Yang-Lee Edge singularity and

and the structure of QCD phase diagram

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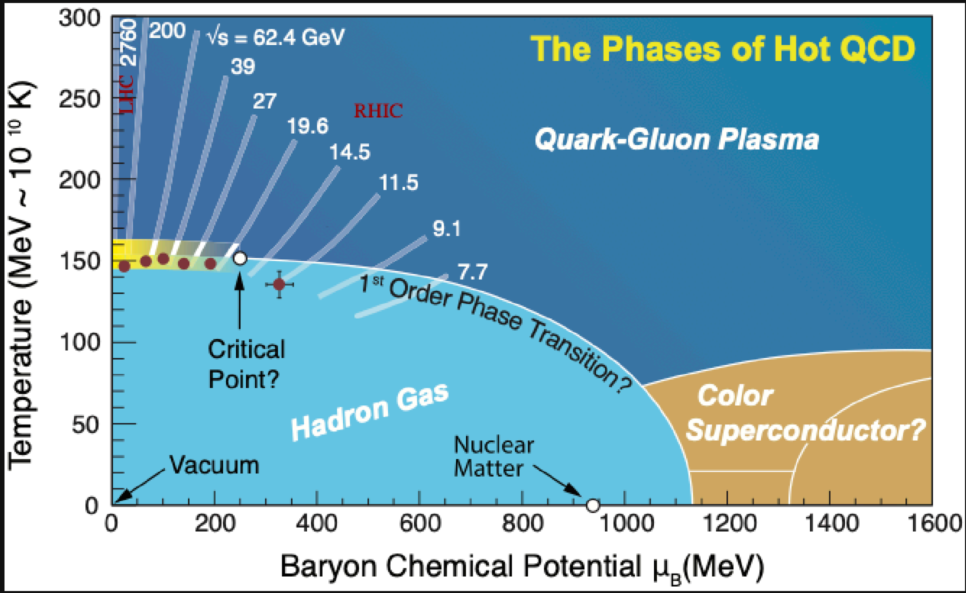
Outline

- Introduction: why analytic structure? and what is Yang-Lee edge singularity?
- Universal location of YLE for most relevant universality classes
- Tracing YLE in QCD
- Conclusions

QCD Phase diagram



- Experiment with relativistic heavy ions: the system is small and has a short lifetime
- Theory: although the underlying theory (QCD) is known, we cannot solve it ✗
- Numerical methods: zero density region only due to the ``sign'' problem ✗
- Indirect methods: Taylor series coefficients/imaginary $\mu \longrightarrow$ non-zero baryon density ✓

$$p/T^4 = \sum_{n=0}^{\infty} \frac{\chi_n}{n!}\left(\frac{\mu}{T}\right)^n; \quad \chi_n = \frac{\partial^n (p/T^4)}{\partial\,(\mu/T)^n} \quad \chi_2 = \frac{\langle(\delta N)^2\rangle}{VT^3} \quad \chi_4 = \frac{\langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2}{VT^3}$$

Taylor series expansion

- Consider an arbitrary function expanded around a regular point

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n x^n$$

- What limits the predictive power of this expansion? e.g. $f(x) = 1/(e^x + 1)$.

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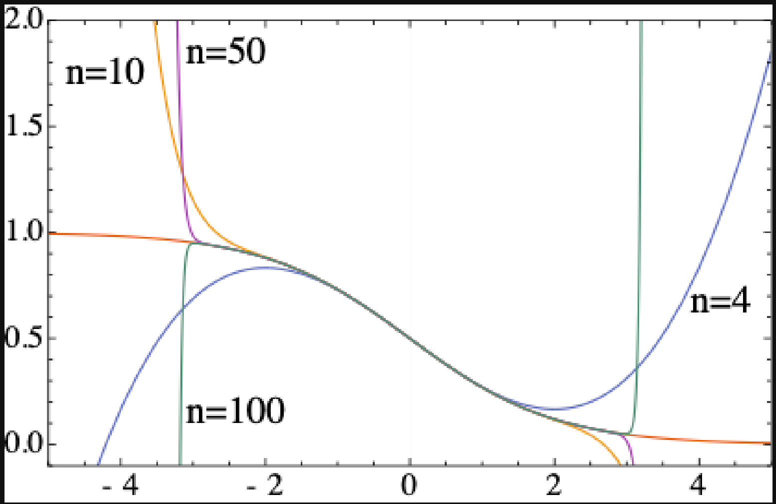

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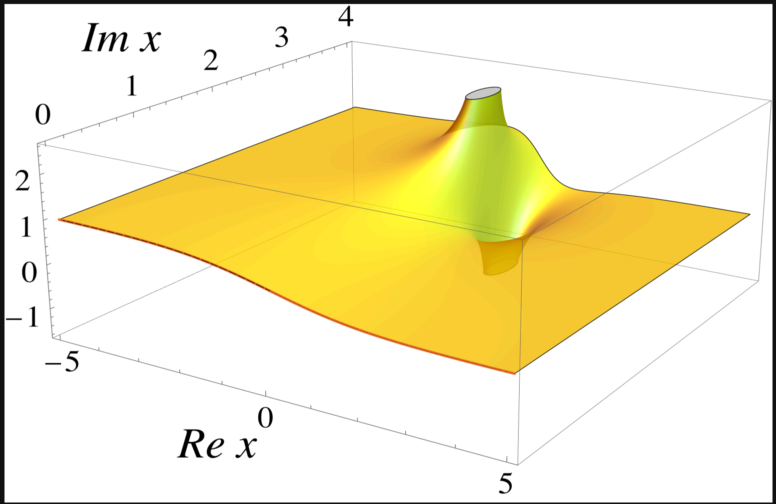

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$$|x| < R_c \equiv \left(\limsup_n \left| f_n^{1/n} \right| \right)^{-1}$$

- R_c is the radius of convergence
- $R_c =$ distance in the *complex* plane from the expansion point to the nearest singularity

Are there singularities associated with critical point/phase transitions?

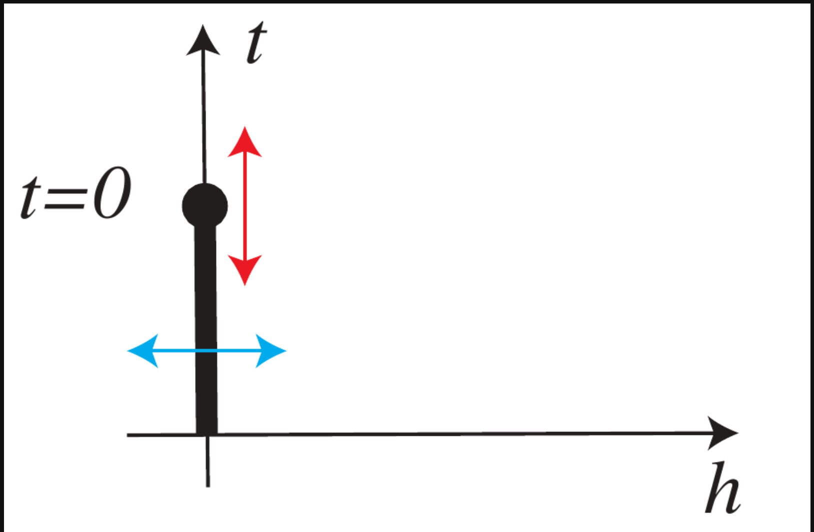
Example: Landau free energy

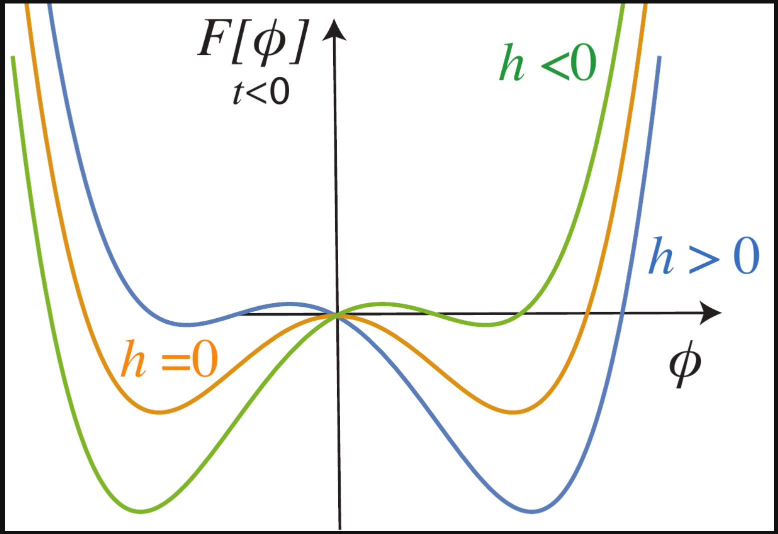
$$F = \int d^d x \left(\frac{1}{2} t\phi^2 + \frac{1}{4} \lambda \phi^4 - h\phi \right)$$

