Top-Quark Decay at Next-to-Next-to-Next-to-Leading Order in QCD

Xiang Chen(Peking University)

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In collaboration with Long Chen, Xin Guan, Yan-Qing Ma
Based on: [arXiv: 2309.01937v1]

Outline

1. **Introduction: Motivation and Background**
2. Computational Techniques
3. Numerical Results
4. Summary and Outlook

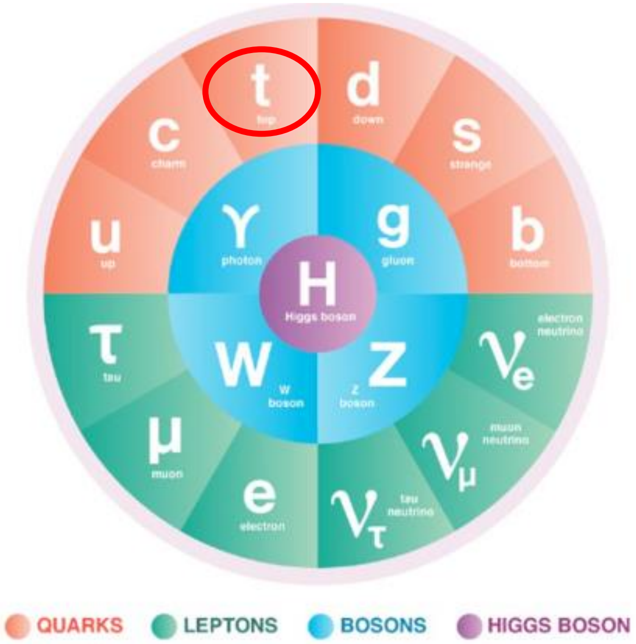
Top Quark

- Standard Model → New Physics? (High precision)
- The heaviest fundamental particle in SM: **Top Quark**

$m_t = 172.69 \pm 0.30\mathrm{GeV}$

Particle Data Group. *(2022)*

- Relevant physical problems:
  - The origin of Higgs mass
  - The quark mixing
  - CP violation···
- **Precision test of SM**, and probes for BSM physics
- $m_t \gg \Lambda_{QCD}$, allows us to use perturbative QCD
- Unique decay phenomenon:
  - Heavy mass: **dominantly decay to** W boson and b-quark
  - Before hadronization: the **information about its spin state** is preserved in distributions of top-quark decay products



Experimental Measurements for Mass and Decay Width

- Top quark mass m_t (PDG average)

$$m_t = 172.69 \pm 0.30\text{GeV}$$

Particle Data Group. *(2022)*

- Current best measurement for Γ_t:

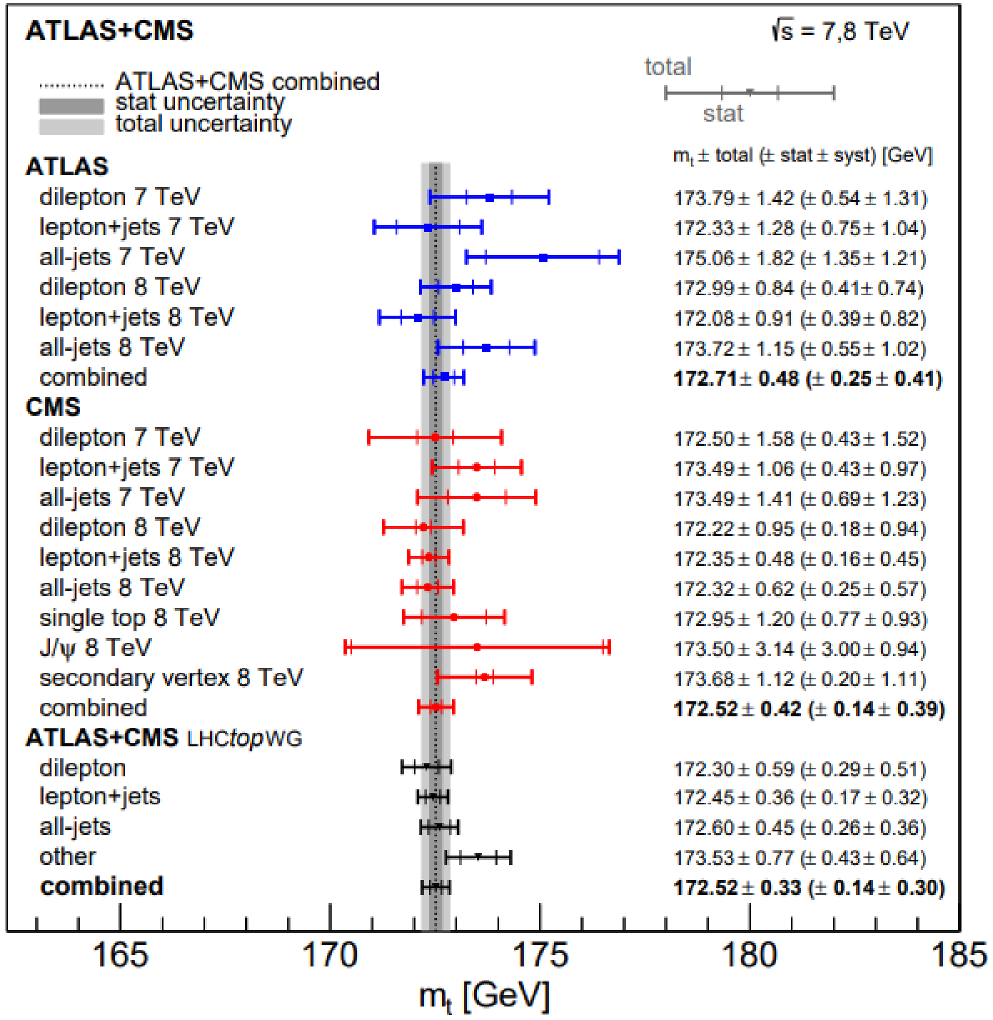
$$\Gamma_t = 1.36 \pm 0.02(\text{stat.})^{+0.14}_{-0.11}(\text{syst.})\text{GeV}$$

