N^3LO and resummed cross sections for on-shell gluon fusion Higgs boson production

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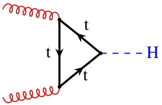
The inclusive Higgs cross section: motivation

I will focus on SM on-shell Higgs ($m_H = 125$ GeV) production at LHC.

$\sim 90\%$ of the inclusive Higgs cross section comes from gluon fusion



A lot of theory activity for decades, due to a number of reasons:

- the LO is loop induced $\to$ perturbative corrections are complicated
- the NLO correction is 130% of the LO $\to$ very slow perturbative convergence
- central for LHC physics $\to$ high precision is required

Topics covered in this talk:

- theory ingredients and state-of-the-art predictions for on-shell ggH
- theory uncertainties on ggH
- codes for ggH (partial and very biased overview)

The LHC H(XS)WG recommendation

The LHCHXSWG Yellow Report 4 recommendation for the ggH XS is based on the result advocated by the Zurich group: (LHC 13 TeV, $m_H = 125$ GeV)

$$\begin{aligned} \sigma = 48.58\,\mathrm{pb} = \quad & 16.00\,\mathrm{pb} && \text{(LO, rEFT)} \\ & +20.84\,\mathrm{pb} && \text{(NLO, rEFT)} \\ & +\ \ 9.56\,\mathrm{pb} && \text{(NNLO, rEFT)} \\ & +\ \ 1.49\,\mathrm{pb} && (\mathrm{N^3LO, rEFT}) \\ & -\ \ 2.05\,\mathrm{pb} && ((t,b,c), \text{ exact NLO}) \\ & +\ \ 0.34\,\mathrm{pb} && (\text{NNLO}, 1/m_t) \\ & +\ \ 2.40\,\mathrm{pb} && \text{(EW, QCD-EW)} \end{aligned}$$

[Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Lazopoulos,Mistlberger 1602.00695]

A long story 1977....2016....

Further developments in recent years, related to the quark mass effects at NNLO and electroweak corrections

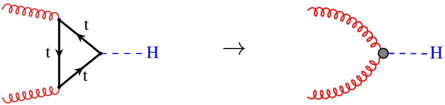
I will also discuss the impact of resummations of classes of logarithmic contributions

rEFT: Born-rescaled Effective Field Theory

In the limit $m_H \ll m_t$ it is possible to use the so-called Higgs Effective Field Theory

$$\mathcal{L}_{\rm EFT} = -\frac{1}{4v}\, C_1\, G^a_{\mu\nu} G^{a\mu\nu} H$$

with effective Hgg vertex



Perturbative corrections in the EFT is much simpler → N^{3}LO achieved

To improve the accuracy, the EFT result is rescaled to the full LO with exact m_t dependence

rescaled EFT (rEFT): $$\sigma^{\rm rEFT} = \frac{\sigma^{\rm exact}_{\rm LO}(m_t)}{\sigma^{\rm EFT}_{\rm LO}}\sigma^{\rm EFT} = \sigma^{\rm exact}_{\rm LO}(m_t) \times K^{\rm EFT}$$
$$\simeq 1.06 \times \sigma^{\rm EFT}$$

The rEFT result represents the bulk of the ggH cross section

Supplement rEFT with t mass effects beyond LO and b, c mass effects at LO and beyond

- NLO: exact result for any quark running in the loop [Spira,Djouadi,Graudenz,Zerwas 1995]
- NNLO:
  - top quark mass effects as an expansion in $1/m_t$ [Pak,Rogal,Steinhauser 2009] [Harlander,(Mantler,Marzani),Ozeren 2009(10)] [Davies,Gröber,Maier,Rauh,Steinhauser 2019]
  - exact top quark mass effects [not in YR4] [Czakon,Harlander,Klappert,Niggetiedt 2021]
  - exact top-bottom interference effects [not in YR4] [Czakon,Eschment,Niggetiedt,Poncelet,Schellenberger 2023]

Numerical impact:

- top mass corrections to rEFT in the $\overline{\text{MS}}$ (OS) scheme

$$\sigma_{\text{exact, only top}} - \sigma_{\text{rEFT}} = \begin{array}{rrl} 0 & (0) & \text{LO} \\ -0.24\,\text{pb} & (-0.32\,\text{pb}) & \text{NLO} \\ +0.34\,\text{pb} & & \text{NNLO } (1/m_t \text{ corrections}) \\ & (+0.15\,\text{pb}) & \text{NNLO (exact)} \end{array}$$

- bottom and charm corrections in the $\overline{\text{MS}}$ (OS) scheme

$$\sigma_{\text{exact, } t+b+c} - \sigma_{\text{exact, only top}} = \begin{array}{rrl} -1.17\,\text{pb} & (-2.23\,\text{pb}) & \text{LO} \\ -0.66\,\text{pb} & (-0.36\,\text{pb}) & \text{NLO} \\ & (+0.43\,\text{pb}) & \text{NNLO } (b\text{-}t \text{ interference}) \end{array}$$

Uncertainties due to heavy quark masses

Uncertainties quoted by the Zurich group in YR4:

δ(scale)	δ(trunc)	δ(PDF-TH)	$\delta(1/m_t)$	$\delta(t,b,c)$	δ(EW)
$^{+0.10\text{ pb}}_{-1.15\text{ pb}}$	$\pm$0.18 pb	$\pm$0.56 pb	$\pm$0.49 pb	$\pm$0.40 pb	$\pm$0.49 pb

[Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Lazopoulos,Mistlberger 1602.00695]

- $\delta(1/m_t)$ represents unknown mass correction terms at NNLO
  $\to$ now gone
- $\delta(t,b,c)$ represents missing b,c mass corrections beyond NLO and t mass corrections beyond NNLO
  It also accounts for scheme dependence ($\overline{\text{MS}}$ vs OS)
  $\to$ reduced by recent results, although difficult to quantify reliably residual uncertainty