E.g.:

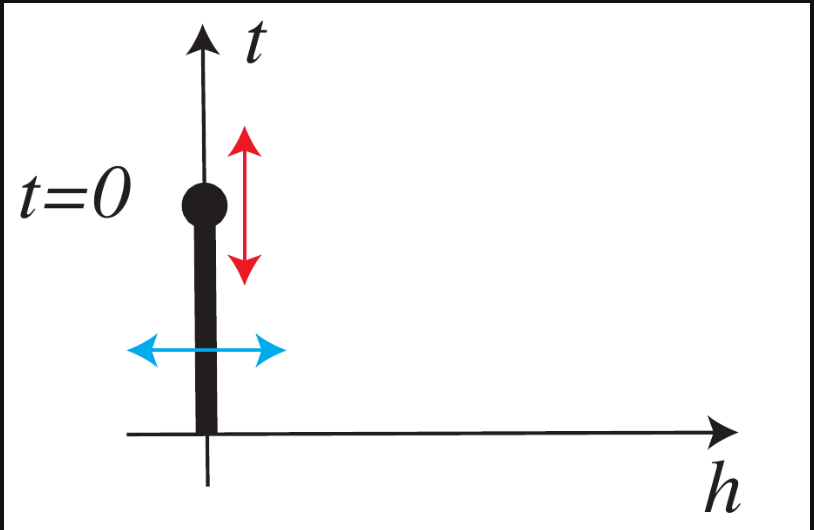
- near chiral limit: $t \propto T - T_c + \kappa\mu^2$, $h \propto m_{u,d}$
- near CP: $t, h \propto \alpha_{t,h}(T - T_c) + \beta_{t,h}(\mu - \mu_c)$
- near RW: $t \propto T - T_{RW}$, $h \propto \mu_B - i\pi T$

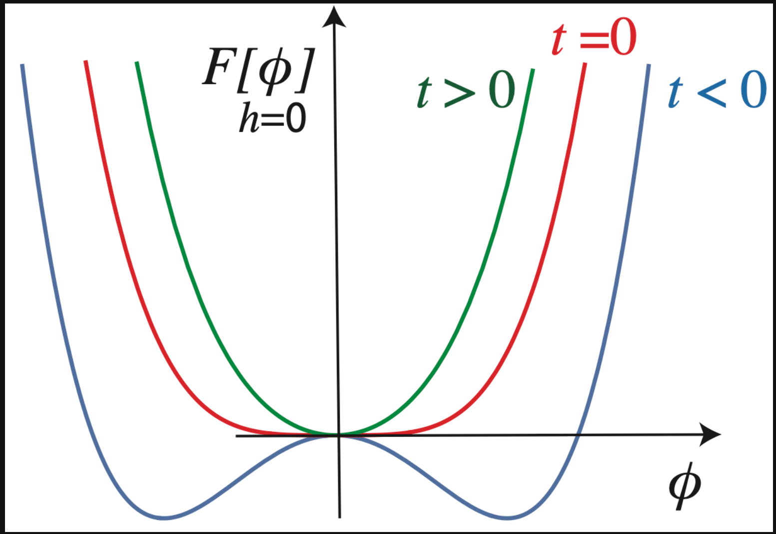
Vary h





Vary t





Magnetic equation of state

$$F = \int d^d x \left(\frac{1}{2} t\phi^2 + \frac{1}{4}\lambda\phi^4 - h\phi \right)$$

Minimize $F[\phi] \rightsquigarrow$ equilibrium order parameter:

- Arbitrary t and h: $t\phi + \lambda\phi^3 = h$
- To simplify math $\lambda \to 1$: $t\phi + \phi^3 = h$
- Ansatz for the solution $\phi = h^{1/3} f_G$

$th^{1/3} f_G + h f_G^3 = h$ or
$\frac{t}{h^{2/3}} f_G + f_G^3 = 1$

- Scaling form of "magnetic equation of state"

$$f_G(z + f_G^2) = 1, \quad z = \frac{t}{h^{\frac{1}{\beta\delta}}} \quad \text{with} \quad \beta = 1/2, \delta = 3$$

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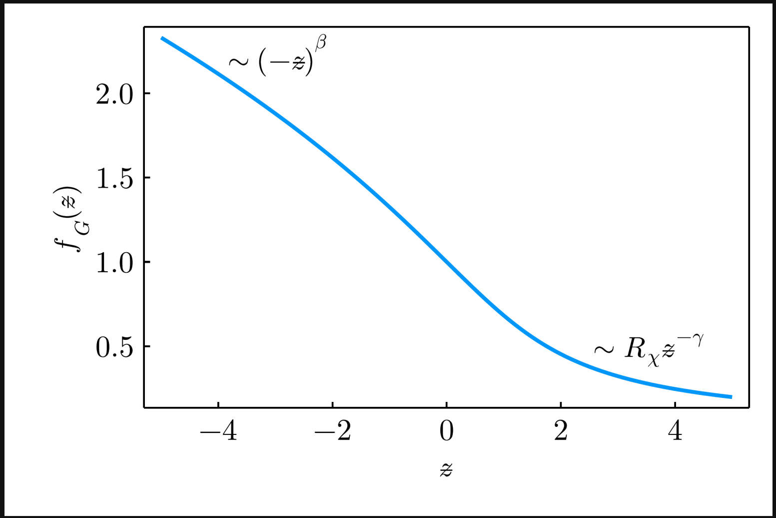