CMS. *(2014)*

- Expected experimental precision at future colliders:

20~26MeV

CLIC. *(2019)*
ILC. *(2013)*
CEPC. *(2023)*



From ATLAS, CMS. *(2024)*

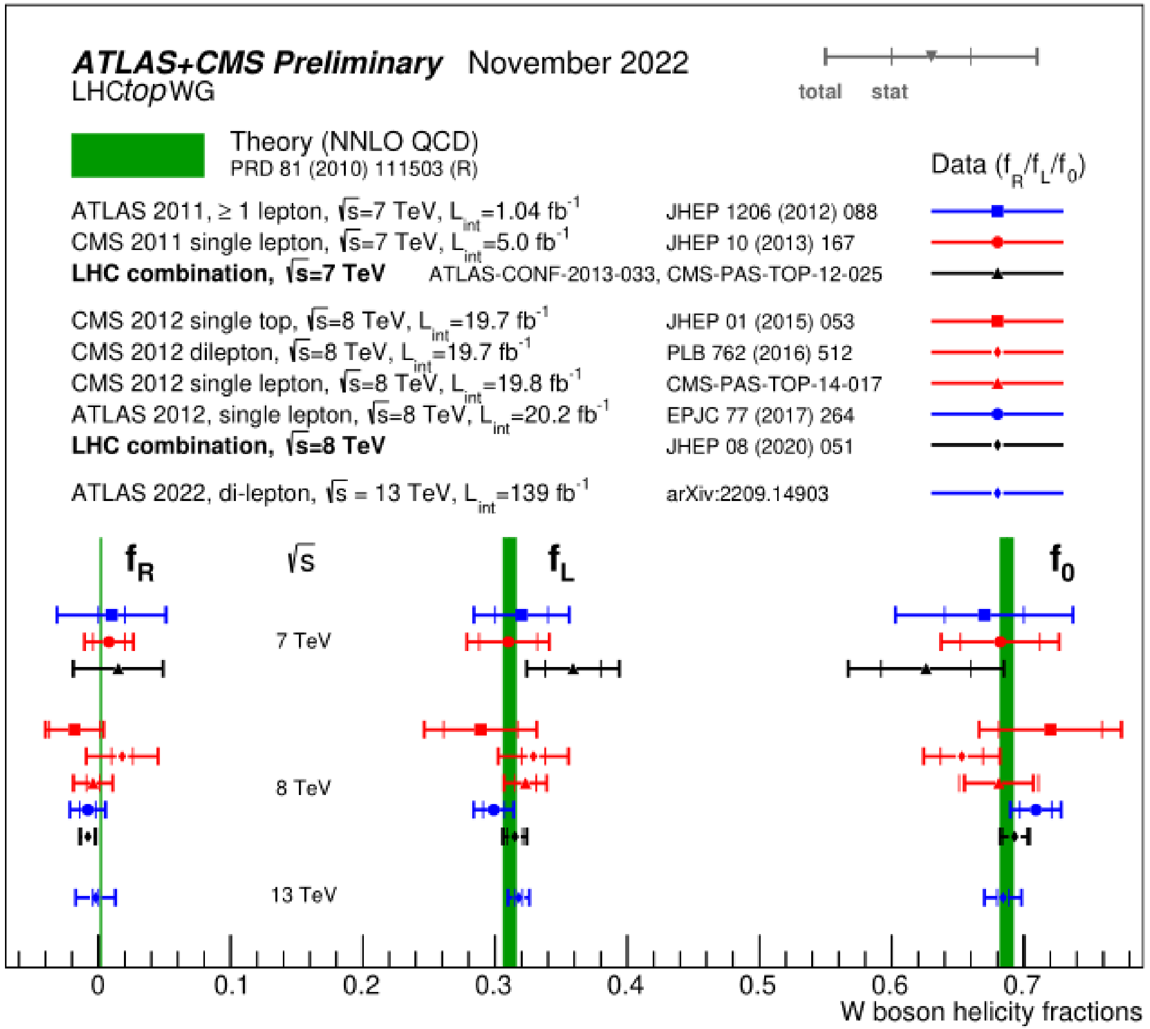
Experimental Measurements for W-Helicity Fractions

