Microscopic Encoding of Macroscopic Universality: Scaling properties of Dirac Eigenspectra near QCD Chiral Phase Transition

*How do universal behaviors at macroscale arise from quarks and gluons?*

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based on Phys. Rev. Lett. 131 (2023) 16, 161903 & work in progress,

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Exploration of QCD phase diagram

Searching for signatures of criticality in Macroscopic quantities

D. Almaalol et al., arXiv:2209.05009

How does criticality at Macroscale arise from Microscopic d.o.f of QCD?
Universal Scaling in QCD Chiral Transition

- Restoration of chiral symmetry:  
  - Crossover at physical point;  
  - True chiral phase transition only in limiting cases

- Role of $U(1)_A$ symmetry in chiral phase transition:
  Broken around $T_c \Rightarrow$ 2nd order $O(4)$ phase transition

Universal $O(2)$ scaling behaviors in staggered discretization scheme

**Order parameter:**
$$M(t, h) = h^{1/\delta} f_1(z) + f_{\text{reg}}(T, H)$$

**Order parameter susceptibility:**
$$\chi_M(t, h) = \partial M / \partial H = h_0^{-1} h^{1/\delta - 1} f_2(z) + f'_{\text{reg}}$$

**Higher order susceptibility:**
$$e.g., \partial^2 \chi_M / \partial H^2 = h_0^{-2} h^{1/\delta - 2} f_3(z) + f''_{\text{reg}}$$

H.-T. Ding et al., [HotQCD], PRL 123 (2019) 062002

A. Bazavov et al., [HotQCD], PLB 795 (2019) 15

Pisarski & Wilczek, PRD 29 (1984) 338
Pelissetto & Vicari, PRD 88 (2013) 105018

S. Ejiri et al., PRD 80, 094505 (2009)

H.-T. Ding, HWP et al., arXiv: 2112.00318

H.-T. Ding et al., arXiv:1504.05274
Hard to Reach Microscopic Origin of Criticality

Dimensionality & Symmetries determine by Scaling behavior around 2nd order phase transition irrespective of Microscopic d.o.f. & Interactions

E.g.: Liquid to superfluid $\lambda$-transition in $^4$He

fundamental d.o.f.: electrons & photons

$C \left( \frac{J}{mol \cdot K} \right)$

$T[K]$

He II

He I

Same dimensionality & symmetry

Same 3-d O(2) scaling behaviors

Chiral phase transition in the chiral limit of light quark using staggered fermion discretization

fundamental d.o.f.: quarks & gluons

This talk:

A first lattice QCD-based understanding of Microscopic Origin of Criticality in QCD!

✓ Able to study via effective theories

✗ No infos about fundamental micro level of d.o.f. via EFT!!
Banks-Casher Relation: Connect Macro to Micro in Chiral Limit

Chiral order parameter:

\[ \langle \bar{\psi} \psi \rangle = \int_0^\infty \frac{4m}{\lambda^2 + m^2} \rho(\lambda, m) d\lambda \rightarrow \pi \rho(\lambda = 0) \]

\[ \rho(\lambda, m) = \frac{T}{V} \langle \rho_U(\lambda) \rangle \equiv \frac{T}{V} \langle \sum_j \delta(\lambda - \lambda_j) \rangle \text{ with } D \psi_j = i \lambda_j \psi_j \]

Connect Macroscopic quantity to Microscopic d.o.f.: \( \langle \bar{\psi} \psi \rangle \Longleftrightarrow \rho_U(\lambda) : \text{energy spectra of massless quarks} \)

Next: More generally…
From Quark Energy Spectra to its Cumulants

1st order cumulant:
\[
\langle \bar{\psi} \psi \rangle = \frac{T}{V} \frac{2 \text{Tr} [DU] + m}{(2m)^{-1}}
\]

1-point correlation of quark energy spectra:
\[
\frac{4m \rho_U(\lambda)}{\lambda^2 + m^2}
\]

Generalization: from \textit{generating functional} of cumulants

- \textbf{Generating functional}: \( \mathcal{G}(m; \epsilon) = \ln \left\langle \exp \left\{ -m \bar{\psi} \psi(\epsilon) \right\} \right\rangle_0 = \ln \left\langle \exp \left\{ -m \int_0^\infty P_U(\lambda; \epsilon) \, d\lambda \right\} \right\rangle_0 \)

