Critical dynamics of phase transition in the QCD phase diagram within the real-time fRG approach

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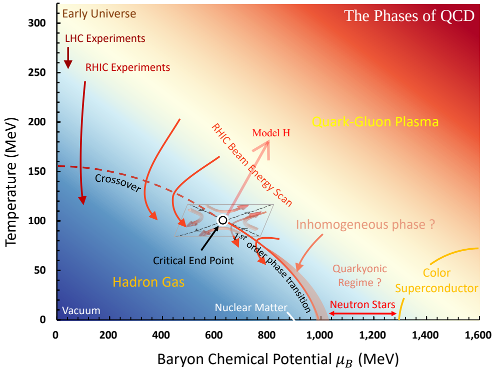
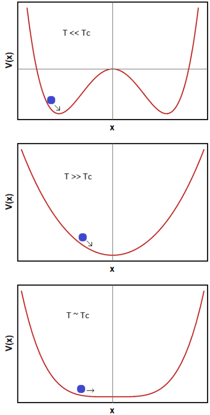
Based on YRC, Y.-y. Tan, and W.-j. Fu, PRD 109, 094044
YRC, Y.-y. Tan, and W.-j. Fu, arXiv:2406.00679

Outline

Introduction



QCD phase diagram[1]



critical slowing down[2]

Relaxation time: $\tau = \xi^{z} f(k\xi)$

z: dynamic critical exponent

[1] **临界现象与泛函重整化群. 核技术**, **2023**, **46(4)**.

[2] **Anshul Kogar**.

Dynamic universality classes[4]

Model A:a non-conserved order parameter

$$\partial_t \phi_\alpha(x,t) = -\Gamma_0 \frac{\delta \mathscr{H}[\phi]}{\delta \phi_\alpha(x,t)} + \Gamma_0 h_\alpha(x,t) + \eta_\alpha(x,t)$$

Model H:critical end point in QCD[3], $Z(2)$, static

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \lambda_0 \boldsymbol{\nabla}^2 \frac{\delta F}{\delta \phi} - g_0 \boldsymbol{\nabla}\phi \cdot \frac{\delta F}{\delta \boldsymbol{j}} + \theta(x,t)\,, \\ \frac{\partial \boldsymbol{j}}{\partial t} &= \Pi^{\perp}\left(\eta_0 \boldsymbol{\nabla}^2 \frac{\delta F}{\delta \boldsymbol{j}} + g_0 \boldsymbol{\nabla}\phi \frac{\delta F}{\delta \phi} + \zeta(x,t)\right)\end{aligned}$$

[3] **D. T. Son and M. A. Stephanov(2004)**.

[4] **Hohenberg and Halperin(1977)**.

Closed time path

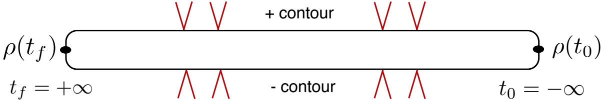
Green functions in non-equilibrium

$$\langle T\phi(x)\phi(y)\rangle \equiv \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)\,T\phi(x)\phi(y)]$$
$$= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)\,T_P\phi_I(x)\phi_I(y)\,U_{CTP}(t_0)]$$

$U_{CTP}(t_0)$ is the unitary evolution operator defined on a closed-time path

$$U_{CTP}(t_0) \equiv T_P\left[exp\left(-i\int_{CTP} dt H_I(t)\right)\right]$$
$$= T_P\left[exp\left(-i\int_{t_0}^{\infty} dt_+ H_I(t_+) + i\int_{t_0}^{\infty} dt_- H_I(t_-)\right)\right]$$

Only states at t_0 is involved which lead to the CTP



fRG in the Schwinger-Keldysh field theory

The Keldysh generating functional

$$Z[J_c, J_q] = \int [d\Phi_c][d\Phi_q] \exp\left\{ i\Big(S[\Phi] + \Delta S_k[\Phi] + (J_c^a \Phi_{a,q} + J_q^a \Phi_{a,c}) \Big) \right\}$$

with the two-point regulator term

$$\Delta S_k[\Phi] = \frac{1}{2}(\Phi_{a,c}, \Phi_{a,q}) \begin{pmatrix} 0 & R_k^{ab} \\ R_k^{*ab} & 0 \end{pmatrix} \begin{pmatrix} \Phi_{b,c} \\ \Phi_{b,q} \end{pmatrix} = \frac{1}{2}(\Phi_{a,c} R_k^{ab} \Phi_{b,q} + \Phi_{a,q} R_k^{*ab} \Phi_{b,c})$$

Keldysh rotation

$$\begin{cases} \Phi_{a,+} = \frac{1}{\sqrt{2}}(\Phi_{a,c} + \Phi_{a,q}), \\ \Phi_{a,-} = \frac{1}{\sqrt{2}}(\Phi_{a,c} - \Phi_{a,q}) \end{cases}$$

The flow equation for the effective action

$$\partial_\tau \Gamma_k[\Phi] = \frac{1}{2}(iG^R_{k,ab})\partial_\tau R_k^{ab} + \frac{1}{2}(iG^A_{k,ab})\partial_\tau R_k^{*ab}$$