Electroweak corrections

$$\sigma = 48.58\,\text{pb} = \quad \ldots.. + (2.40 \pm 0.49)\,\text{pb} \qquad (\text{EW, QCD-EW})$$

Cross section gets EW corrections as well:

$$\sigma = \sigma_0 \left[1 + \alpha_s \sigma_1 + \alpha_s^2 \sigma_2 + \alpha_s^3 \sigma_3 + \ldots + \alpha \lambda_{\text{EW}} (1 + \alpha_s s_1 + \ldots) + \alpha^2 \cdots \right]$$

- Additive approach + mixed QCD-EW (Zurich group):
  Estimate s_1 from an EFT ($m_H \ll m_{Z,W}$) [Anastasiou,Boughezal,Petriello 2008]
  Gives a $+4.9\%$ effect
- Complete factorization approach: [Actis,Passarino,Sturm,Uccirati 2008]

$$\sigma = \sigma_0 (1 + \alpha \lambda_{\text{EW}}) \left[1 + \alpha_s \sigma_1 + \alpha_s^2 \sigma_2 + \alpha_s^3 \sigma_3 + \ldots \right]$$

  Gives a $+5.1\%$ effect

Uncertainty estimated by varying s_1 and/or by comparing the complete factorized result to the additive one

Recent exact computation of mixed QCD-EW correction s_1 (light quark contribution)
[Becchetti,Bonciani,DelDuca,Hirschi,Moriello,Schweitzer 2010.09451]

$$\sigma_{\text{EW}} = (2.19 \pm 0.26)\,\text{pb} \qquad [\text{not in YR4}]$$

rEFT

Where do these numbers come from?

$$
\begin{aligned}
\sigma = 48.58\,\mathrm{pb} = \quad & \,16.00\,\mathrm{pb} && \text{(LO, rEFT)} \\
& +20.84\,\mathrm{pb} && \text{(NLO, rEFT)} \\
& +\ \ 9.56\,\mathrm{pb} && \text{(NNLO, rEFT)} \\
& +\ \ 1.49\,\mathrm{pb} && \text{(N}^3\text{LO, rEFT)} \\
& -\ \ 2.05\,\mathrm{pb} && ((t,b,c),\ \text{exact NLO}) \\
& +\ \ 0.34\,\mathrm{pb} && (\text{NNLO},\ 1/m_t) \\
& +\ \ 2.40\,\mathrm{pb} && \text{(EW, QCD-EW)}
\end{aligned}
$$

A long journey

- LO [Wilczek 1977] [Georgi,Glashow,Machacek,Nanopoulos 1977]
- NLO [Dawson 1991] [Djouadi,Spira,Zerwas 1991]
- NNLO [Harlander,Kilgore 2002] [Anastasiou,Melnikov 2002]
- Approximate N^3LO
  - soft approximation (only log terms) [Moch,Vogt 2005]
  - soft + high-energy approximation [Ball,MB,Forte,Marzani,Ridolfi 2013]
  - soft + next-to-soft approximation [deFlorian,Mazzitelli,Moch,Vogt 2014]
- Full N^3LO

  Wilson coefficient at N^3LO [Chetyrkin,Kniehl,Steinhauser 1997] three loops [Baikov, Chetyrkin,Smirnov,Smirnov,Steinhauser 2009] [Lee,Smirnov,Smirnov 2010] [Gehrmann, Glover,Huber,Ikizlerli,Studerus 2010] one emission at two loops [Gehrmann,Jaquier,Glover, Koukoutsakis 2012] [Duhr,Gehrmann 2013] [Li,Zhu 2013] one emission at one loop [Anastasiou,Duhr,Dulat,Herzog,Mistlberger 2013] [Kilgore 2013] three emissions (soft expansion) [Anastasiou,Duhr,Dulat,Mistlberger 2013] scale dependent terms [Anastasiou, Bühler,Duhr,Herzog 2012] [Höschele,Hoff,Pak,Steinhauser,Ueda 2012] [Bühler,Lazopoulos 2013] two emissions at one loop [Li,vonManteuffel,Schabinger,Zhu 2014] all soft and next-to-soft terms at N^3LO [Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Mistlberger 2014]

  37 terms in the soft expansion [Anastasiou,Duhr,Dulat,Herzog,Mistlberger 2015]

  exact qq' [Anzai,Hasselhuhn,Höschele,Hoff,Kilgore,Steinhauser,Ueda 2015]

  exact [Mistlberger 2018]

Scale dependence

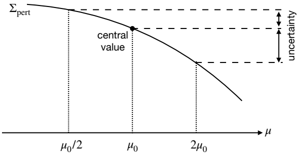
Once $m_H = 125$ GeV and the collider energy 13 TeV are fixed, the ggH cross section still depends on unphysical scales:

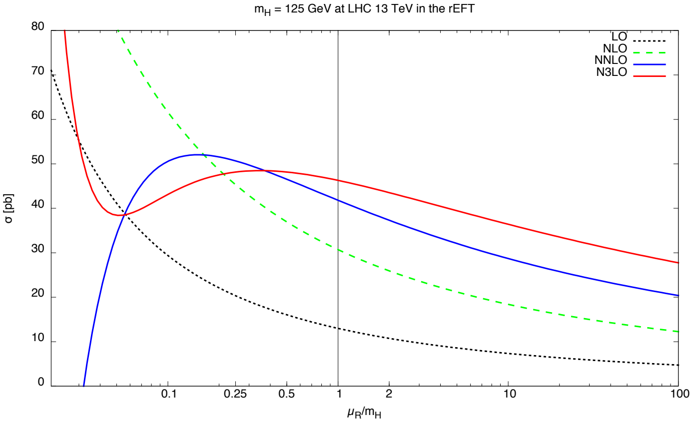
- the factorization scale $\mu_{\rm F}$
- the rinormalization scale $\mu_{\rm R}$

It turns out that for on-shell Higgs the dependence on the factorization scale $\mu_{\rm F}$ is very mild.
Conversely, the cross section depends strongly on the renormalization scale $\mu_{\rm R}$
$\rightarrow$ related to a badly convergent perturbative expansion!

