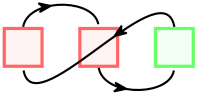
Spin Alignment Induced by Curved Freeze-Out Hypersurface

Speaker: Zhong-Hua Zhang (张衷华)

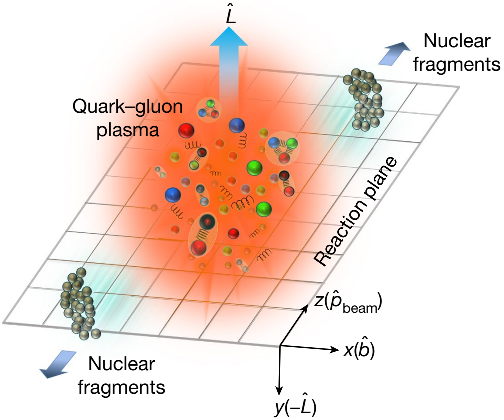
Collaborator: Xu-Guang Huang

Fudan University, Shanghai

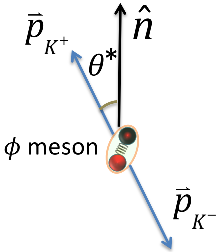
July 19th @ XQCD2024



Spin Alignment



Heavy Ion Collision



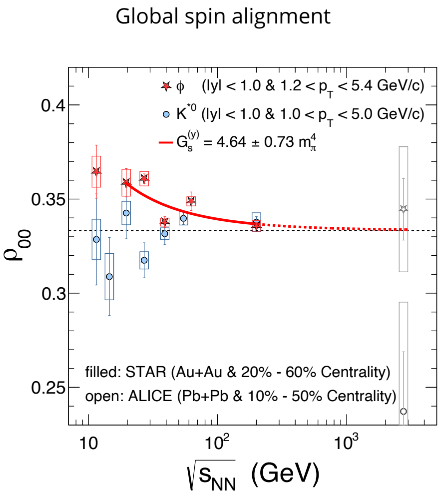
$$\frac{1}{N}\frac{dN}{d\cos\theta^*} = \frac{1}{2} + \frac{3}{4}(3\cos^2\theta^* - 1)\left(\Theta_{00} - \frac{1}{3}\right)$$

spin alignment

Liang, Wang, PLB 629, 20 (2005)

STAR, Nature 614, 244-248 (2023)

Global Spin Alignment



- ϕ meson $\Theta_{00} > 1/3$ and too big

$$\Theta_{00} - \frac{1}{3} \sim P_{\Lambda}^{2} \sim 10^{-4}$$

- K^{*0} different from ϕ

Figure: STAR, Nature 614, 244-248 (2023)

Physical Mechanisms

$$\phi \text{ meson: } \delta\Theta_{00} = \Theta_{00} - \frac{1}{3} \approx +c_\Lambda + c_B + c_s + c_F + c_L + c_H + c_\phi + c_g + \cdots$$

Physical mechanism	$\delta\Theta_{00}$
c_Λ: Quark coalescence + vorticity [1]	magnitude $\sim -10^{-4}$
c_B: Quark coalescence + EM-field [1]	magnitude $\sim 10^{-4}$
c_S: Spectrum splitting [2]	unclear
c_F: Quark fragmentation [3]	magnitude $\sim 10^{-5}$
c_L: Local spin alignment [4]	magnitude $\sim -10^{-2}$
c_H: Second-order hydro fields [5]	unclear
c_φ: Vector meson field [6]	> 0, fit to data
c_g: Glasma fields [7]	< 0, magnitude unclear

[1]. Liang, Wang, PLB 629, 20 (2005);
Yang *et al.*PRC 97, 034917 (2018);
Xia *et al.*PLB 817, 136325 (2021);
Becattini *et al.*PRC 88, 034905 (2013).
[2]. Liu, Li, arXiv: 2206.11890;
Sheng *et al.*, Eur.Phys.J.C 84, 299 (2024);
Wei, Huang, Chin.Phys.C 47,104015 (2023).
[3]. Liang, Wang PLB 629, 20 (2005);
[4]. Xia *et al.*PLB 817, 136325 (2021);
Gao, PRD 104, 076016 (2021).
[5]. Kumar, Yang, Gubler, PRD 109, 054038(2024);
Gao, Yang, Chin.Phys.C 48, 053114 (2024);
ZZ, Huang, Becattini, Sheng, 2024.
[6]. Sheng *et al.*, PRD 101, 096005 (2020);
Sheng *et al.*, PRD 102, 056013 (2020);
Sheng *et al.*, PRL 131, 042304 (2023).
[7]. Muller, Yang, PRD 105, L011901 (2022);

Kumar *et al.*, Phy. Rev. D108, 016020 (2023).

Spin Density Matrix and Wigner function

- Free Lagrangian for neutral vector bosons

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$$

- Future time-like (particle) Wigner function:

$$\widehat{W}_{+}^{\mu\nu}(x,k) = \frac{1}{2\pi}\int d^4s\, e^{ik\cdot s}\widehat{A}^{\nu}(x-\frac{s}{2})\widehat{A}^{\mu}(x+\frac{s}{2})\theta(k^2)\theta(k^0)$$