Two-point Green functions

$$iG_k = \begin{pmatrix} iG_k^K & iG_k^R \\ iG_k^A & 0 \end{pmatrix}$$

Critical dynamics of Model A

Langevin equation:

$$\partial_t \phi_\alpha(x,t) = -\Gamma \frac{\delta \mathscr{H}[\phi]}{\delta \phi_\alpha(x,t)} + \Gamma h_\alpha(x,t) + \eta_\alpha(x,t)$$

Time reversal symmetry

$$\begin{cases} t \to -t \\ \phi_c \to \phi_c \\ \phi_q \to \phi_q - \frac{1}{\Gamma}\partial_t \phi_q \end{cases}$$

fluctuation–dissipation theorem

correlation response

$$C(k,\omega) = \frac{2T}{\omega}\mathrm{Im}\chi(k,\omega)$$

Hamiltonian:

$$\mathscr{H}[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2}|\nabla\phi(x)|^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 \right)$$

The classical action(average over the Langevin noise):

$$S = \int_{x,t} \phi_{\alpha,q} i\partial_t \phi_{\alpha,c} + i\Gamma \phi_{\alpha,q} \frac{\delta \mathscr{H}[\phi]}{\delta \phi_{\alpha,c}} - i2\Gamma \phi_q^2$$

The effective action of Model A

Substituting the Hamiltonian with general form

$$\mathscr{H}[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_\phi(\rho)(\partial_i \phi_a)(\partial_i \phi_a) + V(\rho) - c\sigma \right)$$

The effective action of Model A

$$\begin{aligned}\Gamma_k = \int_{x,t} & \phi_{\alpha,q} Z_{t,k} i\partial_t \phi_{\alpha,c} - i\phi_{\alpha,q} Z_{\phi,k}(\rho_c)\nabla^2 \phi_{\alpha,c} \\ & - \frac{i}{2} Z'_{\phi,k}(\rho_c)\phi_{\alpha,q}\phi_{\beta,c}\nabla\phi_{\alpha,c}\nabla\phi_{\beta,c} + \frac{i}{4} Z'_{\phi,k}(\rho_c)\phi_{\alpha,q}\phi_{\alpha,c}\nabla\phi_{\beta,c}\nabla\phi_{\beta,c} \\ & + iV'_k(\rho_c)\phi_{\alpha,q}\phi_{\alpha,c} - i\sqrt{2}c\sigma_q - i2Z_{t,k}T\phi^2_{\alpha,q}\end{aligned}$$

- derivative expansion to $O(\partial^2)$
- $Z_{t,k}$: kinetic coefficient
- $Z_{\phi,k}(\rho_c)$: wave function renormalization with field dependent
- $V'_k(\rho_c)$: the effective potential, ϕ has N components

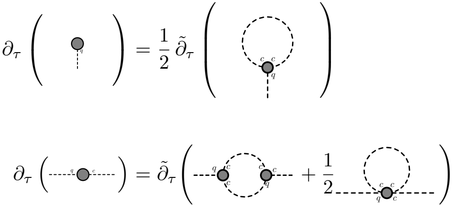
Flow of $V'_k(\rho)$, $Z_{\phi,k}(\rho)$ and $Z_{t,k}$

Flow equation:

$$\partial_\tau V'_k(\rho) = \frac{1}{2} \lim_{\substack{p_0 \to 0 \\ \boldsymbol{p} \to 0}} \frac{\delta \partial_\tau \Gamma_k}{\delta \phi_q(p,\omega)}$$

$$\partial_\tau Z_{\phi,k}(\rho) = \lim_{\substack{p_0 \to 0 \\ \boldsymbol{p} \to 0}} (-i) \frac{\partial}{\partial \boldsymbol{p}^2} \frac{\delta^2 \partial_\tau \Gamma_k[\Phi]}{\delta \phi_{a,q}(-p) \delta \phi_{a,c}(p)}$$

$$\partial_\tau Z_{t,k} = \lim_{\substack{p_0 \to 0 \\ \boldsymbol{p} \to 0}} \frac{\partial}{\partial p_0} \frac{\delta^2 \partial_\tau \Gamma_k[\Phi]}{\delta \phi_{a,q}(-p) \delta \phi_{a,c}(p)}$$



$$\eta = -\frac{\partial_\tau Z_{\phi,k}}{Z_{\phi,k}}, \quad \eta_t = -\frac{\partial_t Z_{t,k}}{Z_{t,k}}$$

Dynamic critical exponent

$$z = 2 - \eta + \eta_t$$

Dimensionless, truncation scheme

Dimensionless:

$$\begin{cases} \bar{\rho} = Z_{\phi,k} T^{-1} k^{2-d} \rho \\ u(\bar{\rho}) = T^{-1} k^{-d} V_k(\rho) \\ z_\phi(\bar{\rho}) = Z_{\phi,k}(\rho)/Z_{\phi,k} \end{cases}$$