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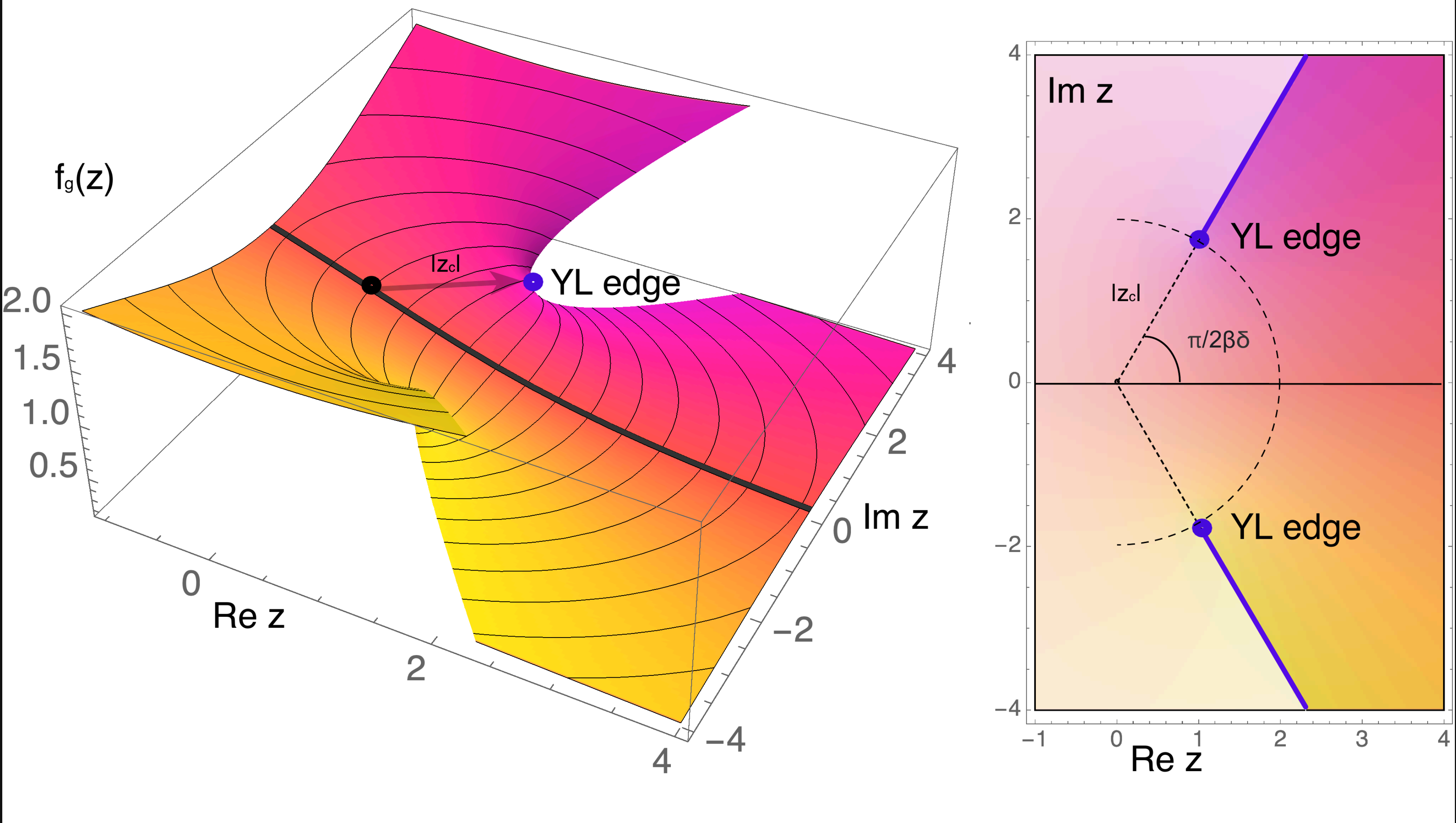
Yang-Lee edge singularity



Near YLE singularity

- f_G is singular

$$f_G - f_G^c \propto (z - z_c)^{\sigma_{\text{YLE}}}$$

- The critical exponent σ_{YLE} is independent of the underlying universality class
- From conformal bootstrap, $\sigma_{\text{YLE}}^{d=3} = 0.085(1)$
- Mean-field approximation gets it wrong: $\sigma_{\text{YLE}}^{\text{MF}} = \frac{1}{2}$
- Not surprisingly, mean-field gets z_c wrong as well
- z_c is universal: for $O(N)$, z_c depends only on N and d

Type of critical point:	critical	tricritical
Number of relevant variables:	2	4

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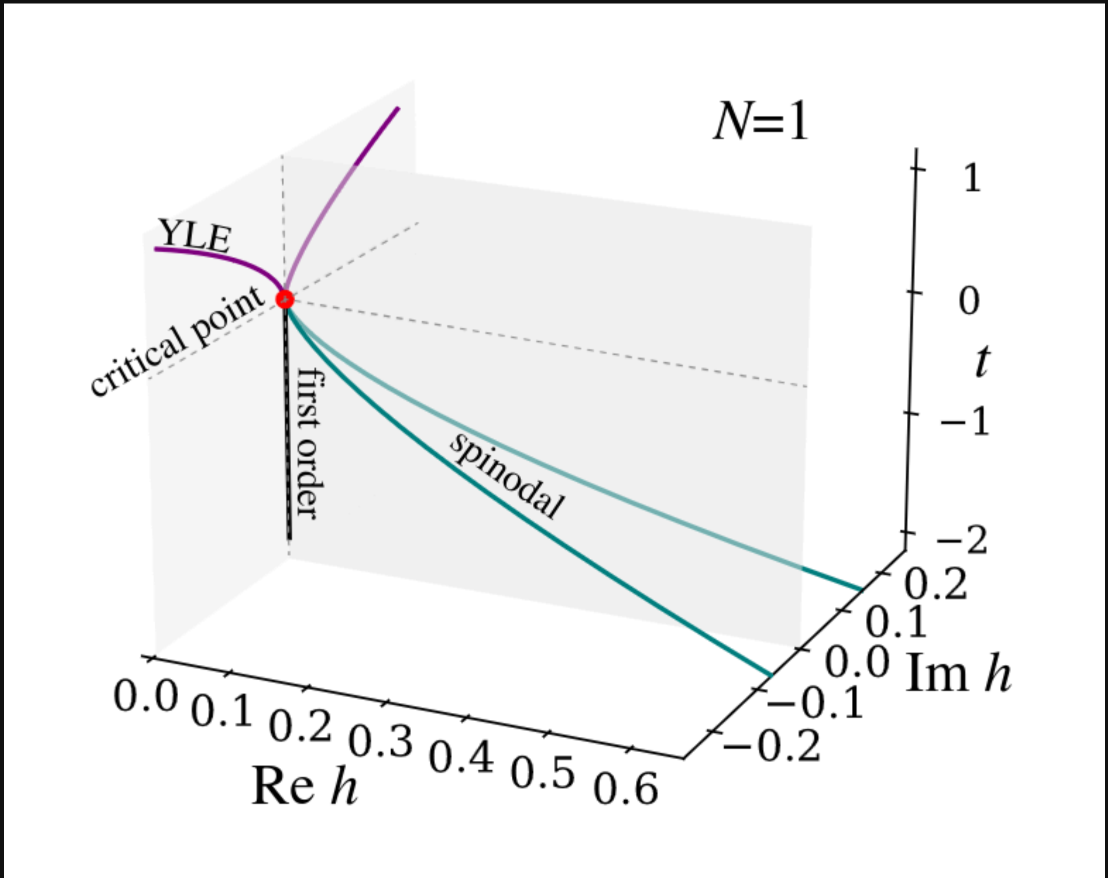
Type of critical point:	protocritical = YLE	critical	tricritical
Number of relevant variables:	1	2	4

1 independent crit. exp., c.f. standard critical point with 2 independent crit. exp.

Illustration in Ising model: $h = i|z_c|^{-\beta\delta} t^{\beta\delta}$



F. Rennecke, G. Johnson, and V.S., Phys.Rev.D 107 (2023) 11, 116013

- In contrast to the critical point, YLE form lines
- YLE are continuously connected to critical point

Universal location of YLE

- The phase of $z_c = |z_c| e^{\pm \frac{i\pi}{2\beta\delta}}$ is defined by the critical exponents of the underlying universality class. How to find $|z_c|$?
- Ordinary, we rely on two methods: ε-expansion and lattice
  - ε-expansion breaks down: YLE is described by ϕ^3 with upper critical dimension $d_c = 6$, while underlying universality class has $d_c = 4$

M. Fisher, "Yang-Lee Edge Singularity and φ3 Field Theory", Phys. Rev. Lett. 40 1610 (1978)

Only leading order under perturbative control

$$|z_c| \approx |z_c^{\mathrm{MF}}| \left[1 + \frac{27\ln\left(\frac{3}{2}\right) - (N-1)\ln 2}{9(N+8)}\epsilon\right] + \epsilon^2 \log\epsilon \times (\cdots).$$

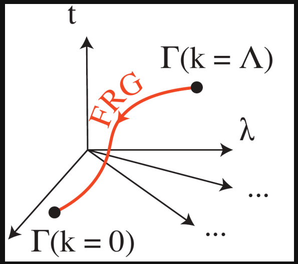
  - lattice: direct calculations at complex values of parameters are impossible due to sign problem; indirectly lattice can provide information about YLE

F. Karsch, C. Schmidt, S. Singh, 2311.13530

- Functional Renormalization group provides most precise $|z_c|$ in $d = 3$

Functional/Exact Renormalization Group

- Start with bare classical action at small distances/large momentum $S_{k=\Lambda}$
- Gradually include fluctuations of larger size/smaller momentum
- Continue until fluctuations of all possible sizes/momenta are accounted for



- Equation that does it: Functional Renormalization Group equation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{STr}\left[\left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \cdot \partial_k R_k \right]$$

Wetterich, 1993

Pros: Exact, non-perturbative, no sign problem. **Cons**: requires truncation.