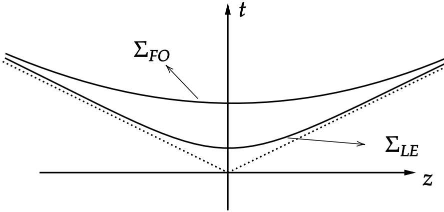
- Spin density matrix:

$$\Theta_{rs}(\boldsymbol{k}) \equiv \frac{\mathrm{Tr}\,(\widehat{\rho}\,\widehat{a}_{\boldsymbol{k}}^{s\dagger}\widehat{a}_{\boldsymbol{k}}^{r})}{\sum_r \mathrm{Tr}\,(\widehat{\rho}\,\widehat{a}_{\boldsymbol{k}}^{r\dagger}\widehat{a}_{\boldsymbol{k}}^{r})} = \frac{\int d\Sigma\cdot k\,\Theta_{rs}(x,k)f(x,k)}{\int d\Sigma\cdot k\, f(x,k)}$$

- Phase space distribution:

$$f(x,k) = \sum_r \epsilon_r^{\mu*}(k)\epsilon_r^{\nu}(k)W_{\mu\nu}(x,k)$$

$$\Theta_{rs}(x,k) = \epsilon_r^{\mu*}(k)\epsilon_s^{\nu}(k)W_{\mu\nu}(x,k)/f(x,k)$$



Freeze-out hypersurface

LEDO

- Local equilibrium density operator (LEDO)

$$\hat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left\{ -\int_\Sigma d\Sigma_\mu(y) \left[\overbrace{\hat{T}^{\mu\nu}(y)}^{\text{stress tensor}} \underbrace{\beta_\nu(y)}_{\text{thermal current } u_\nu/T \sim \mathcal{O}(1)} - \frac{1}{2} \overbrace{\hat{S}^{\mu\rho\sigma}(y)}^{\text{spin tensor}} \underbrace{\Omega_{\rho\sigma}(y)}_{\text{spin potential } \sim \mathcal{O}(\partial)} \right] \right\}$$

Canonical currents: $T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L}$, $S^{\mu\rho\sigma} = -F^{\mu\rho}A^\sigma + F^{\mu\sigma}A^\rho$

Integral measure: $d\Sigma_\mu(y) = d\Sigma(y)n_\mu(y)$, normal vector $n_\mu(y)$

- LEDO maximizes the entropy of $\hat{\rho}_{\rm LE}$ on hypersurface Σ:

$$S[\hat{\rho}_{\rm LE}] = -\,\mathrm{Tr}(\hat{\rho}_{\rm LE}\ln\hat{\rho}_{\rm LE}),$$

under matching conditions $n_\mu T^{\mu\nu}(x) = n_\mu \left\langle \hat{T}^{\mu\nu}(x) \right\rangle_{\rm LE}[\beta,\Omega], \quad n_\mu S^{\mu\rho\sigma}(x) = n_\mu \left\langle \hat{S}^{\mu\rho\sigma}(x) \right\rangle_{\rm LE}[\beta,\Omega]$

Zubarev, Prozorkevich, Smolyanskii, Theo. and Math. Phys. 40, 821 (1979)

van Weert, Ann. of Phys. 140, 133 (1982)

Becattini, Buzzegoli, Palermo, Particles 2, 197 (2019)

Cumulant Expansion

- LEDO $\widehat{\rho}_{\mathrm{LE}} = \dfrac{1}{Z_{\mathrm{LE}}} \exp\left\{ - \int_\Sigma d\Sigma_\mu(y) \left[\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \dfrac{1}{2}\widehat{S}^{\mu\rho\sigma}(y)\Omega_{\rho\sigma}(y) \right] \right\}$

  “Gaussian" term $\widehat{A} = -\beta_\nu(x)\widehat{P}^\nu = -\beta_\nu(x)\int_\Sigma d\Sigma_\mu(y)\widehat{T}^{\mu\nu}(y)$

  “Perturbative" terms $\widehat{B} = -\int_\Sigma d\Sigma_\mu(y) \left[\widehat{T}^{\mu\nu}(y)(\beta_\nu(y) - \beta_\nu(x)) - \dfrac{1}{2}\widehat{S}^{\mu,\rho\sigma}(y)\Omega_{\rho\sigma}(y) \right]$

- Cumulant expansion $e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}} \sum_{n=0}^\infty \widehat{B}_n$, with $\widehat{B}_0 = 1$, $\widehat{B}_n \sim \mathcal{O}(\partial^n)$,

$$\boxed{W_{\mu\nu}(x,k) = \frac{\sum_{n=0}^\infty \left\langle \widehat{B}_n \widehat{W}_{\mu\nu}(x,k) \right\rangle_0}{\sum_{n=0}^\infty \left\langle \widehat{B}_n \right\rangle_0}} \quad \sim \quad \boxed{\begin{aligned} &\langle\Omega|\, T\,\phi(x_1)\phi(x_2)\,|\Omega\rangle \\ \approx &\frac{\langle 0|\, T\, e^{i\int d^4z\mathcal{L}_{int}(z)}\phi(x_1)\phi(x_2)\,|0\rangle}{\langle 0|\, T\, e^{i\int d^4z\mathcal{L}_{int}(z)}\,|0\rangle} \end{aligned}}$$

with $\left\langle \widehat{O} \right\rangle_0 = \mathrm{Tr}\left(e^{\widehat{A}}\widehat{O}\right) \Big/ \mathrm{Tr}\left(e^{\widehat{A}}\right)$.