A common choice to improve convergence: $\mu_{\rm R} = m_H/2$

Customarily, scale dependence is used to estimate the uncertainty from missing higher orders (MHO), by varying the scale about the central choice by a factor of 2 up and down

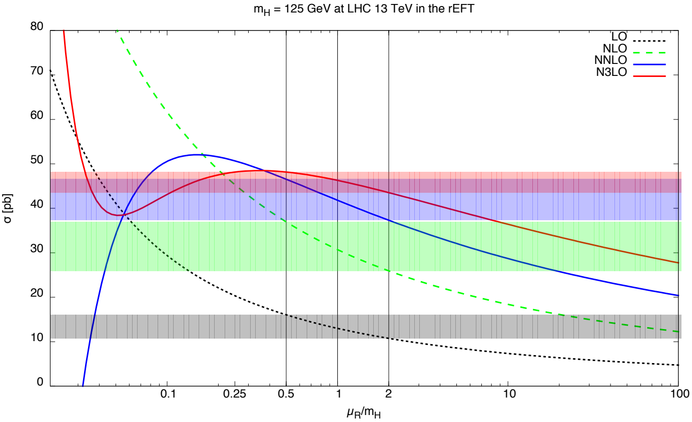


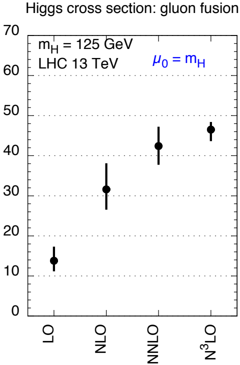


Computed with ggHiggs

Other codes for fixed order: `ihixs`, `SuSHi`, `HIGLU`, ...

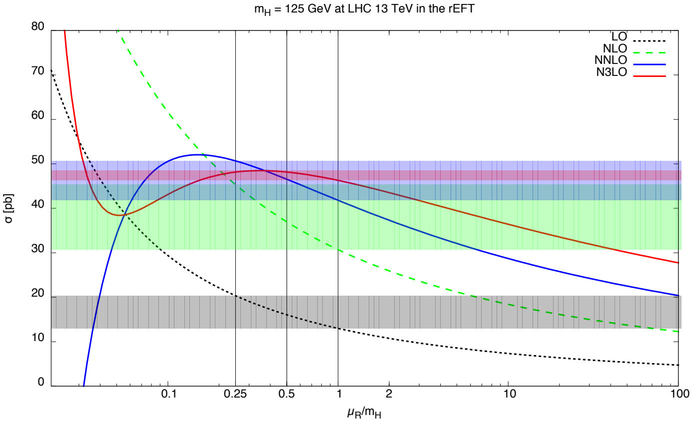
Higgs cross section: perturbative (in)stability

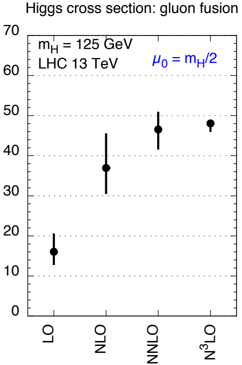




$$1/2 < \mu_R/m_H < 2$$

Higgs cross section: perturbative (in)stability





$$1/4 < \mu_R/m_H < 1$$

Note the very asymmetric N^3LO band, due to the presence of a stationary point

Observations

- The canonical scale variation uncertainty depends on the central scale, and clearly underestimates the actual MHO uncertainty at low orders
  Why should we trust it?

- Moreover, what do these uncertainties mean?
  - LHCHXSWG interpretation: 100% c.l. flat interval
  - LHCHXSWG alternative interpretation: 68% c.l. gaussian interval

  Either interpretation is arbitrary — no probabilistic foundation!

- Perturbative corrections are large, and several orders are required to see some convergence
  Understanding the origin of these large corrections helps improving the convergence
  $\rightarrow$ **resummation of threshold logarithms**

Threshold resummation

Gluon luminosity, peaked at small x, enhances the partonic coefficient at large z

$$\sigma_{gg} = \tau \int_\tau^1 \frac{dz}{z} \mathscr{L}_{gg}\left(\frac{\tau}{z}\right) C_{gg}(z), \qquad \tau = \frac{m_H^2}{s}$$

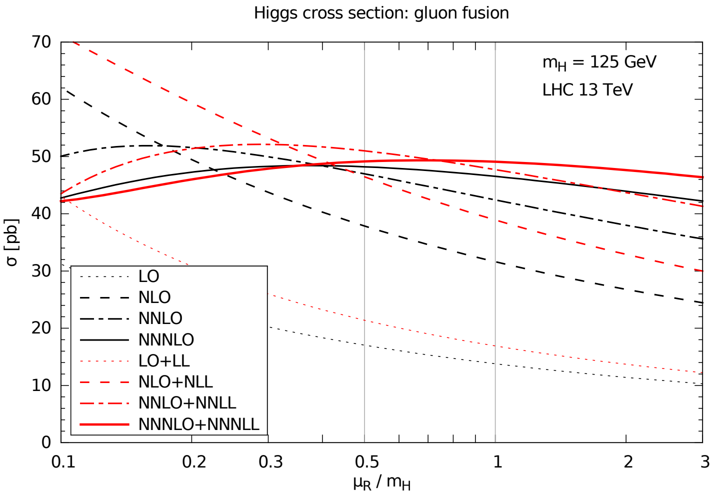
The threshold $z \to 1$ region dominates

The partonic cross section $C_{gg}(z)$ contains $\log(1-z)$ terms that are enhanced in the threshold region

It is possible to stabilise the perturbative expansion by resumming these large logarithmic contributions to all orders in α_s (thanks to welle established techniques)

