}{\sqrt{3}} \cdot \sqrt{A}$	$\sqrt{1 + 3(3+\beta)\sigma_{\mathrm{n}}^2}$	1.1547
hexagon	$\frac{\sqrt{5} \cdot 3^{1/4}}{6} \cdot \sqrt{A}$	$\sqrt{1 + 3\left(\frac{6}{5} + \beta\right)\sigma_{\mathrm{n}}^2}$	0.4905

- **Hexagons are best** (least sensitive to noise!)

BACK TO COG
NOW WITH NOISE

Problems with Centroid

- Resolution for small σ is bad → make f(x) wide

- BUT: Summing up many strips creates increasing noise

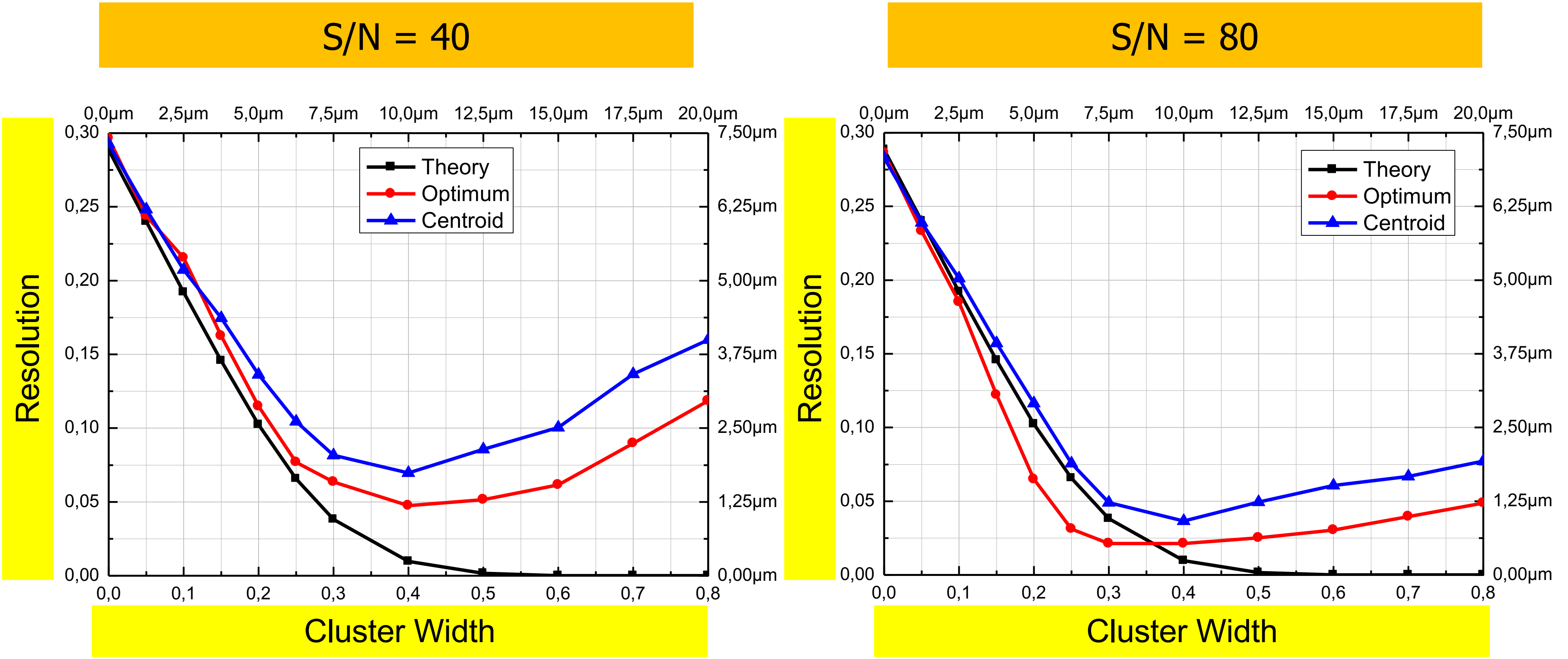
- Must chose N small but such that reconstruction is ‘just’ ok.

