Thermalization of the Wigner function

— a real time, non-perturbative quantum simulation based on the Schwinger model

Shile Chen

In collaboration with Shuzhe Shi and Li Yan

2024/07/18 XQCD in Lanzhou

Outline

- Motivation
- The real-time simulation by quantum computing algorithm
- Thermalization of different systems
- ETH analysis
- Conclusion

Thermalization of QGP

QGP in heavy-ion collisions:

How does it thermalize/isotropize?

Kinetic theory	weak coupled/dilute
Hydrodynamics	local equilibrium+small perturbation

Thermalization of QGP

QGP in heavy-ion collisions:

How does it thermalize/isotropize?

Kinetic theory	weak coupled/dilute
Hydrodynamics	local equilibrium+small perturbation

- Strong coupling quantum system
- Real time Quantum Simulation

Thermalization of QGP

QGP in heavy-ion collisions:

How does it thermalize/isotropize?

Kinetic theory	weak coupled/dilute
Hydrodynamics	local equilibrium+small perturbation

- Strong coupling quantum system
- Real time Quantum Simulation
- Quantum many body system thermalize

of

Quantum Distribution function

Schwinger model

- 1+1 QED

Mimic QCD / Chiral condensate/Confinement

Chiral Condensate

Christoph Adam, Massive Schwinger Model within Mass Perturbation Theory, Ann. Phys. (N.Y.) 259, 1 (1997).

C. Adam, Normalization of the chiral condensate in the massive Schwinger model, Phys. Lett. B 440, 117 (1998).

Confinement

Giuseppe Magnifico, Marcello Dalmonte, Paolo Facchi Saverio Pascazio, Francesco V. Pepe, and Elisa Ercolessi, Real Time Dynamics and Confinement in the Zn Schwinger-Weyl lattice model for 1+1 QED, Quantum 4, 281 (2020).

CP violation

L. Funcke, K. Jansen, and S. Kuhn, "Exploring the CP-violating Dashen phase in the Schwinger model with tensor networks," Phys. Rev. D 108 no. 1, (2023) 014504, arXiv:2303.03799 [hep-lat].

Schwinger model

- 1+1 QED

Mimic QCD / Chiral condensate/Confinement

Topological θ angle

J. C. Halimeh, I. P. McCulloch, B. Yang, and P. Hauke, "Tuning the topological θ-angle in cold-atom quantum simulators of gauge theories," PRX Quantum 3 (Nov, 2022) 040316.

String breaking mechanism

Lee, Kyle and Mulligan, James and Ringer, Felix and Yao, Xiaojun, Liouvillian dynamics of the open Schwinger model: String breaking and kinetic dissipation in a thermal medium, Phys. Rev. D, 108, 9, 094518 (2023)

Schwinger model

- 1+1 QED

Mimic QCD / Chiral condensate/Confinement

Lagrangian $\mathscr{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

⇩

Hamiltonian $H = \int \left(\bar{\psi}\left(\gamma^1(-i\partial_z - gA_1) + m\right)\psi + \frac{1}{2}\mathscr{E}^2 \right) \mathrm{d}z$

⇩ Reducing infinite d.o.f with

Gauss law	$\mathscr{E}(x) = g\int_0^x \bar{\psi}\gamma^0\psi$
Gauge fixing	$A_1 = 0$

$$H = \int \left(-\bar{\psi}i\gamma^1\partial_z + m\psi + \frac{1}{2}\mathscr{E}^2 \right) \mathrm{d}z$$

Schwinger model

- **Discretization** of $H = \int \left[-\bar{\psi}(i\gamma^1\partial_z + m)\psi + \frac{1}{2}\left(g\int_0^x \bar{\psi}\gamma^0\psi \right)^2 \right] \mathrm{d}z$

Non-physical energy scale $\frac{1}{a}$

Energy scales

1+1 D QED Chain

fermion

anti-fermion

Dimension of fermion sector	2^N
Dimension of electric field sector	M
Dimension of total Hilbert space	$2^N \times M$

Hamiltonian with periodic boundary condition

$$H_{PBC} = \sum_{n=1}^{N} \left(-\frac{i}{2}\frac{1}{a}(\chi_n^\dagger e^{i\phi_n}\chi_{n+1} - \chi_{n+1}^\dagger e^{-i\phi_n}\chi_n) + (-1)^n m_0 \chi_n^\dagger \chi_n + \frac{ag^2}{2}\varepsilon_n^2 \right)$$

Schwinger model

- Gate representation

$$\chi_n = \frac{X_n - iY_n}{2}\Pi_{m=1}^{n-1}(-iZ_m)$$

$$\chi_n^\dagger = \frac{X_n + iY_n}{2}\Pi_{m=1}^{n-1}(iZ_m)$$

$$X_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y_n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Schwinger model

- Gate representation

$$\chi_n = \frac{X_n - iY_n}{2} \Pi_{m=1}^{n-1}(-iZ_m)$$

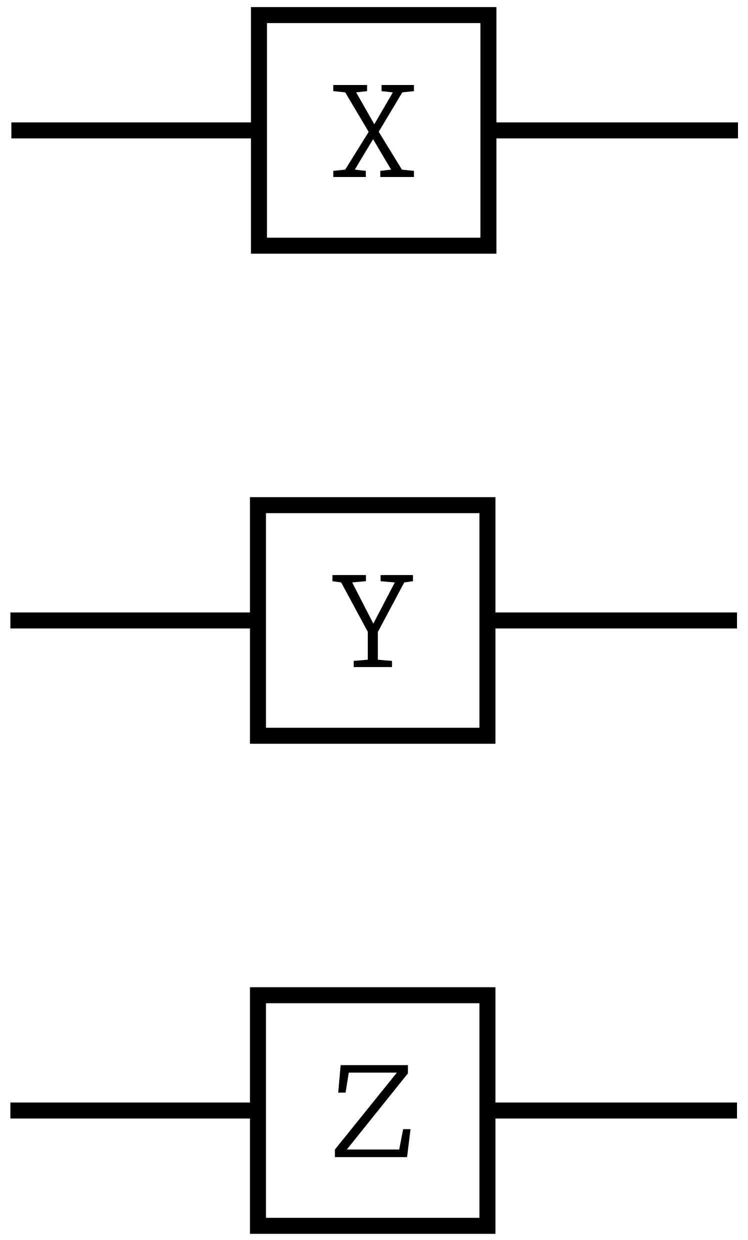
$$\chi_n^\dagger = \frac{X_n + iY_n}{2} \Pi_{m=1}^{n-1}(iZ_m)$$

$$X_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y_n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One qubit gate



Time evolution of a quantum state

- A Closed System

  Energy-eigenstates $\{|n\rangle\}$

  - Initial pure state $|\Psi\rangle_0 = \sum_n c_n |n\rangle$

    Operator $\mathcal{O}$ For estimation

Time evolution of a quantum state

- A Closed System Energy-eigenstates $\{|n\rangle\}$
  - Initial pure state $|\Psi\rangle_0 = \sum_n c_n |n\rangle$

    Operator $\mathcal{O}$ For estimation

Real-time evolution

Time evolving state $$|\Psi\rangle_t = \sum_n c_n e^{-iE_n t} |n\rangle$$

Time evolving expectation value $$\langle \Psi_t | \mathcal{O} | \Psi \rangle_t = \sum_{n,n'} c_n c_{n'}^* e^{i(E_{n'} - E_n)t} \langle n' | \mathcal{O} | n \rangle$$

Long time average $$\langle \mathcal{O} \rangle_{LTA} = \sum_n |c_n|^2 \langle n | \mathcal{O} | n \rangle$$

"Thermal" average

- **A Closed System** — Energy-eigenstates $\{|n\rangle\}$

  - Initial pure state $|\Psi\rangle_0 = \sum_n c_n |n\rangle$

    Operator $\mathcal{O}$ — For estimation

Closed system + Unitary Time evolution → Energy conservation

Canonical Ensemble average

Assuming thermal equilibrium → Inverse T β → CE of any operator

Closed system + Energy Spectrum → Density of energy state

Micro-canonical Ensemble average

→ MCE of any operator

Wigner function

- **Equal-time Wigner function** Quantum distribution function

$$W_{ab}(t,z,p) = \int \langle \Psi_t | \bar{\psi}_a(z+\frac{y}{2})\psi_b(z-\frac{y}{2}) | \Psi_t \rangle e^{ipy} dy$$