- \textbf{n-th order cumulant of} \( \bar{\psi} \psi \):
\[
\mathbb{K}_n[\bar{\psi} \psi(m)] = \left. \frac{T}{V} \frac{(-1)^n}{\partial m^n} \mathcal{G}(m; \epsilon) \right|_{\epsilon = m}
\]

\( K_1[X_1, X_2, \ldots, X_n] \) denotes 1st order joint cumulant of \( n \)-variables

H.-T. Ding, HWP et al., PRL 131, 161903
Microscopic Encoding of Macroscopic Criticality

\[ \kappa_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \ldots, P_U(\lambda_n; m)] \prod_{i=1}^n d\lambda_i \equiv \int_0^\infty P_n(\lambda) \ d\lambda \]

**n-th order cumulant of the chiral order parameter**

**n-point correlation of the quark energy spectra**

Chiral condensate:

\[ \kappa_1[\bar{\psi}\psi] = \frac{T}{V} \langle \bar{\psi}\psi(m) \rangle = \int_0^\infty P_1(\lambda) \ d\lambda \quad \text{Around } T_c \quad \sim m^{1/\delta} f_1(z) \]

Disconnected susceptibility:

\[ \kappa_2[\bar{\psi}\psi] = \frac{T}{V} \left\langle \left[ \bar{\psi}\psi(m) - \langle \bar{\psi}\psi(m) \rangle \right]^2 \right\rangle = \int_0^\infty P_2(\lambda) \ d\lambda \quad \text{Around } T_c \quad \sim m^{1/\delta-1} f_2(z) \]

\[ \cdots \cdots \]

\[ \kappa_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda) \ d\lambda \quad \text{Around } T_c \quad \sim m^{1/\delta-n} f_n(z) \]

How does the criticality of \( \kappa_n[\bar{\psi}\psi] \) arise from \( P_n(\lambda) \)?

H.-T. Ding, HWP et al., PRL 131, 161903
Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit:

\[
P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}
\]

\[
P_U(\lambda; m \to 0) = 2\pi\rho_U(\lambda)\delta(\lambda)
\]

**Generalized Banks-Casher relation:**

\[
\lim_{m \to 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), \rho_U(0), \ldots, \rho_U(0)]\delta(\lambda)
\]

\[
\Rightarrow \quad \lim_{m \to 0} K_n[\bar{\psi}\psi] = (2\pi)^n K_n[\rho_U(0)]
\]

\(n = 1\) back to Banks-Casher relation!

Criticality in \(\lim_{m \to 0} K_n[\bar{\psi}\psi]\) must arise from universal behaviors of \(\lambda\)-independent \(K_n[\rho_U(0)]\)

\[
K_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda) \, d\lambda \quad \text{Around } T_c \quad \sim m_1^{1/\delta-n+1}f_n(z)
\]

**Conjecture:**

\[
P_n(\lambda) = m^{1/\delta-n+1}f_n(z)g_n(\lambda)
\]

Scaling arise from \(P_n(\lambda)\) at deep infrared \(\lambda\) region

Include all system-specific \(\lambda\)-dependence

H.-T. Ding, HWP et al., PRL 131, 161903
Highly improved staggered quarks and tree-level Symanzik gauge action

Lattice size: $N_\tau = 8, \ N_\sigma = 32, 40, 56$

Quark mass: $m_s^{\text{phy}}/m_l = 27, 40, 80, 160$

($m_\pi \approx 140, 110, 80, 55 \text{ MeV}$)

Temperatures: $T \in (135, 176) \text{ MeV}$

$\rho_U(\lambda)$ computed via Chebyshev filtering technique


HotQCD configurations; measurements carried out on NSC$^3$ at CCNU, Wuhan Supercomputing Center & BNL
$P_n(\lambda)$ around $T_c$

$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$

$\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$

$\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$

$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda})$ and $\hat{P}_3(\hat{\lambda})$

- Infrared lambda region dominates;
- Significant dependence on quark mass and temperature

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1}f_n(z)g_n(\hat{\lambda})$
Rescaled $P_n(\lambda)$ around $T_c$

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$

\[
\begin{align*}
\hat{P}_1(\hat{\lambda}) &/ (m_l/m_s)^{1/\delta} f_1(z) \\
\hat{P}_2(\hat{\lambda}) &/ (m_l/m_s)^{1/\delta - 1} f_2(z) \\
\hat{P}_3(\hat{\lambda}) &/ (m_l/m_s)^{1/\delta - 2} f_3(z)
\end{align*}
\]