Zeroth-Order Result

- Cumulant expansion at 0th order: $W^{(0)}_{\mu\nu}(x,k) = \left\langle \widehat{W}_{\mu\nu}(x,k) \right\rangle_0$
- “Free" distribution: $\left\langle \widehat{a}^{s\dagger}_{\boldsymbol{k}} \widehat{a}^{r}_{\boldsymbol{q}} \right\rangle_0 = (2\pi)^3 \delta^{rs} \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})(2E_{\boldsymbol{k}})\, n_B(\beta(x)\cdot k)$

Bose-Einstein distribution: $n_B(x) = 1/(e^x - 1)$

- Zeroth-order Wigner fn.: nothing polarized.

$$
\begin{aligned}
W^{(0)}_{\mu\nu}(x,k) = &\int \frac{d^3\boldsymbol{p}_1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_2}{(2\pi)^3}(2\pi)^3 \frac{1}{2E_{\boldsymbol{p}_1}}\frac{1}{2E_{\boldsymbol{p}_2}} e^{-i(p_1-p_2)\cdot x} \\
&\times \sum_{a_1,a_2} \delta^{(4)}\left(k - \frac{p_1}{2} - \frac{p_2}{2}\right) \left\langle \widehat{a}^{a_2\dagger}_{\boldsymbol{p}_2} \widehat{a}^{a_1}_{\boldsymbol{p}_1} \right\rangle_0 \epsilon^{\mu}_{a_1}(p_1)\epsilon^{\nu*}_{a_2}(p_2). \\
= &- \delta(k^2 - m^2)\theta(k^0)\, \Delta^{(k)}_{\mu\nu}\, n_B(\beta(x)\cdot k)
\end{aligned}
$$

projection $\perp$ to k

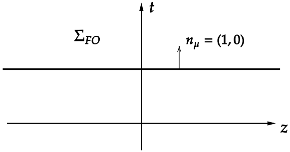
$\Delta^{(k)}_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu / m^2$

- Recall that $\Theta_{rs}(x,k) \propto \epsilon^{\mu*}_r(k)\epsilon^{\nu}_s(k) W_{\mu\nu}(x,k)$, which leads to $\Theta^{(0)}_{rs} = \delta_{rs}$

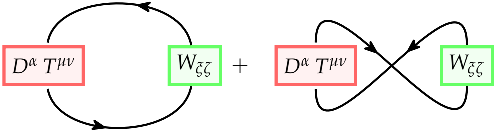
Flat Hypersurface: First-Order Gradient Expansion

Cumulant expansion: $W^{(1)}_{\mu\nu}(x,k) = \left\langle \widehat{B}_1 \widehat{W}_{\mu\nu}(x,k) \right\rangle_{0,c} = \left\langle \widehat{B}_1 \widehat{W}_{\mu\nu}(x,k) \right\rangle_0 - \left\langle \widehat{B}_1 \right\rangle_0 \left\langle \widehat{W}_{\mu\nu}(x,k) \right\rangle_0$

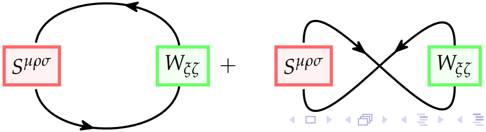
$$\begin{aligned}\widehat{B}_1 = &- n_\mu \partial_\alpha \beta_\nu(x) \int_0^1 d\lambda \int_P d^3\boldsymbol{y}\, (y-x)^\alpha \widehat{T}^{\mu\nu}(y - i\lambda\beta(x)) \\ &+ \frac{1}{2} n_\mu \Omega_{\rho\sigma}(x) \int_0^1 d\lambda \int_P d^3\boldsymbol{y}\, \widehat{S}^{\mu\rho\sigma}(y - i\lambda\beta(x))\end{aligned}$$



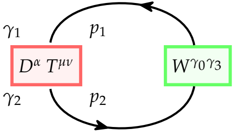
$$\int_0^1 d\lambda \int_P d^3\boldsymbol{y}\, (y-x)^\alpha \left\langle \widehat{T}^{\mu\nu}(y - i\lambda\beta(x)) \widehat{W}_{\xi\zeta}(x,k) \right\rangle_{0,c} =$$



$$\int_0^1 d\lambda \int_P d^3\boldsymbol{y}\, \left\langle \widehat{S}^{\mu\rho\sigma}(y - i\lambda\beta(x)) \widehat{W}_{\xi\zeta}(x,k) \right\rangle_{0,c} =$$



First-Order Gradient Expansion: Diagram Rules



$$= \int_0^1 d\lambda \int_P d^3\boldsymbol{y}(y-x)^\alpha \prod_{i=0}^{3}\left(\sum_{a_i}\int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}_i}}\right)(2\pi)^3\delta^{(4)}\left(k-\frac{p_1}{2}-\frac{p_2}{2}\right)\Big[t^{\mu\nu}{}_{\gamma_1\gamma_2}(p_1,-p_2)e^{i(p_0-p_3)\cdot x}$$

$$\times\left\langle \widehat{a}^{a_1}_{\boldsymbol{p}_1}\widehat{a}^{a_0\dagger}_{\boldsymbol{p}_0}\right\rangle_0\left\langle \widehat{a}^{a_2\dagger}_{\boldsymbol{p}_2}\widehat{a}^{a_3}_{\boldsymbol{p}_3}\right\rangle_0 \epsilon^{a_0 *}_{\gamma_0}(p_0)\epsilon^{a_1}_{\gamma_1}(p_1)\epsilon^{a_2 *}_{\gamma_2}(p_2)\epsilon^{a_3}_{\gamma_3}(p_3)e^{-i(p_1-p_2)\cdot(y_1-i\lambda_1\beta(x))}\Big]$$