Wigner function

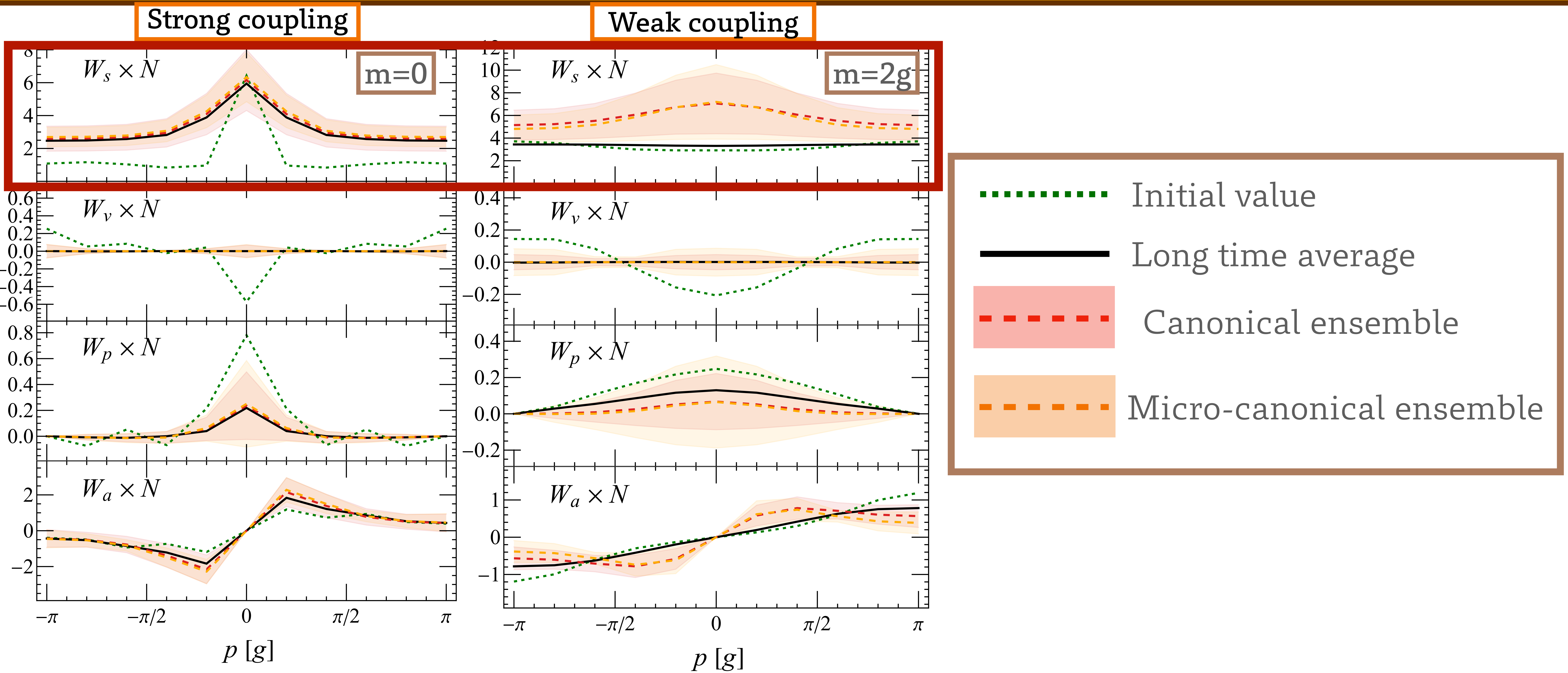
- **Equal-time Wigner function** Quantum distribution function

$$W_{ab}(t,z,p) = \int \langle \Psi_t | \bar{\psi}_a(z+\frac{y}{2})\psi_b(z-\frac{y}{2}) | \Psi_t \rangle e^{ipy} dy$$

Decomposition $W = W_s + W_v\,\gamma^0 + W_a\,\gamma^1 - iW_p\,\gamma^5$

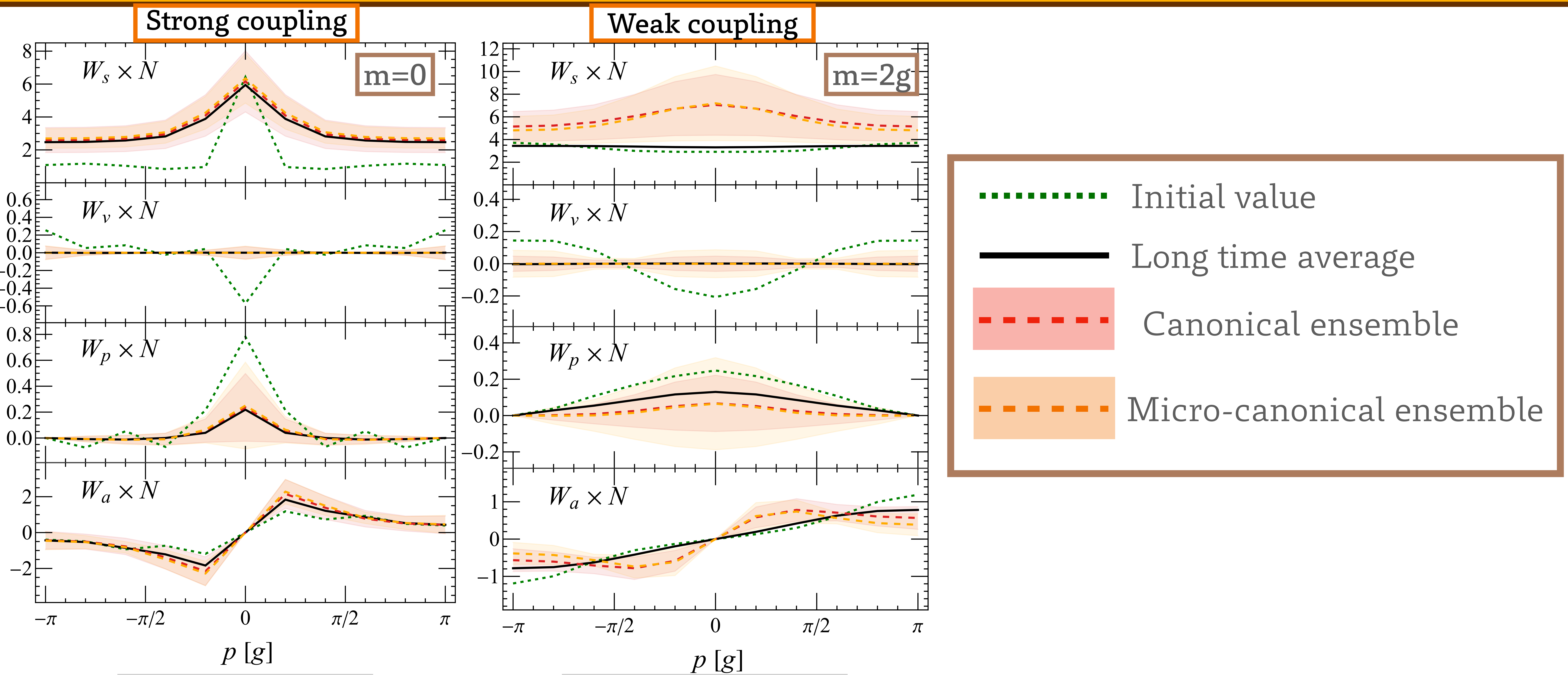
W_s :	$n_f + n_{\bar{f}}$	W_v :	$n_f - n_{\bar{f}}$
W_a :	$\chi_f + \chi_{\bar{f}}$	W_p :	$\chi_f - \chi_{\bar{f}}$

Time evolution of Wigner function



Distribution function in momentum space

Time evolution of Wigner function



Thermalize

Not Thermalize

Eigenstate Thermalization Hypothesis

- ETH

A chaotic quantum system in a finitely excited energy eigenstate **behaves thermally** when probed by typical operators

Eigenstate Thermalization Hypothesis

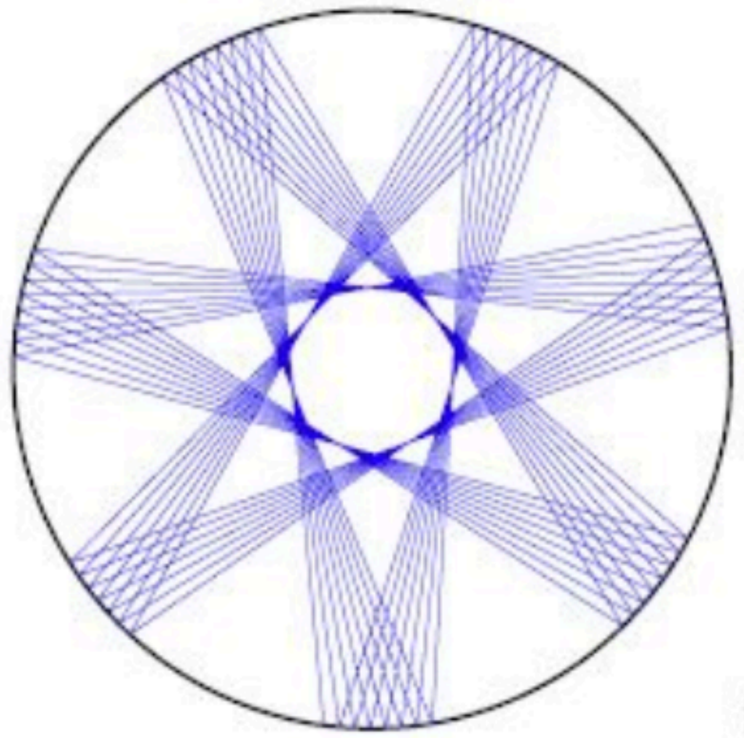
- ETH

A chaotic quantum system in a finitely excited energy eigenstate **behaves thermally** when probed by typical operators