Truncation: derivative expansion

- Near critical point: long wave excitations $\rightsquigarrow$ expansion around the uniform field
- First-order derivative expansion

$$\Gamma_k[\phi] = \int d^d x \left(U_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_i \phi)^2 \right)$$

  - The average potential

$$\partial_t U_k(\rho) = \frac{1}{2} \int \bar{d}^d q \partial_t R_k \left(q^2\right) \left[G_k^{\parallel} + (N-1) G_k^{\perp} \right], \quad \rho = \frac{\phi^2}{2}$$

with

$$G_k^{\perp} = \frac{1}{Z_k^{\perp}(\rho) q^2 + U_k'(\rho) + R_k(q^2)}, \quad G_k^{\parallel} = \frac{1}{Z_k^{\parallel}(\rho) q^2 + U_k'(\rho) + 2\rho U_k''(\rho$$

Truncation: derivative expansion

Wave function renormalization:

$$\partial_t Z_\parallel(\phi) = \int \bar{d}^d q \partial_t R_k(q^2) \Bigg\{ G_\parallel^2 \Big[\gamma_\parallel^2 \big(G'_\parallel + 2G''_\parallel \frac{q^2}{d}\big) 2\gamma_\parallel Z'_\parallel(\phi) \big(G_\parallel + 2G'_\parallel \frac{q^2}{d}$$

$$+ (Z'_\parallel(\phi))^2 G_\parallel \frac{q^2}{d} \frac{1}{2} Z''_\parallel(\phi) \Big]$$

$$+ (N-1) G_\perp^2 \Big[\gamma_\perp^2 \big(G'_\perp + 2G''_\perp \frac{q^2}{d}\big) 4\gamma_\perp Z'_\perp(\phi) G'_\perp \frac{q^2}{d} (Z'_\perp(\phi))^2 G_\perp \frac{q}{d}$$

$$+ 2 \frac{Z_\parallel(\phi) - Z_\perp(\phi)}{\phi} \gamma_\perp G_\perp \frac{1}{2} \left(\frac{1}{\phi} Z'_\parallel(\phi) \frac{2}{\phi^2} (Z_\parallel - Z_\perp) \right) \Big] \Bigg\}$$

with $\gamma_\parallel = q^2 Z'_\parallel(\phi) + U^{(3)}(\phi), \quad \gamma_\perp = q^2 Z'_\perp(\phi) + \frac{\partial}{\partial\phi}\left(\frac{1}{\phi} U'(\phi)\right), \quad G' = \frac{\partial G}{\partial q^2}, \dots$

G. Johnson, F. Rennecke, and V.S., Phys.Rev.D 107 (2023) 11, 116013

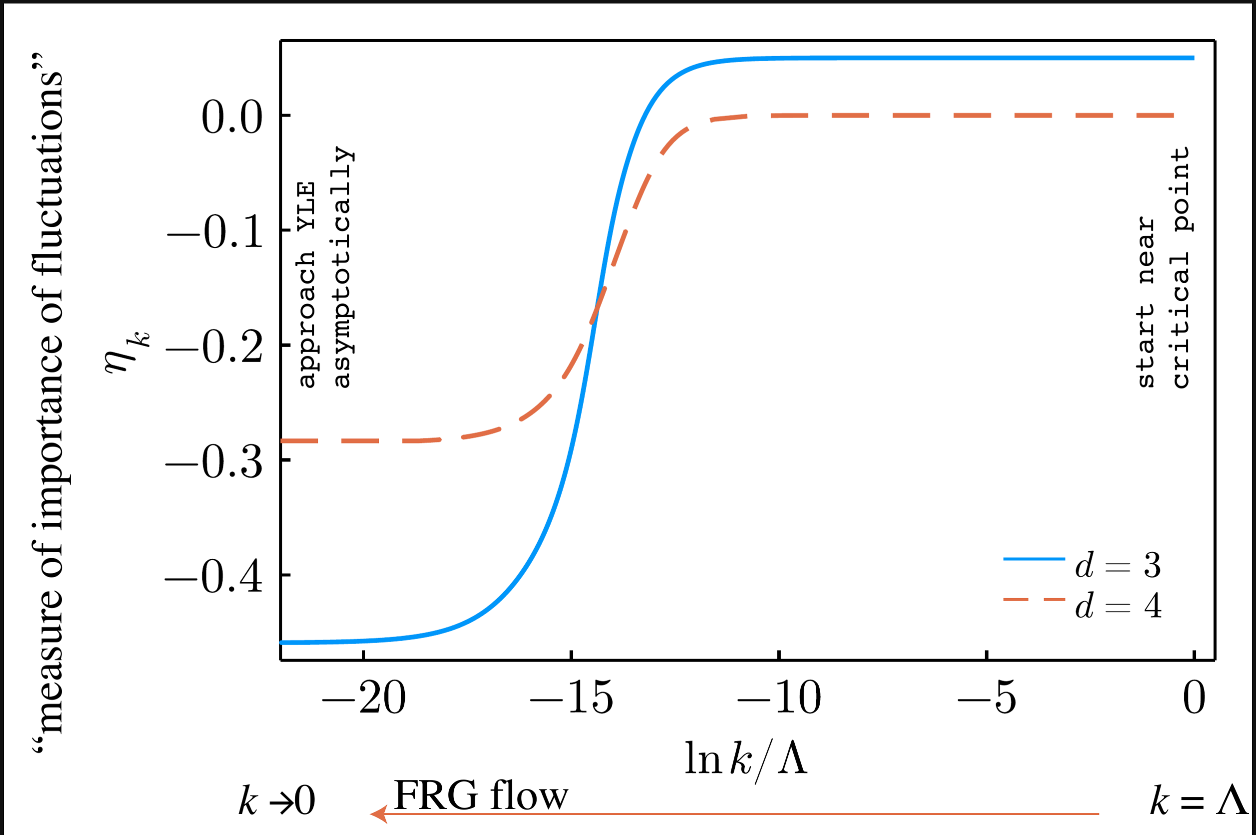
Truncation: series expansion

- Taylor series expansion of $U_k(\phi)$ and $Z_k(\phi)$ (orders 12 and 6 respectively)
  - Traditionally: expand near k-dependent minimum: $U_k'[\phi_k] = h =$ const.
  - To locate YLE: expand near $U_k''[\phi_k] = m^2 \to 0$.