Bose-Einstein distributions from lines

momentum conservation

$$= \delta(k^0-E_{\boldsymbol{k}})\int_0^1 d\lambda\; D^\alpha t^{\mu\nu}{}_{\gamma_1\gamma_2}(p_1,-p_2)\;(2n\cdot p_1)(n_B(\beta\cdot p_1)+1)\;\;(2n\cdot p_2)n_B(\beta\cdot p_2)\;\left(-\Delta^{\gamma_0\gamma_1}_{(p_1)}\right)\;\left(-\Delta^{\gamma_2\gamma_3}_{(p_2)}\right)\Big|_{p_1=p_2=k}$$

vertex

spins sums from lines

$$t^{\mu\nu}{}_{\gamma_1\gamma_2}(p_1,p_2) = \frac{e^{-\lambda(p_1+p_2)\cdot\beta}}{(2n\cdot p_1)(2n\cdot p_2)}\left[p_1^\mu p_2^\nu\eta_{\gamma_1\gamma_2} - p_{2,\gamma_2}p_2^\nu\eta^\mu{}_{\gamma_1} - \frac{1}{2}(p_1\cdot p_2)\eta^{\mu\nu}\eta_{\gamma_1\gamma_2} + \frac{1}{2}p_{1,\gamma_2}p_{2,\gamma_1}\eta^{\mu\nu} - \frac{1}{2}m^2\eta^{\mu\nu}\eta_{\gamma_1\gamma_2}\right]$$

$$D^\alpha = \left(-\frac{i}{2}\right)\left(\eta^{\alpha\zeta}-n^\alpha n^\zeta\right)\left(\frac{\partial}{\partial p_1^\zeta}-\frac{\partial}{\partial p_2^\zeta}\right)$$

ZZ, Huang, Becattini, Sheng, to appear

- Adaptability (higher-order results, additional operators in LEDO, fermions)

Flat Hypersurface: First-Order Gradient Expansion

- Recall that: $W^{(1)}_{\mu\nu}(x,k) = -n_\xi \partial_\alpha \beta_\lambda(x) \int_0^1 d\lambda \int_P d^3\boldsymbol{y}(y-x)^\alpha \left\langle \widehat{T}^{\xi\lambda}(y - i\lambda\beta(x))\widehat{W}_{\mu\nu}(x,k)\right\rangle_{0,c}$ + spin potential contribution

$$W^{(1)}_{\perp,\mu\nu}(x,k) = -i\delta(k^2-m^2)\theta(k^0)n_B(1+n_B)\Delta^{(k)}_{\mu\rho}\Delta^{(k)}_{\nu\sigma}\left[\overset{\substack{\text{thermal vorticity}\\ \downarrow}}{\boxed{\varpi^{\rho\sigma}}} - \Xi_\alpha{}^{[\rho}\left(\overset{\substack{\text{thermal shear}\\ \downarrow}}{\boxed{\xi^{\sigma]\alpha}}} + \underset{\substack{\uparrow\\ \text{net spin potential: } \Omega-\varpi}}{\boxed{\delta\Omega^{\sigma]\alpha}}}\right)\right]$$

with: $\Xi^{\mu\nu} = \eta^{\mu\nu} - \dfrac{k^\mu n^\nu}{(n\cdot k)}$

- Spin alignment at leading order:

$$\delta\Theta_{00}(x,k) \approx \frac{1}{3n_B\delta(k^2-m^2)\theta(k^0)}\left(\epsilon^\mu_y(k)\epsilon^\nu_y(k) + \frac{1}{3}\Delta^{\mu\nu}_{(k)}\right)W_{\mu\nu}(x,k)$$

- Space-time reversal odd: $\boxed{W^{(1)}_{\mu\nu} = -W^{(1)}_{\nu\mu} \;\Rightarrow\; \delta\Theta_{00} = 0 + \mathcal{O}(\partial^2)}$

ZZ, Huang, Becattini, Sheng, to appear

Flat Hypersurface: Second-Order Gradient Expansion

- Following the cumulant expansion and the diagram rules:

$$W^{(2)}_{\perp,\mu\nu}|_{\varpi^2} = \frac{1}{2}\delta(k^2-m^2)\theta(k^0)n_B(1+n_B)(1+2n_B)\left(\eta_{\rho\sigma}-\frac{k_\rho k_\sigma}{2m^2}\right)\ \varpi_{\mu\rho}(x)\ \ \varpi_{\sigma\nu}(x)\ +\Delta^{(k)}_{\mu\nu}\cdots$$

$$W^{(2)}_{\perp,\mu\nu}|_{\partial\varpi} = \delta(k^2-m^2)\theta(k^0)n_B(1+n_B)(1+2n_B)\frac{1}{2(n\cdot k)}\ \partial_\alpha\varpi_{\rho\sigma}(x)$$

$$\times\left\{\eta^{\alpha\rho}\hat{k}^\sigma n_\mu n_\nu - n^\rho\hat{k}^\sigma\eta^\alpha{}_{(\mu}n_{\nu)} - 2\eta^{\alpha\rho}\eta^\sigma{}_{(\mu}n_{\nu)} + 2n^\rho\eta^\sigma{}_{(\mu}\eta_{\nu)}{}^\alpha\right\} + \Delta^{(k)}_{\mu\nu}\cdots$$

$$\vdots$$

- PT even: $W^{(2)}_{\mu\nu}(x,k) = W^{(2)}_{\nu\mu}(x,k)$

ZZ, Huang, Becattini, Sheng to appear

Space-time Reversal Property

- Space-time reversal w.r.t. x, $PT:\ y \to y' = 2x - y$

$$\hat{\rho} = \frac{1}{Z}\exp\left\{-\int_{\Sigma} d\Sigma_\mu(y)\left[\hat{T}^{\mu\nu}(y)\,\beta_\nu(y) - \frac{1}{2}\hat{S}^{\mu\rho\sigma}(y)\,\Omega_{\rho\sigma}(y)\right]\right\}$$