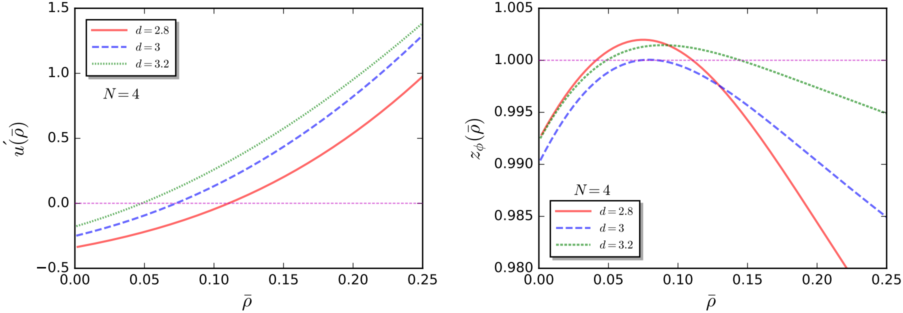
Truncation schemes:

- LPA,$\partial_\tau Z_{\phi,k} = 0 \quad (\eta = 0)$
- LPA′,$\partial_\tau Z_{\phi,k} \neq 0 \quad (\eta \neq 0)$
- $O(\partial^2)$,consider flows of $\partial_\tau u'(\bar{\rho})$ and $\partial_\tau z_\phi(\bar{\rho})$

For the fixed-point equation of $u'(\bar{\rho})$ and $z_\phi(\bar{\rho})$

- the dynamic part decouples from the statics
- the flows of $u'(\bar{\rho})$ and $z_\phi(\bar{\rho})$ are same as Euclidean space calculation

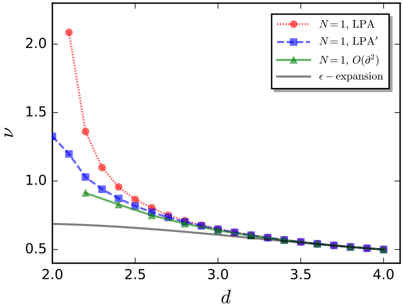
Field dependence of $u'(\bar{\rho})$ and $z_\phi(\bar{\rho})$

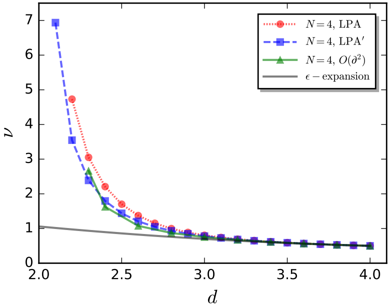


The fixed-point solution of $u'(\bar{\rho})$ and $z_\phi(\bar{\rho})$

- zero point $\bar{\rho}_0$ increases with the decrease of dimension d
- field dependence of $z_\phi(\bar{\rho})$ should be considered with the decrease of dimension d

ν under three truncation schemes

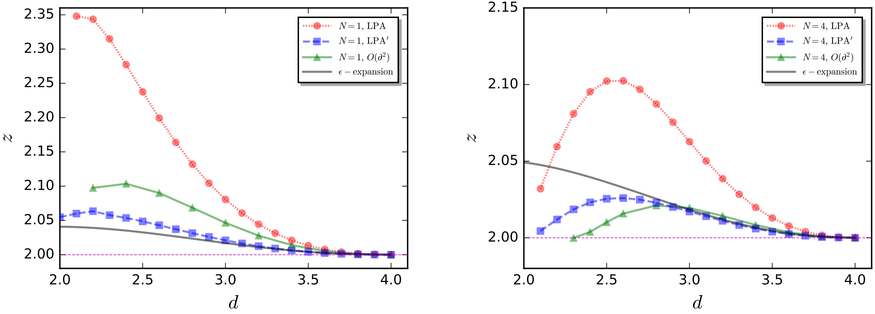




critical exponent ν compare with ϵ expansion in the order of $O(\epsilon^3)$

- ν corresponds to the negative eigenvalue of the stability matrix M

Dynamic critical exponent z



- For $d \gtrsim 3.5$, LPA′ and $O(\partial^2)$ are comparable with the ϵ expansion
- $N = 1$, $z \geq 2$ in the whole range of $2 \leq d \leq 4$
- $N = 4$, $z < 2$ for the derivative expansion when $d < 2.5$

Critical dynamics of Model H

Langevin equation of Model H:

$$\frac{\partial \phi}{\partial t} = \lambda_0 \boldsymbol{\nabla}^2 \frac{\delta \mathscr{H}}{\delta \phi} - g_0 \boldsymbol{\nabla} \phi \cdot \frac{\delta \mathscr{H}}{\delta \boldsymbol{j}} + \theta(x, t),$$
$$\frac{\partial \boldsymbol{j}}{\partial t} = \Pi^{\perp} \left(\eta_0 \boldsymbol{\nabla}^2 \frac{\delta \mathscr{H}}{\delta \boldsymbol{j}} + g_0 \boldsymbol{\nabla} \phi \frac{\delta \mathscr{H}}{\delta \phi} + \boldsymbol{\zeta}(x, t) \right).$$

Hamiltonian:

$$\mathscr{H} = \int \mathrm{d}^d x \left(\frac{1}{2} |\nabla \phi(x)|^2 + \frac{r}{2} \phi^2 + \frac{u}{4} \phi^4 + \frac{1}{2} \boldsymbol{j}^2 \right)$$