  $\rightsquigarrow U_k'[\phi_k] = h_k \neq$ const

  $\rightsquigarrow$ Calculations in the broken phase are not feasible
- 18-26 coupled stiff differential equations
  - Mathematica to obtain equations (multiple Gb)
  - Implicit solvers for ODE's
  - Months on an HPC

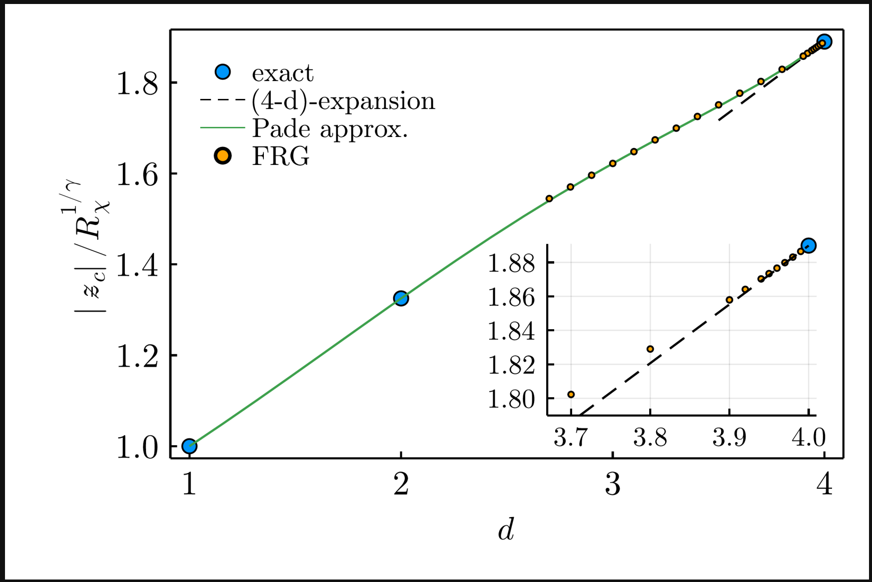
Results: importance of fluctuations (N=1)



F. Rennecke and V. S, Annals Phys. 444 (2022) 169010

Results: Ising universality class $N = 1$

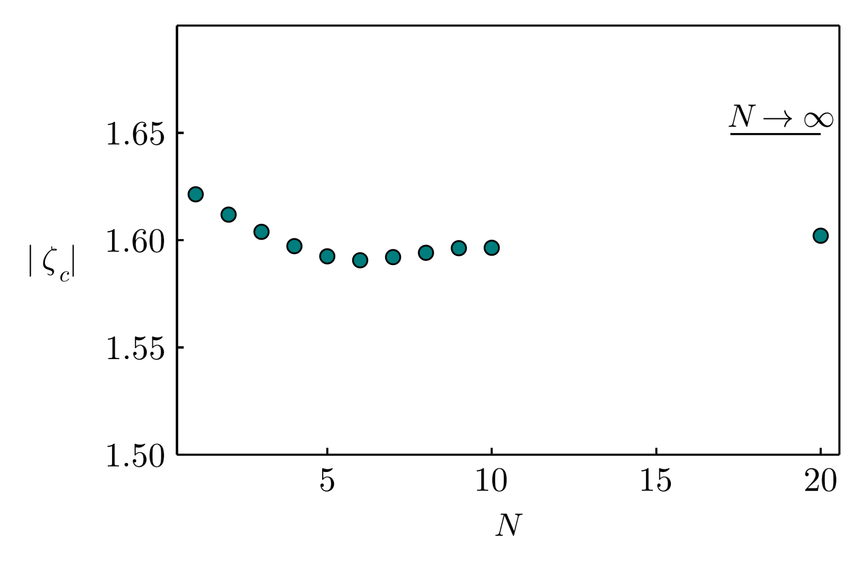
d does not have to be integer in FRG



d	1	2	3	4
$\|z_c\|/R_\chi^{1/\gamma}$	1	1.32504(2)	1.621(4)	$3/2^{2/3}$

G. Johnson, F. Rennecke, and V. S, Phys.Rev.D 107 (2023) 11, 116013
F. Rennecke and V. S, Annals Phys. 444 (2022) 169010
A. Connelly, G. Johnson, F. Rennecke, and V. S, Phys.Rev.Lett. 125 19, 191602 (2020)
$d = 2$: H.-L. Xu and A. Zamolodchikov, JHEP 08 (2022) 057 H.-L. Xu and A. Zamolodchikov, 2304.07886