- The choice is fairly arbitrary
- And:
  - In real system, there is often a **threshold** (hits below this are not read out)
  - The reconstructed amplitude is wrong (signals below threshold are lost)
  - Broken pixels need special treatment

Monte Carlo Simulation

- Many possibilities… I do not go in details

Ignore red curves…

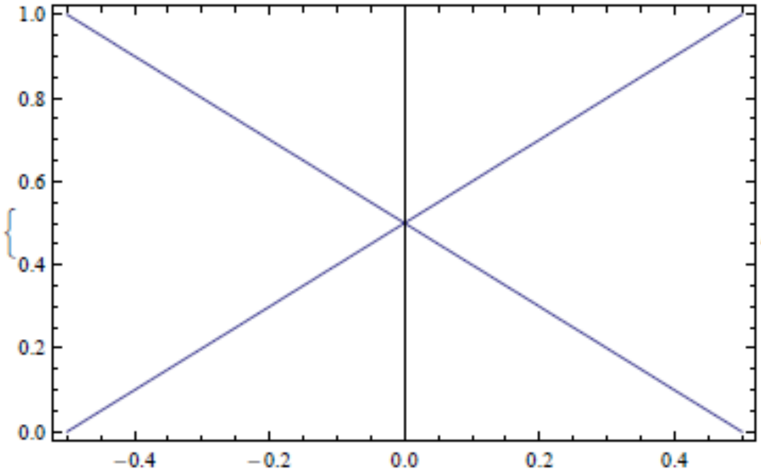


- Error does not go to 0 for wide signals when we have noise.
- The **optimum signal width is still close to FWHM = a!**

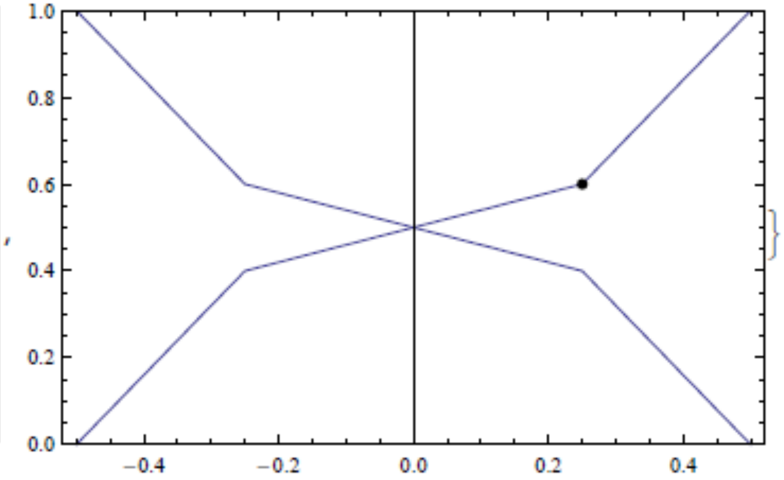
ETA FUNCTION

Motivation

- Often the Signals Distribution function (e.g. on 2 strips) is not linear.
- This is related to the ‘famous’ eta-function.



Linear Charge Sharing



Arbitrary function

- The position then cannot be calculated by CoG, but by using the inverse function (or the ‘eta’-lookup table)
- Question: How does resolution depend on f(x)?

1D Case: Reconstruction with Inverse Function

- The signals on the two strips shall be

$$\begin{aligned} S_1(x) &= Q\, f(x) \\ S_2(x) &= Q - S_1(x) = Q\,(1 - f(x)) \end{aligned}$$

(we assume no signal is lost, i.e. we require S_1+S_2 = Q

- We require
  - f(x) is strictly monotonic (obvious)
  - f(x) shall be symmetric in x (may not always be the case)

- Obviously

$$x_{rek} = f^{-1}\left[\frac{S_1}{S_1 + S_2}\right]$$

Adding Noise

- With Noise on S_1 and S_2 we get

$$
\begin{aligned}
x_{rek} &= f^{-1}\left[\frac{f(x)+n_1}{1+n_1+n_2}\right] \\
&\approx f^{-1}\left[(f(x)+n_1)(1-n_1-n_2)\right] \\
&\approx f^{-1}\left[f(x)+n_1(1-f(x))-n_2 f(x)\right] \\
&\approx x+\boxed{\left.\frac{\mathrm{d}f^{-1}(s)}{\mathrm{d}s}\right|_{f(x)}}\cdot\left[n_1(1-f(x))-n_2 f(x)\right]
\end{aligned}
$$

$$
x_{err} = \frac{n_1(1-f(x))-n_2 f(x)}{\boxed{f'(x)}}
$$

Add noise

Taylor (as before)