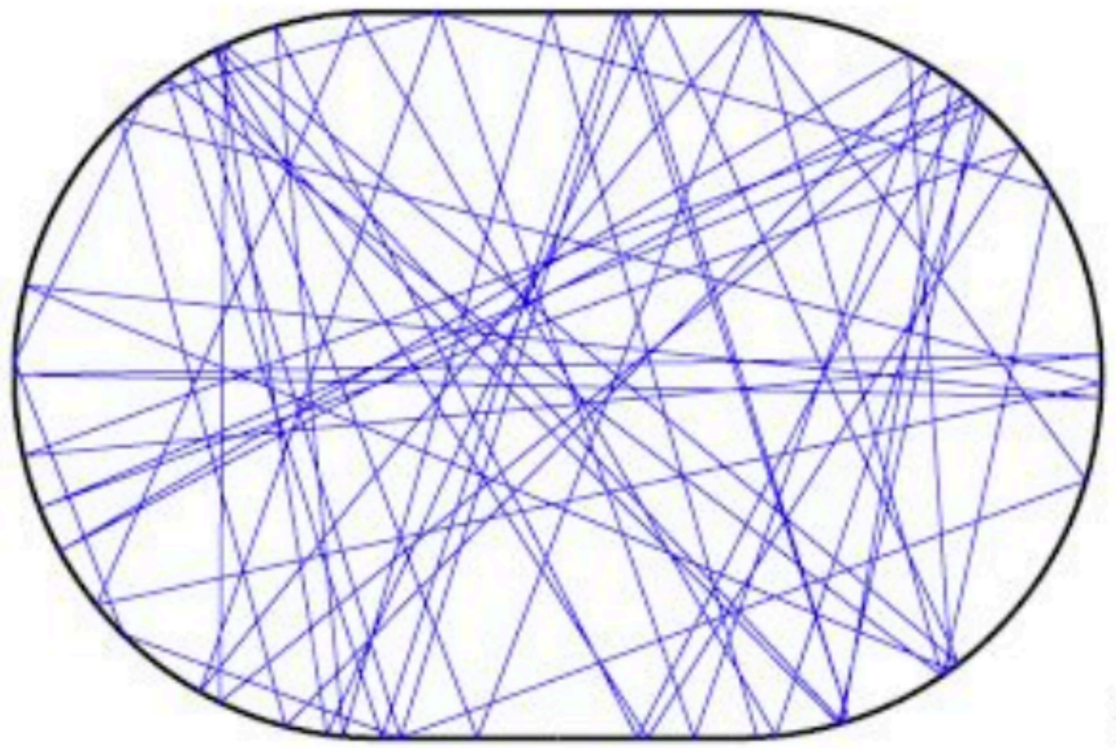
Classical point of view

Trajectories of a bouncing particle in a cavity

- Integral system
- Non-ergodic
- Non-chaotic



(a)



(b)

- Non-integral system
- Ergodic
- Chaotic Bunimovich stadium

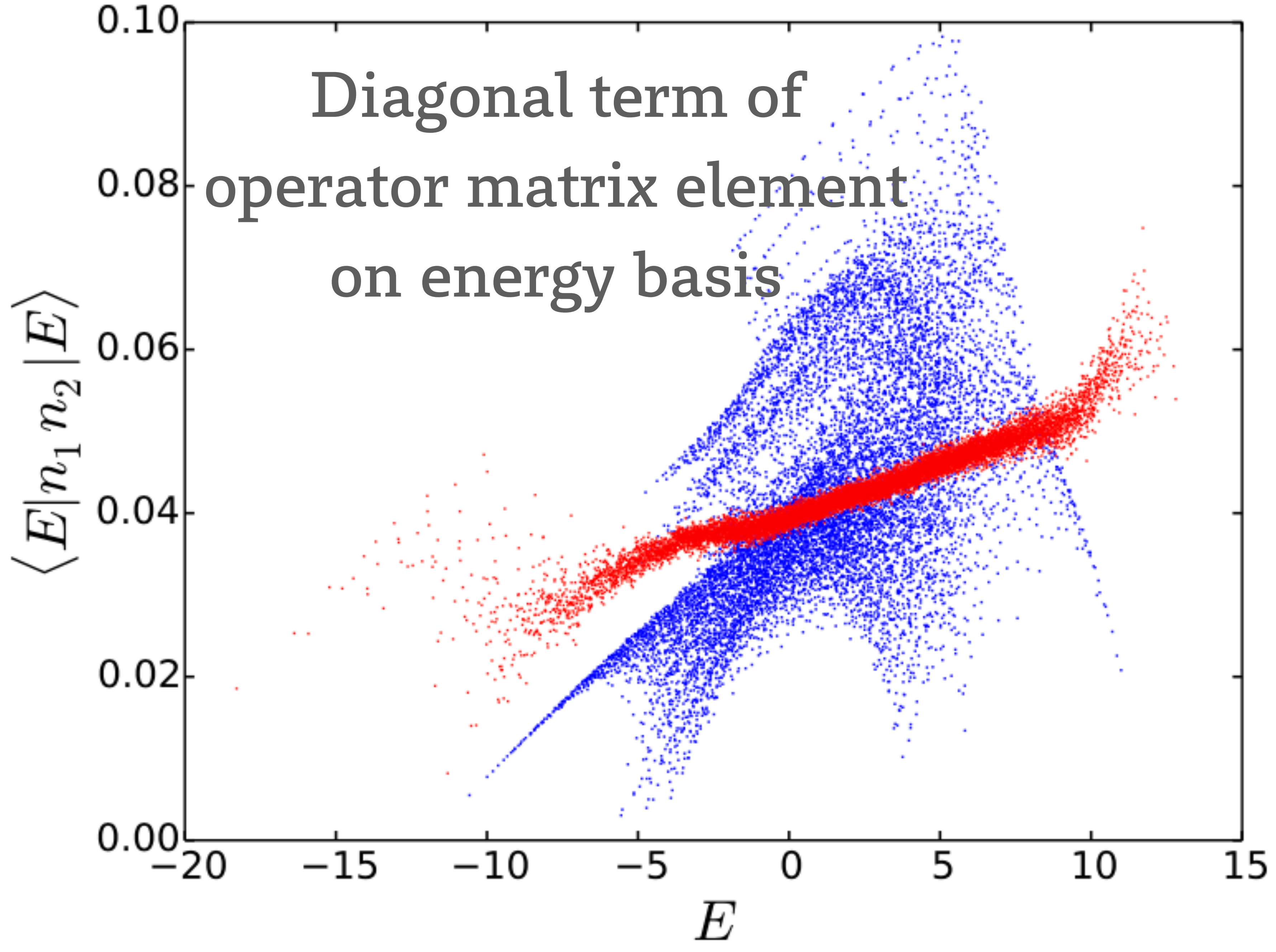
Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, Adv.Phys. 65 (2016) 3, 239-362

Eigenstate Thermalization Hypothesis

- ETH

A chaotic quantum system in a finitely excited energy eigenstate **behaves thermally** when probed by typical operators

Quantum point of view



Matrix element in energy basis

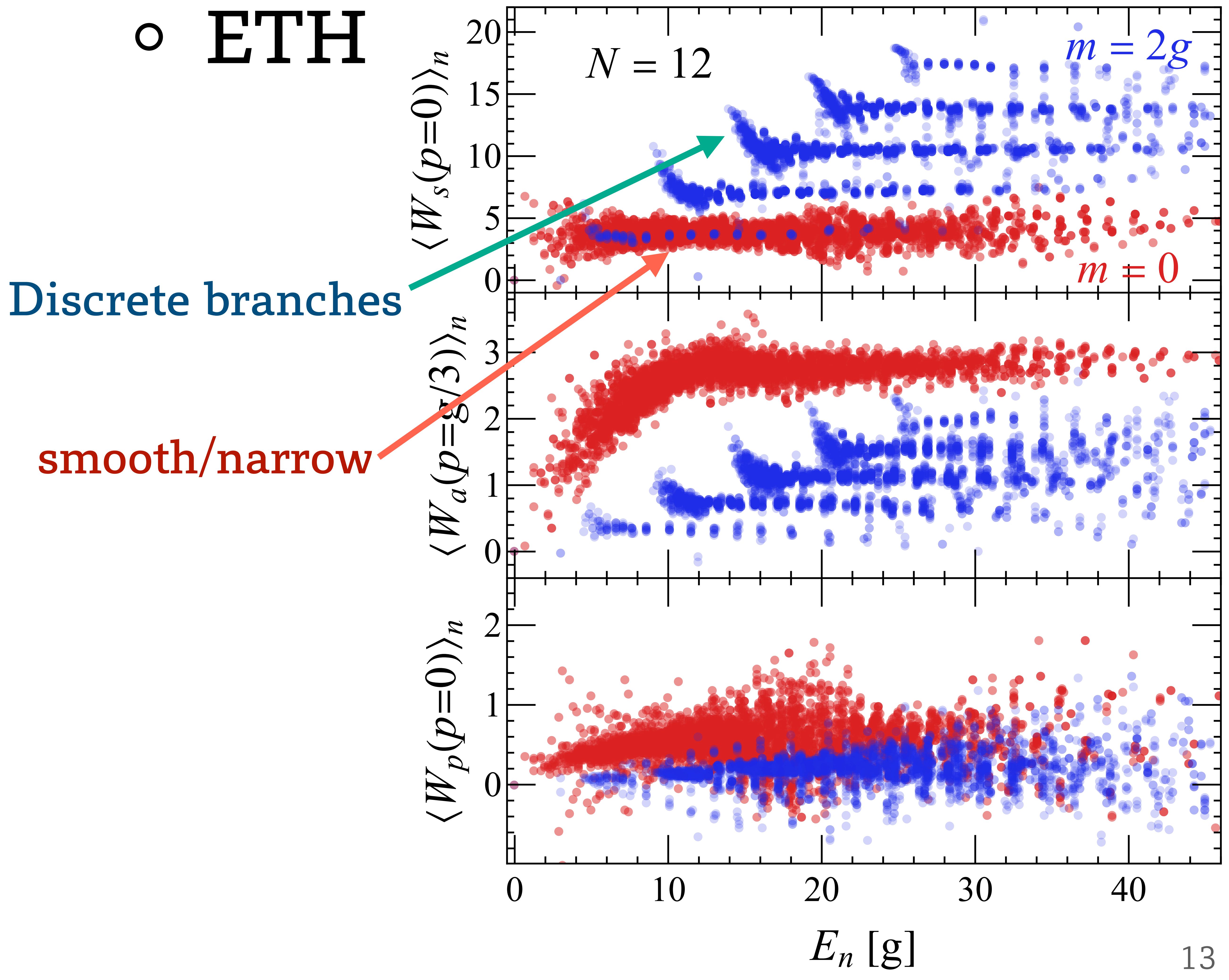
$$\langle E_a | \mathcal{O} | E_b \rangle = f_{\mathcal{O}}(E)\delta_{ab} + \Omega^{-1/2}(E) r_{ab} \qquad E = \frac{E_a + E_b}{2}$$