$$\Downarrow\ (PT)\hat{T}^{\mu\nu}(y)(PT)^{-1} = \hat{T}^{\mu\nu}(y'),\quad (PT)\hat{S}^{\mu\rho\sigma}(y)(PT)^{-1} = -\hat{S}^{\mu\rho\sigma}(y')$$

$$(PT)\hat{\rho}(PT)^{-1} = \frac{1}{Z}\exp\left\{-\int_{\underset{\overset{\uparrow}{\Sigma\to\Sigma'}}{\Sigma'}} d\Sigma_\mu(y')\left[\hat{T}^{\mu\nu}(y')\,\underset{\overset{\uparrow}{\beta_\nu(y)}}{\beta'_\nu(y')} - \frac{1}{2}\hat{S}^{\mu\rho\sigma}(y')\,\underset{\overset{\uparrow}{-\Omega_{\rho\sigma}(y)}}{\Omega'_{\rho\sigma}(y')}\right]\right\}$$

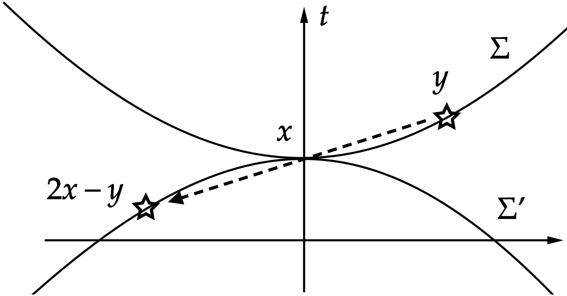
- Wigner function: $W_{\mu\nu}(x,k) = \mathrm{Tr}\left(\hat{\rho}\widehat{W}_{\mu\nu}(x,k)\right) = \mathrm{Tr}\left((PT)\hat{\rho}(PT)^{-1}(PT)\widehat{W}_{\mu\nu}(x,k)(PT)^{-1}\right)$
  $(PT)\widehat{W}_{\mu\nu}(x,k)(PT)^{-1} = \widehat{W}_{\nu\mu}(x,k)$
- For a hyperplane, $\Sigma' = \Sigma$: $W_{\mu\nu}(x,k)[\beta_\nu,\Omega_{\rho\sigma}] = W_{\nu\mu}(x,k)[\beta'_\nu,\Omega'_{\rho\sigma}]$
- Recall the power counting rules: $\beta_\nu \sim \mathcal{O}(1)$, $\Omega_{\rho\sigma} \sim \mathcal{O}(\partial) \Rightarrow \boxed{W^{(n)}_{\mu\nu}(x,k) = (-1)^n W^{(n)}_{\nu\mu}(x,k)} \sim \mathcal{O}(\partial^n)$

Break PT Symmetry of the System

How to break $W^{(1)}_{\mu\nu} = -W^{(1)}_{\nu\mu}$ to get $\delta\Theta_{00} \propto \partial_\alpha \beta_\nu, \Omega_{\rho\sigma}$?

- Dissipative process → Time reversal symmetry broken W. Dong *et al.*, PRD 109, 056025 (2024); ...
- Chiral symmetry breaking → Parity symmetry broken not clear
- Freeze-out hypersurface curved this work

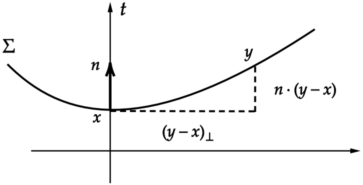
Recall that $W^{(1)}_{\mu\nu} = -W^{(1)}_{\nu\mu}$ based on $\Sigma = \Sigma'$



Geometry of Curved Hypersurface

- Assume Σ is a space-like 3D hypersurface:
  $F(x) = 0, \ \forall x \in \Sigma$
  Choose a proper F, s.t. $v_0 > 0$, with $v_\mu = \partial F(x)/\partial x^\mu$
  Normal vector of the hypersurface: $n_\mu = v_\mu / \sqrt{v \cdot v}$
- Taylor expansion around x upto second order:

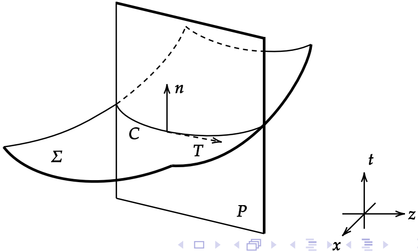
$$n \cdot (y - x) = \frac{1}{2} B_{\mu\nu} (y-x)^\mu (y-x)^\nu + \mathcal{O}\left(((y-x)_\perp)^3 \right)$$



- Curvature tensor:

$$\boxed{B_{\mu\nu}(x) \equiv -\frac{1}{\sqrt{v \cdot v}} \overbrace{\Delta^{(n)}_{\mu\rho} \quad \Delta^{(n)}_{\nu\sigma}}^{\text{projection } \perp \text{ to } n} \frac{\partial^2 F(x)}{\partial x_\rho \partial x_\sigma}}$$

$$B_{\mu\nu} T^\mu T^\nu = \overset{\text{curvature of curve } C}{\boxed{\kappa_C}} \left(- \overset{\Delta^{(n)}_{\mu\nu} T^\mu T^\nu}{\boxed{T_\perp^2}} \right)$$