- For years the LHCHXSWG recommendation was based on NNLO+NNLL'
  [Catani,deFlorian,Grazzini,Nason 2003] [deFlorian,Grazzini 2012]
- NNLO+N^{3}LL' also available
  dQCD: [MB,Marzani 2014] [Schmidt,Spira 2015]
  SCET: [Ahrens,Becher,Neubert,Yang 2008] [MB,Rottoli 2014]
- N^{3}LO+N^{3}LL' most accurate result
  [MB,Marzani,Muselli,Rottoli 2016]

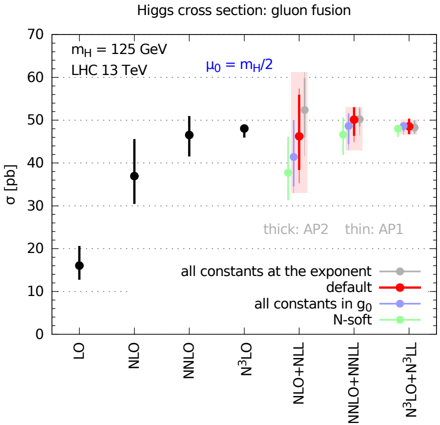
Scale dependence with threshold resummation

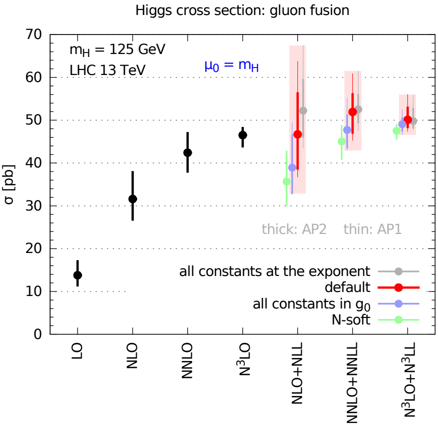


[MB,Marzani 2014]

Computed with TROLL

Other codes for threshold resummation: RGHiggs, …





Perturbative convergence sped up! [MB,Marzani,Muselli,Rottoli 2016]

Reduction of theory uncertainty increasing the order

Less sensitivity to central scale

More robust uncertainty estimate (probabilistic interpretation still missing...)

Uncertainties from MHOs

Beyond canonical scale variation

We have seen that canonical scale variation has a number of limitations:

- the result depends on the central scale chosen
- the variation by a factor of 2 is arbitrary
- it underestimates the actual uncertainty (for ggH and other processes as well)
- no probabilistic interpretation

New definition of theory uncertainties from missing higher orders:

- reliable
- less dependent on arbitrary assumptions
- probabilistically well defined

Ideally, theory uncertainty from MHO should be a **probability distribution**

A probabilistic definition in this context can only be based on a **Bayesian approach**

The breakthrough: the Cacciari-Houdeau model

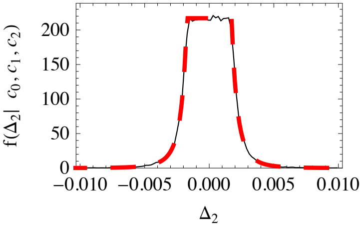
Cacciari and Houdeau [1105.5152] proposed a probabilistic model for the interpretation of theory uncertainties, based on the behaviour of the perturbative expansion

$$\Sigma = \sum_k c_k \alpha_s^k$$

"We make the assumption that all the coefficients c_k in a perturbative series share some sort of upper bound $\bar{c} > 0$ to their absolute values, specific to the physical process studied. The calculated coefficients will give an estimate of this $\bar{c}$, restricting the possible values for the unknown c_k."

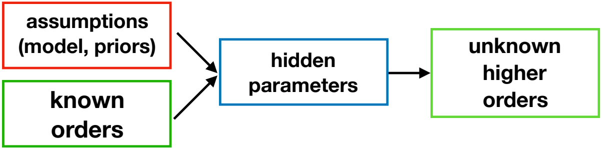
In other words, the model assumes that

$$|c_k| \leq \bar{c} \qquad \forall k$$



Bayesian inference

Inference scheme



Inference on the unknown coefficients c_k

$$P(\text{unknown } c_k|\text{known } c_k) = \int d\text{pars}\, P(\text{unknown } c_k|\text{pars}) P(\text{pars}|\text{known } c_k)$$

in terms of the posterior distribution of the hidden parameters

$$P(\text{pars}|\text{known } c_k) \propto P(\text{known } c_k|\text{pars}) P_0(\text{pars})$$

which depends on the prior distribution $P_0(\text{pars})$ and on the model through the likelihood $P(c_k|\text{pars})$

Cacciari-Houdeau: $P(c_k|\bar{c}) \propto \theta(\bar{c} - |c_k|)$, $\quad P_0(\bar{c}) \propto 1/\bar{c}$

- CH probabilistic framework is good (probably the only way to define probabilistically a theory uncertainty from missing higher orders)
- better model assumptions on the behaviour of the expansion
- do not forget scale dependence:
  - **as a tool**, to gain further information on missing higher orders (as in canonical scale variation)
  - **as an issue**, due to the need of choosing a scale

Model 1:
geometric behaviour model

Model 2:
scale variation model

Other models:
variants, combinations, ...

a unified probabilistic way to deal with scale dependence

New model (1): Geometric behaviour model (aka improved CH)

More general expansion

$$\Sigma = \Sigma_{\rm LO}(\mu) \sum_{k\geq 0} \delta_k(\mu) \qquad\qquad \Sigma_{\rm LO}(\mu)\delta_k(\mu) = c_k(\mu)\alpha_s^k(\mu)$$

CH model assumes that δ_k behave as α_s^k

$$\Sigma_{\rm LO}(\mu)\, |\delta_k(\mu)| \leq \bar{c}\, \alpha_s^k$$

Power growth of the coefficients $c_k \sim \eta^k$ is very likely:

- Cacciari-Houdeau proposed a modified version with η accounted for
- in [Bagnaschi,Cacciari,Guffanti,Jenniches 1409.5036] η is determined from a survey
- in an alternative approach [Forte,Isgrò,Vita 1312.6688] the value of η is fitted

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My proposal: geometric behaviour model

$$|\delta_k(\mu)| \leq c\, a^k$$

depends on two hidden parameters c, a, it accounts for a possible power growth of the coefficients within the model

Asymmetric variant, called abc model, proposed in [Duhr,Huss,Mazeliauskas,Szafron 2106.04585]

Constructing a "scale-independent" result

The method just described still needs to chose a renormalization scale μ: if I change the scale, the result changes

How can we get rid of the scale?