Only 1st order in noise

Taylor Series for f^{-1} around f(x)

A ‘forgotten’ math theorem:
The derivative of the inverse function is the inverse of the derivative

Sigma – Averaging over Noise

- To get

$$x_{err} = \frac{n_1(1-f(x)) - n_2 f(x)}{f'(x)}$$

$$\sigma^2_{err} = \langle x^2_{err} \rangle - \cancel{\langle x_{err} \rangle}^2$$

- we average first over noise. We get

$$\sigma^2_{err} = \sigma^2_n \left\langle \frac{1-2f+2f^2}{f'^2} \right\rangle + 2\,\langle n_1 n_2 \rangle \left\langle \frac{f^2-f}{f'^2} \right\rangle$$

  - Coefficients depend on the shape of the response function
  - They are small where the response function is steep (obvious..)
  - Vice versa: Flat parts in eta are bad.
- For uncorrelated noise, only the first term matters

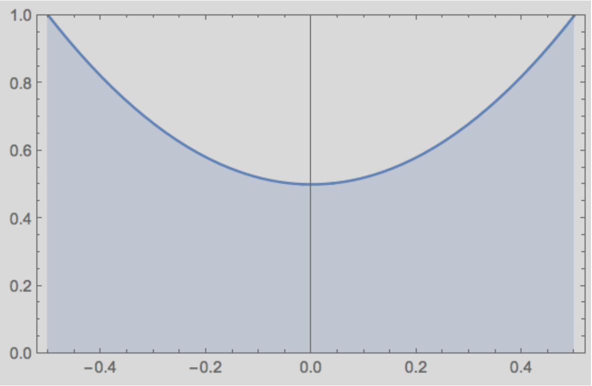
$$\frac{\sigma^2_{err}}{\sigma^2_n} = \left\langle \frac{1-2f+2f^2}{f'^2} \right\rangle$$

Average over position

Back to linear Interpolation

- What does this mean for linear interpolation, f(x) = x+0.5 ?
- Let us first look at the *position dependent* error

$$\frac{\sigma_{err}^2}{\sigma_n^2} = \frac{1 - 2f + 2f^2}{f'^2} = \frac{1}{2} + 2x^2$$



- This is NOT constant. It doubles at the edges !!!
  - When we reconstruct in the middle, we **know** the error is smaller!
- The average error is

$$\frac{\sigma_{err}^2}{\sigma_n^2} = \int_{-1/2}^{1/2} \left(\frac{1}{2} + 2x^2 \right) = \frac{2}{3}$$

as before.

Finding New Distribution functions

- Very exciting: Can we find a f(x) such that the integral is ***better*** than with linear interpolation
    - Probably not (?) But let's see…

- Easier: Can we find a distribution function so that the error is *independent of position*?
- One line of Mathematica is enough:

$$\frac{1}{4}\left(2+\left(1+\sqrt{2}\right)^{-2\,x}-\left(1+\sqrt{2}\right)^{2\,x}\right)$$