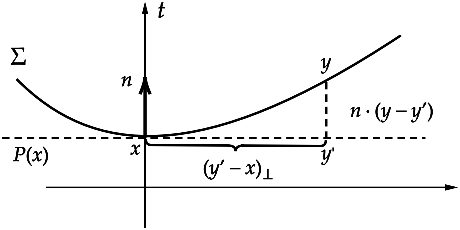
Curved Hypersurface: First-Order Gradient Expansion

- Cumulant expansion: $W^{(1)}_{\mu\nu}(x,k) = \left\langle \widehat{B}_1 \widehat{W}_{\mu\nu}(x,k) \right\rangle_{0,c}$

$$\widehat{B}_1 = -\partial_\alpha \beta_\nu(x) \int_0^1 d\lambda \int_{y\in\Sigma} d\Sigma_\mu(y)\, (y-x)^\alpha \widehat{T}^{\mu\nu}(y - i\lambda\beta(x)) + \frac{1}{2}\Omega_{\rho\sigma}(x) \int_0^1 d\lambda \int_{y\in\Sigma} d\Sigma_\mu(y)\, \widehat{S}^{\mu\rho\sigma}(y - i\lambda\beta(x))$$

- Taylor expansion at x: For $y \in \Sigma$, $y' \in P(x)$, $(y-y')_\mu = \frac{1}{2}B_{\rho\sigma}(y'-x)^\rho (y'-x)^\sigma n_\mu + \cdots$

$$\begin{aligned} \int_{y\in\Sigma} &\to \int_{y'\in P(x)} \\ y_\mu &\to y'_\mu + n\cdot(y-y')n_\mu \\ d\Sigma_\mu(y) &\to d^3\boldsymbol{y}'_\perp \left[n_\mu(x) - B_{\mu\nu}(x)(y'-x)^\nu\right] \\ \widehat{T}^{\mu\nu}(y - i\lambda\beta(x)) &\to \widehat{T}^{\mu\nu}(y' - i\lambda\beta(x)) \\ &\quad + n\cdot(y-y')(n\cdot\partial)\widehat{T}^{\mu\nu}(y'-i\lambda\beta(x)) \end{aligned}$$



- Valid when $1/\kappa \gg \lambda$

]

Curved Hypersurface: First-Order Gradient Expansion

- Cumulant expansion: $W^{(1)}_{\mu\nu}(x,k) = \left\langle \widehat{B}_1 \widehat{W}_{\mu\nu}(x,k) \right\rangle_{0,c}$

$$\widehat{B}_1 = -\partial_\alpha \beta_\nu(x) \int_0^1 d\lambda \int_{y\in\Sigma} d\Sigma_\mu(y)\, (y-x)^\alpha \widehat{T}^{\mu\nu}(y - i\lambda\beta(x)) + \frac{1}{2}\Omega_{\rho\sigma}(x) \int_0^1 d\lambda \int_{y\in\Sigma} d\Sigma_\mu(y)\, \widehat{S}^{\mu\rho\sigma}(y - i\lambda\beta(x))$$

- Taylor expansion at x: For $y \in \Sigma$, $y' \in P(x)$, $(y-y')_\mu = \frac{1}{2} B_{\rho\sigma}(y'-x)^\rho (y'-x)^\sigma n_\mu + \cdots$

$$\begin{aligned}
\widehat{B}_1 = & -\partial_\alpha \beta_\nu(x) n_\mu(x) \int_0^1 d\lambda \int_{y'\in P} d^3\boldsymbol{y}'_\perp (y'-x)^\alpha \widehat{T}^{\mu\nu}(y' - i\lambda\beta(x)) \longrightarrow \text{calculated previously} \\
& + \partial_\alpha \beta_\nu(x) B_{\mu\rho}(x) \int_0^1 d\lambda \int_{y'\in P} d^3\boldsymbol{y}'_\perp (y'-x)^\rho (y'-x)^\alpha \widehat{T}^{\mu\nu}(y' - i\lambda\beta(x)) \\
& - \frac{1}{2} \partial_\alpha \beta_\nu(x) B_{\rho\sigma}(x) n_\mu \int_0^1 d\lambda \int_{y'\in P} d^3\boldsymbol{y}'_\perp (y'-x)^\alpha (y'-x)^\rho (y'-x)^\sigma (n\cdot\partial) \widehat{T}^{\mu\nu}(y' - i\lambda\beta(x)) \\
& - \frac{1}{2} \partial_\alpha \beta_\nu(x) B_{\rho\sigma}(x) n^\alpha n_\mu \int_0^1 d\lambda \int_{y'\in P} d^3\boldsymbol{y}'_\perp (y'-x)^\rho (y'-x)^\sigma \widehat{T}^{\mu\nu}(y' - i\lambda\beta(x)) + \mathcal{O}(\partial, B^2) \\
& + \cdots \longrightarrow \text{spin potential contributions}
\end{aligned}$$

Curved Hypersurface: Results

- New terms at first order of gradient:

$$
\begin{aligned}
W_{\perp}^{(1)\mu\nu}|_B(x,k) =& \delta(k^2-m^2)\theta(k^0) n_B(1+n_B) \frac{B_{\alpha\beta}}{2(n\cdot k)} \Bigg\{ \xi_{\rho\sigma}(x) \left(\eta^{\rho\sigma} + \frac{k^\rho k^\sigma}{2m^2}\right) \Xi^{\alpha(\mu}\Xi^{\beta\nu)} \\
&+ \Big(\delta\Omega_{\rho\sigma}(x) - \xi_{\rho\sigma}(x)\Big) \Xi^{\alpha\rho}\Xi^{\beta(\mu}\Delta_{(k)}^{\nu)\sigma} + \left(\Delta_{(k)}^{\mu\nu}\cdots\right) \Bigg\}
\end{aligned}
$$

projected Wigner fn.