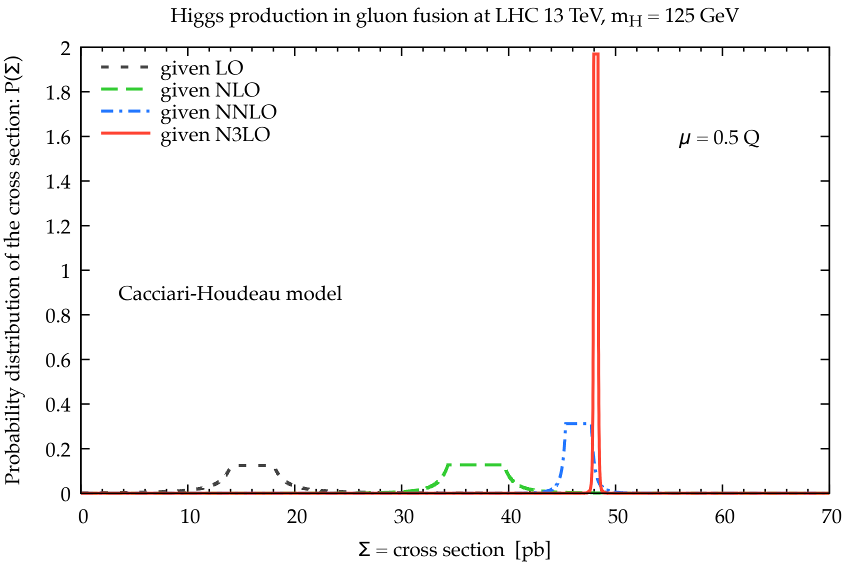
Basic idea: treat the unphysical scale μ as a parameter of the model, and simply marginalize over it

$$P(\Sigma|\delta_0, ..., \delta_n) = \int d\mu \; P(\Sigma|\delta_0, ..., \delta_n, \mu) \, P(\mu|\delta_0, ..., \delta_n)$$

where $P(\mu|\delta_0, ..., \delta_n)$ is the posterior distribution for μ given the known orders (which depends on the model)

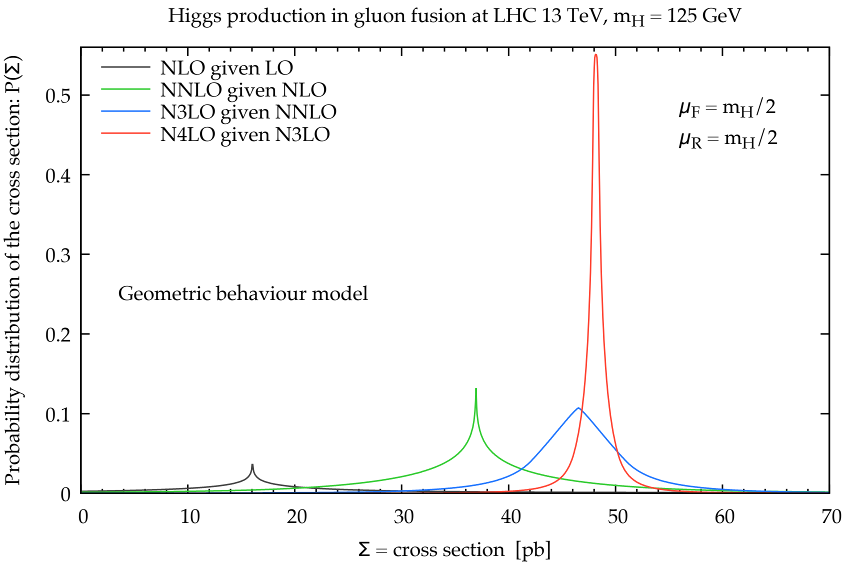
The prior $P_0(\mu)$ contains our prejudices on what are the most appropriate scales, but the results are largely independent of the precise form and size of the prior
$\Rightarrow$ **a lot of arbitrariness is removed!**

Higgs in gluon fusion at LHC: probability distributions

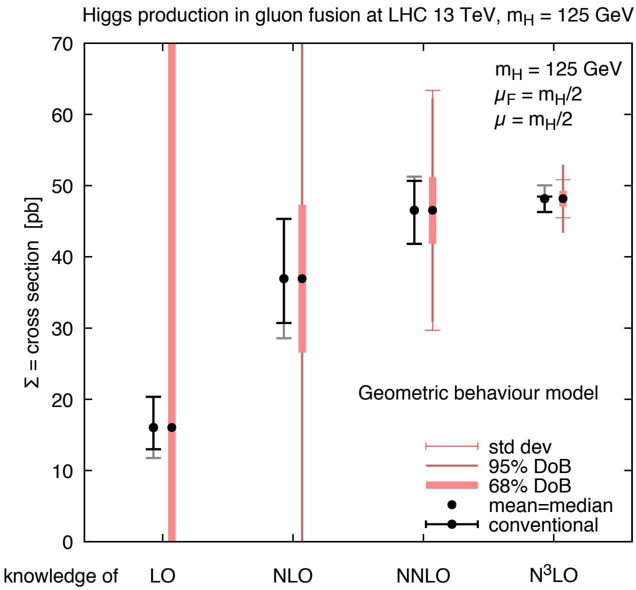


Computed with THunc (see also `miho`)