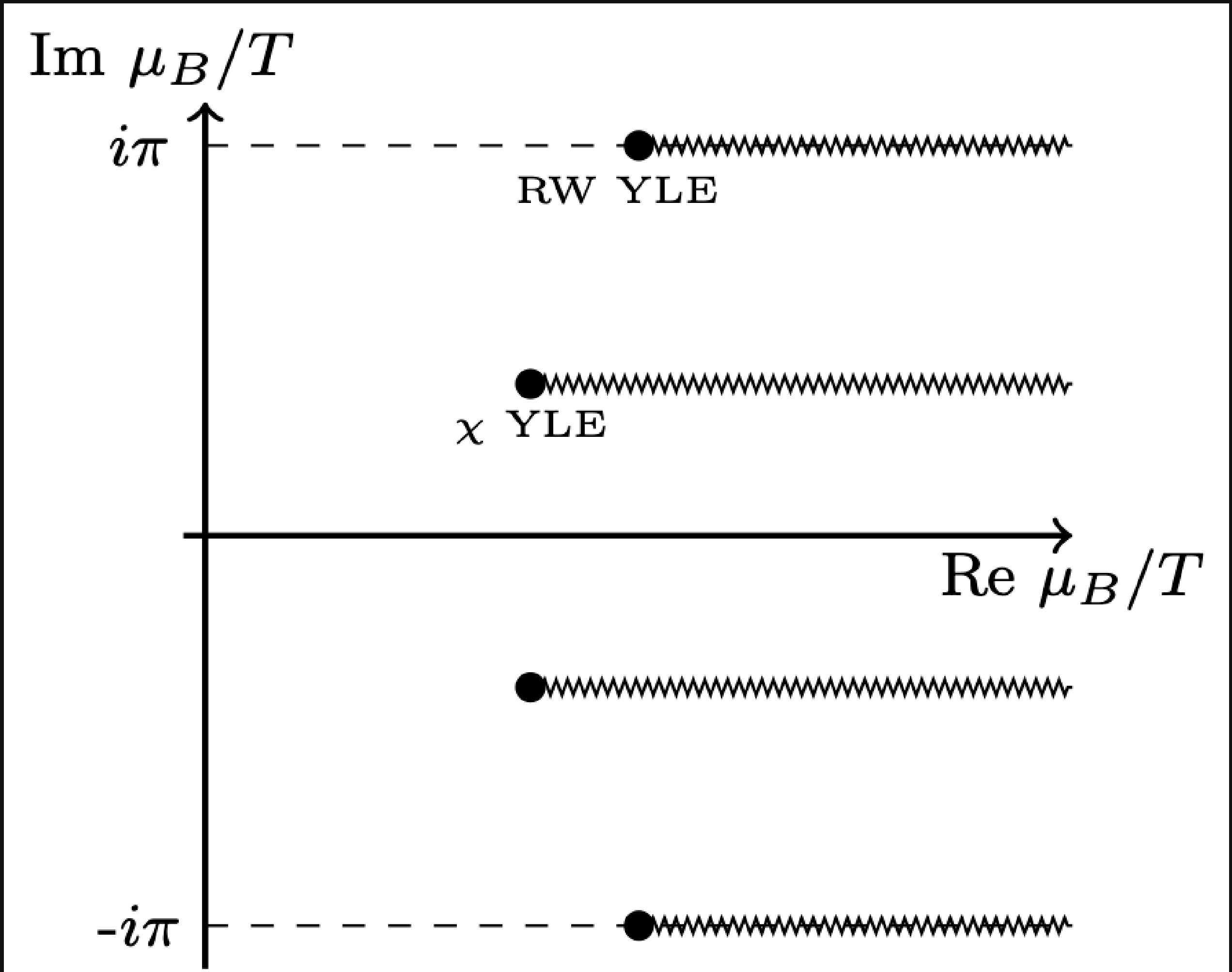
Arbitrary $N, d = 3$



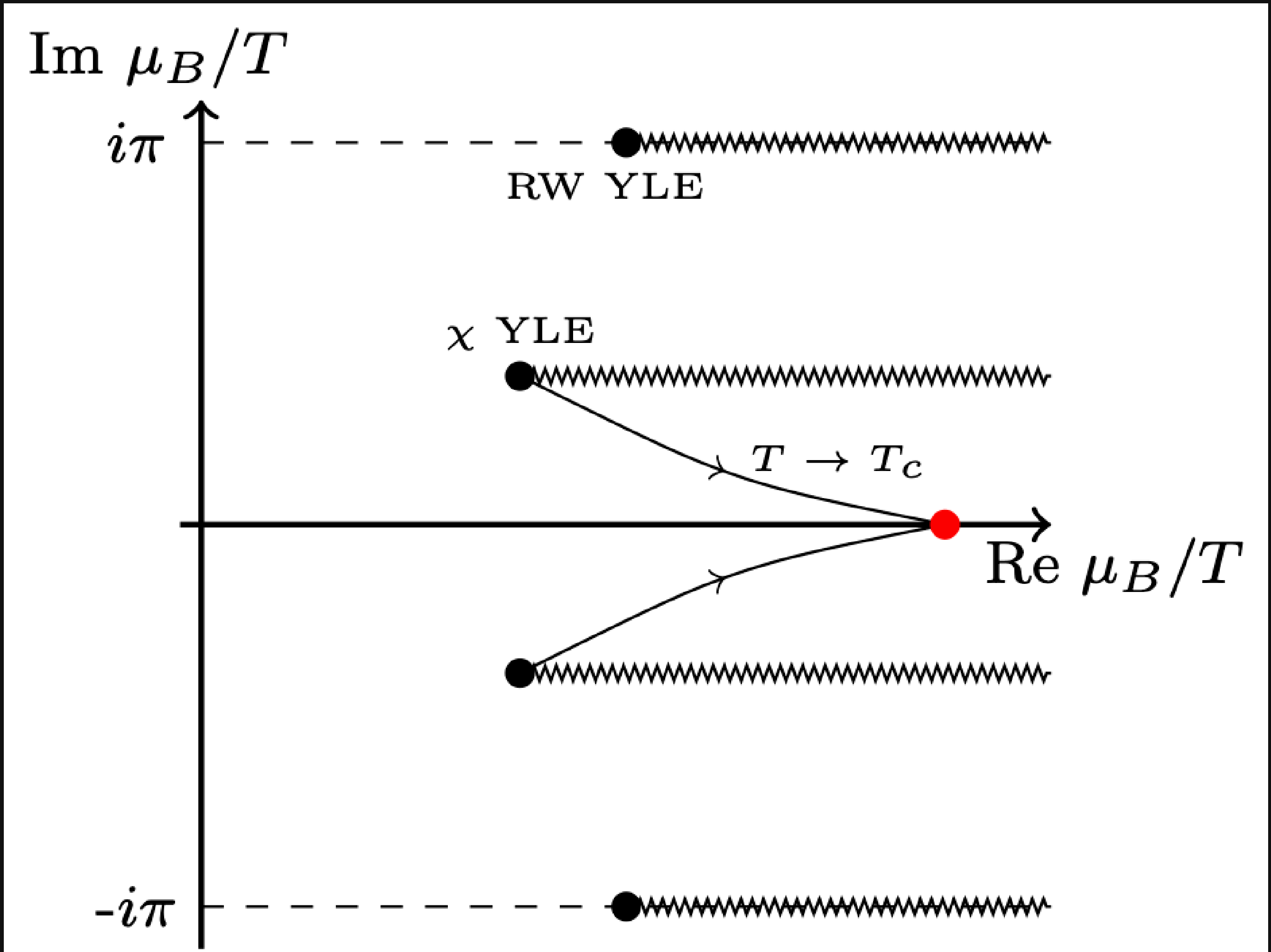
N	1	2	3	4	5
$\lvert z_c \rvert / R_\chi^{1/\gamma}$	1.621(4)(1)	1.612(9)(0)	1.604(7)(0)	1.597(3)(0)	1.5925(2)(1)

G. Johnson, F. Rennecke, and V. S, Phys.Rev.D 107 (2023) 11, 116013
c.f. F. Karsch, C. Schmidt, and S. Singh Phys.Rev.D 109 (2024) 1, 014508