- The $\boldsymbol{W}$ **boson** from decay process $t \to W + b$ is **polarized** even if the top quark is unpolarized.

- The current best measurements for $f_\lambda (= \frac{\Gamma_\lambda}{\Gamma_t})$

$$
\begin{aligned}
f_0 &= 0.684 \pm 0.005(stat.) \pm 0.014(syst.)\\
f_L &= 0.318 \pm 0.003(stat.) \pm 0.008(syst.)\\
f_R &= -0.002 \pm 0.002(stat.) \pm 0.014(syst.)
\end{aligned}
$$

ATLAS. (2022)

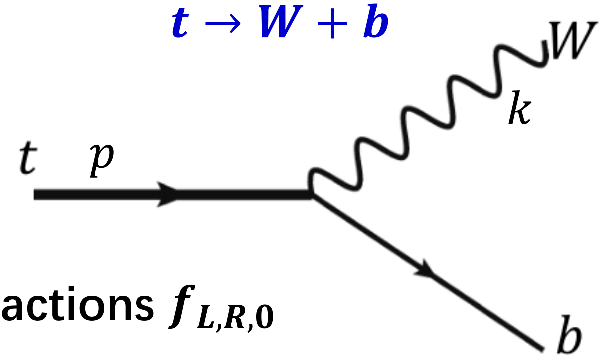


Theoretical Efforts in Literature

- Improvement of **experimental accuracy** ➡ Need more **precise theoretical prediction**

- **Total decay width Γ_t**
  - NLO(QCD)
    M. Jezabek, J. H. Kuhn. *(1988)*
    A. Czarnecki. *(1990)*
    C.-S. Li, R. J. Oakes, T. C. Yuan. *(1990)*
  - NNLO(QCD)
    A. Czarnecki, K. Melnikov. *(1998)*
    K. G. Chetyrkin, R. Harlander, T. Seidensticker, M. Steinhauser. *(1999)*
    M. Fischer, S. Groote, J. G. Korner, M. C. Mauser. *(2001)*
    I. Blokland, A. Czarnecki, M. Slusarczyk, F. Tkachov. *(2004)*
    I. Blokland, A. Czarnecki, M. Slusarczyk, F. Tkachov. *(2005)*
    A. Czarnecki, J. G. Korner, J. H. Piclum. *(2010)*
    J. Gao. C.-S. Li. H.-X. Zhu. *(2012)*
    M. Brucherseifer, F. Caola, K. Melnikov. *(2013)*
    R.-Q. Meng, S.-Q. Wang, T. Sun. C.-Q. Luo, J.-M. Shen, X.-G. Wu. *(2022)*
    L.-C. Chen, H.-T. Li, J. Wang Y.-F. Wang. *(2022)*
  - NLO(EW)
    A. Denner, T. Sack. *(1990)*

$t \to W + b$



- ***W*-helicity fractions $f_{L,R,0}$**
  - NNLO(QCD)
    A. Czarnecki, J. G. Korner, J. H. Piclum. *(2010)*
    J. Gao. C.-S. Li. H.-X. Zhu. *(2012)*
    M. Brucherseifer, F. Caola, K. Melnikov. *(2013)*
  - NLO(EW)
    H. S. Do, S. Groote, J. G. Korner, M. C. Mauser. *(2002)*

- **Differential distributions**
  - NLO(QCD)
    M. Fischer, S. Groote, J. G. Korner, M. C. Mauser. *(2001)*
  - NNLO(QCD)
    J. Gao. C.-S. Li. H.-X. Zhu. *(2012)*
    M. Brucherseifer, F. Caola, K. Melnikov. *(2013)*

Theoretical uncertainty of Γ_t is about $1\%(13\mathrm{MeV})$ at NNLO!

