Nuclear Science
Computing Center at CCNU

Microscopic Encoding of Macroscopic Universality: Scaling properties of Dirac Eigenspectra near QCD Chiral Phase Transition

How do universal behaviors at macroscale arise from quarks and gluons?

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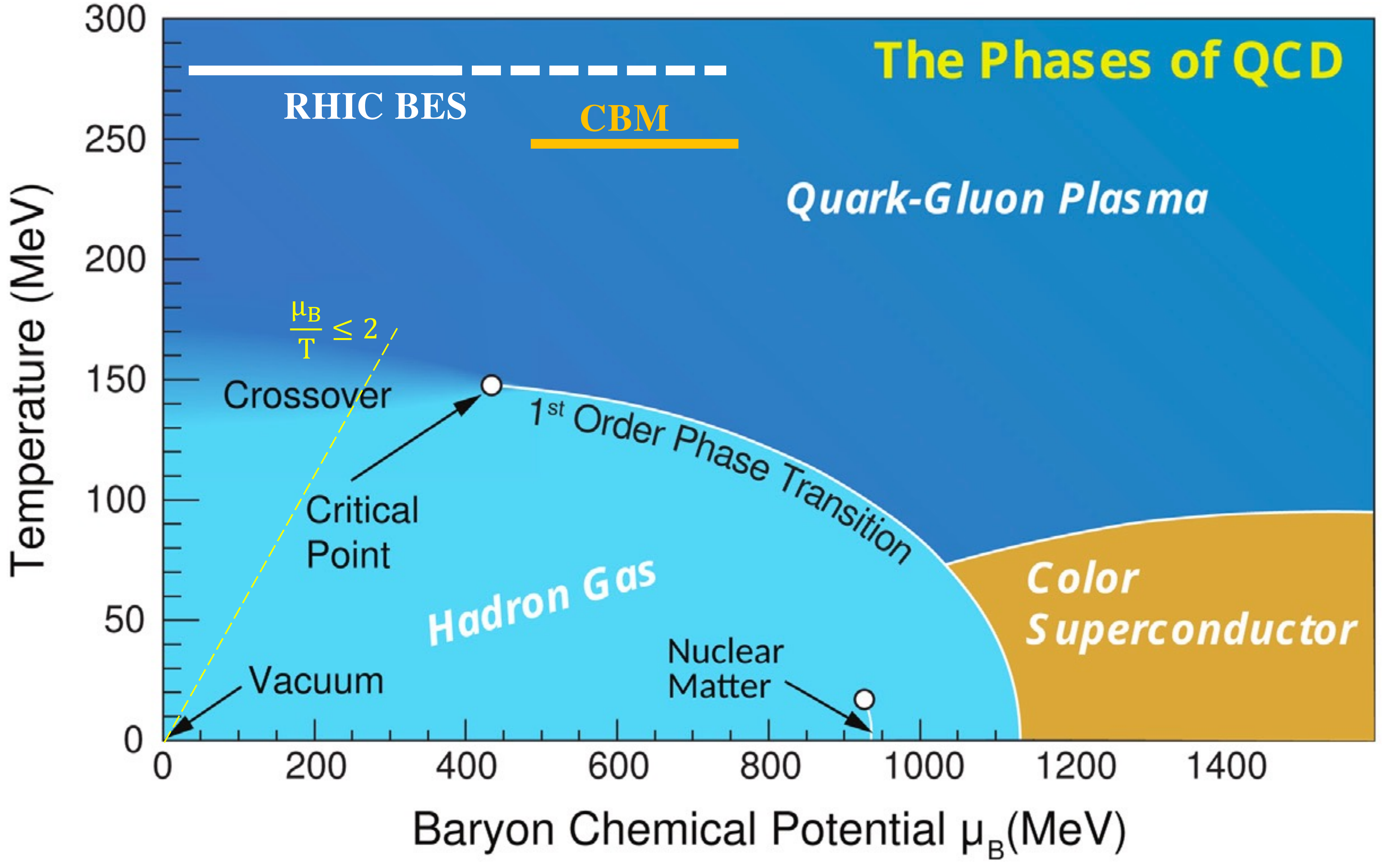
based on Phys. Rev. Lett. 131 (2023) 16, 161903 & work in progress,
in collaboration with
Heng-Tong Ding (丁亨通), Swagato Mukherjee, Peter Petreczky, Yu Zhang (张瑜)

XQCD 2024, July 17 - 19 @ Lanzhou, China

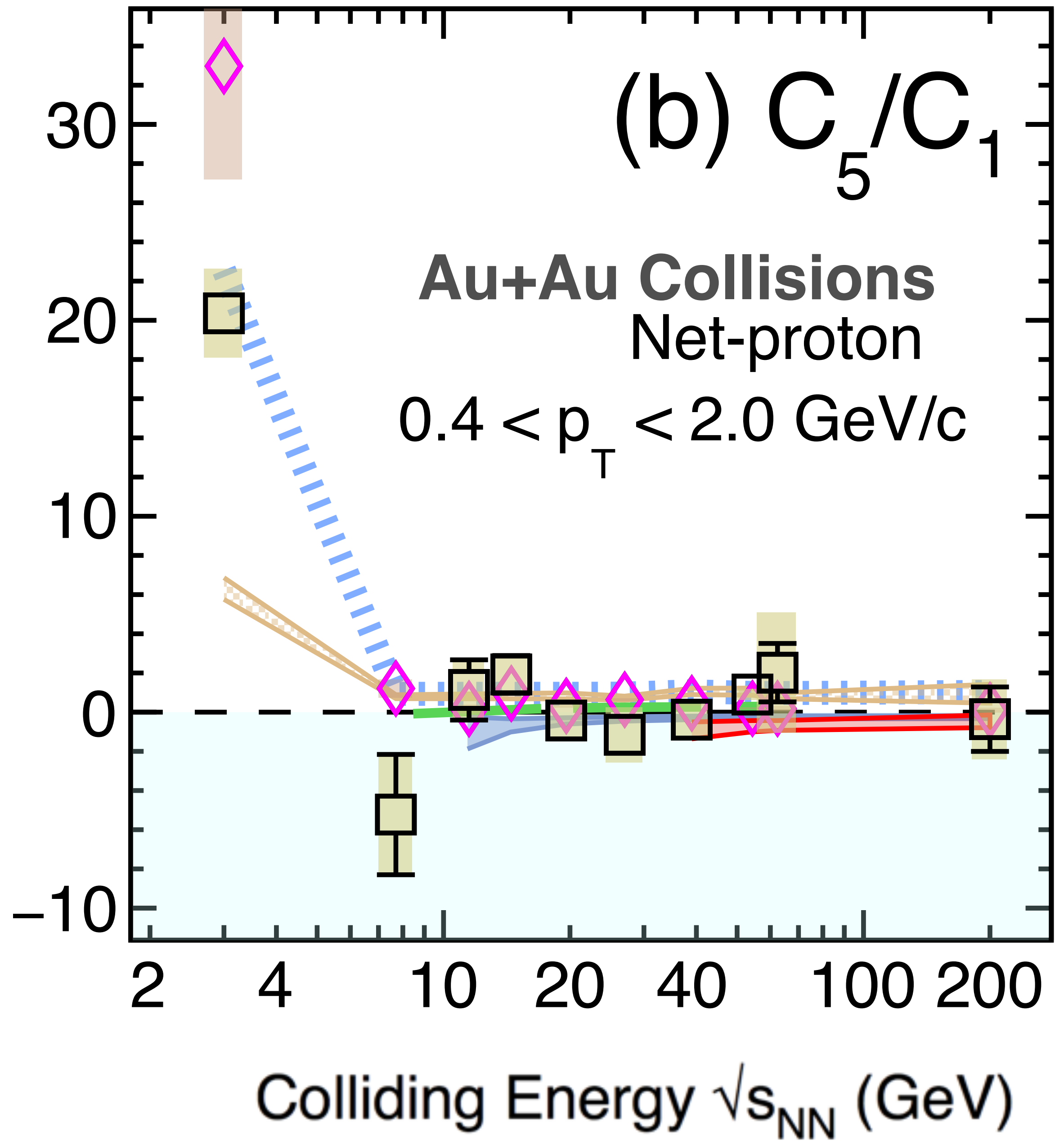
Search for Criticality in QCD

Exploration of QCD phase diagram



D. Almaalol et al., arXiv:2209.05009

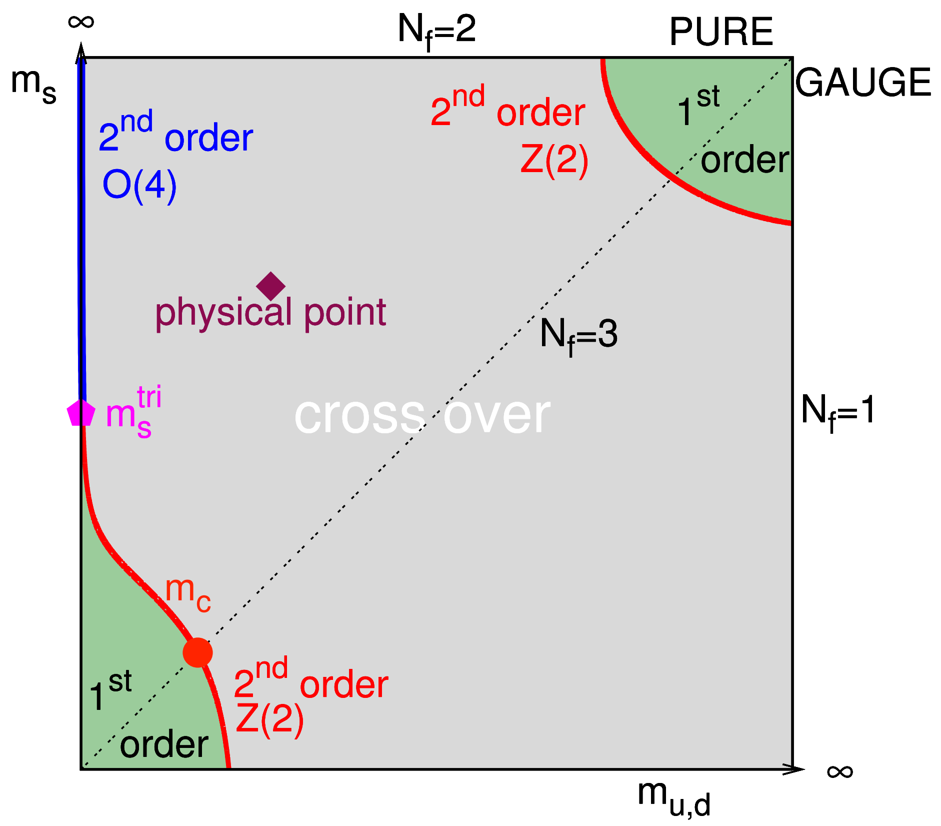
Searching for signatures of criticality in Macroscopic quantities



STAR, Phys. Rev. Lett. 130, 082301 (2023)

How does criticality at Macroscale arise from Microscopic *d.o.f* of QCD ?