The correlation function for the noise term

$$\langle \theta(x, t) \theta(x', t') \rangle = -2\lambda_0 \boldsymbol{\nabla}^2 \delta(x - x') \delta(t - t'),$$
$$\langle \zeta_\mu(x, t) \zeta_\nu(x', t') \rangle = -2\eta_0 \boldsymbol{\nabla}^2 \delta(x - x') \delta(t - t') \delta_{\mu\nu}.$$

Critical dynamics of Model H

characteristic frequency of the order parameter[6]

$$\omega_\phi(k) = Dk^2\Omega(k\xi) = D_0\xi^{2-z}k^2\Omega(k\xi)$$

D comes from "Kawasaki-Stokes" relation[5]

$$D = \lambda/\chi_\phi = Rk_BT/\eta\xi$$

Dynamic critical exponent is determined by

$$z = 4 - \eta_\phi - x_\lambda$$

ϵ-expansion results for x_λ and x_η[6]

$$x_\lambda = \frac{18}{19}\epsilon[1 - 0.033\epsilon + O(\epsilon^2)]\,,$$
$$x_\eta = \frac{1}{19}\epsilon[1 + 0.238\epsilon + O(\epsilon^2)].$$

$$x_\lambda = -\frac{\partial_\tau\lambda_k}{\lambda_k}\,, \quad x_\eta = -\frac{\partial_\tau\eta_k}{\eta_k}$$

[5] **Ann. Phys. 61, 1(1970)**.

[6] **Phys.Rev.B 13, 2110(1976)**.

Model H within the real-time fRG approach

The effective action of Model H

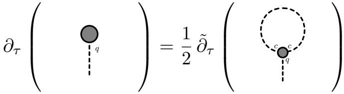
$$\Gamma_k[\Phi] = i\int \mathrm{d}^d x \mathrm{d}t\, \phi_q \left(\frac{\partial}{\partial t}\phi_c + \lambda_k \boldsymbol{\nabla}^2\boldsymbol{\nabla}^2\phi_c\right) - \lambda_k\phi_q\boldsymbol{\nabla}^2\frac{\delta V_k(\rho_c)}{\delta\phi_c} + g_k\phi_q\boldsymbol{\nabla}\phi_c\cdot\boldsymbol{j_c} + 2\lambda_k\phi_q\boldsymbol{\nabla}^2\phi_q + \boldsymbol{j}_{\boldsymbol{q},\alpha}\Pi^{\perp}_{\alpha\beta}\left[\left(\frac{\partial}{\partial t}\boldsymbol{j}_{\boldsymbol{c},\beta} - \eta_k\boldsymbol{\nabla}^2\boldsymbol{j}_{\boldsymbol{c},\beta}\right) - g_k\boldsymbol{\nabla}\phi_c(-\boldsymbol{\nabla}^2\phi_c) + 2\eta_k\boldsymbol{\nabla}^2\boldsymbol{j}_{\boldsymbol{q},\beta}\right].$$

- ϕ:order parameter, $\boldsymbol{j}$:momentum density
- λ_k:transport coefficient, η_k:shear viscosity, g_k:mode couplings
- scaling behavior:$\lambda_k \sim k^{-x_\lambda}, \quad \eta_k \sim k^{-x_\eta}$
- transverse projection:$\Pi^{\perp}_{\alpha\beta} = (\delta^{\alpha\beta} - \boldsymbol{q}_\alpha\boldsymbol{q}_\beta/\boldsymbol{q}^2)$

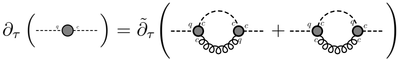
Flow equation for the correlation function

Flow equation:

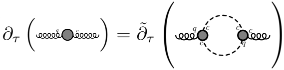
$$\partial_\tau V_k'(\rho) = \int_{\omega,q} \frac{1}{2}\Gamma^{(3)}_{\phi_q\phi_c\phi_c,k}\tilde{\partial}_\tau G^K_{\phi\phi,k}/(-\lambda_k \boldsymbol{\nabla}^2)$$



$$\partial_\tau \lambda_k = \lim_{\substack{p_0\to 0\\ \boldsymbol{p}\to 0}} (-i)\frac{\partial}{\partial \boldsymbol{p}^4}\frac{\delta^2\partial_\tau\Gamma_k[\Phi]}{\delta\phi_q(-p)\delta\phi_c(p)}\bigg|_{\Phi_{\mathrm{EoM}}},$$



$$\partial_\tau \eta_k = \frac{1}{d-1}\lim_{\substack{p_0\to 0\\ \boldsymbol{p}\to 0}} \frac{\partial}{\partial \boldsymbol{p}^2}\frac{(-i)\delta^2\partial_\tau\Gamma_k[\Phi]}{\delta \boldsymbol{j}_{q,\alpha}(-p)\delta \boldsymbol{j}_{c,\beta}(p)}\bigg|_{\Phi_{\mathrm{EoM}}}\Pi^\perp_{\alpha\beta}$$



Flow equation for the coupling constant f

The combination $\bar{\lambda}\bar{\eta}$ whose exponent determines an exact scaling law[7]
Define a new dimensionless coupling constant