From L.-C. Chen, H.-T. Li, J. Wang Y.-F. Wang. *(2022)*

Full computation of NNNLO QCD correction is still indispensable!

Challenges for Perturbation Theory

➢ **Challenges** in high-order perturbative calculations:

- The number of Feynman diagrams increases rapidly
  - ✓ Loops
  - ✓ External legs
- The number of loops of the Feynman integrals increases
  - ✓ 1-loop: has been solved analytically
  - ✓ Beyond 1-loop: no systematic and analytic method for all situations
- The number of mass scales increases
  - ✓ Mass of particles in SM (W, Z, Higgs, etc)
  - ✓ Mandelstam variables

- Require systematic approach
- Difficult to calculate analytically, calculate numerically

Equipped with the-state-of-art techniques, the high-order corrections are available now!!

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Work-flow

- Work-flow for high-order computation:

Generating Integrands
($\mathcal{O}$(few minutes to hours))

$$\mathcal{A} = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i$$

Feynman diagrams

Integrals Reduction
($\mathcal{O}$(hours to days))

$$I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} M_j$$

Integrate-By-Part(IBP) method

Evaluating master integrals
($\mathcal{O}$(days) typically in frontier)

$$I_i(p_i \cdot p_j, m_i^2) = \int \frac{d^D l_1 \cdots d^D l_h}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}}$$

Differential equations

Formula

- To calculate the **semi-inclusive** distribution, Γ_t is expressed as

$$\Gamma_t = \frac{1}{2m_t}\int \frac{d^{D-1}k}{(2\pi)^{D-1}2E}\mathcal{W}_{tb}^{\mu\nu}\sum_{\lambda}^{L,R,0}\varepsilon_\mu^*(k,\lambda)\varepsilon_\nu(k,\lambda)$$

- Hadronic tensor $\mathcal{W}_{tb}^{\mu\nu}$ can be decomposed to 5 linearly-independent Lorentz-tensor structures:

$$\mathcal{W}_{tb}^{\mu\nu}(p,k) = W_1 g^{\mu\nu} + W_2 p^\mu p^\nu + W_3 k^\mu k^\nu + W_4(p^\mu k^\nu + k^\mu p^\nu) + W_5 i\epsilon^{\mu\nu\rho\sigma}p_\rho k_\sigma$$

- $\mathcal{W}_{tb}^{\mu\nu}$
  - directly → The energy distribution of $\boldsymbol{W}$ **boson**
  - $\varepsilon_\mu^*(k,\lambda)\varepsilon_\nu(k,\lambda) \rightarrow \mathbb{P}_\lambda^{\mu\nu}$, and $\int dE$ ... manually → Total decay width Γ_t and partial width $\Gamma_\lambda(\lambda = L, R, 0)$

- Projectors for polarized W boson

$$\begin{cases} \mathbb{P}^{\mu\nu} = -g^{\mu\nu} + \dfrac{k^\mu k^\nu}{m_W^2} \\ \mathbb{P}_0^{\mu\nu} = \dfrac{m_W^2}{m_t^2}\dfrac{1}{\left|\vec{k}\right|^2}\left(p^\mu + \dfrac{p\cdot k}{m_W^2}k^\mu\right)\left(p^\nu - \dfrac{p\cdot k}{m_W^2}k^\nu\right) \\ \mathbb{P}_F^{\mu\nu} = -\dfrac{1}{m_t}\dfrac{1}{\left|\vec{k}\right|}i\epsilon^{\mu\nu\rho\sigma}p_\rho k_\sigma \end{cases} \Longrightarrow \begin{cases} \mathbb{P}_L^{\mu\nu} = (\mathbb{P}^{\mu\nu} - \mathbb{P}_0^{\mu\nu} - \mathbb{P}_F^{\mu\nu})/2 \\ \mathbb{P}_R^{\mu\nu} = (\mathbb{P}^{\mu\nu} - \mathbb{P}_0^{\mu\nu} + \mathbb{P}_F^{\mu\nu})/2 \end{cases}$$

K. G. Chetyrkin, A. Retey. *(2000)*

Computational Details for $\mathcal{W}_{tb}^{\mu\nu}$

- Generating integrands:
  - Phase-space integrals $\xrightarrow{\text{reverse unitarity}}$ Loop integrals

$$\delta(x) \to \frac{1}{2\pi i}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right)$$