$z = z_0(m_l/m_s)^{-\frac{1}{\delta}}(T - T_c)/T_c : O(2)$ scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of $T_c$, $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in $\hat{P}_n(\hat{\lambda})$ extend up to physical light quark mass
$P_n(\lambda)$ and Rescaled $P_n(\hat{\lambda})$ away from $T_c$

Away from $T_c$, no scaling behaviors are observed in $\hat{P}_n(\hat{\lambda})$
Transition from scaling to dilute instanton gas behaviors

\[ \frac{\partial^n \rho}{\partial m_i^n} \propto \text{combinations of } P_n \]

For \( T \sim T_c \):
Governed by scaling behaviors

For high \( T \sim 1.6T_c \):
Consistent with dilute instanton gas picture

\[ m_i^{-1} \frac{\partial \rho}{\partial m_i} \approx \frac{\partial^2 \rho}{\partial m_i^2} \quad \text{and} \quad \frac{\partial^3 \rho}{\partial m_i^3} \approx 0 \]

\[ \Rightarrow \rho(\lambda \to 0, m_i \to 0) \propto m_i^2 \delta(\lambda) \]
Away from critical window: an unexplored region

\[ \partial^n \rho / \partial m_i^n \text{ at } T \in [171, 207] \text{ MeV} \]

\[ m_i^{-1} \partial \rho / \partial m_i \approx \partial^2 \rho / \partial m_i^2 \quad \& \quad \partial^3 \rho / \partial m_i^3 \approx 0 \quad \text{do NOT recover simultaneously} \]

⇒ Other kinds of mass dep. besides \( m^2 \) ? Hidden mechanism?

Work in progress with H.-T. Ding, Y. Zhang, et al.
We establish a novel relation:
\[
\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \ldots, P_U(\lambda_n; m)] \prod_{i=1}^n d\lambda_i \equiv \int_0^\infty P_n(\lambda) \, d\lambda.
\]

A generalization of the Banks-Casher relation is obtained:
\[
\lim_{m \to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)].
\]

Microscopic encoding of macroscopic criticality
\[
P_n(\lambda) = m_1^{1/\delta - n + 1} f_n(z) g_n(\lambda).
\]

Universal behaviors manifested in microscopic energy levels of QCD extend up to physical light quark masses.

Transitioning from the dilute instanton gas picture to chiral phase transition … ?
Backup
**Signatures of symmetry restorations**

Susceptibilities defined as integrated two point correlation functions of quark bilinear $J_M(x) = \bar{q}(x)\Gamma_M q(x)$

\[
\chi_M = \int d^4x \left\langle J_M(x)J_M^\dagger(0) \right\rangle
\]

A. Bazavov et al., [HotQCD], PRD 86 (2012) 094503

N. Carabba et al., [HotQCD], PRD 105 (2022) 5, 054034

Related to Dirac eigenvalues:

\[
\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{4m_l^2 \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \quad \chi_\pi - \chi_\sigma = \int_0^\infty d\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{(\lambda^2 + m_l^2)^2} \quad \chi_{\text{disc}} = \int_0^\infty d\lambda \frac{4m_l \cdot \partial \rho / \partial m_l}{\lambda^2 + m_l^2}
\]
Microscopic origin in Dirac eigenvalues

\[
\langle \bar{\psi} \psi \rangle_1 = \int_0^\infty d\lambda \frac{4m_l \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \quad \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{(\lambda^2 + m_l^2)^2} \quad \chi_{\text{disc}} = \int_0^\infty d\lambda \frac{4m_l \cdot \partial \rho / \partial m_l}{\lambda^2 + m_l^2}
\]

- Restoration of $SU(2)_L \otimes SU(2)_R$ symmetry:
  \[\rho(0) = 0\] from Banks-Casher formula
  \[\lim_{m_l \to 0} \lim_{V \to \infty} \langle \bar{\psi} \psi \rangle_1 = \lim_{m_l \to 0} \lim_{V \to \infty} 2\pi \rho(0, m_l)\]
  Banks and Casher, NPB 169 (1980) 103

- Effective restoration of $U(1)_A$ symmetry:
  A sizable gap in the near-zero mode
  Cohen, arXiv:nucl-th/9801061