Higgs in gluon fusion at LHC: probability distributions



Computed with `THunc` (see also `miho`)

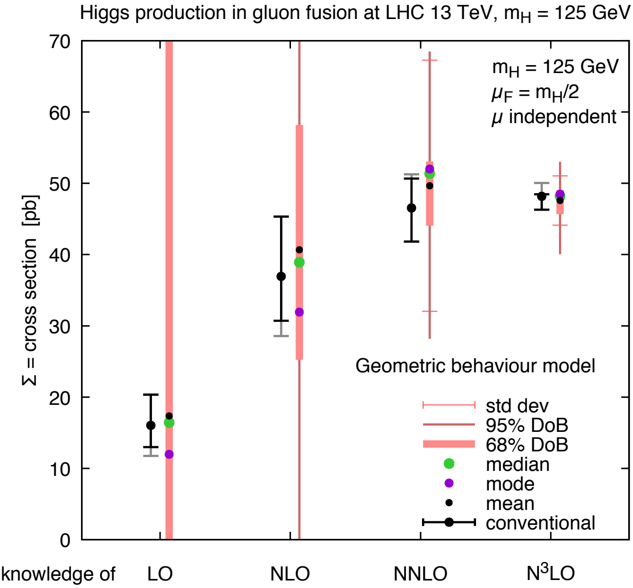


conventional result: canonical scale variation by a factor of 2 about $\mu_R = m_H/2$

new result: geometric behaviour model

Made with THunc

After marginalising over the renormalization scale

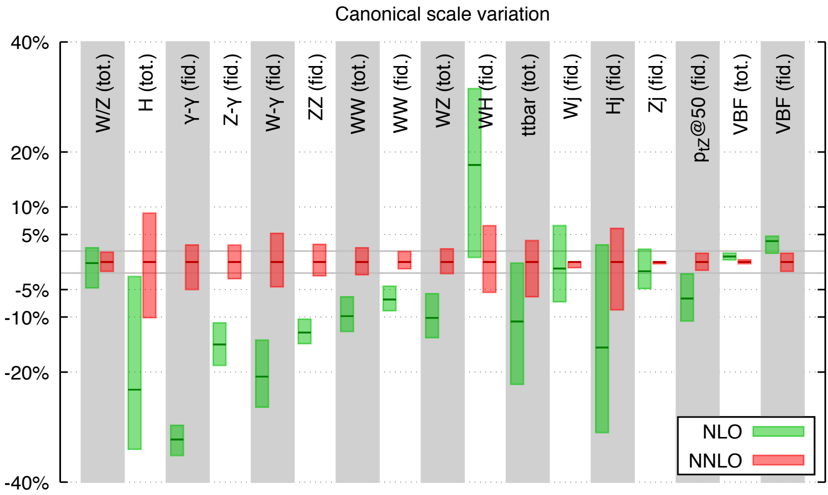


conventional result: canonical scale variation by a factor of 2 about $\mu_R = m_H/2$

new result: geometric behaviour model

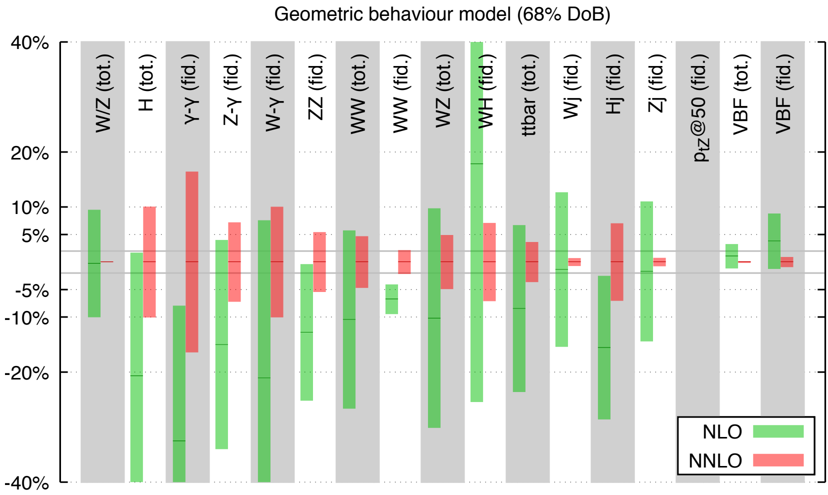
Made with THunc

Canonical scale variation vs Geometric behaviour model



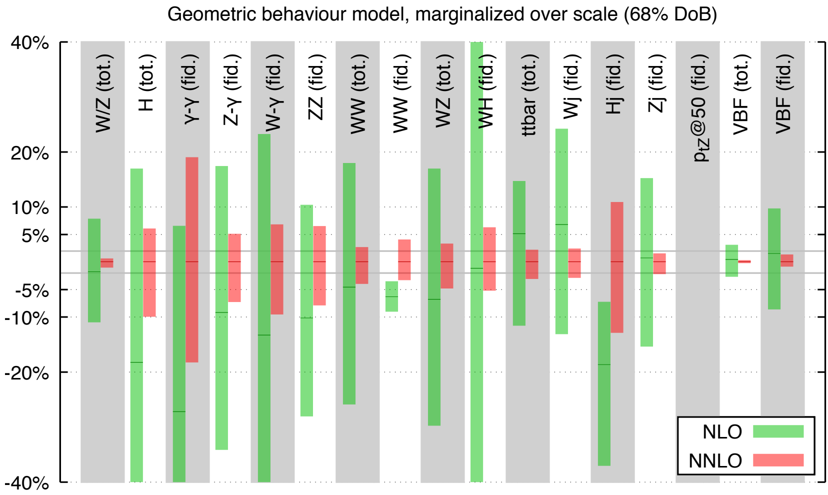
(courtesy of Gavin Salam)

Canonical scale variation vs Geometric behaviour model



Computed with THunc

Canonical scale variation vs Geometric behaviour model



Computed with THunc

Correlations in theory uncertainties from MHOs

Correlations in theory uncertainties from MHOs are expected:

- between different bins of the same observable
- between different observables of the same process
- between different processes (or maybe not?)