C. Anastasiou, K. Melnikov. *(2002)*

  - Complexity of Feynman amplitudes:
    - ✓ The number of integral families: 245
    - ✓ The number of Feynman integrals: $\sim 7 \times 10^4$

- Integrals reduction:
  - Package: **`Blade`** (Integration-By-Parts relations + **Block-Triangular form**)
  - The number of master integrals: 2988
  - Computing resources: $\sim 1.1 \times 10^4 CPU \cdot h$

K. G. Chetyrkin, F. V. Tkachov. *(1981)*
S. Laporta. *(2000)*
A. V. Manteuffel, R. M. Schabinger. *(2015)*

X. Guan, X. Liu, Y.-Q. Ma. *(2020)*

- Evaluating master integrals:
  - Package: **`AMFlow`** (Numerical differential equations + **Auxiliary mass flow method**)
  - Expressed as deeply expanded power-log series
  - Computing resources: $\sim 1.5 \times 10^4 CPU \cdot h$

R. N. Lee, V. A. Smirnov. *(2018)*
R. Bonciani. G. Degrassi, P. P. Giardino, R. Grober. *(2019)*
H. Frellesvig, M. Hidding, L. Maestri, F. Moriello, G. Salvatori. *(2020)*
L. Chen, M. Czakon, M. Niggetiedt. *(2021)*
Martijin Hidding. *(2021)*
X. Liu, Y.-Q. Ma. *(2021)*
M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *(2022)*

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Total Decay Width

- The QCD corrections to Γ_t can be parameterized as

$$\Gamma_t = \Gamma_0 (c_0 + \frac{\alpha_s}{\pi} c_1 + \left(\frac{\alpha_s}{\pi}\right)^2 c_2 + (\frac{\alpha_s}{\pi})^3 c_3)$$

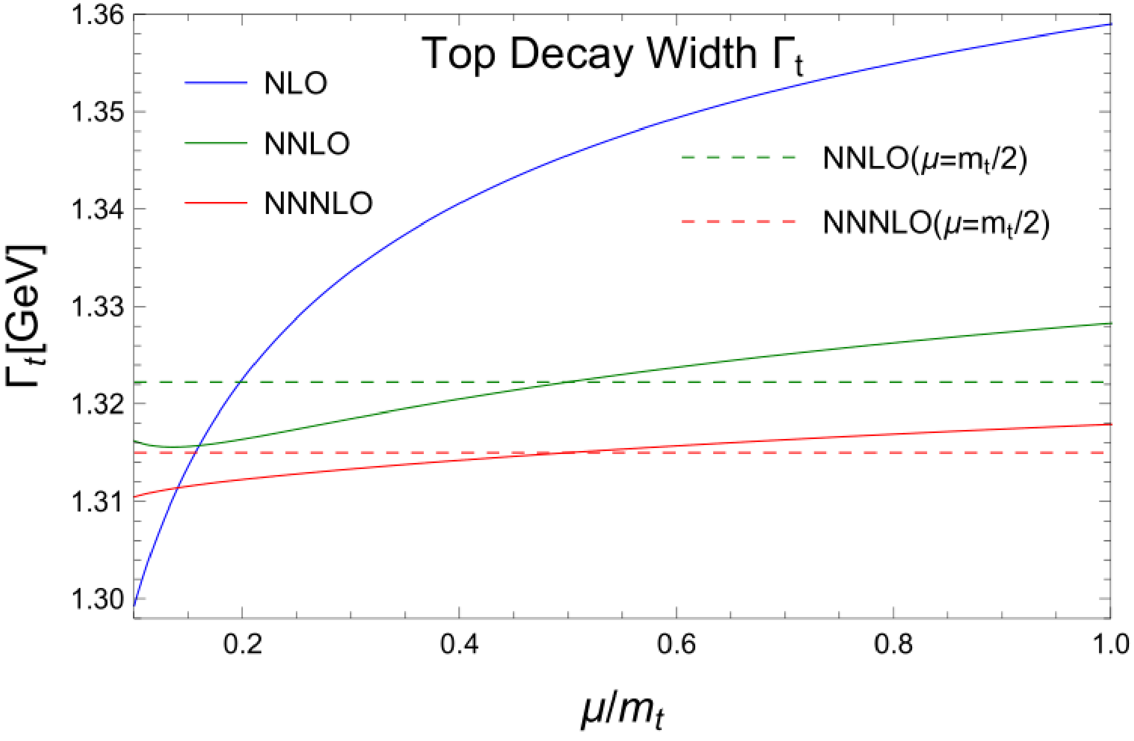
where $\Gamma_0 = \frac{G_F m_W^2 m_t |V_{tb}|^2}{12\sqrt{2}}$.