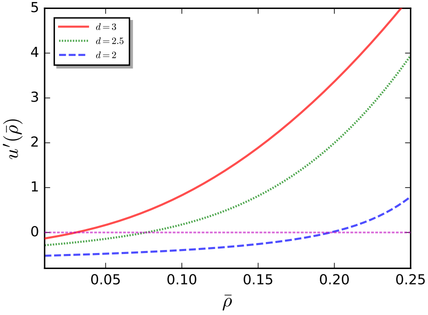
$$f = \nu_d \frac{\bar{g}^2}{\bar{\lambda}\bar{\eta}}$$

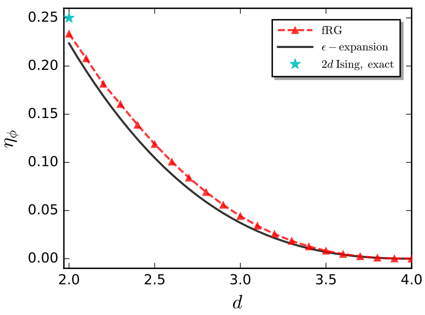
By combining the flow equations for $\bar{\lambda}$, $\bar{\eta}$ and $\bar{g}$ (see backup)

$$\partial_\tau f = (\eta_\phi - (4 - d))f - I_\lambda f^2 - I_\eta f^2$$

[7] **Phys.Rev.B 13, 2110(1976)**.

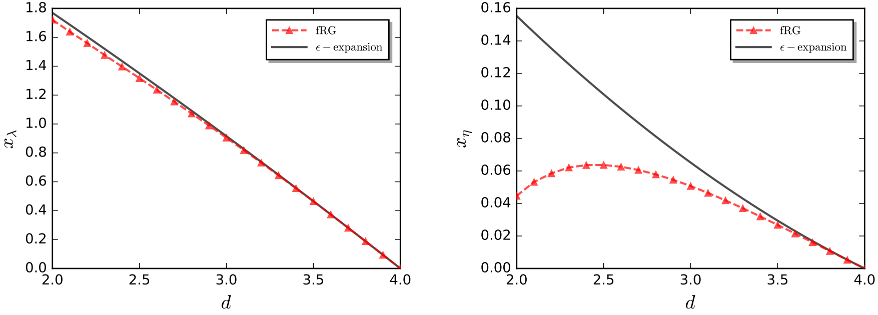
$u'(\bar{\rho})$ and static anomalous dimension η_ϕ





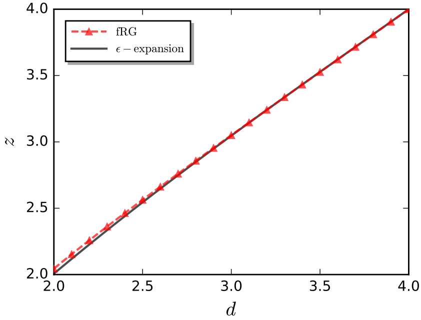
- Field dependence of $u'(\bar{\rho})$ under three different dimensions
- η_ϕ as a function of spatial dimension d

Spatial dimension dependence of x_λ and x_η



- x_λ shows excellent agreement with the perturbative expansion up to ϵ^2
- x_η from fRG calculation and ϵ-expansion are comparable as $d \to 4$

Dynamic critical exponent z

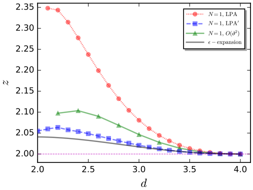


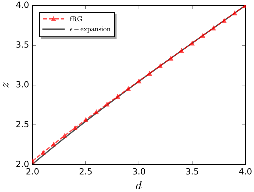
$$z = 4 - \eta_\phi - x_\lambda$$

- z as a function of the spatial dimension d
- ϵ-expansion for η_ϕ and x_λ are also presented as comparison

Summary and Outlook

- Investigate the critical dynamics of Model A and Model H within the real-time fRG appraoch
- The effective action of Model A is expanded to the order of $O(\partial^2)$ in the derivative expansion
- Dynamic critical exponent z are obtained for Model A, Model H as a function of spatial dimension





Time-evolution of critical modes (see Yang-yang Tan's talk)

Thanks for your attentions!

Backup

Order parameter propagators

$iG_k^R =$ c - - - - - - - - - - q , $iG_k^A =$ q - - - - - - - - - - c , $iG_k^K =$ c - - - - - - - - - - c

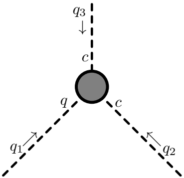
Diagrammatic representation of the propagators

$$iG^R_{k,ab}(q) = \frac{i}{Z_{t,k}\,q_0 + iZ_{\phi,k}(\rho_c)\boldsymbol{q}^2 + iZ_{\phi,k}\boldsymbol{q}^2 r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right) + im^2_{a,k}}\delta_{ab}\,,$$

$$iG^A_{k,ab}(q) = \frac{i}{-Z_{t,k}\,q_0 + iZ_{\phi,k}(\rho_c)\boldsymbol{q}^2 + iZ_{\phi,k}\boldsymbol{q}^2 r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right) + im^2_{a,k}}\delta_{ab}\,,$$

$$iG^K_{k,ab}(q) = \frac{4Z_{t,k}T}{\left(Z_{t,k}\,q_0\right)^2 + \left(Z_{\phi,k}(\rho_c)\boldsymbol{q}^2 + Z_{\phi,k}\boldsymbol{q}^2 r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right) + m^2_{a,k}\right)^2}\delta_{ab}\,.$$