It is possible to treat them within the Bayesian model, but never done so far.

Crucial observations:

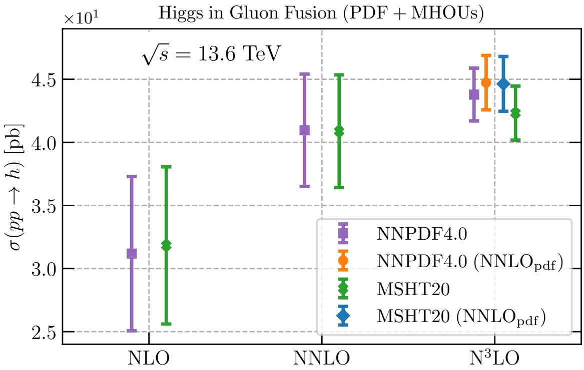
- correlations from MHO are due to similarities in the form of the perturbative expansions
- in a distributions, two adjacent bins tend to be 100% correlated
- in a distributions, two far bins may be characterized by very different perturbative expansions, and so be uncorrelated ...
- ... unless constraints like the knowledge of the total cross section (integral of the distribution) are present (they may also induce anti-correlations)
- it is very difficult to foresee correlations among different processes, unless the underlying mechanism for the dominant perturbative corrections is the same
- certainly, scale dependence is not to be used to generate correlations between different processes and not even different observables of the same process!

Let's go back to the Higgs...

N^{3}LO PDFs

All results presented so far were computed with NNLO PDFs

Very recently, (approximate) N^{3}LO PDFs became available [MSHT 2022] [NNPDF 2024]

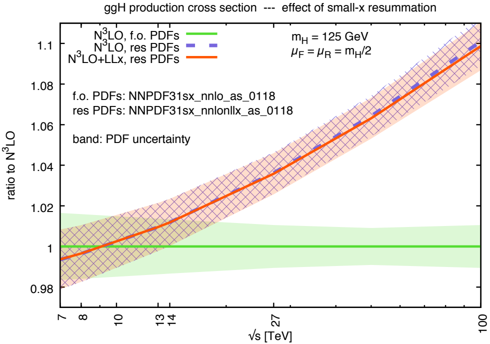


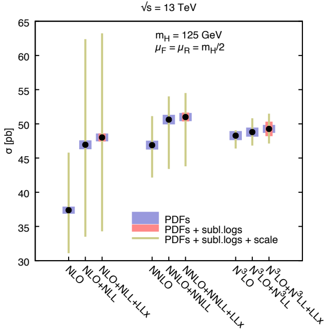
The uncertainty due to missing N^{3}LO PDFs is thus gone now

Note that PDF (and α_s) parametric uncertainties are still present, but should be updated with respect to the YR4 value $\pm 1.56\,\text{pb}$

Small-x resummation

Another resummation can be important for ggH, in the opposite regime of threshold, i.e. at small x. It mostly affect PDFs, giving a much larger gluon at small x





Computed with HELL [MB,Marzani 1802.07758] [MB 1805.08785]

- ggH cross section at FCC-hh $\sim 10\%$ larger than expected!
- At LHC $+1\%$ effect; larger effect expected at differential level
- Becomes less important for high masses (likely negligible for off-shell Higgs)

Summary

State of the art for ggH:

- N^{3}LO QCD large-m_t EFT
- NLO QCD exact
- NNLO QCD top mass corrections
- NLO EW + mixed NLO QCD-EW in the EFT
- N^{3}LL threshold resummation (QCD)

LHCHXSWG YR4 recommendation (LHC13):

$$\sigma = 48.6\,^{+2.2}_{-3.3}\,\mathrm{pb}\,(\text{theory}) \pm 1.56\,\mathrm{pb}\,(\text{PDF}+\alpha_s)$$

Beyond YR4:

- new results on mass corrections at NNLO
- new results on mixed EW-QCD corrections
- N^{3}LO PDFs
- threshold and small-x resummations
- more robust estimates of MHO uncertainties

codes

ggHiggs
SusHi
ihixs
HIGLUE
TROLL
RGHiggs
HELL
THunc

Backup slides

Quark mass effects at resummed level

Threshold resummation

$$C_{gg}(N,\alpha_s) \stackrel{N\to\infty}{=} g_0\left(\alpha_s, \frac{m_H}{m_t}, \frac{m_H}{m_b}, \dots\right) \times \exp \mathcal{S}(\alpha_s, \ln N)$$

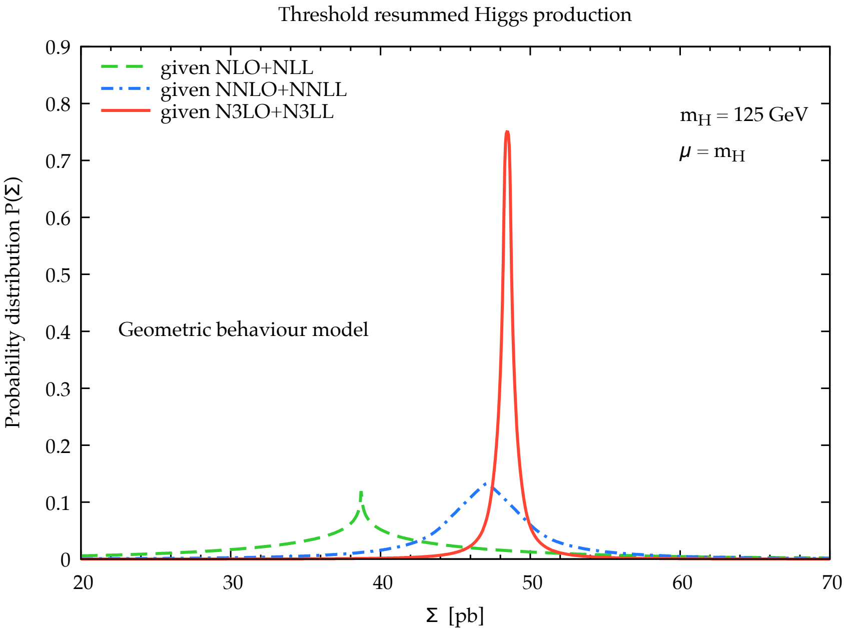
quark mass dependence appears only in g_0, and is determined by matching to fixed order.