- Input values: $G_F = 1.166379 \times 10^{-5} \mathrm{GeV}^{-2}$, $\alpha_s(\mu = m_t/2) = 0.1189$, $m_t = 172.69 \mathrm{GeV}$, $m_W = 80.377 \mathrm{GeV}$, $\mu = m_t/2$ motivated by kinematic energy $m_t - m_W - m_b$

$$\boxed{\begin{aligned}\Gamma_t &= 1.48642 - 0.140877 - 0.023306 - 0.007240\ \mathbf{GeV} \\ &= \mathbf{1.31500\ GeV}\end{aligned}}$$

- The pure $\mathcal{O}(\alpha_s^3)$ correction reduces the NNLO results of Γ_t by about $10\mathrm{MeV}$ (**exceeding NNLO scale variation**).

- The NNLO green curve can **never** cover the NNNLO red curve at any scales less than $\frac{\mu}{m_t} = 0.6$.



Studying the μ dependence at NNLO ⇒ Underestimate the theoretical error!!

Other Effects

- Similar perturbative parametrization: $\Gamma_t = \Gamma_0(c_0 + \frac{\alpha_s}{\pi} c_1 + \left(\frac{\alpha_s}{\pi}\right)^2 c_2 + (\frac{\alpha_s}{\pi})^3 c_3)$
  - **Off-shell W** effects (**effective mass of W** is the **invariant-mass of the lepton pair** produced from W-decay)

$$\frac{\tilde{c}_0 - c_0}{c_0} = -1.54\%, \quad \frac{\tilde{c}_1 - c_1}{c_1} = -1.53\%, \quad \frac{\tilde{c}_2 - c_2}{c_2} = -1.39\%, \quad \frac{\tilde{c}_3 - c_3}{c_3} = -1.23\%$$

  - **Finite b-quark mass** effects, $m_b = 4.78\mathrm{GeV}$

$$\frac{c_1^{m_b} - c_1}{c_1} \approx \frac{c_2^{m_b} - c_2}{c_2} \approx -1.47\%$$

  - **NLO electroweak corrections**: $\delta_{EW} = 1.68\%$

- Final results for decay width Γ_t:

$$\boxed{\boldsymbol{\Gamma_t = 1.3148^{+0.003}_{-0.005} \times |V_{tb}|^2 + 0.027(m_t - 172.69)\mathrm{GeV}}}$$

  - Conservative estimate of the QCD scale uncertainty ($\frac{\mu}{m_t} \in [0.1,1]$)
  - Input t-quark mass value $172.69 \pm 0.30\mathrm{GeV}$

- The error of which **meets the request by future colliders.**

$\boldsymbol{W}$-Helicity Fractions

- Partial width for **a polarized $\boldsymbol{W}$**:

$$\Gamma_\lambda = \frac{1}{2m_t}\int \frac{d^{D-1}k}{(2\pi)^{D-1}2E}\mathcal{W}_{tb}^{\mu\nu}\varepsilon_\mu^*(k,\lambda)\varepsilon_\nu(k,\lambda)$$

- W-helicity fractions: $f_\lambda^{[n]} = \frac{\sum_{i=0}^n \Gamma_\lambda^{[n]}}{\sum_{i=0}^n \Gamma_t^{[n]}}$