Feynman rules of three-point vertices

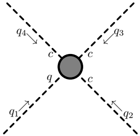


$$= i\Gamma^{(3)}_{k,qcc}(q_1, q_2, q_3)$$

$$i\Gamma^{(3)qcc}_{k,a_1 a_2 a_3}(q_1, q_2, q_3)$$

$$= Z'_{\phi,k}(\rho_c)\rho_c^{1/2}\Big(\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 \delta_{a_1 a_2}\delta_{a_3 0} + \boldsymbol{q}_1 \cdot \boldsymbol{q}_3 \delta_{a_1 a_3}\delta_{a_2 0} + \boldsymbol{q}_2 \cdot \boldsymbol{q}_3 \delta_{a_2 a_3}\delta_{a_1 0}\Big)$$
$$- \rho_c^{1/2} V_k^{(2)}(\rho_c)\Big(\delta_{a_1 a_2}\delta_{a_3 0} + \delta_{a_1 a_3}\delta_{a_2 0} + \delta_{a_2 a_3}\delta_{a_1 0}\Big) - 2\rho_c^{3/2} V_k^{(3)}(\rho_c)\delta_{a_1 0}\delta_{a_2 0}\delta_{a_3 0}\,,$$

Feynman rules of four-point vertices



$$= i\Gamma^{(4)}_{k,qccc}(q_1, q_2, q_3, q_4)$$

$$
\begin{aligned}
&i\Gamma^{(4)}_{k,\mathrm{I}} \\
=&\frac{1}{2}Z'_{\phi,k}(\rho_c)\Big[\big(\boldsymbol{q}_1\cdot\boldsymbol{q}_2+\boldsymbol{q}_3\cdot\boldsymbol{q}_4\big)\delta_{a_1a_2}\delta_{a_3a_4}+\big(\boldsymbol{q}_1\cdot\boldsymbol{q}_3 \\
&+\boldsymbol{q}_2\cdot\boldsymbol{q}_4\big)\delta_{a_1a_3}\delta_{a_2a_4}+\big(\boldsymbol{q}_1\cdot\boldsymbol{q}_4+\boldsymbol{q}_2\cdot\boldsymbol{q}_3\big)\delta_{a_1a_4}\delta_{a_2a_3}\Big] \\
&+\rho_c Z^{(2)}_{\phi,k}(\rho_c)\Big(\boldsymbol{q}_1\cdot\boldsymbol{q}_2\delta_{a_1a_2}\delta_{a_30}\delta_{a_40} \\
&+\boldsymbol{q}_1\cdot\boldsymbol{q}_3\delta_{a_1a_3}\delta_{a_20}\delta_{a_40}+\boldsymbol{q}_1\cdot\boldsymbol{q}_4\delta_{a_1a_4}\delta_{a_20}\delta_{a_30} \\
&+\boldsymbol{q}_2\cdot\boldsymbol{q}_3\delta_{a_2a_3}\delta_{a_10}\delta_{a_40}+\boldsymbol{q}_2\cdot\boldsymbol{q}_4\delta_{a_2a_4}\delta_{a_10}\delta_{a_30} \\
&+\boldsymbol{q}_3\cdot\boldsymbol{q}_4\delta_{a_3a_4}\delta_{a_10}\delta_{a_20}\Big),
\end{aligned}
$$

Feynman rules of four-point vertices

Effective potential contributes to four-point vertices

$$
\begin{aligned}
& i\Gamma^{(4)}_{k,\mathrm{II}} \\
=& -\frac{1}{2}V_k^{(2)}(\rho_c)\Big(\delta_{a_1 a_2}\delta_{a_3 a_4}+\delta_{a_1 a_3}\delta_{a_2 a_4}+\delta_{a_1 a_4}\delta_{a_2 a_3}\Big) \\
& -\rho_c V_k^{(3)}(\rho_c)\Big(\delta_{a_1 a_2}\delta_{a_3 0}\delta_{a_4 0}+\delta_{a_1 a_3}\delta_{a_2 0}\delta_{a_4 0} \\
& +\delta_{a_1 a_4}\delta_{a_2 0}\delta_{a_3 0}+\delta_{a_2 a_3}\delta_{a_1 0}\delta_{a_4 0}+\delta_{a_2 a_4}\delta_{a_1 0}\delta_{a_3 0} \\
& +\delta_{a_3 a_4}\delta_{a_1 0}\delta_{a_2 0}\Big)-2\rho_c^2 V_k^{(4)}(\rho_c)\delta_{a_1 0}\delta_{a_2 0}\delta_{a_3 0}\delta_{a_4 0}\,.
\end{aligned}
$$