Long time average ≈ Thermal average

Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, Adv.Phys. 65 (2016) 3, 239-362

Eigenstate Thermalization Hypothesis

- ETH



- Very large fermion mass
- Approx. conserved quantity
- Particle number \ Chirality

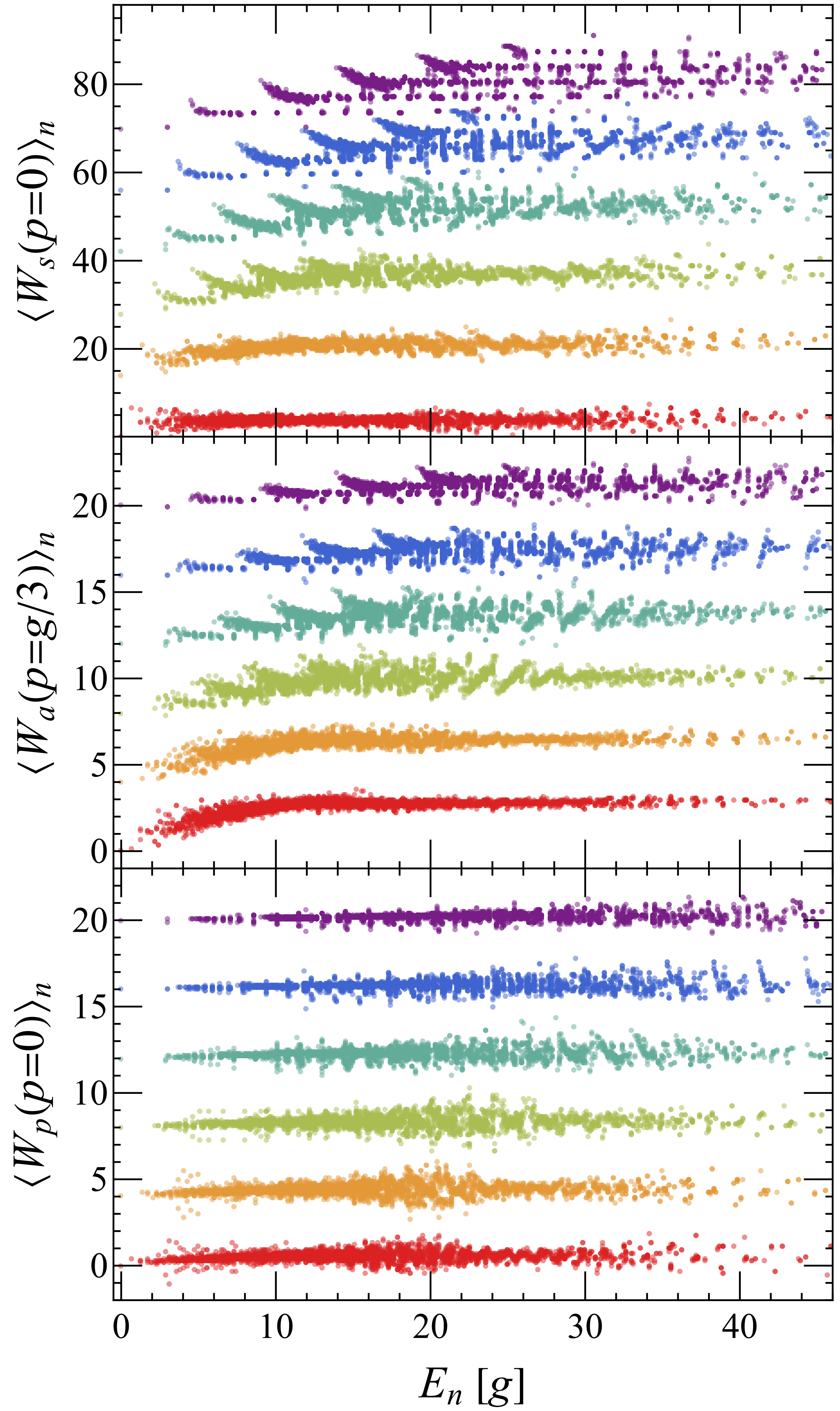
Energy Degeneracy

Not change with Unitary Time Evolution

Localized in Fock space

Eigenstate Thermalization Hypothesis

- ETH



- Very large fermion mass
- Approx. conserved quantity
- Particle number \ Chirality

Energy Degeneracy

Not change with Unitary Time Evolution

Localized in Fock space

Localization vs Thermalization

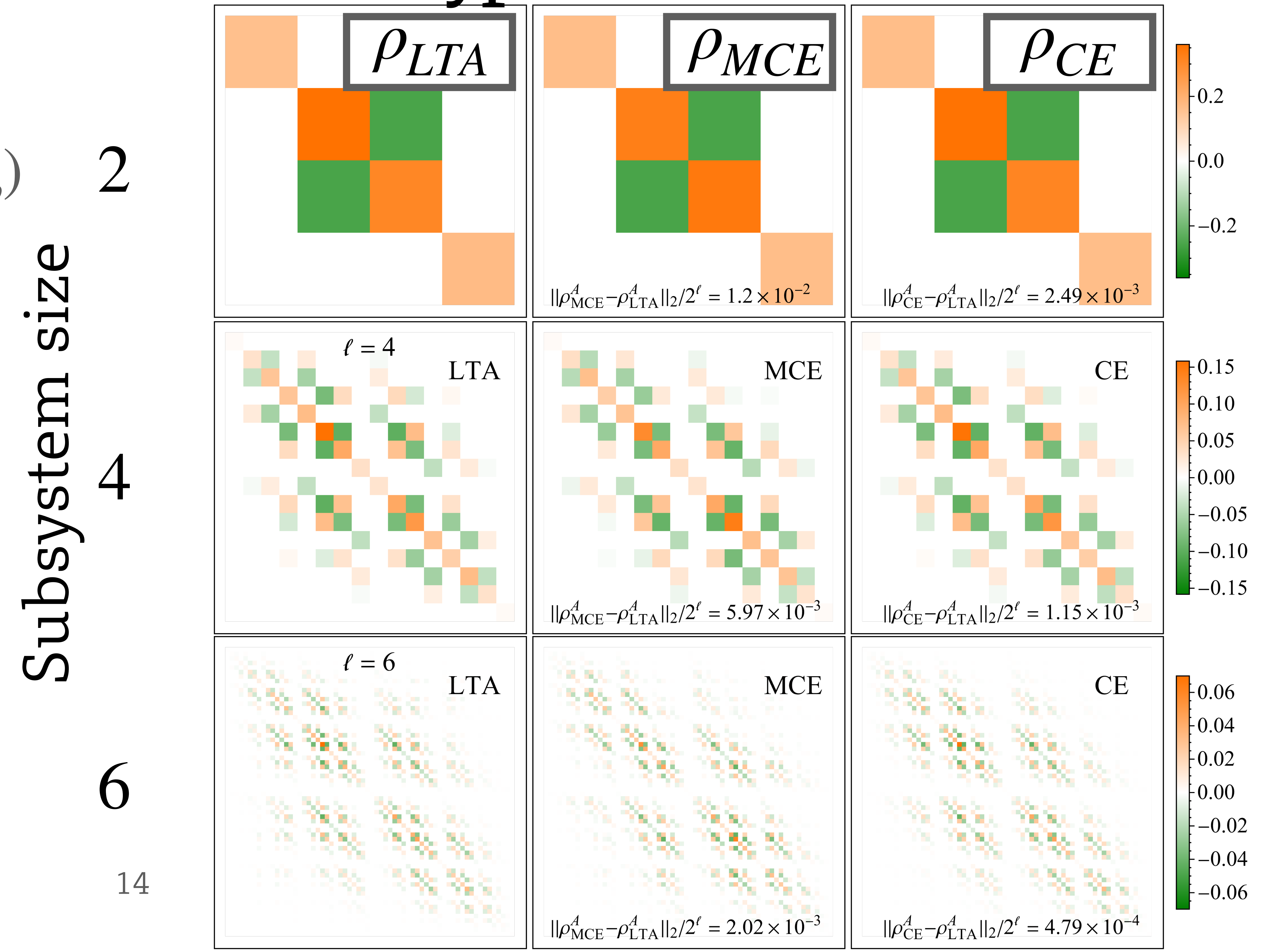
D.A.Abanin, E.Altman, I.Bloch and M.Serbyn, Many-body localization, thermalization, and entanglement," Rev. Mod. Phys. 91, 021001 (2019)