net spin potential: $\Omega - \omega$

thermal shear

no spin polarization

- Recall that:

$$
\Xi^{\mu\nu} = \left(\eta^{\mu\nu} - \frac{k^\mu n^\nu}{(n\cdot k)}\right)
$$

$$
\delta\Theta_{00}(x,k) \approx \frac{1}{3n_B\delta(k^2-m^2)\theta(k^0)} \left(\epsilon_y^\mu(k)\epsilon_y^\nu(k) + \frac{1}{3}\Delta_{(k)}^{\mu\nu}\right) W_{\mu\nu}(x,k)
$$

- Spin alignment at $\mathcal{O}(\partial)$!

Curved Hypersurface: Why Vorticity Disappears?

- The local equilibrium density operator upto $\mathcal{O}(\partial)$:

$$\widehat{\rho}_{\rm LE} \approx \frac{1}{Z}\exp\left\{-\beta(x)\cdot\widehat{P} + \frac{1}{2}\,\varpi_{\rho\sigma}(x)\;\widehat{J}^{\rho\sigma}(x) - \frac{1}{2}\,\xi_{\lambda\nu}(x)\;\widehat{Q}_{\Sigma}^{\lambda\nu}(x) + \frac{1}{2}\,\delta\Omega_{\lambda\nu}(x)\;\widehat{R}_{\Sigma}^{\lambda\nu}(x)\right\}$$

vorticity $\longleftarrow \widehat{J}^{\rho\sigma}(x) = \int_\Sigma d\Sigma_\mu(y)\left[(y-x)^\rho\,\widehat{T}^{\mu\sigma}(y) - (y-x)^\sigma\,\widehat{T}^{\mu\rho}(y) + S^{\mu\rho\sigma}(y)\right]$

shear $\longleftarrow \widehat{Q}_{\Sigma}^{\lambda\nu}(x) = \int_\Sigma d\Sigma_\mu(y)\left[(y-x)^\lambda\,\widehat{T}^{\mu\nu}(y) + (y-x)^\nu\,\widehat{T}^{\mu\lambda}(y)\right]$

net spin potential $\longleftarrow \widehat{R}_{\Sigma}^{\rho\sigma}(x) = \int_\Sigma d\Sigma_\mu(y)\left[\widehat{S}^{\mu\rho\sigma}(y)\right]$

- The total angular momentum (w.r.t. x) $\widehat{J}^{\rho\sigma}(x)$: independent of hypersurface Σ
- The leafing order contribution from vorticity always be:

$$W^{(1)}_{\perp,\mu\nu}(x,k) = -i\delta(k^2-m^2)\theta(k^0)n_B(1+n_B)\Delta^{(k)}_{\mu\rho}\Delta^{(k)}_{\nu\sigma}\;\varpi^{\rho\sigma}$$

Example

- Assume: $k_\mu = mn_\mu$, $\overset{\text{fluid current}}{u_\mu} = n_\mu$, constant temperature on Σ
- Decompose thermal shear tensor:

$$\xi_{\mu\nu} = u_\mu u_\nu (u\cdot\partial)\frac{1}{T} + \frac{1}{3T}\underbrace{\theta}_{\text{scalar expansion } \partial\cdot u}\Delta^{(u)}_{\mu\nu} + \frac{1}{T}\overbrace{A_{(\mu}}^{\text{acceleration } (u\cdot\partial)u_\mu} u_{\nu)} + \frac{1}{T}\underbrace{\sigma_{\mu\nu}}_{\text{shear tensor}}$$

curvature along direction y

$$\delta\Theta_{00}|_{\text{shear}}(x,k) = (1+n_B)\Bigg\{\frac{(u\cdot\partial)(1/T)}{4m}(\kappa_y - H) + \frac{\theta}{9mT}(\kappa_y - H) - \frac{1}{6mT}\left(\epsilon^\alpha_y(k)B_{\alpha\beta}\sigma^\beta{}_\nu\epsilon^\nu_y(k) + \frac{1}{3}B_{\alpha\beta}\sigma^{\alpha\beta}\right)\Bigg\}$$

average curvature

- Curvature along "y" : $\kappa_y = B_{\alpha\beta}\epsilon^\alpha_y(k)\epsilon^\beta_y(k)$, average curvature: $H = -\frac{1}{3}B^\mu{}_\mu$

Example

fluid current

- Assume: $k_\mu = mn_\mu$, $u_\mu = n_\mu$, constant temperature on Σ
- Bjorken expansion: $\sqrt{t^2 - z^2} = \tau_c \approx 5 \sim 10$ fm.

  Curvature along "y" : $\kappa_y = 0$, average curvature: $H = \dfrac{1}{3\tau_c}$, scalar expansion $\theta = \dfrac{1}{\tau_c}$

$$\delta\Theta_{00}|_{\text{expansion}}(x,k) = (1 + n_B)\left\{\frac{\theta}{9mT}\left(\kappa_y - H\right)\right\} \sim -10^{-4}$$

Discussion

Spin alignment more sensitive to the curvature of the freezeout hypersurface?