Flow equation of the effective potential

$$
\begin{aligned}
\partial_\tau u'(\bar\rho) = {} & (-2+\eta_\phi)u'(\bar\rho) + (-2+d+\eta_\phi)\bar\rho u''(\bar\rho) \\
& - \frac{\nu_d}{2}\Big(\int \mathrm{dx}\, x^{\frac{d}{2}-1} \frac{z'_\phi(\bar\rho)x + 3u''(\bar\rho) + 2\bar\rho u^{(3)}(\bar\rho)}{l_\sigma(x)^2} s(x) \\
& + (N-1)\int \mathrm{dx}\, x^{\frac{d}{2}-1} \frac{z'_\phi(\bar\rho)y + u''(\bar\rho)}{l_\pi(x)^2} s(x)\Big)
\end{aligned}
$$

Flow equation of the $z(\bar{\rho})$

$$
\begin{aligned}
\partial_\tau z_\phi(\bar\rho) &= \eta z_\phi(\bar\rho) + (-2+d+\eta)\bar\rho z'_\phi(\bar\rho) \\
&+ \frac{2}{d}\bar\rho\left(z'_\phi(\bar\rho)\right)^2 \nu_d \times \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}} s(x) \left(\frac{1}{L_\pi(x) L_\sigma^2(x)} + \frac{1}{L_\pi^2(x) L_\sigma(x)} \right) \\
&+ 4\bar\rho z'_\phi(\bar\rho) u^{(2)}(\bar\rho)\nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} \frac{s(x)}{L_\pi(x) L_\sigma(x)^2} \\
&- 4\bar\rho\left(u^{(2)}(\bar\rho)\right)^2 \nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} \frac{s(x)}{L_\pi^2(x) L_\sigma^2(x)} \left(\partial_x L_\pi(x)\right) \\
&+ \frac{8}{d}\bar\rho\left(u^{(2)}(\bar\rho)\right)^2 \nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}} \left(\frac{1}{L_\pi^2(x) L_\sigma^3(x)} + \frac{1}{L_\pi^3(x) L_\sigma^2(x)} \right) \left(\partial_x L_\pi(x)\right)^2 s(x) \\
&- \nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}} \frac{8}{d}\bar\rho\left(u^{(2)}(\bar\rho)\right)^2 \frac{1}{L_\pi^2(x) L_\sigma^2(x)} \left(\partial_x^2 L_\pi(x)\right) s(x) \\
&- \left(z'_\phi(\bar\rho) + 2\bar\rho z_\phi^{(2)}(\bar\rho)\right)\nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} \frac{1}{L_\sigma^2(x)} s(x) \\
&- (N-1) z'_\phi(\bar\rho)\nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} \frac{1}{L_\pi^2(x)} s(x)
\end{aligned}
$$

Flow equation of the $Z_{t,k}$

$$\partial_\tau Z_{t,k} = - Z_{t,k} 2\bar{\rho}\big(u^{(2)}(\bar{\rho})\big)^2 \nu_d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} s(x) \times \frac{L_\pi^2(x) + 4L_\pi(x)L_\sigma(x) + L_\sigma^2(x)}{L_\pi^2(x)L_\sigma^2(x)\big[L_\pi(x) + L_\sigma(x)\big]^2}$$

with

$$\begin{aligned} s(x) =& \big[2 - \eta(1-x)\big]\Theta(1-x)\,, \\ L_\pi(x) =& z_\phi(\bar{\rho})x + (1-x)\Theta(1-x) + u'(\bar{\rho})\,, \\ L_\sigma(x) =& z_\phi(\bar{\rho})x + (1-x)\Theta(1-x) + u'(\bar{\rho}) + 2\bar{\rho}u^{(2)}(\bar{\rho}) \end{aligned}$$

Order parameter propagators

$iG_k^R = \underset{c}{-}\!-\!-\!-\!-\!-\!\underset{q}{-}$, $iG_k^A = \underset{q}{-}\!-\!-\!-\!-\!-\!\underset{c}{-}$, $iG_k^K = \underset{c}{-}\!-\!-\!-\!-\!-\!\underset{c}{-}$

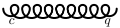
The expression for order parameter propagators

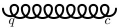
$$iG^R_{\phi,k}(q) = \frac{i}{q_0 + i\lambda_k \boldsymbol{q}^2 \left(\boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right) + m^2_{\phi,k}\right)},$$

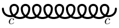
$$iG^A_{\phi,k}(q) = \frac{i}{q_0 - i\lambda_k \boldsymbol{q}^2 \left(\boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right) + m^2_{\phi,k}\right)},$$

$$iG^K_{\phi,k}(q) = \frac{4\lambda_k \boldsymbol{q}^2}{q_0^2 + \left[\lambda_k \boldsymbol{q}^2 \left(\boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right) + m^2_{\phi,k}\right)\right]^2}.$$

Momentum density propagators

$iG^R_{j,k} =$ 

, $iG^A_{j,k} =$ 

, $iG^K_{j,k} =$ 

The expression for momentum density propagators

$$iG^R_{j,k}(q) = \frac{i}{q_0 + i\eta_k \boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right)}\Pi^\perp ,$$

$$iG^A_{j,k}(q) = \frac{i}{q_0 - i\eta_k \boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right)}\Pi^\perp ,$$

$$iG^K_{j,k}(q) = \frac{4\eta_k \boldsymbol{q}^2}{q_0^2 + \left[\eta_k \boldsymbol{q}^2\left(1 + r_B\left(\frac{\boldsymbol{q}^2}{k^2}\right)\right)\right]^2}\Pi^\perp .$$