Entanglement & reduced density matrix

- Subsystem eigenstate thermalization hypothesis

$$||\rho_a^A - \rho^A(E=E_a)|| \sim O[\Omega^{-1/2}(E_a)]$$

$$||\rho_{ab}^A|| \sim O[\Omega^{-1/2}(E)], \quad E = \frac{1}{2}(E_a + E_b)$$

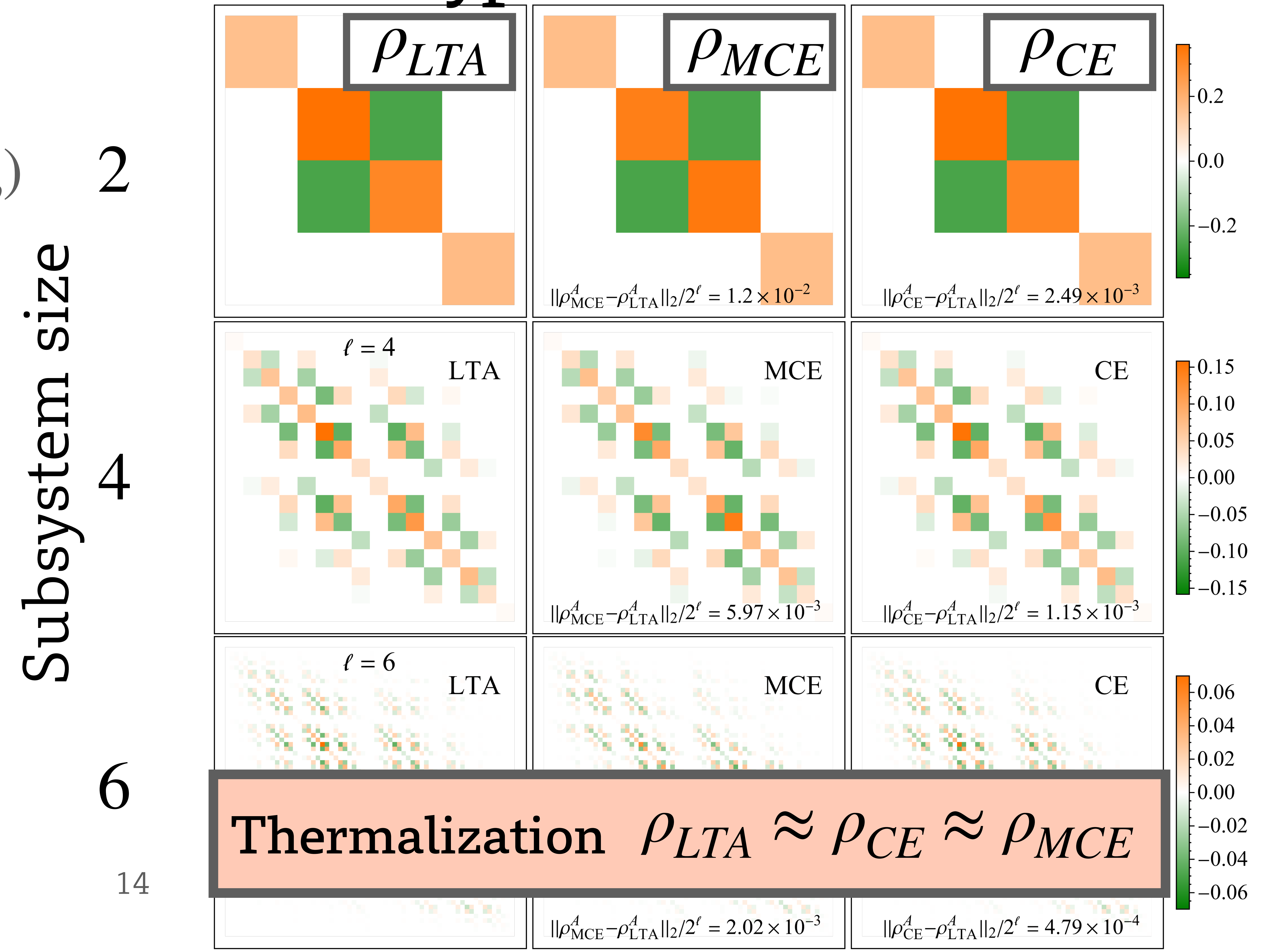


Entanglement & reduced density matrix

- Subsystem eigenstate thermalization hypothesis

$$||\rho_a^A - \rho^A(E=E_a)|| \sim O[\Omega^{-1/2}(E_a)]$$

$$||\rho_{ab}^A|| \sim O[\Omega^{-1/2}(E)], \quad E = \frac{1}{2}(E_a + E_b)$$



Thermalization $\rho_{LTA} \approx \rho_{CE} \approx \rho_{MCE}$

Entanglement & reduced density matrix

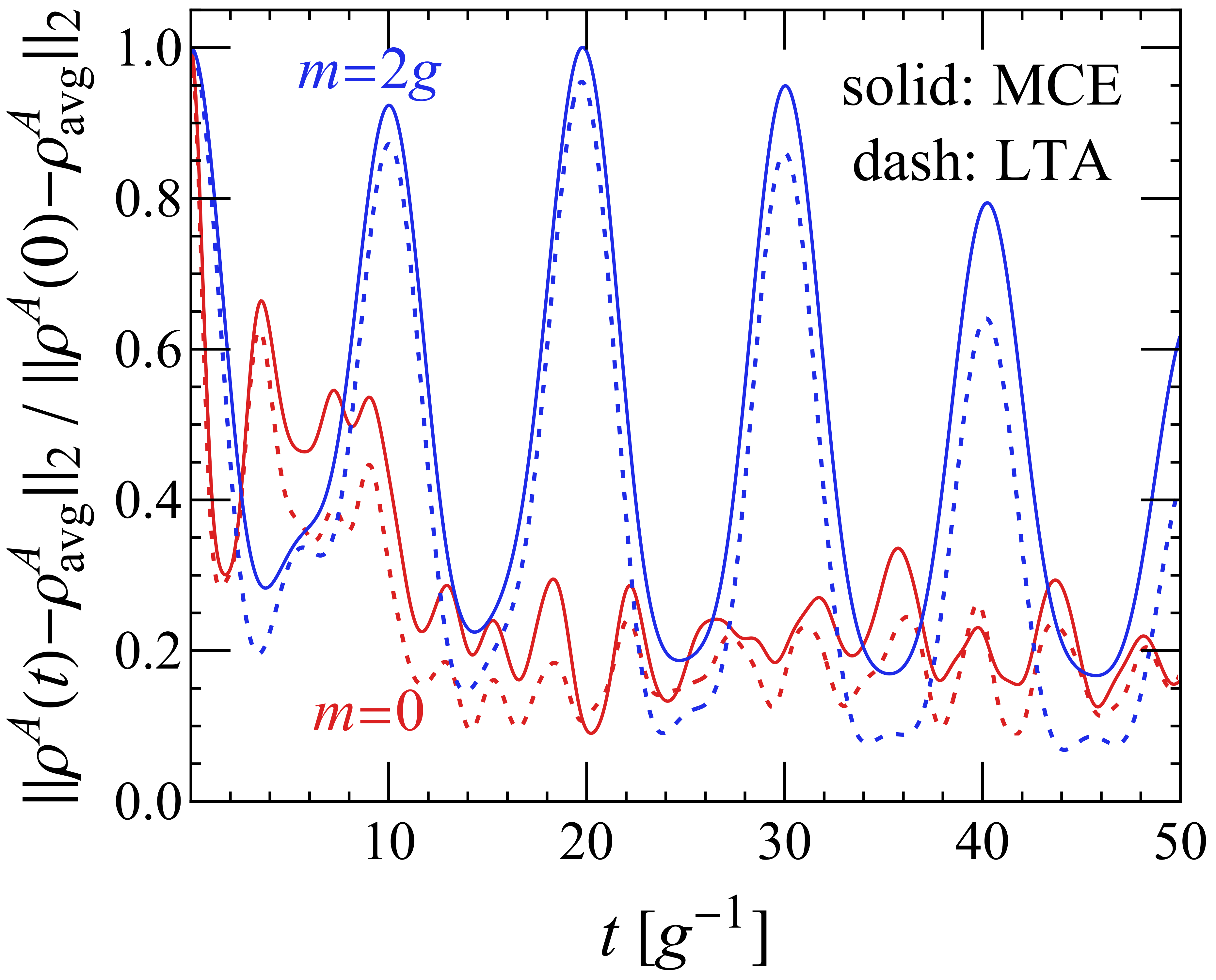
- Subsystem eigenstate thermalization hypothesis