- Underlying structure of $\rho(\lambda, m_l)$ responsible for symmetry restorations:
  \[
  \rho(\lambda, m_l) = c_0 + c_1 \lambda + c_2 m_l^2 \delta(\lambda) + c_3 m_l + c_4 m_l^2 + \ldots
  \]
  \[
  \langle \bar{\psi} \psi \rangle = 2c_0 \pi - 4c_1 m_l \ln(m_l) + 2c_2 m_l + 2c_3 \pi + 2\pi c_4 m_l^2
  \]
  \[
  \chi_\pi - \chi_\delta = 2c_0 \pi / m_l + 4c_1 + 4c_2 + 2c_3 \pi + 2c_4 \pi m_l
  \]
  \[
  c_0 \text{ & } c_1 \text{ term: break both symmetries}
  \quad c_2 \text{ term: dilute instanton gas predicts}
  \quad c_3 \text{ term: break } U(1)_A \text{ symmetry}
  \quad c_4 \text{ term: make } U(1)_A \text{ anomaly unmanifested in 2-pt correlators}
  \]

Mass derivatives of $\rho$ are needed to determine the microscopic origin

Gross et al., RMP 81'
HotQCD, PRD86(2012)094503
Aoki et al., PRD86(2012)114512
Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

Mode number: \( n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[ \sum_{j=0}^{p} g_j^p \gamma_j v_k^T T_j(A) v_k \right] \)

\( T_j \) : Chebyshev polynomial
\( \gamma_j \) & \( g_j^p \) : expansion coefficients
\( n_v \) : number of random vectors
\( P \) : number of polynomial orders

Eigenvalue spectrum: \( \rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2,\lambda+\delta/2]}}{2\delta\lambda} \)

1/4 : Staggered Fermion Discretization Scheme
1/2 : positive and negative eigenvalue pairs
\( \delta\lambda \) : bin-size


Cossu et al., arXiv: 1601.00744
\[ \frac{\partial^n \rho}{\partial m_l^n} \text{ and Quark Energy Spectra} \]

Eigenvalue spectrum for (2+1)-flavor QCD:

\[
\rho \left( \lambda, m_l \right) = \frac{T}{VZ[U]} \int D[U] e^{-S_{G[U]}} \det \left[ \mathcal{D}[U] + m_s \right] \times \left( \det \left[ \mathcal{D}[U] + m_l \right] \right)^2 \rho_U(\lambda)
\]

Partition function:

\[
Z[U] = \int D[U] e^{-S_{G[U]}} \det \left[ \mathcal{D}[U] + m_s \right] \times \left( \det \left[ \mathcal{D}[U] + m_l \right] \right)^2 = \exp \left( \int_0^\infty d\lambda \rho_U(\lambda) \text{ln} \left[ \lambda^2 + m_l^2 \right] \right)
\]

Mass derivative of \( \rho(\lambda, m_l) \):

\[
\frac{V}{T} \frac{\partial \rho(\lambda, m_l)}{\partial m_l} = \int_0^\infty d\lambda_2 \frac{4m_l C_2}{\lambda_2^2 + m_l^2} \quad C_2(\lambda, \lambda_2) = \left\langle \rho_U(\lambda) \rho_U(\lambda_2) \right\rangle - \left\langle \rho_U(\lambda) \right\rangle \left\langle \rho_U(\lambda_2) \right\rangle
\]

Eigenvalue spectrum for a given configuration:

\[
\rho_U(\lambda) = \sum_j \delta \left( \lambda - \lambda_j \right)
\]

\( m_l \) dependence enters \( \rho \):
Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

Open symbols: computation via inversions of the fermion matrix $\text{Tr}M^{-1}$
Filled symbols: reconstructed from $P_n(\lambda)$

Cumulants related to $P_n(\lambda)$ can successfully reproduce their corresponding results from inverse fermion matrix.
Criticality in Macroscopic Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ 

$O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_f = 8) = 144.2(6)$ MeV


- For $|z/z_0| \lesssim 0.2$, $K_n(z)/K_n(z = 0)$ with $n = 1, 2, 3$ can be well described by $O(2)$ scaling function $\frac{f_n(z)}{f_n(z = 0)}$

- For $|z/z_0| \lesssim 0.2$, $K_n$ rescaled by $H^{1/\delta-n+1}$ show small quark dependence