Universal Scaling in QCD Chiral Transition



H.-T. Ding et al., arXiv:1504.05274

✦ Restoration of chiral symmetry: — Crossover at physical point;

— True chiral phase transition only in limiting cases

A. Bazavov et al., [HotQCD], PLB 795 (2019) 15

H.-T. Ding et al., [HotQCD], PRL 123 (2019) 062002

✦ Role of $U(1)_A$ symmetry in chiral phase transition:

Broken around $T_c \Rightarrow$ 2^{nd} order O(4) phase transition

Pisarski & Wilczek, PRD 29 (1984) 338

Pelissetto & Vicari, PRD 88 (2013) 105018

Universal *O*(2) scaling behaviors in *staggered* discretization scheme

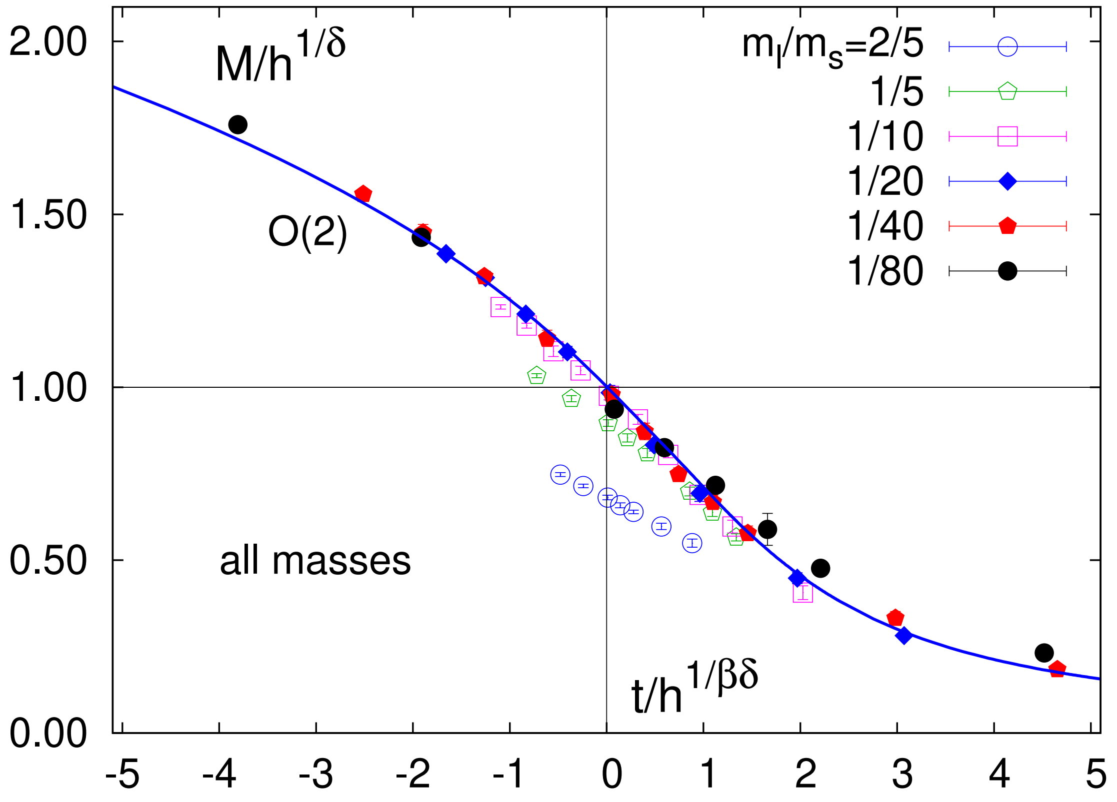
$$z = t/h^{1/\beta\delta}$$

$$t = \frac{T - T_c}{t_0 T_c}$$

$$h = \frac{m_l}{h_0 m_s}$$

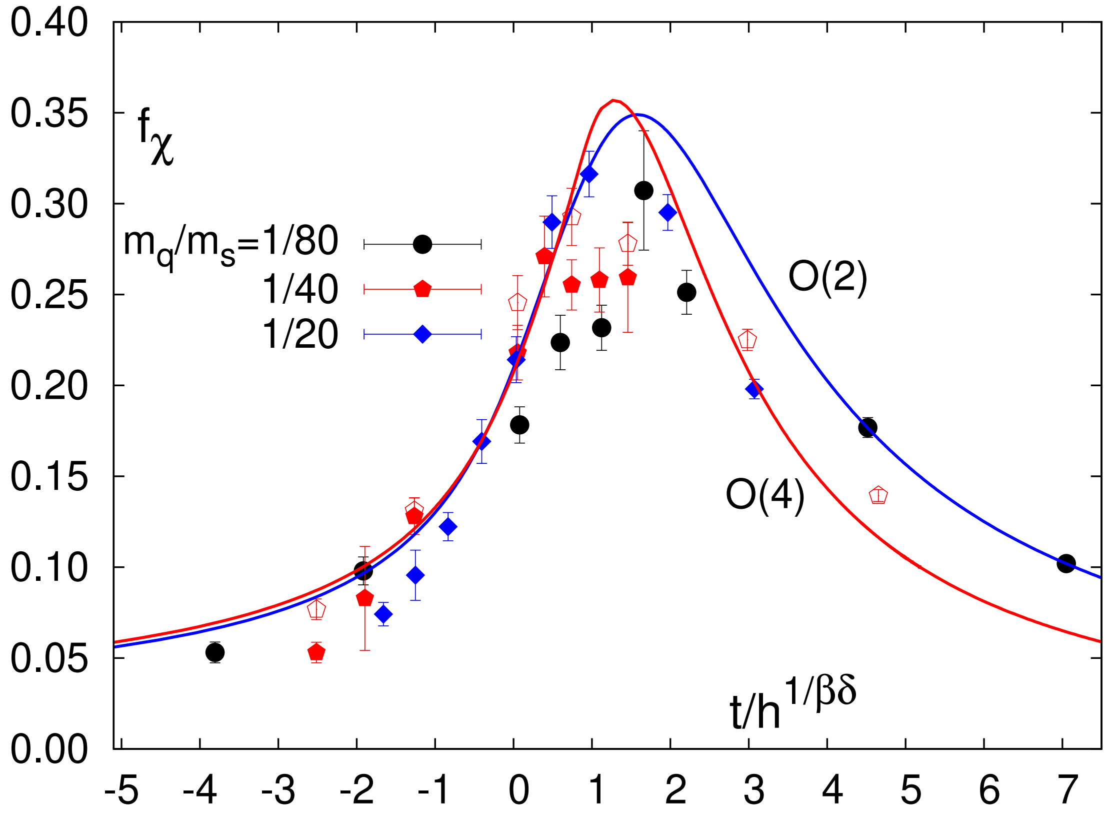
Order parameter :

$$M(t,h) = h^{1/\delta} f_1(z) + f_{\mathrm{reg}}(T,H)$$



Order parameter susceptibility :

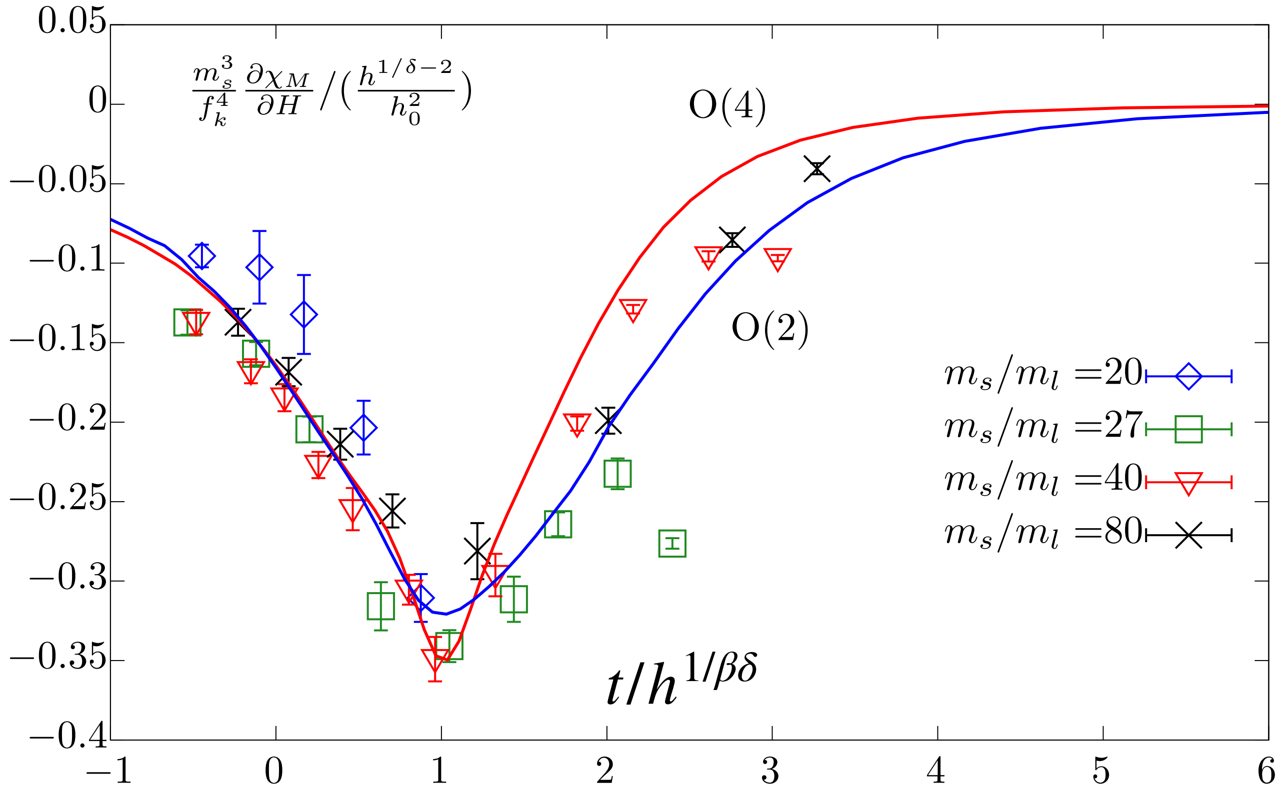
$$\chi_M(t,h) = \partial M/\partial H = h_0^{-1} h^{1/\delta - 1} f_2(z) + f'_{\mathrm{reg}}$$



S. Ejiri et al., PRD 80, 094505 (2009)

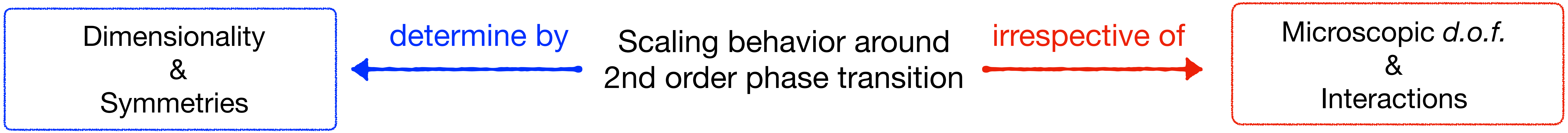
Higher order susceptibility :

$$\text{e.g., } \partial\chi_M/\partial H = h_0^{-2} h^{1/\delta - 2} f_3(z) + f''_{\mathrm{reg}}$$