Entanglement & reduced density matrix

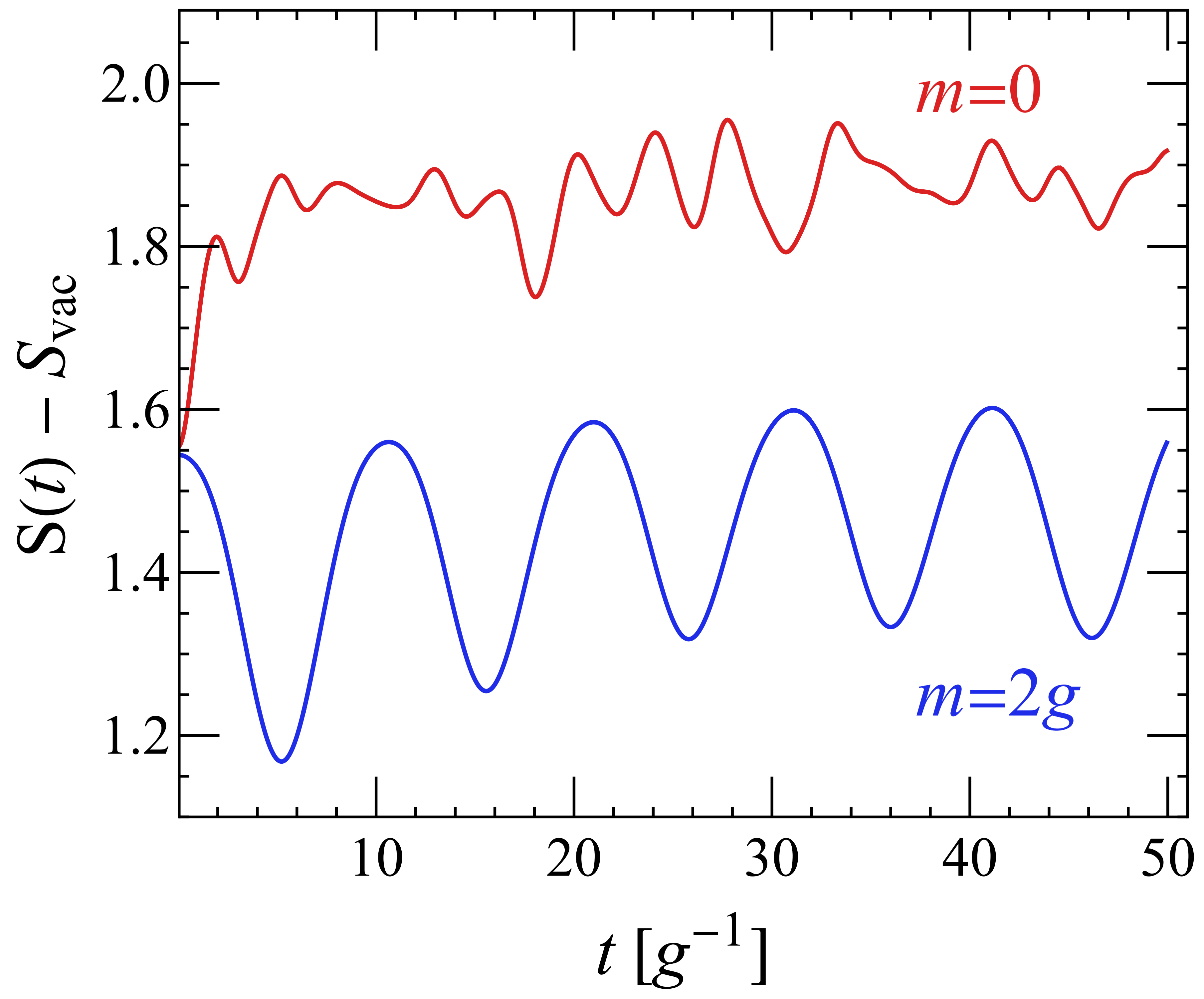
- Time evolution

$$S = -\rho^A \ln \rho^A$$

RDM for pure state ➞ entanglement entropy



Density matrix difference



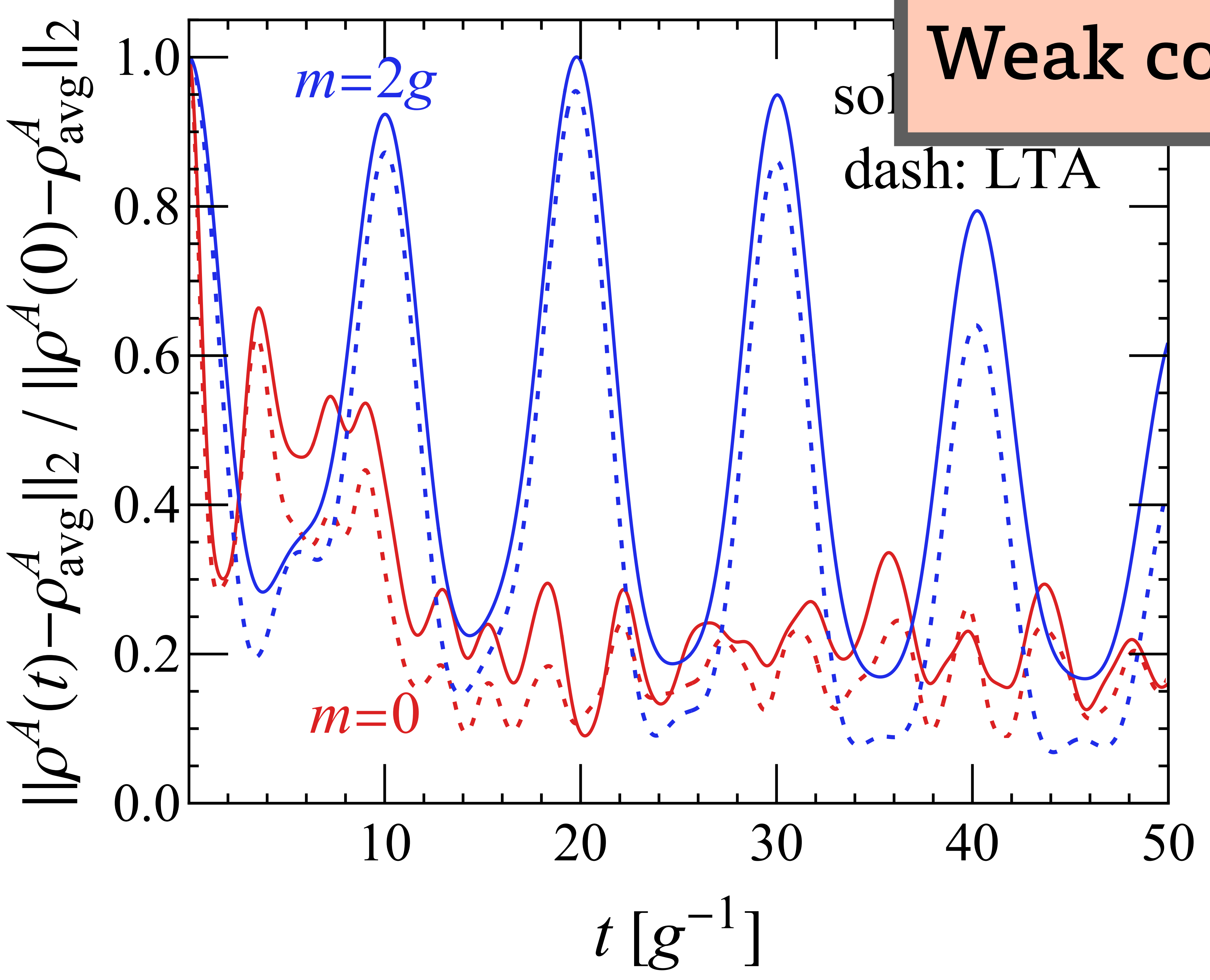
Entanglement entropy

Entanglement & reduced density matrix

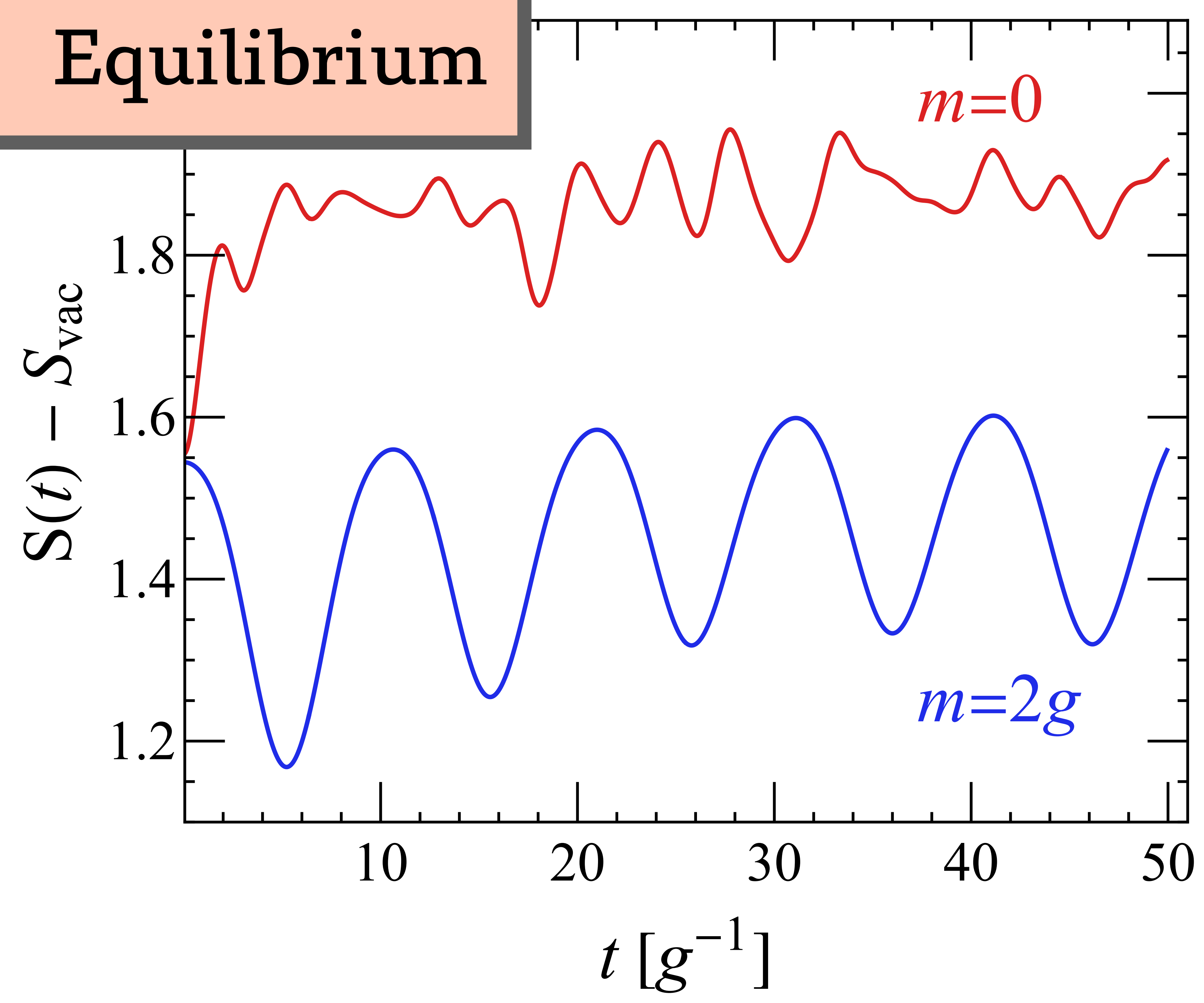
- Time evolution

Strong coulp. → Equilibrium

Weak coulp. ↛ Equilibrium



Density matrix difference



Entanglement entropy

Conclusion

- We simulate the real time evolution of a **closed** many body system with qc algorithm.
- Find the momentum distribution function will thermalize when the system satisfies **ETH**.
- Reduced density matrix of a subsystem is thermalized for the strong coupled system.