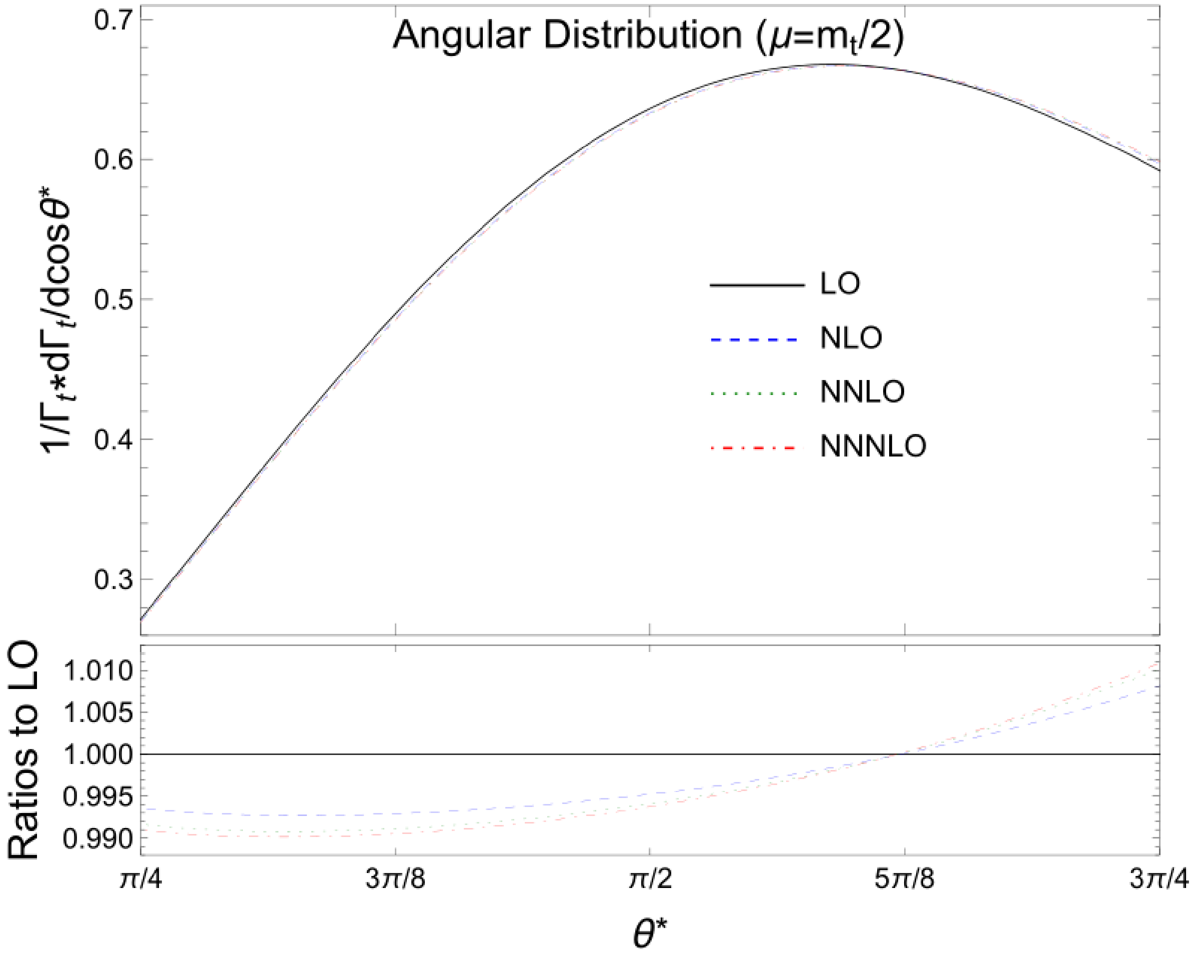
$$\begin{aligned} f_0^{[3]} &= 0.697706 - 0.008401 - 0.001954 - 0.000613 \\ &= 0.686737 \\ f_L^{[3]} &= 0.302294 + 0.007254 + 0.001799 + 0.000586 \\ &= 0.311933 \\ f_R^{[3]} &= 0. + 0.001147 + 0.000155 + 0.000027 \\ &= 0.001330 \end{aligned}$$

- After incorporating the **NLO electroweak** correction and $\boldsymbol{m_b}$ **effects**, the final results:

$$\boxed{f_0^{[3]} = 0.686_{-0.003}^{+0.002}, \qquad f_L^{[3]} = 0.312_{-0.002}^{+0.001}, \qquad f_R^{[3]} = 0.00157_{-0.00002}^{+0.00002}}$$