H.-T. DIng, **HWP** et al., arXiv: 2112.00318

Hard to Reach Microscopic Origin of Criticality



Dimensionality
&
Symmetries

determine by

Scaling behavior around
2nd order phase transition

irrespective of

Microscopic *d.o.f.*
&
Interactions

E.g.: Liquid to superfluid λ-transition in ^{4}He

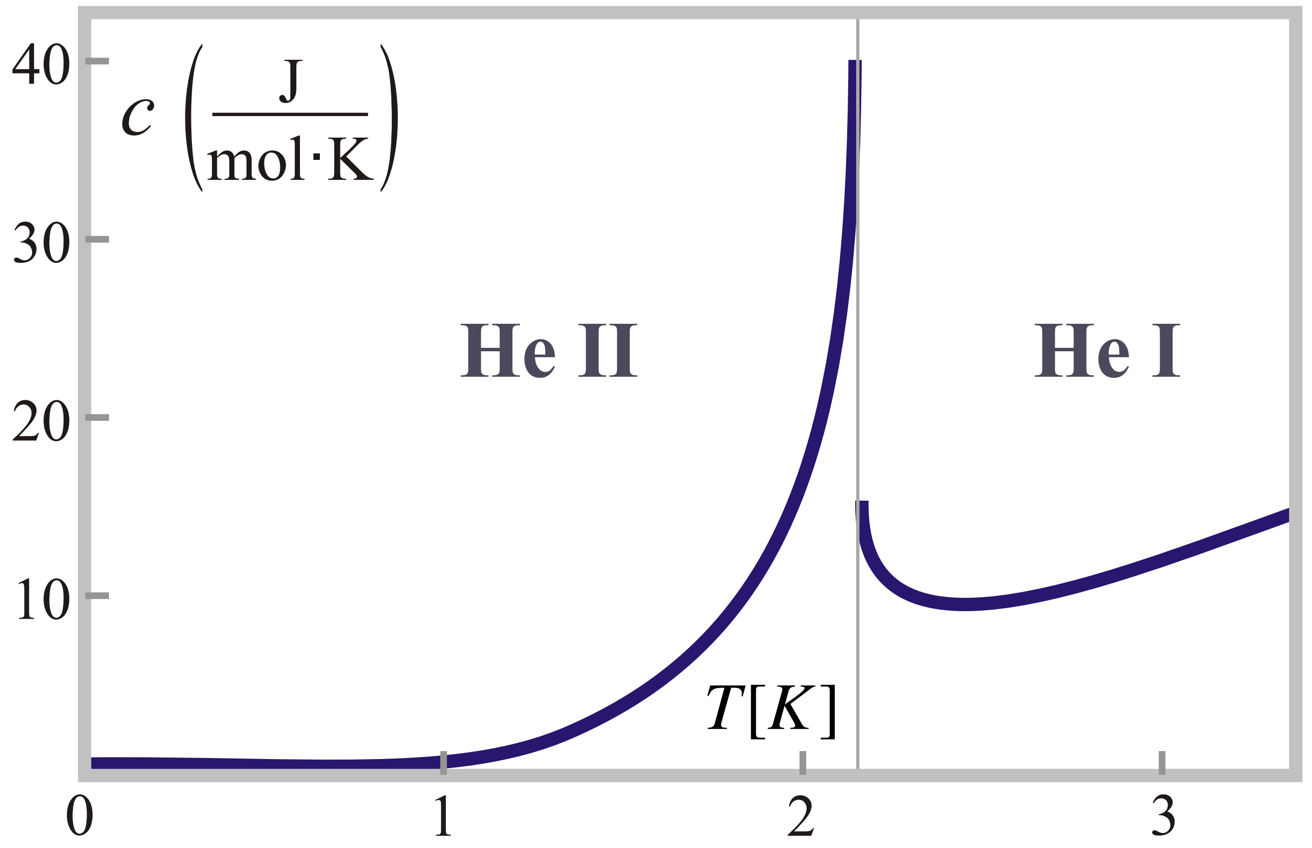
fundamental d.o.f. : electrons & photons

Same dimensionality
& symmetry

Same 3-d O(2)
scaling behaviors

Chiral phase transition
in the chiral limit of light quark
using staggered fermion discretization

fundamental d.o.f. : quarks & gluons



This talk:

A first lattice QCD-based understanding of Microscopic Origin of Criticality in QCD !

✓***Able to study via effective theories***

× ***No infos about fundamental micro level of d.o.f. via EFT !!***

Banks-Casher Relation: Connect Macro to Micro in Chiral Limit

Chiral order parameter : $\langle \bar{\psi}\psi \rangle = \int_0^\infty \frac{4m\,\rho(\lambda, m)}{\lambda^2 + m^2}\, d\lambda \xrightarrow{m\to 0} \pi\rho(\lambda = 0)$

Banks & Casher, NPB 169(1980) 103

$$\rho(\lambda, m) = \frac{T}{V}\langle \rho_U(\lambda) \rangle \equiv \frac{T}{V}\langle \sum_j \delta(\lambda - \lambda_j) \rangle \quad \text{with} \quad \not{D}\psi_j = i\lambda_j \psi_j$$

Calculable by Lattice QCD !!

Connect Macroscopic quantity to Microscopic *d.o.f.* : $\langle \bar{\psi}\psi \rangle \Longleftrightarrow \rho_U(\lambda)$: ***energy spectra of massless quarks***

Next:
More generally…

From Quark Energy Spectra to its Cumulants

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \frac{4m\ \rho(\lambda, m)}{\lambda^2+m^2}\, d\lambda = \frac{T}{V}\int_0^\infty \left\langle \frac{4m\ \rho_U(\lambda)}{\lambda^2+m^2} \right\rangle d\lambda$$

H.-T. Ding, **HWP** et al., PRL 131, 161903

1st order cumulant :

$$\langle\bar{\psi}\psi\rangle = \frac{T}{V}\langle 2\ \mathrm{Tr}(\not{D}[U]+m)^{-1}\rangle$$

1-point correlation of quark energy spectra $\frac{4m\ \rho_U(\lambda)}{\lambda^2+m^2}$

Generalization : from ***generating functional*** of cumulants

— ***Generating functional*** : $\mathbb{G}(m;\epsilon) = \ln\left\langle \exp\left\{-m\ \bar{\psi}\psi(\epsilon)\right\}\right\rangle_0 = \ln\left\langle \exp\left\{-m\int_0^\infty P_U(\lambda;\epsilon)\, d\lambda\right\}\right\rangle_0$

$\langle\cdots\rangle_0$: average over QCD partition function in the chiral limit

Probe operator with valance quark mass ϵ :
$\bar{\psi}\psi(\epsilon) \equiv 2\ \mathrm{Tr}(\not{D}[U]+\epsilon)^{-1}$

Defining $P_U(\lambda;\epsilon) \equiv \frac{4\epsilon\rho_U(\lambda)}{\lambda^2+\epsilon^2}$

— ***n-th order cumulant of*** $\bar{\psi}\psi$: $\mathbb{K}_n[\bar{\psi}\psi] = \frac{T}{V}(-1)^n \frac{\partial^n \mathbb{G}(m;\epsilon)}{\partial m^n}\Big|_{\epsilon=m}$

$$\mathbb{K}_n[\bar{\psi}\psi(m)] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)]\prod_{i=1}^{n} d\lambda_i$$

$K_1[X_1, X_2, \cdots, X_n]$ denotes 1st order joint cumulant of n-variables

Microscopic Encoding of Macroscopic Criticality

H.-T. Ding, **HWP** et al., PRL 131, 161903

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda$$

n-th order cumulant of the chiral order parameter

n-point correlation of the quark energy spectra

Chiral condensate : $\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\langle\bar{\psi}\psi(m)\rangle = \int_0^\infty P_1(\lambda)\ d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta} f_1(z)$

$$z = z_0(\frac{m_l}{m_s})^{-\frac{1}{\beta\delta}}\frac{T-T_c}{T_c}$$
$$f_1(z) \equiv f_G(z)$$
$$f_2(z) \equiv f_\chi(z)$$

Disconnected susceptibility : $\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\left\langle\left[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle\right]^2\right\rangle = \int_0^\infty P_2(\lambda)\ d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta-1} f_2(z)$

……

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\ d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

How does the criticality of $\mathbb{K}_n[\bar{\psi}\psi]$ arise from $P_n(\lambda)$?

Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit :

$$P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}$$

$$P_U(\lambda; m \to 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$$

Generalized Banks-Casher relation :

$$\lim_{m\to 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), \underbrace{\rho_U(0), \cdots, \rho_U(0)}_{(n-1)\ \text{terms}}]\delta(\lambda)$$

$$\Longrightarrow \lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

$n = 1$ back to Banks-Casher relation !

Criticality in $\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi]$ must arise from universal behaviors of ***λ-independent*** $\mathbb{K}_n[\rho_U(0)]$