- Outlook

How does the system reach the thermal equilibrium:

Hydrodynamics?

Attractor?

Open quantum system!

Thanks for listening!!

Backup Quantum simulation

> If you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem because it does not look so easy.
> —— Richard Feynman, 1982

- Lattice QCD

  lattice gauge theory imaginary time evolution sign problem

- Quantum computing

  finite Hilbert space real time evolution

Back up "Thermal" average

- A Closed System Energy-eigenstates $\{|n\rangle\}$
  - Initial pure state $|\Psi\rangle_0 = \sum_n c_n |n\rangle$

Operator $\mathcal{O}$ For estimation

Inverse Temperature $\beta := \{ \sum_n |c_n|^2 E_n = \frac{\sum_n e^{-\beta E_n} E_n}{\sum_n e^{-\beta E_n}} \}$

Canonical average $\langle \mathcal{O} \rangle_\beta = tr(\rho_T \mathcal{O}) = \frac{\sum_n e^{-\beta E_n} \langle n | \mathcal{O} | n \rangle}{\sum_n e^{-\beta E_n}}$

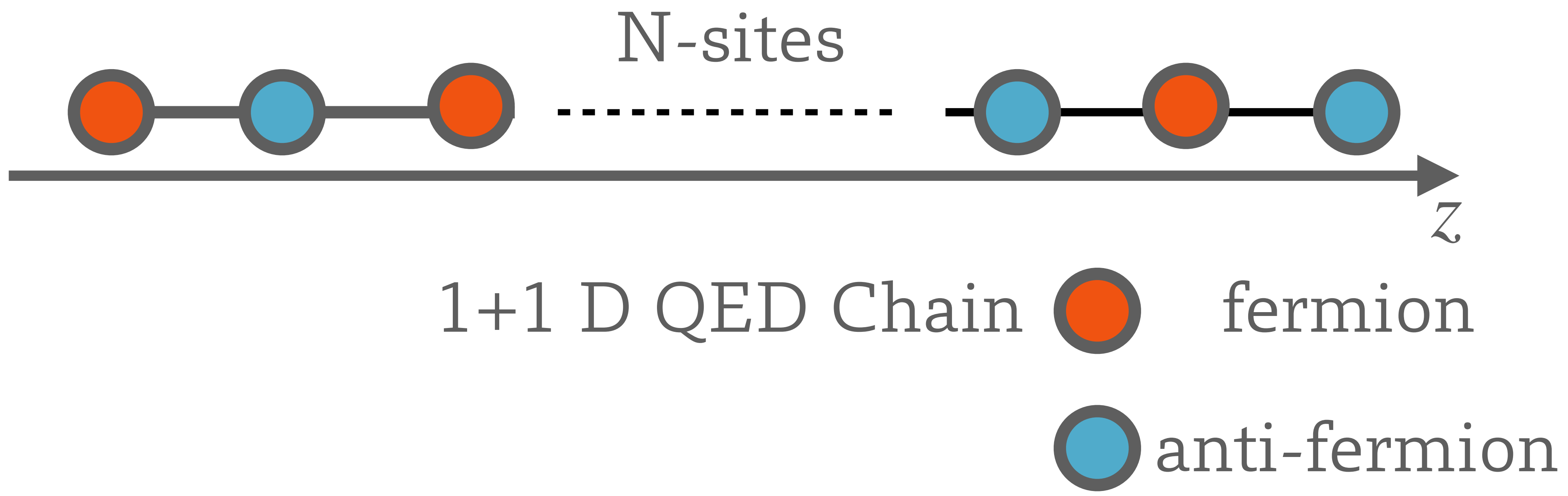
Micro-canonical average $\langle \mathcal{O} \rangle_{MC} = \frac{\sum_{n:|E_n - E| \leq \Delta E} \mathcal{O}_{n,n}}{\sum_{n:|E_n - E| \leq \Delta E}}$

Back up-Schwinger model

- **Discretization** of $H = \int \left(-\bar{\psi} i \gamma^1 \partial_z + m\psi + \frac{1}{2}\mathscr{E}^2 \right) dz$

Energy scale of mass and coupling constant

m=0 theory with 1/a -> not analytical/chiral symmetry



Dimension of fermion sector	2^N
Dimension of electric field sector	M
Dimension of total Hilbert space	$2^N \times M$

$$z_n = na$$

$$\varepsilon_n = g^{-1} E(z_n) \quad \varepsilon_{n+1} - \varepsilon_n = \chi_n^\dagger \chi_n$$