Flow equation for $u'(\bar{\rho})$

$$\partial_\tau u'(\bar{\rho}) = (-2+\eta_\phi)u'(\bar{\rho}) + (-2+d+\eta_\phi)\bar{\rho}u^{(2)}(\bar{\rho}) - \frac{2\nu_d}{d}\left(1-\frac{\eta_\phi}{d+2}\right)\left(\frac{3u^{(2)}(\bar{\rho})+2\bar{\rho}u^{(3)}(\bar{\rho})}{(1+u'(\bar{\rho})+2\bar{\rho}u^{(2)}(\bar{\rho}))^2}\right),$$

Dimensionless

$$\begin{cases} \bar{\lambda} = k^{4-z}\lambda_k \\ \bar{\eta} = k^{2-z}\eta_k \\ \bar{g} = k^{1+\frac{d}{2}-z}g_k \\ \bar{\rho} = k^{2-d}\rho \\ u(\bar{\rho}) = k^{-d}V_k(\rho) \end{cases}$$

Flow equation for transport coefficient $\bar{\lambda}$

$$
\begin{aligned}
\partial_\tau \bar{\lambda} =& -(z-4+\eta_\phi)\bar{\lambda} \\
& - \frac{2\bar{g}^2}{\bar{\eta}} \frac{1}{(1+\bar{m}^2)^2} \int \mathrm{d}_{\mathrm{d}} (-1+\cos^2\theta)\cos^2\theta \\
& + \frac{2}{d} \frac{2\bar{g}^2}{\bar{\eta}} \frac{1}{(1+\bar{m}^2)^2} \int \mathrm{d}_{\mathrm{d}} (2-6\cos^2\theta+4\cos^4\theta) \\
& - \frac{2\bar{g}^2}{\bar{\eta}} \frac{1}{1+\bar{m}^2} \int \mathrm{d}_{\mathrm{d}} (\cos^4\theta-\cos^2\theta) \\
& + \frac{6\bar{g}^2\bar{\lambda}}{\bar{\eta}^2} \int \mathrm{d}_{\mathrm{d}} (-1+\cos^2\theta)\cos^2\theta \\
& + \frac{2}{d+2} \frac{6\bar{g}^2\bar{\lambda}}{\bar{\eta}^2} (1+\bar{m}^2) \int \mathrm{d}_{\mathrm{d}} (-1+\cos^2\theta) \\
& - \frac{3\bar{g}^2\bar{\lambda}}{\bar{\eta}^3} \int \mathrm{d}_{\mathrm{d}} (4\cos^2\theta-4\cos^4\theta) \\
& + \frac{2}{d+4} \frac{32\bar{g}^2\bar{\lambda}^2}{\bar{\eta}^3} (1+\bar{m}^2) \int \mathrm{d}_{\mathrm{d}} (\cos^2\theta-\cos^4\theta)\,,
\end{aligned}
$$

Flow equation for $\bar{\eta}$ and $\bar{g}$

$$
\begin{aligned}
\partial_\tau \bar{\eta} = & -(z-2)\bar{\eta} \\
& + \frac{2}{d}\frac{\bar{g}^2}{\bar{\lambda}}\frac{1}{(1+\bar{m}^2)^3}\int \mathrm{d}_{\mathrm{d}}(1-3\cos^2\theta+2\cos^4\theta)\,,
\end{aligned}
$$

$$
\partial_\tau \bar{g} = -(z-3+(4-d)/2)\bar{g}\,.
$$

Here we neglect the loop diagram correction for the coupling g_k.
Define a new dimensionless coupling constant

$$
f = \nu_d \frac{\bar{g}^2}{\bar{\lambda}\bar{\eta}}
$$

Flow equation for the coupling constant f

By combining the flow equations for $\bar{\lambda}$, $\bar{\eta}$ and $\bar{g}$

$$\partial_\tau f = (\eta_\phi - (4-d))f - I_\lambda f^2 - I_\eta f^2$$

Function I_λ and I_η

$$\begin{aligned} I_\lambda =& -\frac{2}{(1+\bar{m}^2)^2}\int \mathrm{d}\ _\mathrm{d}(-1+\cos^2\theta)\cos^2\theta \\ &+ \frac{4}{d}\frac{1}{(1+\bar{m}^2)^2}\int \mathrm{d}\ _\mathrm{d}(2-6\cos^2\theta+4\cos^4\theta) \\ &- \frac{2}{1+\bar{m}^2}\int \mathrm{d}\ _\mathrm{d}(\cos^4\theta-\cos^2\theta)\,, \\ I_\eta =& \frac{2}{d}\frac{1}{(1+\bar{m}^2)^3}\int \mathrm{d}\ _\mathrm{d}(1-3\cos^2\theta+2\cos^4\theta)\,. \end{aligned}$$