$\boldsymbol{\cos\theta^*}$ Angular Distribution

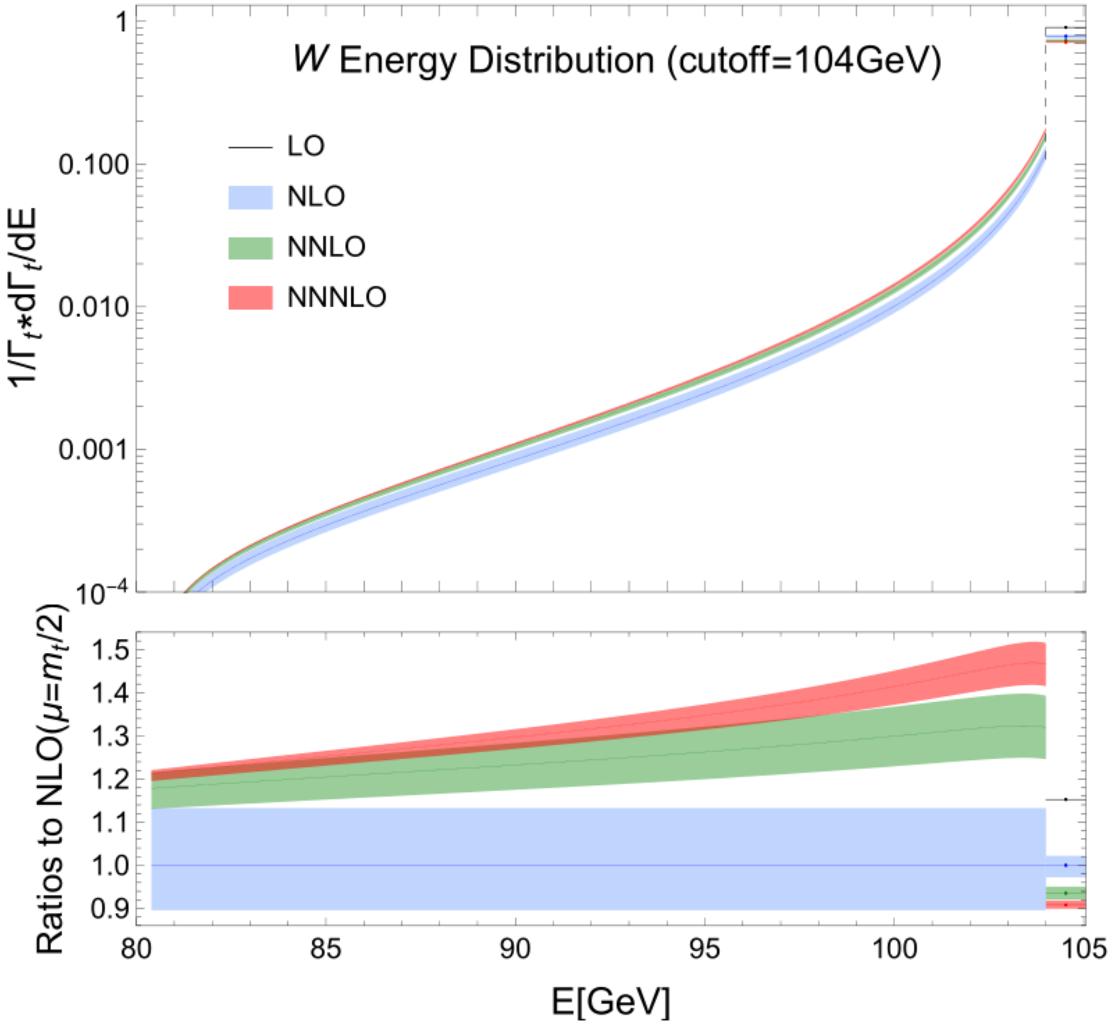
- $\boldsymbol{\cos\theta^*}$ angular distribution (θ^* is the anger between the **momentum direction of the charged lepton** from the W-decay in the W-rest frame and the **$\boldsymbol{W}$-momentum direction** in the t-quark rest frame)



$$\frac{1}{\Gamma_t}\frac{d\Gamma_t}{d\cos\theta^*} = \frac{3}{4}(\sin\theta^*)^2 f_0 + \frac{3}{8}(1-\cos\theta^*)^2 f_L + \frac{3}{8}(1+\cos\theta^*)^2 f_R$$

W-Energy Distribution

- W-energy distribution



- In the continuous region, the QCD corrections are **positive and quite sizable**. The **pure $\mathcal{O}(\alpha_s^3)$** correction modifies the lowest order result by about $\mathbf{7\sim14\%}$ for $\boldsymbol{E \in [94, 104]\mathrm{GeV}}$.

- In the rightmost end, the QCD corrections up to $\mathcal{O}(\alpha_s^3)$ **decrease** the Born-level result.

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Summary and Outlook

- **We have provided the to-date most-precise high-precision theoretical prediction** for top-quark decay width, the error of which meets the request by future colliders.

$$\boxed{\Gamma_t = 1.3148^{+0.003}_{-0.005} \times |V_{tb}|^2 + 0.027(m_t - 172.69)\mathrm{GeV}}$$

- We determined W-helicity fractions, $cos\,\theta^*$ angular distribution and W-energy distribution at $\boldsymbol{\mathcal{O}(\alpha_s^3)}$ for the first time.

- To achieve a real sub-percent level, we also need the NNLO mixed QCD-EW corrections.

- The approach can be readily applied to the decay of polarized t-quark at $\mathcal{O}(\alpha_s^3)$.

Thanks for your attention!