$$\chi_{2n} = a^{1/2} \psi_\uparrow (z_{2n})$$

$$\chi_{2n+1} = a^{1/2} \psi_\downarrow (z_{2n+1})$$

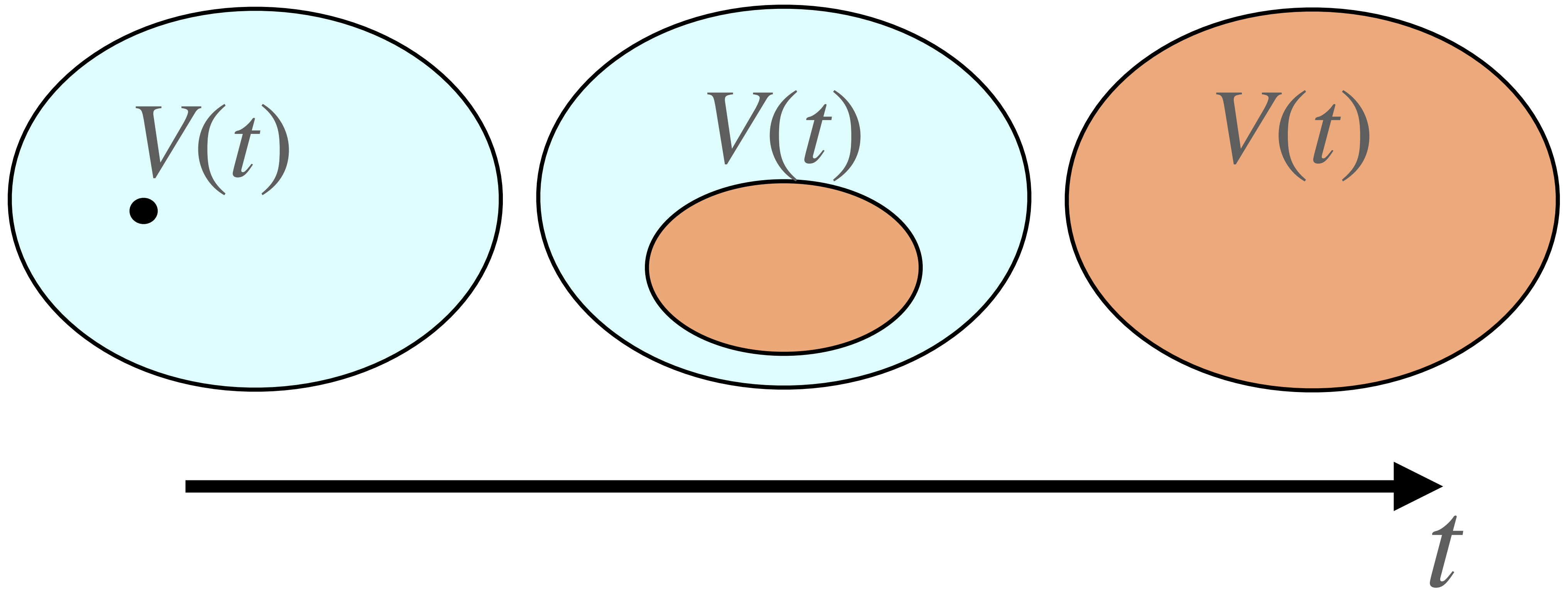
$$\phi_n = agA_0(z_n)$$

Hamiltonian with periodic boundary condition

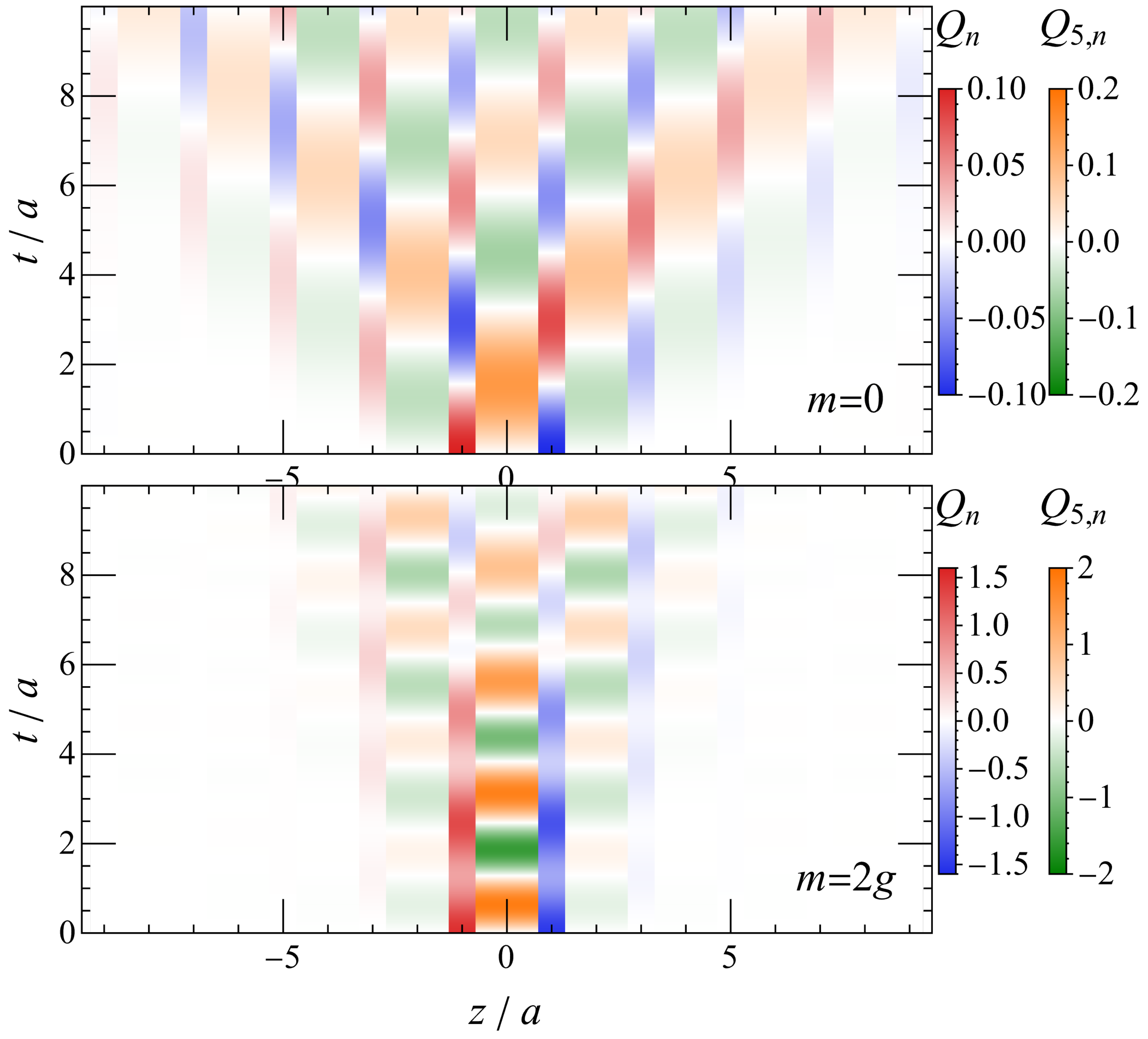
$$H_{PBC} = \sum_{n=1}^{N} \left(-\frac{i}{2}\frac{1}{a}(\chi_n^\dagger e^{i\phi_n}\chi_{n+1} - \chi_{n+1}^\dagger e^{-i\phi_n}\chi_n) + (-1)^n m_0 \chi_n^\dagger \chi_n + \frac{ag^2}{2}\varepsilon_n^2 \right)$$

Localization

- Many-body localization



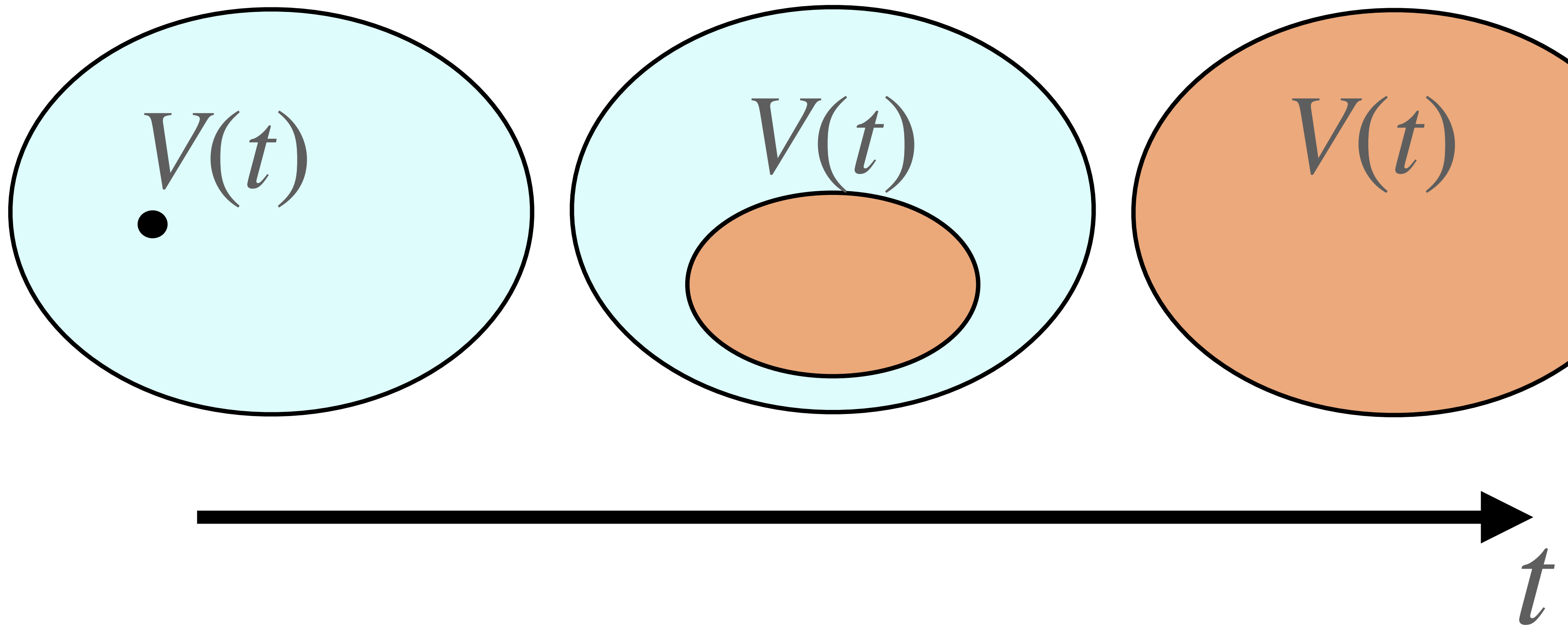
Scrambling of operator



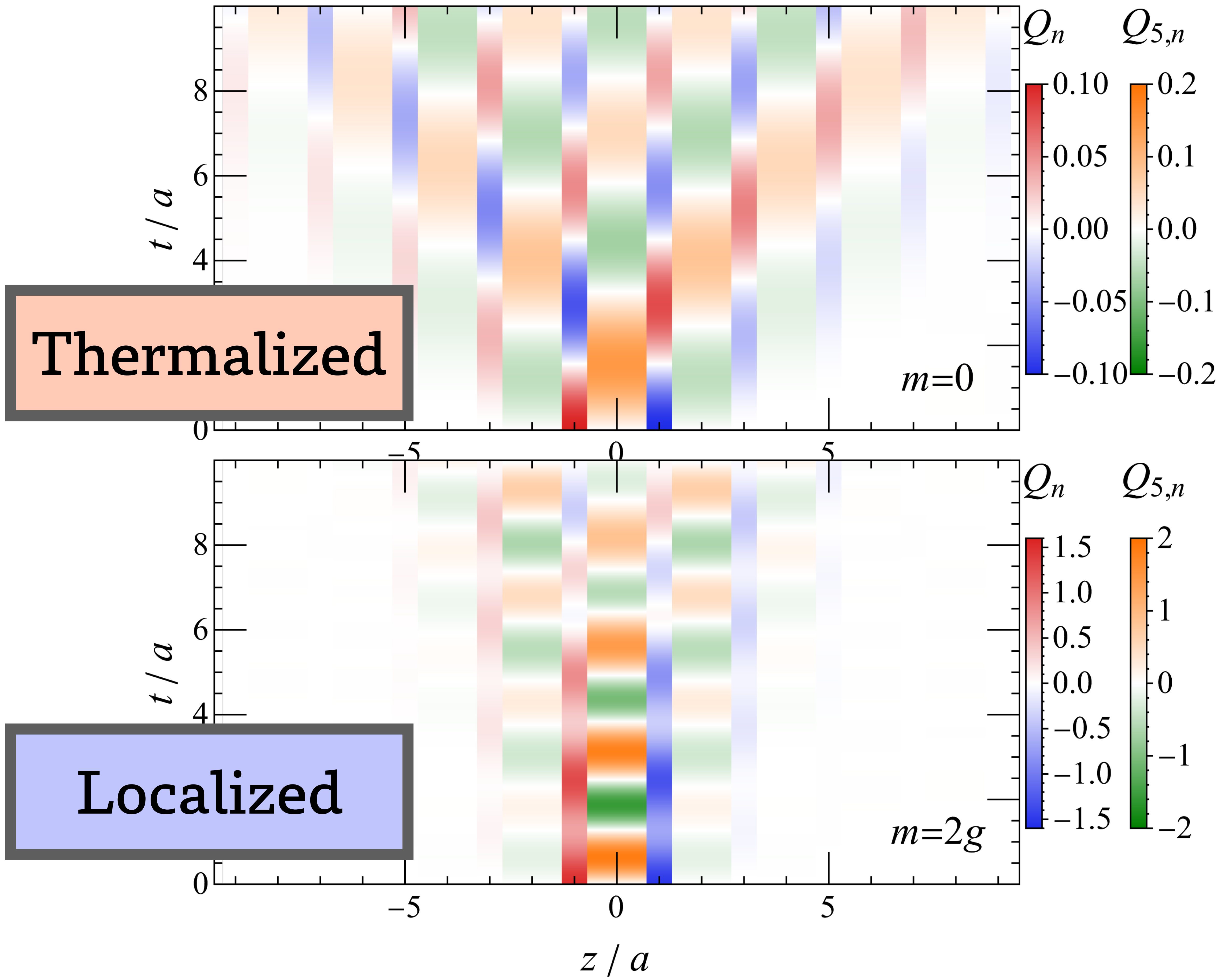
Ikeda, Kazuki and Kharzeev, Dmitri E. and Shi, Shuzhe,Phys. Rev. D",108 7, 074001(2023)

Localization

- Many-body localization



Scrambling of operator



Ikeda, Kazuki and Kharzeev, Dmitri E. and Shi, Shuzhe,Phys. Rev. D",108 7, 074001(2023)