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta - n + 1} f_n(z)$$

Conjecture: $P_n(\lambda) = m^{1/\delta - n + 1} f_n(z) g_n(\lambda)$

Scaling arise from
$P_n(\lambda)$ at ***deep infrared*** λ region

Include ***all*** system-specific λ-dependence

H.-T. Ding, **HWP** et al., PRL 131, 161903

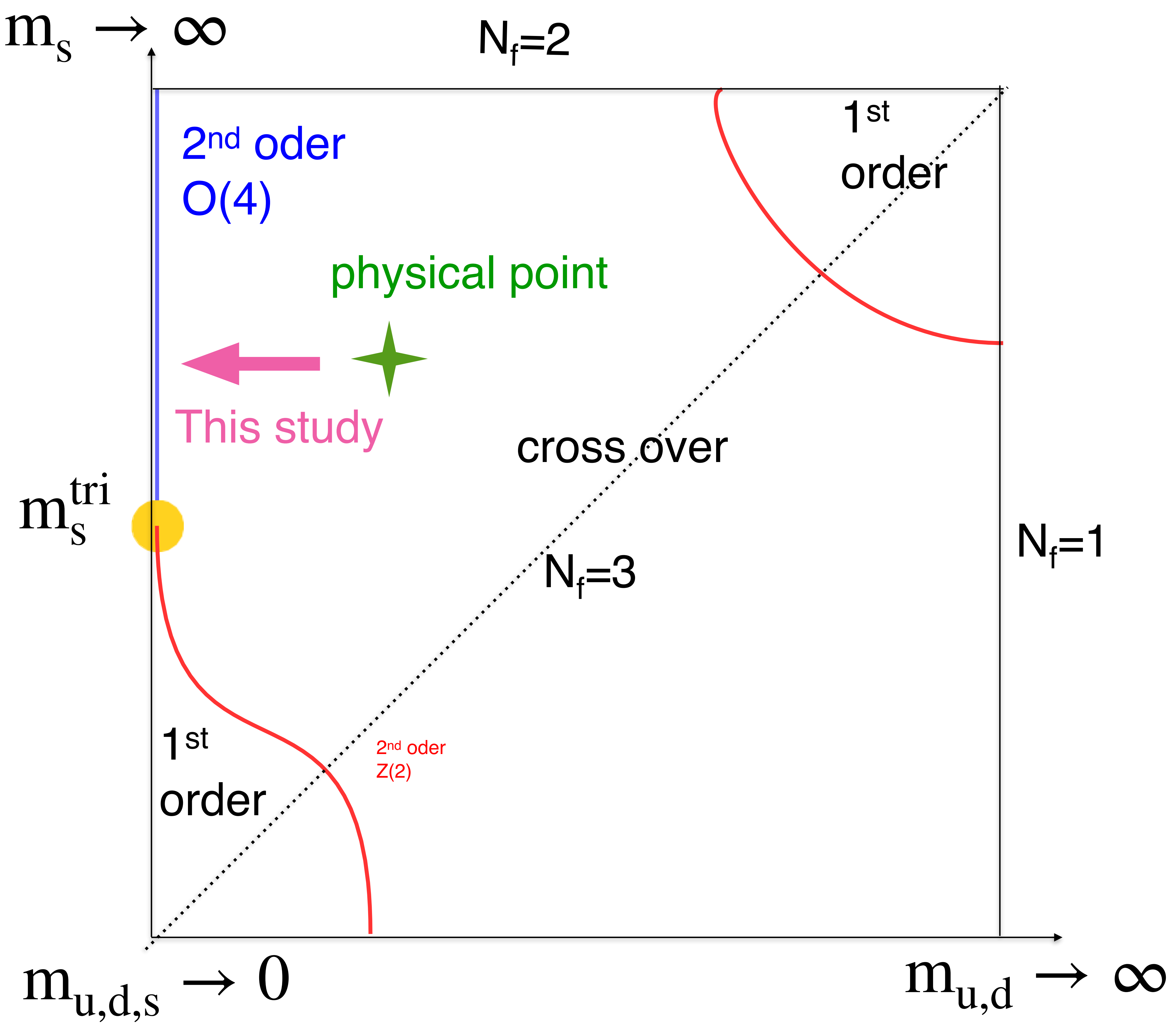
Lattice Setup

- Actions: Highly improved staggered quarks and tree-level Symanzik gauge action
- Lattice size: $N_\tau = 8, \quad N_\sigma = 32, 40, 56$
- Quark mass: $m_s^{\text{phy}}/m_l = 27, 40, 80, 160$

  $(m_\pi \approx 140, 110, 80, 55 \text{ MeV})$
- Temperatures: $T \in (135{,}176)$ MeV
- $\rho_U(\lambda)$ computed via Chebyshev filtering technique

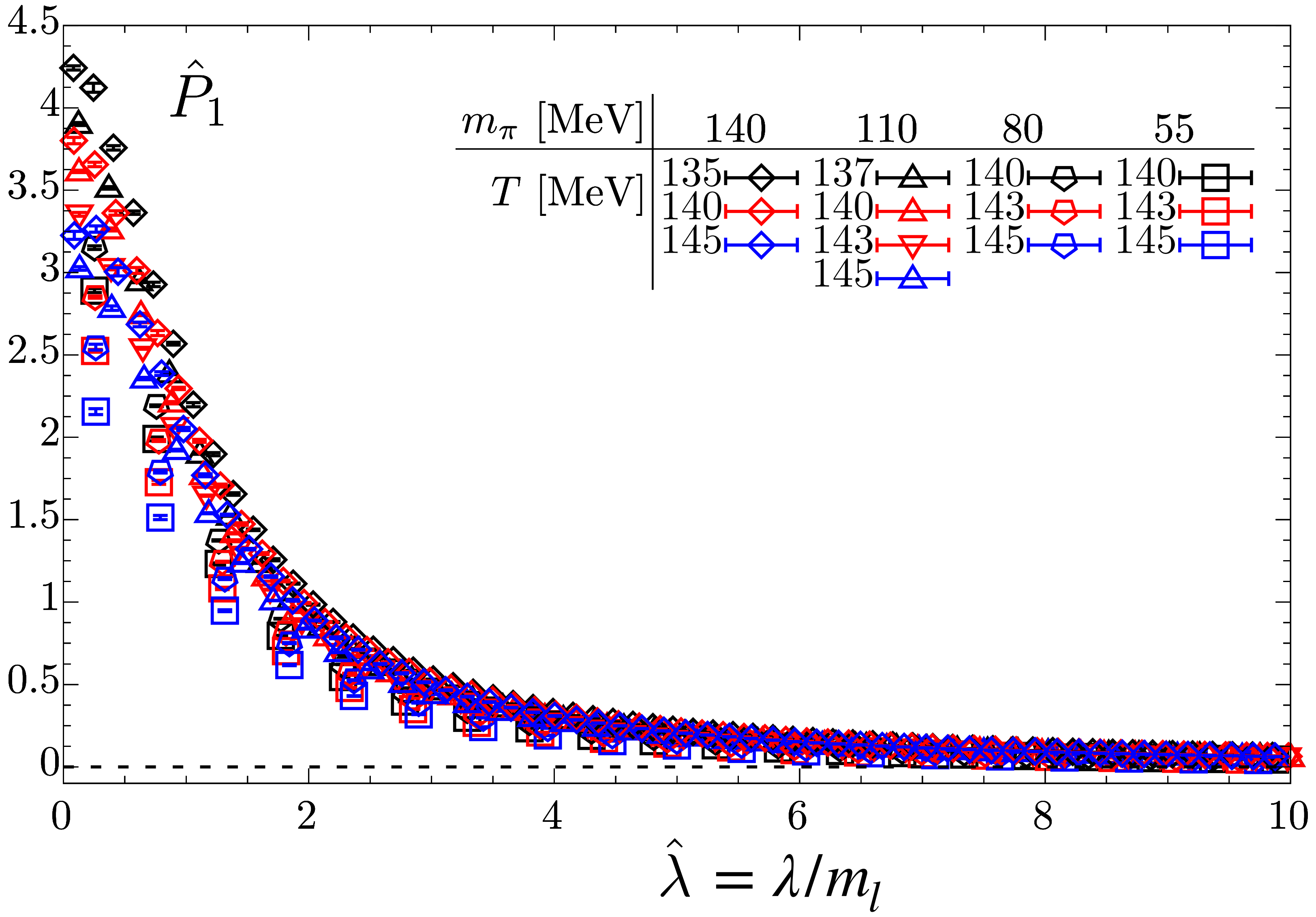
  H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)
- HotQCD configurations; measurements carried out on NSC^3 at CCNU, Wuhan Supercomputing Center & BNL