A massive spin-1/2 Dirac fermion:

$$S^{\mu}(x,k) = -\frac{1}{8m}\left\{\epsilon^{\mu\nu\rho\sigma}\Omega_{\rho\sigma}k_{\nu} + 2\frac{\epsilon^{\mu\nu\rho\sigma}k_{\rho}n_{\sigma}}{k\cdot n}\left[\left(\xi_{\nu\lambda} + \delta\Omega_{\nu\lambda}\right)k^{\lambda} - \partial_{\nu}\alpha\right]\right\} + \mathcal{O}(\partial^2, B)$$

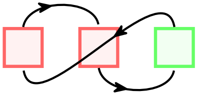
Liu, Huang, Sci. China-Phys. Mech. Astron. 65, 272011 (2022)

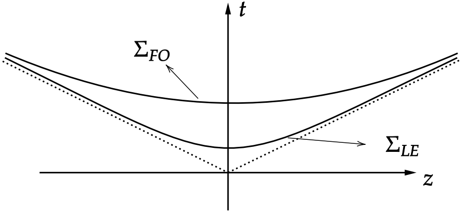
Buzzegoli, PRC 105, 044907 (2021)

Conclusion and Outlook

- Curved freeze-out hypersurface
  $\Rightarrow$ spin alignment $\propto$ thermal shear ...

- Spin alignment at first order of gradient:
  Dissipation;
  Chiral symmetry breaking...





Thank you!

What is Tensor Polarization?

A massive vector boson's spin density matrix

$$\rho_{rs} = \begin{pmatrix} \rho_{1,1} & \rho_{1,0} & \rho_{1,-1} \\ \rho_{0,1} & \rho_{0,0} & \rho_{0,-1} \\ \rho_{-1,1} & \rho_{-1,0} & \rho_{-1,-1} \end{pmatrix}$$

$$\rho = \frac{1}{3}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{3} P_i S_i + \sum_{m=-2}^{2} (-1)^m T_{2,m} S_{2,-m} \tag{1}$$

$\langle S^i \rangle$: Vector polarization (3 DoFs) $\langle S_{2,m} \rangle$: Tensor polarization (5 DoFs)

with

$$S^1 = S_x = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = S_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^3 = S_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

And $S_{2,m} = \sum_{m1,m2} \langle 1, m1; 1, m2 | 2, m \rangle \, S_{1,m1} S_{1,m2}$ the rank-2 spherical irreducible tensor.

How to Detect Tensor Polarization?

Strong decay, parity even $\Rightarrow T_{2,m}$ only:

$$\frac{1}{N}\frac{dN}{d\Omega}(\theta^*,\phi^*) = \frac{1}{4\pi} - \sqrt{\frac{3}{10\pi}}\sum_{m=-2}^{2}(-1)^m T_{2,m} Y_{2,-m}(\theta^*,\phi^*)$$

For now, in the experiment, *spin alignment* only:

$$\frac{1}{N}\frac{dN}{d\cos\theta^*} = \frac{1}{2} + \frac{3}{4}(3\cos^2\theta - 1)\left(\rho_{00} - \frac{1}{3}\right)$$

$T_{2,\pm1}$, $T_{2,\pm2} \rightarrow \phi^*$ distribution (off-diagonal polarization)

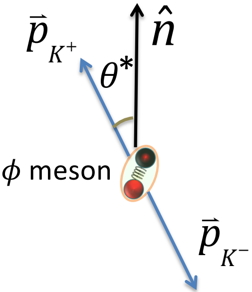


Figure: STAR, Nature.614.244-248 (2023)

Cumulant Expansion

Cumulant expansion:

$$e^{\widehat{A}+x\widehat{B}} = e^{\widehat{A}} \sum_{n=0}^{\infty} x^n \widehat{B}_n, \quad x \to 0$$

$$\begin{aligned}
\widehat{B}_0 &= 1 && (2)\\
\widehat{B}_1 &= \int_0^1 d\lambda_1 \widehat{B}(\lambda_1) && (3)\\
&\cdots \\
\widehat{B}_n &= \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_n - 1} d\lambda_n \widehat{B}(\lambda_1)\widehat{B}(\lambda_2)\cdots\widehat{B}(\lambda_n) && (4)
\end{aligned}$$

with $\widehat{B}(\lambda_i) = e^{-\lambda_i \widehat{A}} \widehat{B} e^{\lambda_i \widehat{A}}$.

From MVSD to Tensor Polarization

- Space-time reversal property caused by power counting rules

$$\boxed{f^{(n)}_{r,s} = (-1)^{r+s+n} f^{(n)}_{-s,-r}} \tag{5}$$

with $f^{(n)}_{r,s}$ the n-order MVSD.

- Spin density matrix in phase space $\rho_{rs}(x,k)$

$$f_{rs}(x,k) = \Theta_{rs}(x,k) f(x,k) \tag{6}$$

with $f(x,k) = \sum_r f_{rr}(x,k)$ the scalar distribution

- The leading order of polarization (y-axis as the spin axis)

$$\{P_x, P_y, P_z\} = \frac{2}{3n_B}\left\{\sqrt{2}\,\mathrm{Im}\, f^{(1)}_{01},\, f^{(1)}_{11},\, \sqrt{2}\,\mathrm{Re}\, f^{(1)}_{01}\right\}$$

$$\{T_{2,0}, T_{2,1}, T_{2,2}\} = \frac{1}{3n_B}\left\{\sqrt{\frac{2}{3}}(f^{(2)}_{11} - f^{(2)}_{00}),\, -\sqrt{2} f^{(2)}_{01},\, f^{(2)}_{-11}\right\}$$

$$\boxed{\Theta_{00} - \frac{1}{3} = \frac{2}{9n_B}(f^{(2)}_{00} - f^{(2)}_{11})} \tag{7}$$