Analytic structure in QCD: $T_c < T < T_{RW}$



Analytic structure in QCD: $T \to T_c$



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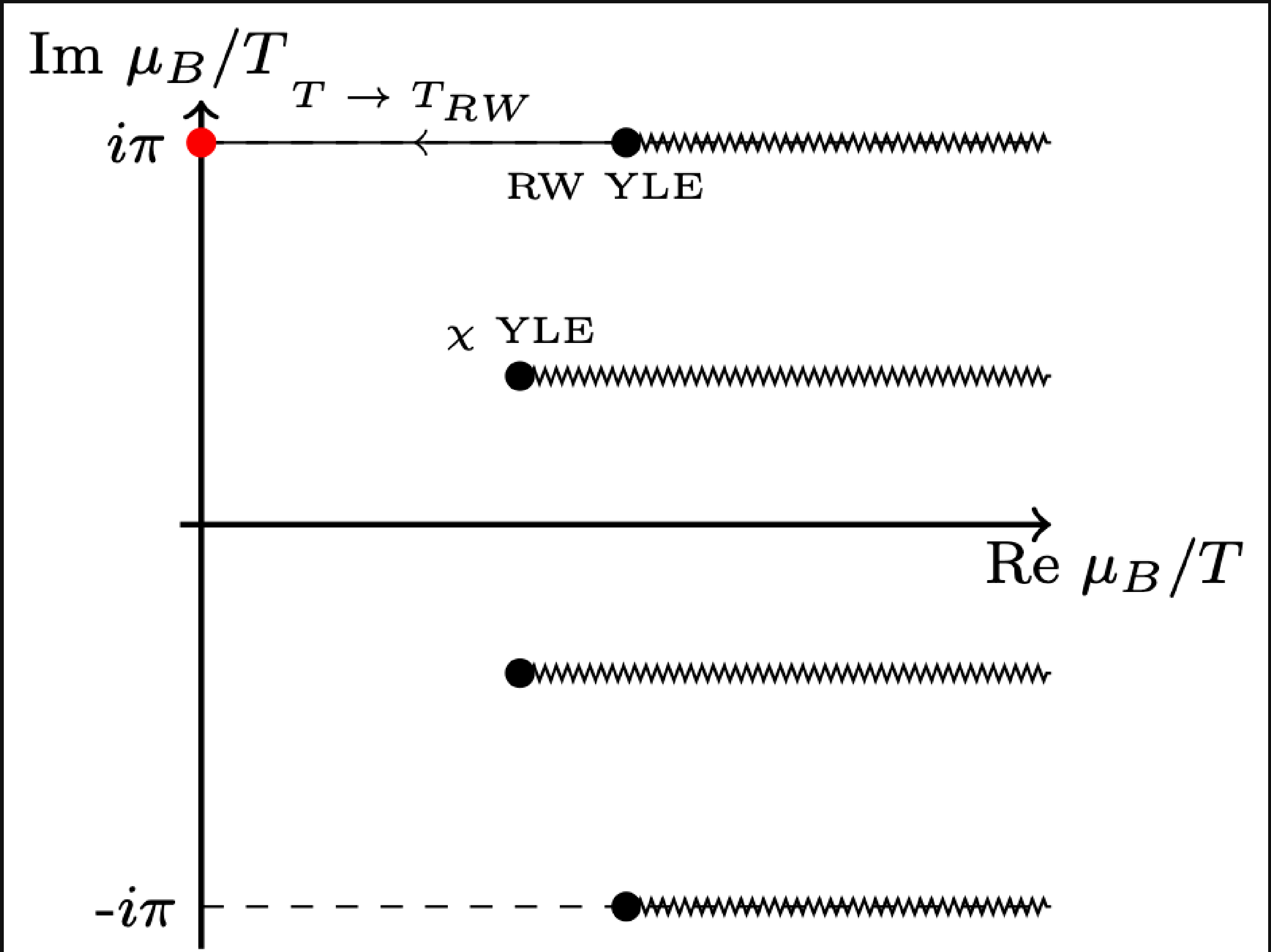
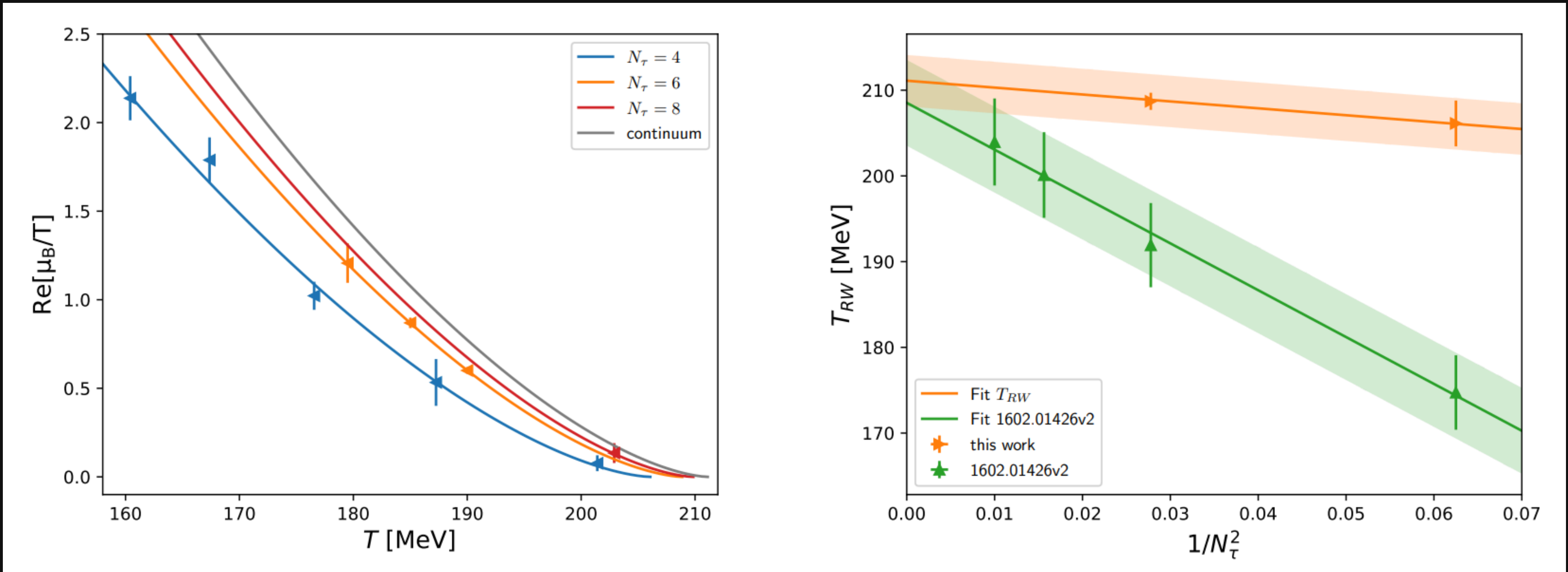


Tracing YLE singularity: RW critical point

Lattice QCD and indirect methods to locate YLE:

input from Im μ & analytic continuation

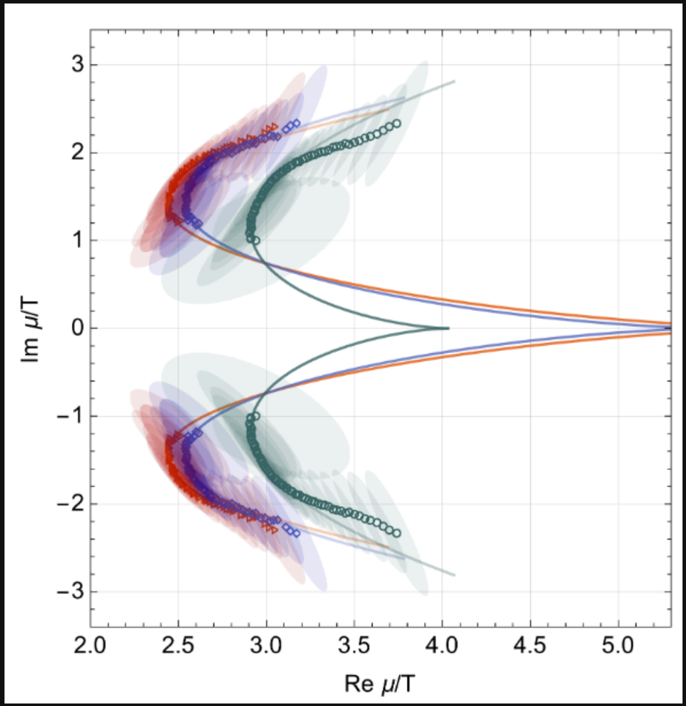


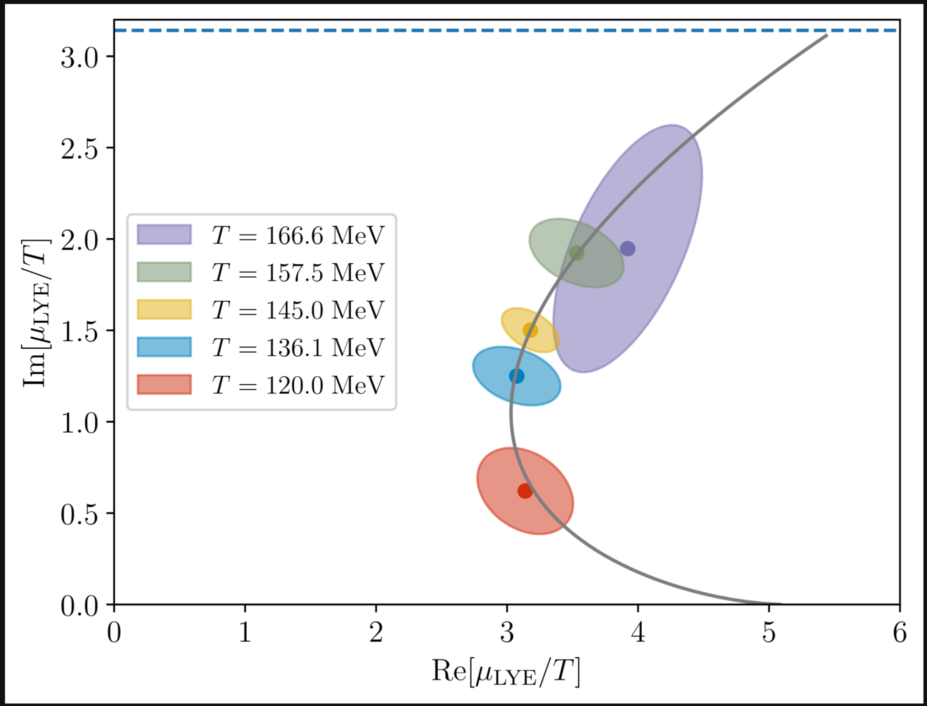
$$z = z_c \to \mathrm{Re}\mu_{YLE} \propto (T_{RW} - T)^{\beta\delta} \quad \rightsquigarrow T_{RW} = 211.1 \pm 3.1\mathrm{MeV}.$$

Christian Schmidt, et. al., 2401.07790

Tracing YLE singularity: chiral critical point

Lattice input from Taylor series coeff. at $\mu = 0$ or Im μ & analytic continuation