Nuclear Science Computing Center at CCNU

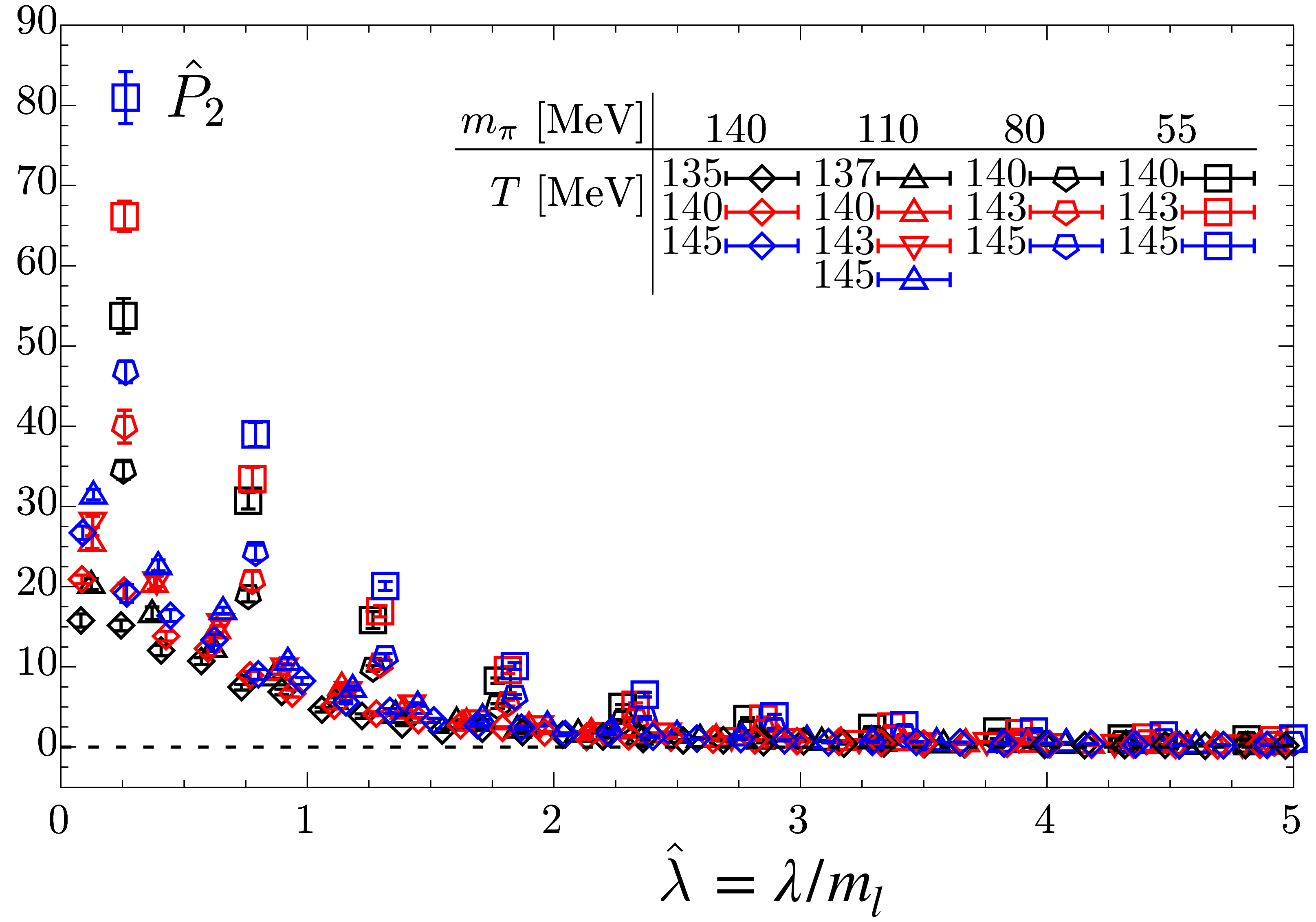


$P_n(\lambda)$ around T_c

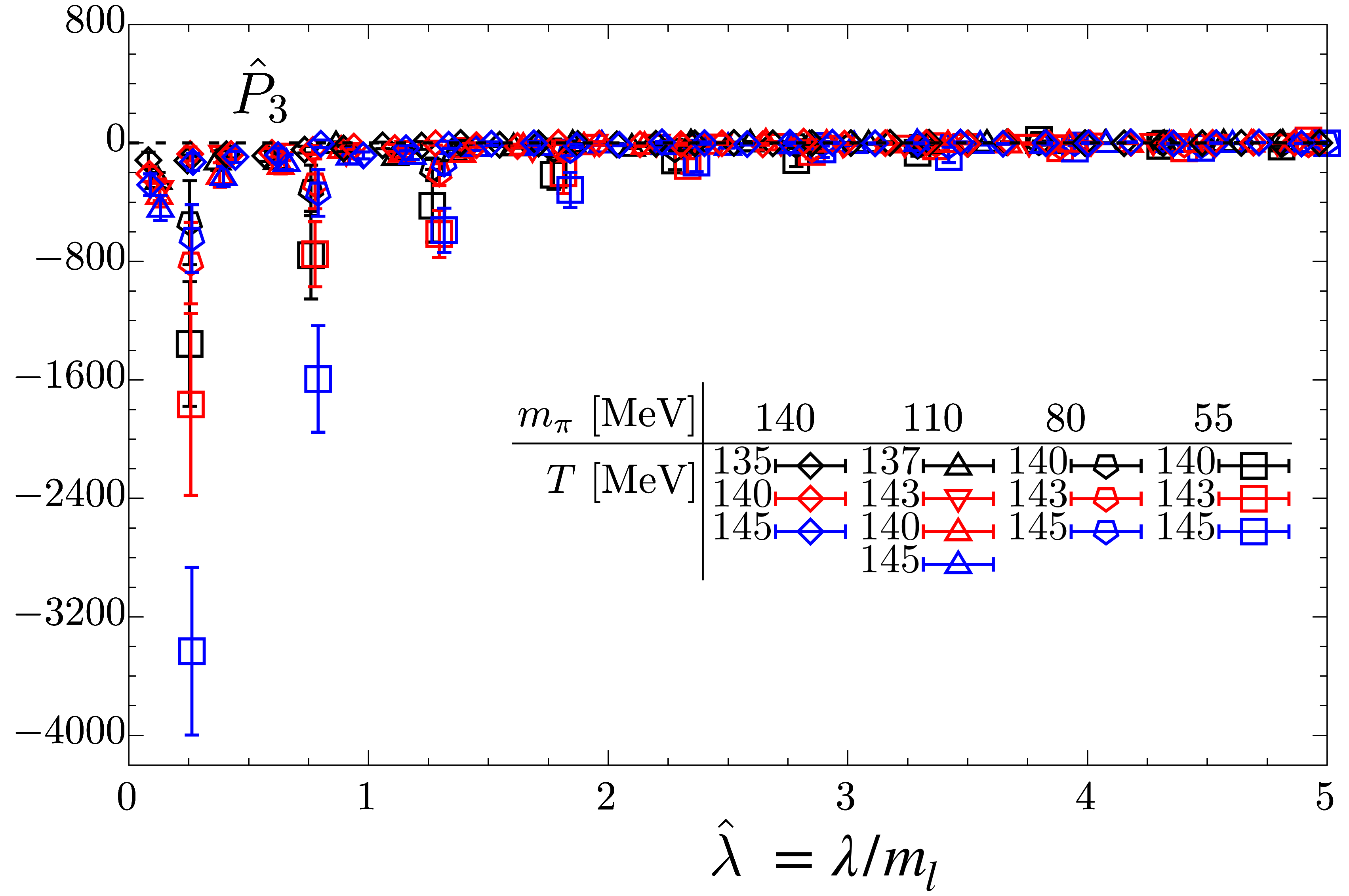
$$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$$



$$\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$$



$$\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$$



$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda})$ and $\hat{P}_3(\hat{\lambda})$

— Infrared lambda region dominates;

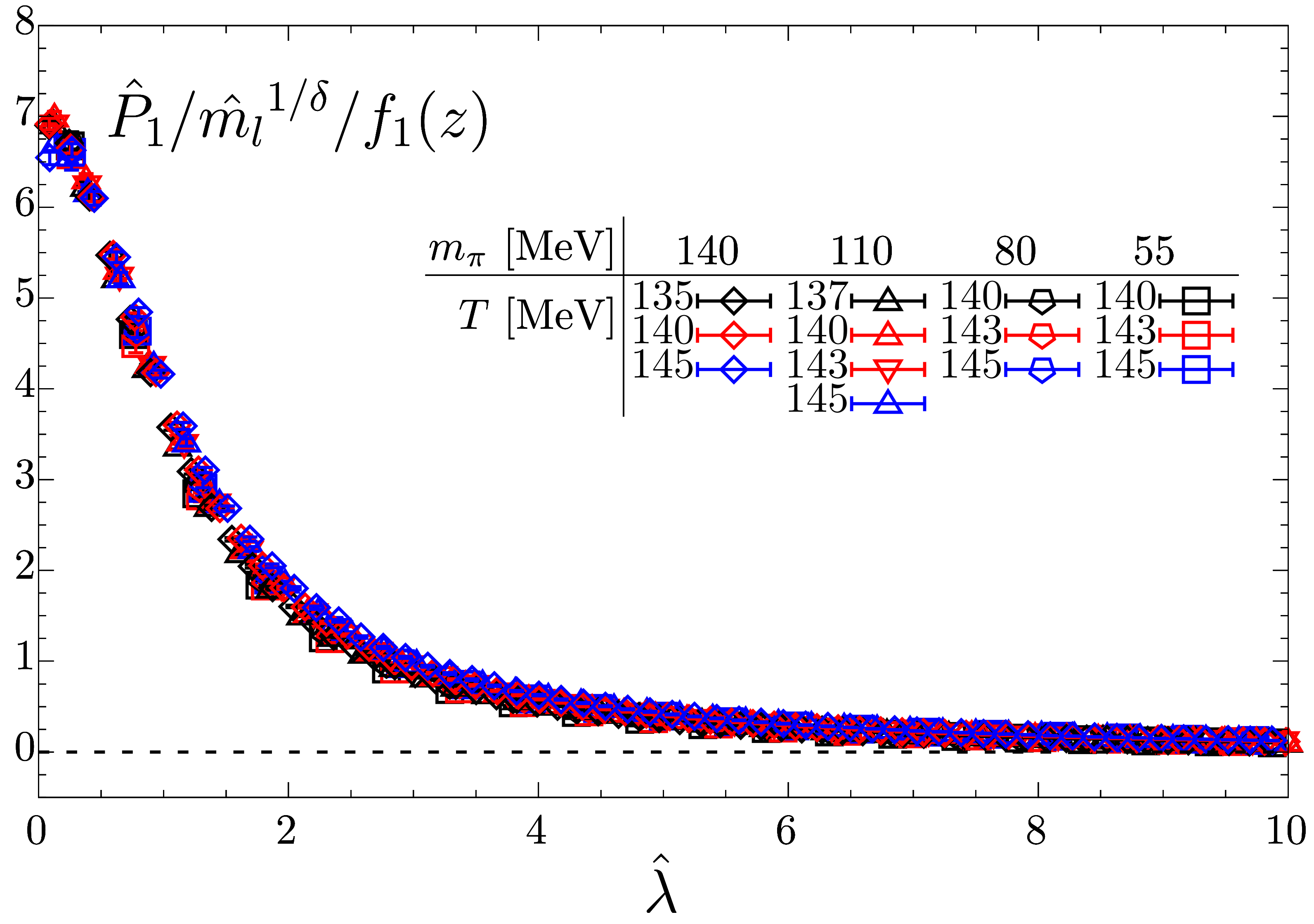
— Significant dependence on quark mass and temperature

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$

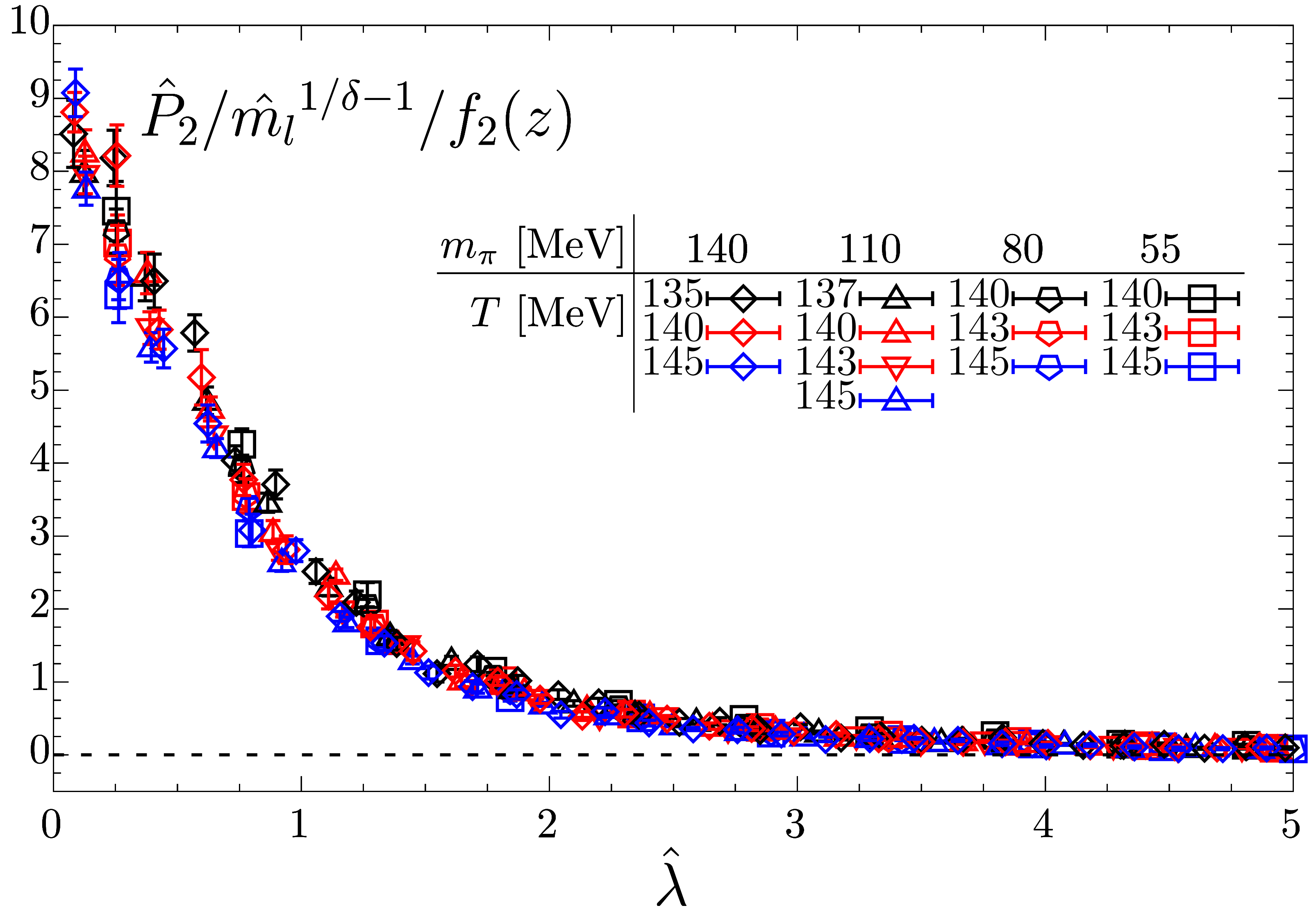
Rescaled $P_n(\lambda)$ around T_c

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n+1} f_n(z) g_n(\hat{\lambda})$