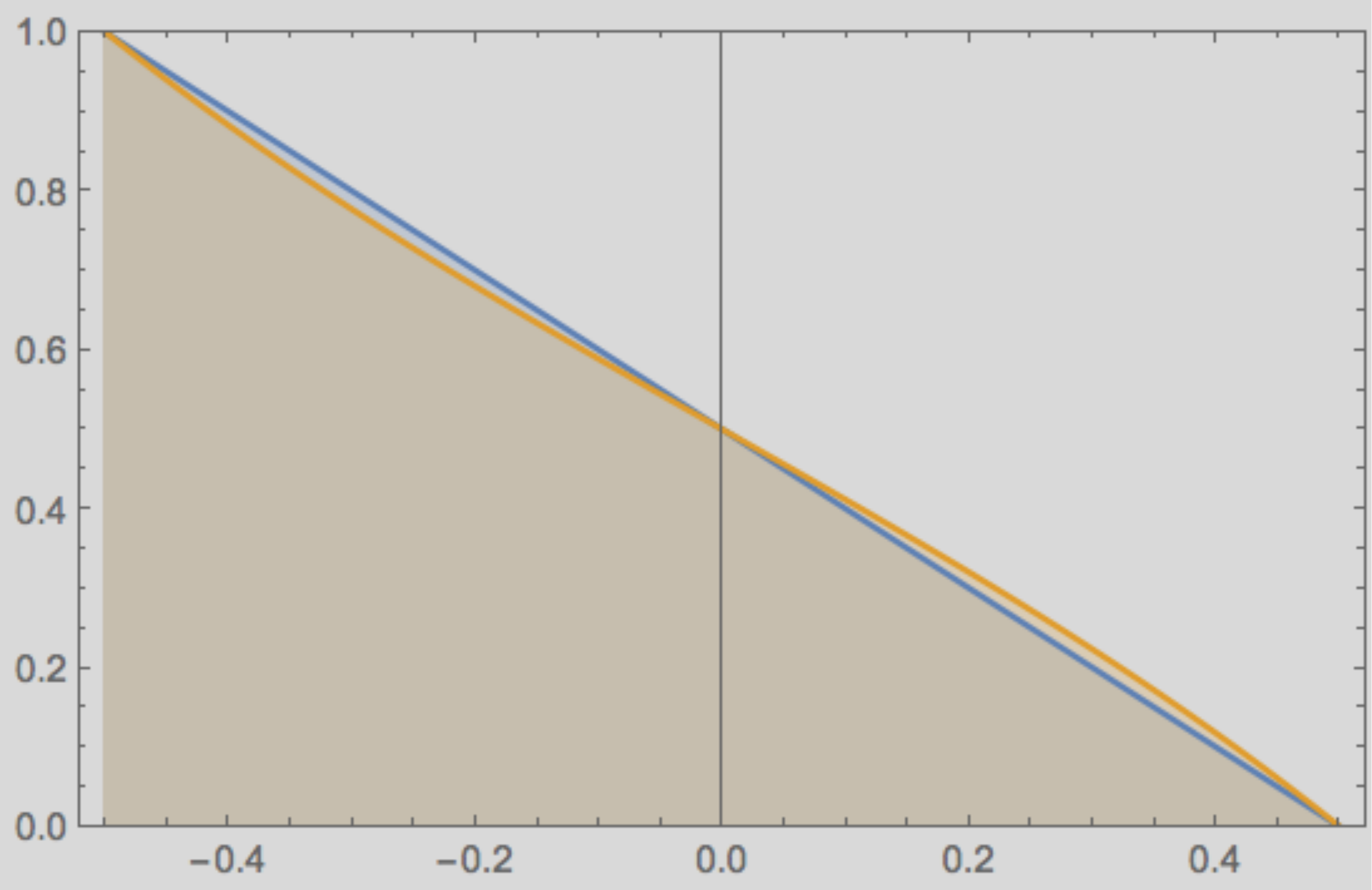
$$f_{flat}(x) = \frac{1}{2}\left(1 - \mathrm{Sinh}[2x\mathrm{ArcSinh}(1)]\right)$$

The average σ^2 is 0.643, which is (a little bit) ***better*** than 2/3=0.66 !!

We found a distribution which is better than linear interpolation!

(it is less noise sensitive)

Better! (but just a little…)



- Are there better functions???

Summary: What did we learn ?

- Basic Algebra is fun…..

- CoG is ‘perfect’ as soon as signal width >≈ strip width
- Wider (too wide) signals are more sensitive to noise
- Ideal κ for strips is 0.816
- Analogue readout for S/N<6 is useless.
- Noise shape (distribution) does not matter for S/N > 10
- Correlated noise is less harmful
- Hexagons have better res. and are less sensitive to noise
- Linear interpolation has more error at the edges (on the stips)
- There ***is*** a better reconstruction function than linear
  - but the difference is negligible….
  - I did not find better so far…

Thank you for your attention!