G. Basar, 2312.06952
D. Clarke et. al., 2405.10196

$$z = z_c \to \mathrm{Re}(\mu - \mu_c) = c_1(T - T_c) + c_2(T - T_c)^2 \text{ and } \mathrm{Im}\mu = c_3(T - T_c)^{\beta\delta}$$

$$\rightsquigarrow T_c \approx 110 \text{ MeV}, \mu_c \approx 650 \text{ MeV}$$

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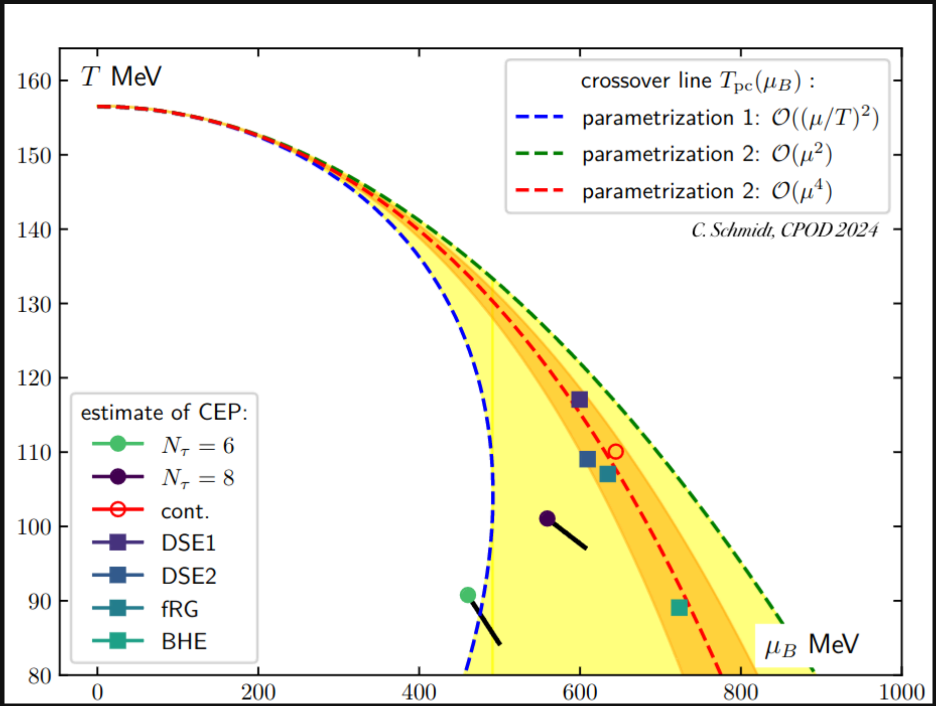

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Taking it further

- Properties of YLE, e.g. $\sigma_{\rm YLE}$ can be use to validate indirect methods of locating YLE in QCD, e.g. volume scaling of the density of zeros

- Moreover,

  - YLE defines the behavior of the higher order Taylor expansion coefficients (Darboux's theorem). E.g. for of $f_G(z)$:

$$f_G^{(n)} \sim 2B_0|z_c|^{-n}\frac{n^{\sigma-1}}{\Gamma(\sigma)}\cos\left(\beta_0 - \frac{\pi n}{2\beta\delta}\right), \quad B_0\exp(i\beta_0) = \lim_{z\to z_c}\frac{f_G(z)-f_G(z_c)}{(1-z/z_c)^\sigma}$$

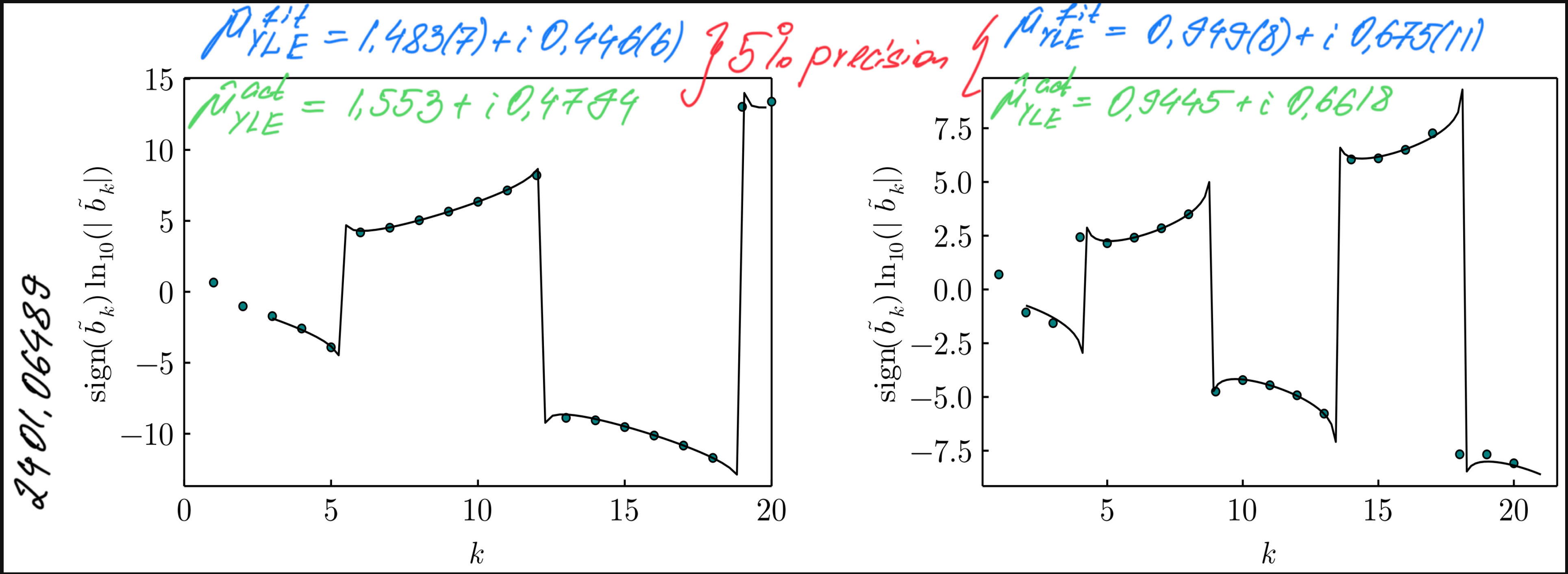
- Fourier coefficients are exponentially sensetive to YLE

$$b_{k\gg1} \approx |\tilde{A}_{\rm YLE}|\frac{e^{-\hat{\mu}_r^{\rm YLE}k}}{k^{1+\sigma}}\cos(\hat{\mu}_i^{\rm YLE}k+\phi_a^{\rm YLE})$$

$$+|\hat{A}_{\rm RW}|(-1)^k\frac{e^{-\hat{\mu}_r^{\rm RW}k}}{k^{1+\sigma}}$$

M. Bryant, C. Schmidt, V. S., 2401.06489

Fourier coefficients



M. Bryant, C. Schmidt, V. S., 2401.06489

Conclusions

- Universal location of YLE was one of not many unknown universal quantities
  - FRG allowed us to find the universal location of YLE for $d > 2.7$ and arbitrary N
  - Xu and Zamolodchikov determined location of YLE in Ising Field Theory, $d = 2$ and $N = 1$
- To map universal location to QCD, one requires non-universal metric factors. They are generically are not known.
- Nevertheless properties of YLE singularities might be useful in establishing existence/location of QCD critical point
  - YLE is continuously connected to critical point;
  - Two distinct approaches based on lattice input from Taylor coefficients and imaginary $\mu \rightsquigarrow$ approximately the same T_c
  - Critical exponent σ at YLE is universal and independent of N; it predicts the behaviour of Lee-Yang zeroes and their scaling with volume
  - Associated analytic structure of complex μ-plane constraints the behaviour of Fourier coefficients