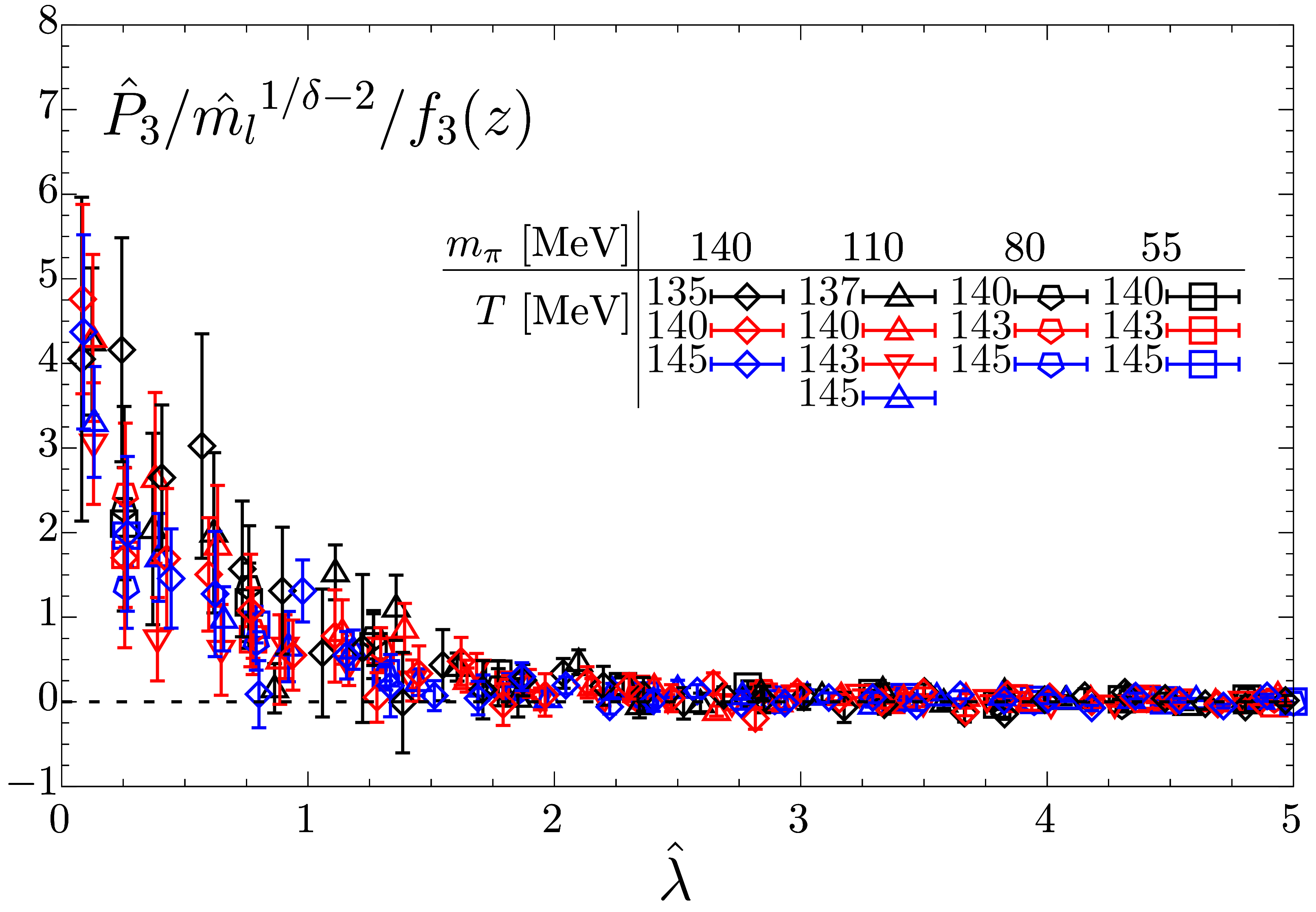
$\hat{P}_1(\hat{\lambda})/(m_l/m_s)^{1/\delta}/f_1(z)$



$\hat{P}_2(\hat{\lambda})/(m_l/m_s)^{1/\delta-1}/f_2(z)$



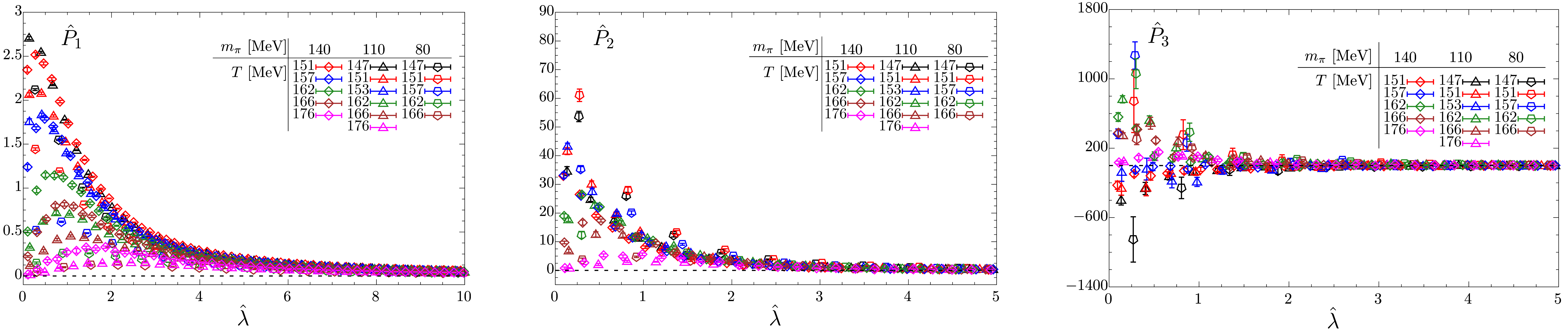
$\hat{P}_3(\hat{\lambda})/(m_l/m_s)^{1/\delta-2}/f_3(z)$



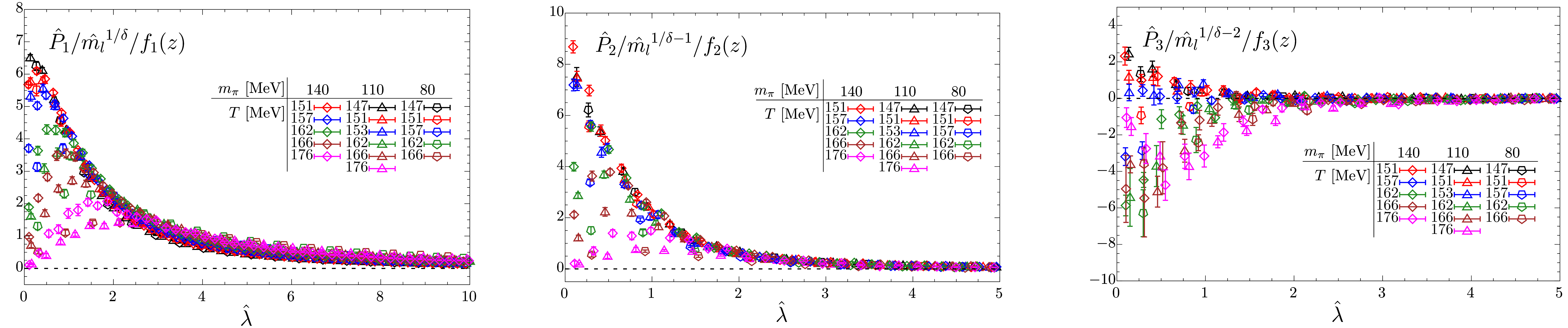
$z = z_0(m_l/m_s)^{-\frac{1}{\beta\delta}}(T - T_c)/T_c$: $O(2)$ scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of T_c , $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n+1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in $\hat{P}_n(\hat{\lambda})$ extend up to physical light quark mass

$P_n(\lambda)$ and Rescaled $P_n(\lambda)$ away from T_c

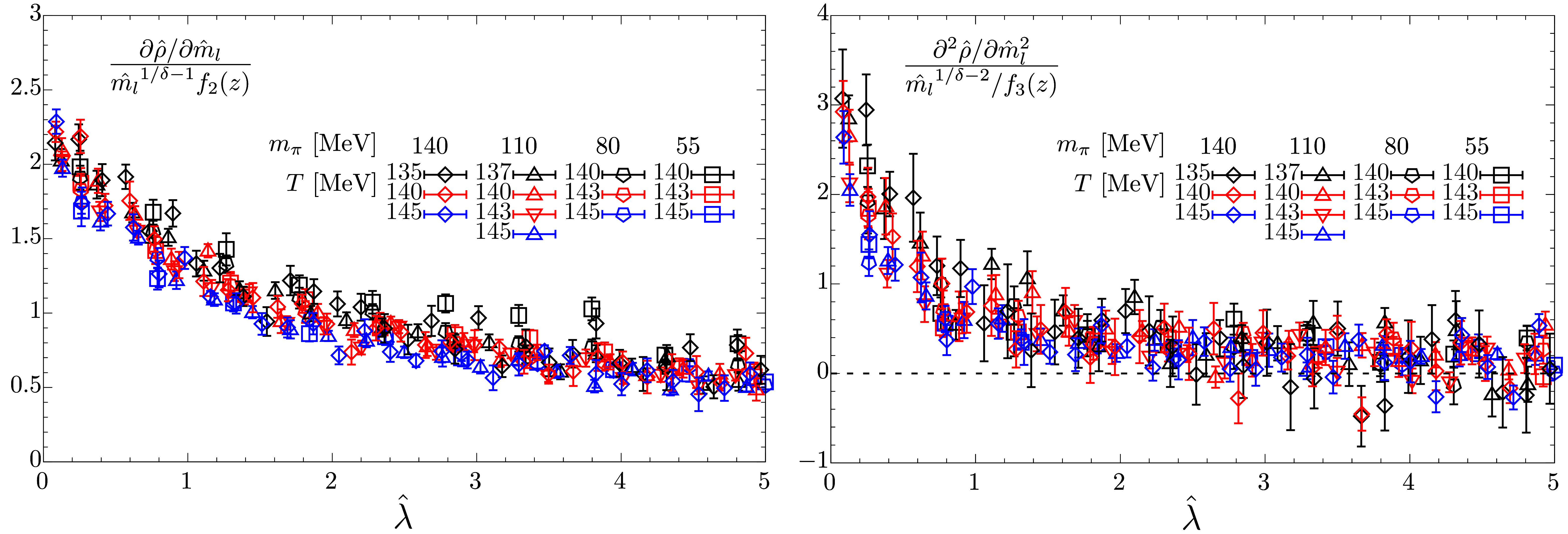


Away from T_c , no scaling behaviors are observed in $\hat{P}_n(\hat{\lambda})$

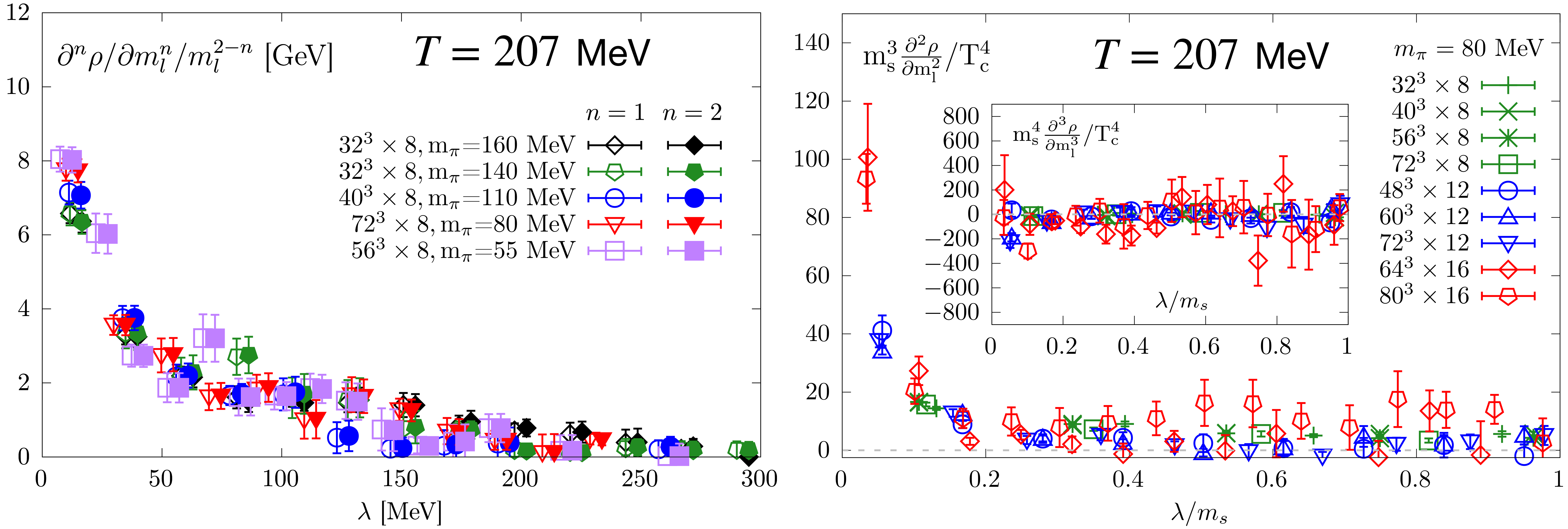


Transition from scaling to dilute instanton gas behaviors

$\partial^n\rho/\partial m_l^n \propto$ combinations of P_n



Work in progress with H.-T. Ding,Y. Zhang, et al.



H.-T. Ding, Y. Zhang et al., Phys. Rev. Lett. 126, 082001 (2021)

For $T \sim T_c$:
Governed by scaling behaviors

For high $T \sim 1.6T_c$:
Consistent with dilute instanton gas picture

$$m_l^{-1}\ \partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2\ \&\ \partial^3\rho/\partial m_l^3 \approx 0$$

$$\Rightarrow\ \rho(\lambda\to0,\ m_l\to0) \propto m_l^2\delta(\lambda)$$

Away from critical window: an unexplored region

$\partial^n\rho/\partial m_l^n$ at $T \in [171,\ 207]$ MeV

$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$ & $\partial^3\rho/\partial m_l^3 \approx 0$ do *NOT* recover simultaneously

$\Rightarrow$ Other kinds of mass dep. besides m^2 ? Hidden mechanism?

Work in progress with H.-T. Ding,Y. Zhang, et al.

Summary

☑ We establish a novel relation

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \cdots, P_U(\lambda_n; m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda\,.$$

n-th order cumulant of the chiral condensate

n-point correlation of the quark energy spectra

☑ A generalization of the Banks-Casher relation is obtained:

$$\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]\,.$$

☑ Microscopic encoding of macroscopic criticality

$$P_n(\lambda) = m_l^{1/\delta - n+1} f_n(z) g_n(\lambda)\,.$$

☑ Universal behaviors manifested in microscopic energy levels of QCD extend up to physical light quark masses.

☐ Transitioning from the dilute instanton gas picture to chiral phase transition … ?

Backup

Signatures of symmetry restorations

Susceptibilities defined as integrated two point correlation functions of quark bilinear $J_M(x) = \bar{q}(x)\Gamma_M q(x)$

$$\chi_M = \int d^4x \left\langle J_M(x) J_M^\dagger(0) \right\rangle$$

A. Bazavov et al., [HotQCD], PRD 86 (2012) 094503

N. Carabba et al., [HotQCD], PRD 105 (2022) 5, 054034

$$\begin{tikzcd}
\chi_{5,\mathrm{conn}} & \pi:\ \bar{q}\gamma_5\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU(2)_L\otimes SU(2)_R"] \arrow[d, leftrightarrow, "U(1)_A"'] & \sigma:\ \bar{q}q \arrow[d, leftrightarrow, "U(1)_A"] & \chi_{\mathrm{conn}}+\chi_{\mathrm{disc}} \\
\chi_{\mathrm{conn}} & \delta:\ \bar{q}\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU(2)_L\otimes SU(2)_R"'] & \eta:\ \bar{q}\gamma_5 q & \chi_{5,\mathrm{conn}}-\chi_{5,\mathrm{disc}}
\end{tikzcd}$$

Restoration of $SU(2)_L \otimes SU(2)_R$

$$\left.\begin{array}{l}\chi_\pi - \chi_\sigma = 0 \\ \chi_\delta - \chi_\eta = 0\end{array}\right. \Rightarrow \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}}$$

Effective restoration of $U(1)_A$

$$\left.\begin{array}{l}\chi_\pi - \chi_\delta = 0 \\ \chi_\sigma - \chi_\eta = 0\end{array}\right. \Rightarrow \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}} = 0$$

Related to Dirac eigenvalues:

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

Microscopic origin in Dirac eigenvalues

$$\langle\bar{\psi}\psi\rangle_l = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

- Restoration of $SU(2)_L \otimes SU(2)_R$ symmetry:

$\rho(0) = 0$ from Banks-Casher formula $\lim_{m_l\to 0}\lim_{V\to\infty}\langle\bar{\psi}\psi\rangle_l = \lim_{m_l\to 0}\lim_{V\to\infty} 2\pi\rho\left(0{,}m_l\right)$

Banks and Casher, NPB 169 (1980) 103

- Effective restoration of $U(1)_A$ symmetry:

A sizable gap in the near-zero mode

Cohen, arXiv:nucl-th/9801061

- Underlying structure of $\rho(\lambda, m_l)$ responsible for symmetry restorations:

$$\rho(\lambda, m_l) = \mathrm{c}_0 + \mathrm{c}_1\lambda + \mathrm{c}_2 m_l^2\delta(\lambda) + \mathrm{c}_3 m_l + \mathrm{c}_4 m_l^2 + \cdots$$

$$\Rightarrow \begin{aligned} &\langle\bar{\psi}\psi\rangle = 2\mathrm{c}_0\pi - 4\mathrm{c}_1 m_l \ln(m_l) + 2\mathrm{c}_2 m_l + 2\mathrm{c}_3\pi + 2\pi\mathrm{c}_4 m_l^2 \\ &\chi_\pi - \chi_\delta = 2\mathrm{c}_0\pi/m_l + 4\mathrm{c}_1 + 4\mathrm{c}_2 + 2\mathrm{c}_3\pi + 2\mathrm{c}_4\pi m_l \end{aligned}$$

c_0 & c_1 term: break both symmetries — Gross et al., RMP 81'

c_2 term: dilute instanton gas predicts — HotQCD, PRD86(2012)094503

c_3 term: break $U(1)_A$ symmetry — Aoki et al., PRD86(2012)114512

c_4 term: make $U(1)_A$ anomaly unmanifested in 2-pt correlators

Mass derivatives of ρ are needed to determine the microscopic origin

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

Mode number : $$n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[\sum_{j=0}^{p} g_j^p \gamma_j v_k^T T_j(A) v_k \right]$$

T_j : Chebyshev polynomial

γ_j & g_j^p : expansion coefficients

n_v : number of random vectors

p : number of polynomial orders

eigenvalue spectrum : $$\rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2, \lambda+\delta/2]}}{2\delta\lambda}$$

$1/4$: Staggered Fermion Discretization Scheme

$1/2$: positive and negative eigenvalue pairs

$\delta\lambda$: bin-size

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Yu Zhang, Lattice 19', arXiv: 2001.05217

Cossu et al., arXiv: 1601.00744

$\partial^n\rho/\partial m_l^n$ and Quark Energy Spectra

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho\left(\lambda, m_l\right) = \frac{T}{VZ[U]}\int D[U]e^{-S_G[U]}\det\left[\not{D}[U]+m_s\right]\times\left(\det\left[\not{D}[U]+m_l\right]\right)^2\rho_U(\lambda)$$

Eigenvalue spectrum for a given configuration:

$$\rho_U(\lambda) = \sum_j \delta\left(\lambda - \lambda_j\right)$$

m_l dependence enters ρ:

$$\det\left[\not{D}[U]+m_l\right] = \prod_j\left(+\mathrm{i}\lambda_j + m_l\right)\left(-\mathrm{i}\lambda_j + m_l\right)$$

$$= \exp\left(\int_0^\infty \mathrm{d}\lambda\rho_U(\lambda)\mathrm{In}\left[\lambda^2+m_l^2\right]\right)$$

Partition function:

$$Z[U] = \int D[U]e^{-S_G[U]}\det\left[\not{D}[U]+m_s\right]\times\left(\det\left[\not{D}[U]+m_l\right]\right)^2$$

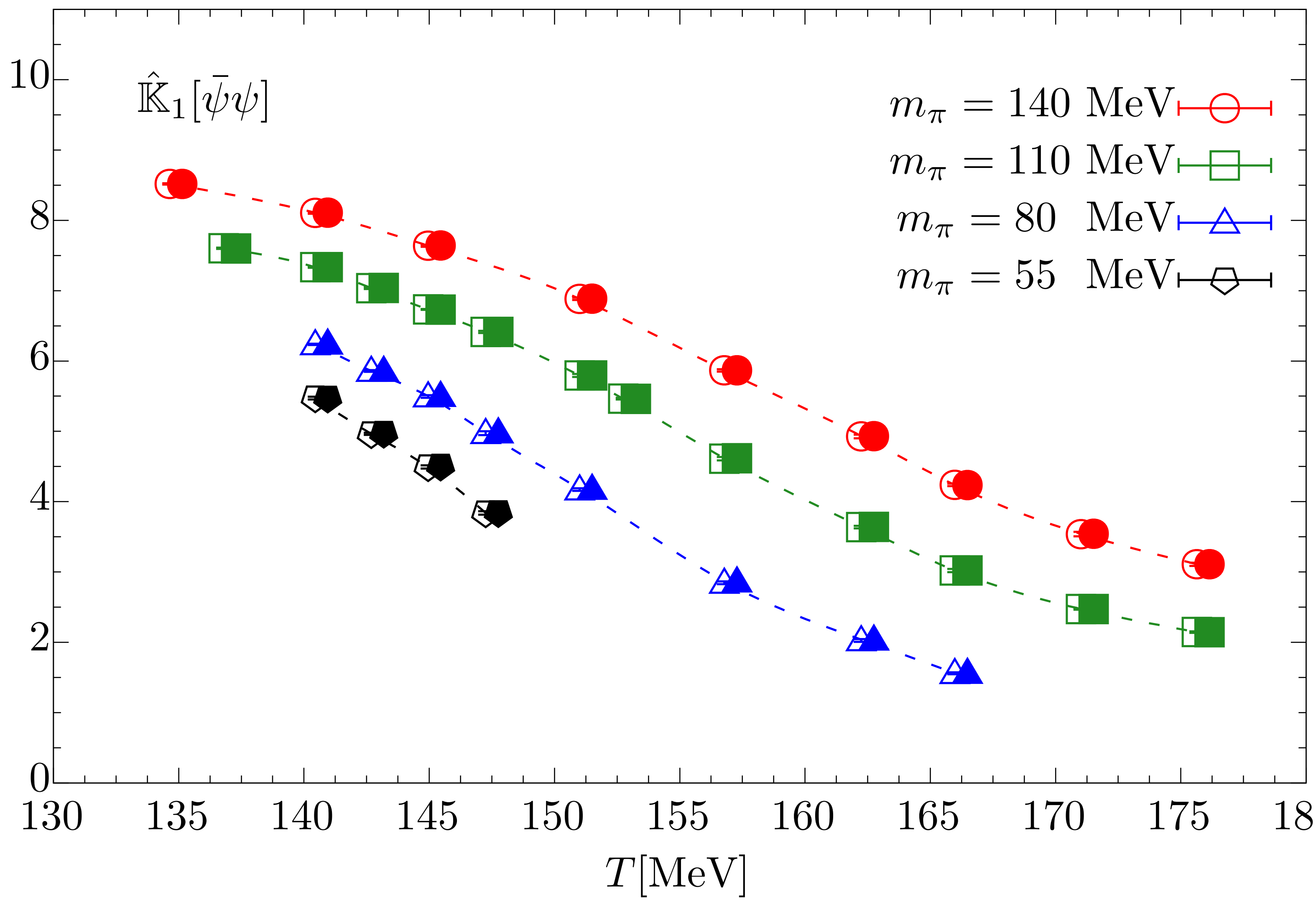
Mass derivative of $\rho(\lambda, m_l)$:

$$\frac{V}{T}\frac{\partial\rho(\lambda, m_l)}{\partial m_l} = \int_0^\infty \mathrm{d}\lambda_2\,\frac{4m_lC_2}{\lambda_2^2+m_l^2} \qquad C_2\left(\lambda, \lambda_2\right) = \left\langle\rho_U(\lambda)\rho_U\left(\lambda_2\right)\right\rangle - \left\langle\rho_U(\lambda)\right\rangle\left\langle\rho_U\left(\lambda_2\right)\right\rangle$$

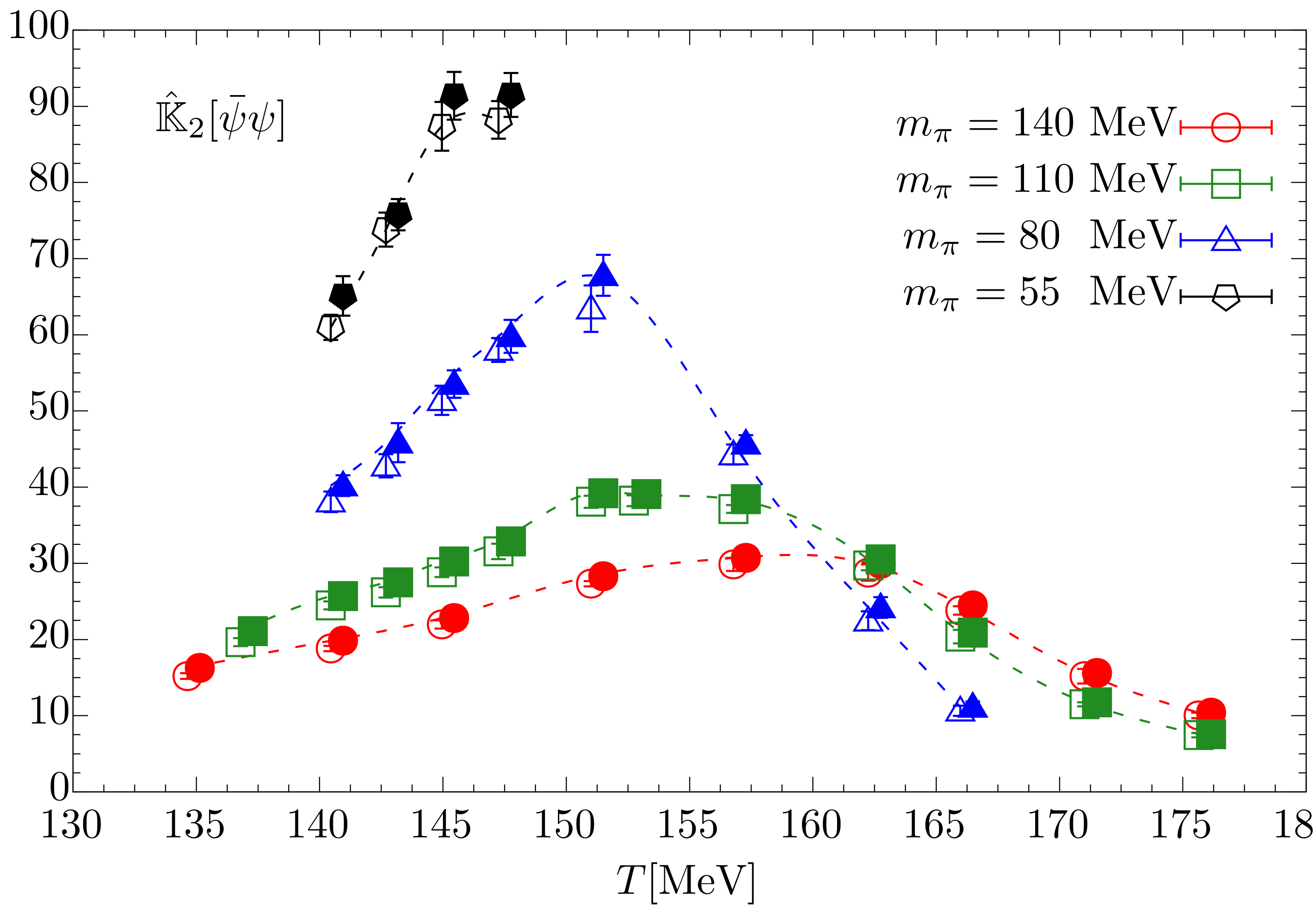
H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

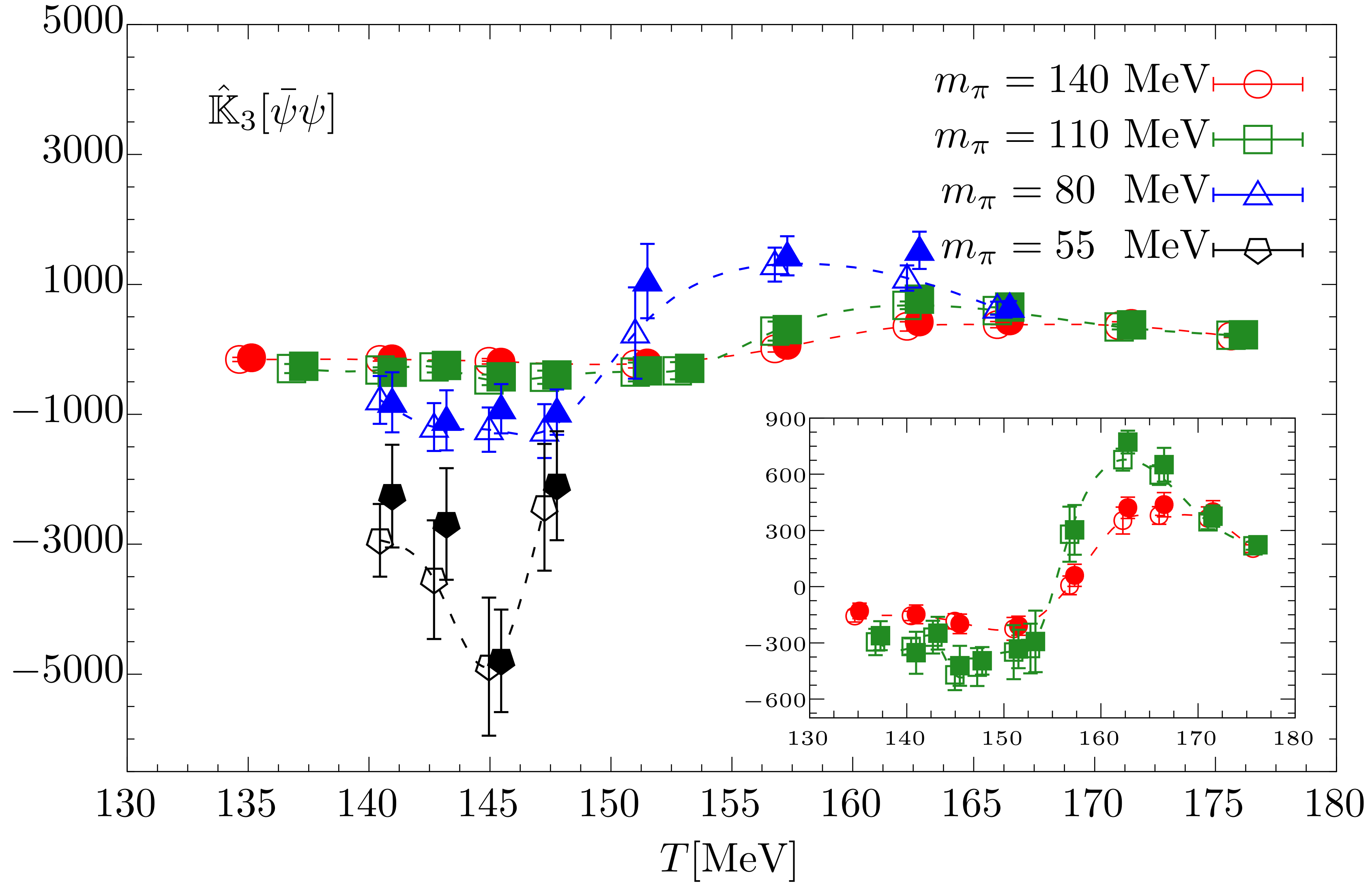
$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_1[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_1(\lambda)\ d\lambda$$



$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_2[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_2(\lambda)\ d\lambda$$



$$\mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_3[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_3(\lambda)\ d\lambda$$



Open symbols: computation via inversions of the fermion matrix $\mathrm{Tr}M^{-1}$

Filled symbols: reconstructed from $P_n(\lambda)$

Cumulants related to $P_n(\lambda)$ can successfully reproduce their corresponding results from inverse fermion matrix

Criticality in Macroscopic Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$

$O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_\tau = 8) = 144.2(6)$ MeV

Parameters adopted from: S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)

- For $|z/z_0| \lesssim 0.2$, $K_n(z)/K_n(z=0)$ with $n = 1,2,3$ can be well described by $O(2)$ scaling function $\dfrac{f_{n-1}(z)}{f_{n-1}(z=0)}$
- For $|z/z_0| \lesssim 0.2$, K_n rescaled by $H^{1/\delta-n+1}$ show small quark dependence