- include in g_0 all known mass dependent terms

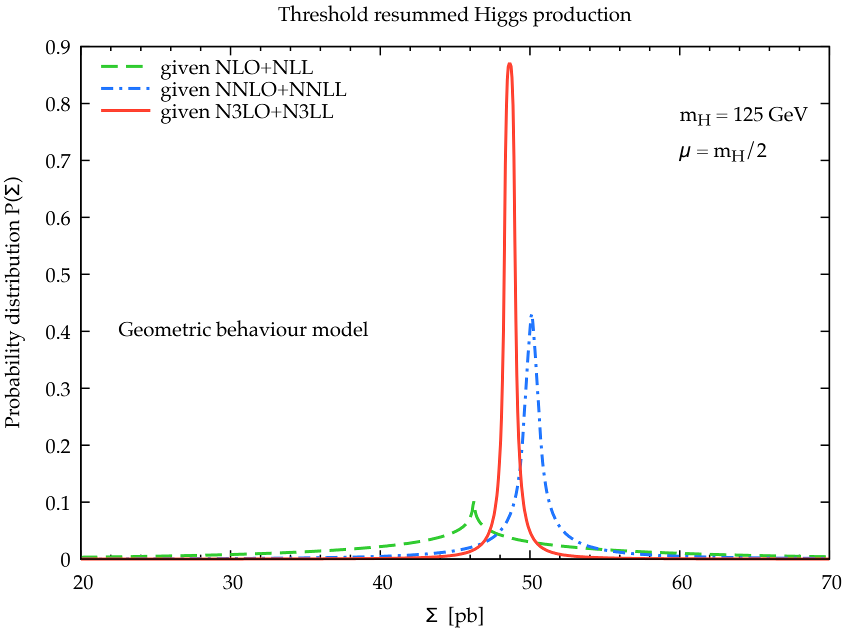
  [deFlorian,Grazzini 2012] [MB,Marzani 2014]

- include only the exact top at NLL only [Schmidt,Spira 2015]
  Motivation: bottom quarks generate additional logarithms in g_0 that are not resummed $\to$ fixed order treatment is preferred

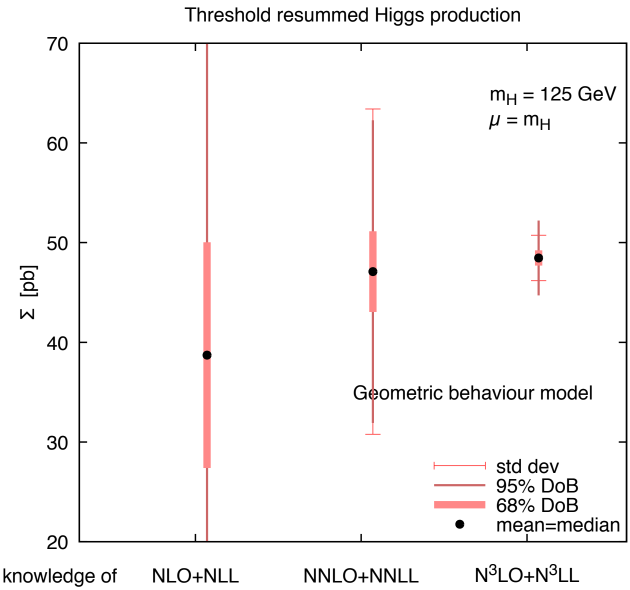
Higgs at resummed level: probability distributions



Higgs at resummed level: probability distributions

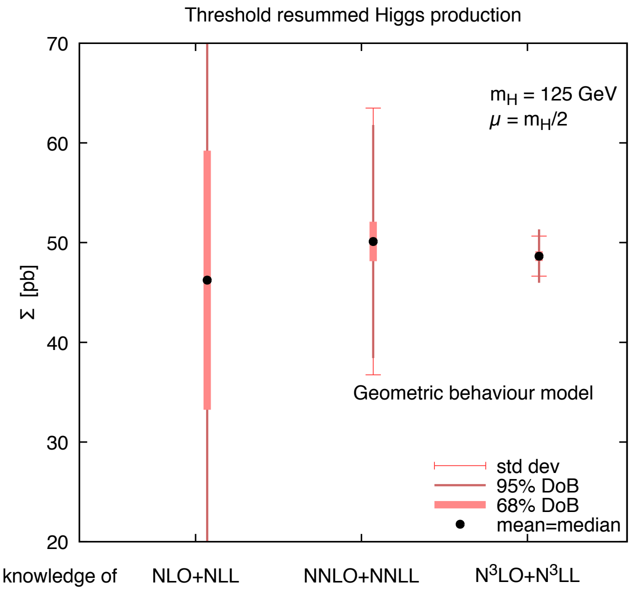


From distributions to statistical estimators



conventional result: canonical scale variation

new result: geometric behaviour model



conventional result: canonical scale variation

new result: geometric behaviour model

New model (2): Scale variation inspired model

Scale dependence probes higher orders... why not using it?

Idea (inspired by canonical scale variation): assume that the size of the higher order is comparable with the size of the scale dependence

Definition: "scale dependence numbers" r_k

$$r_k(\mu) \simeq \left| \mu \frac{d}{d\mu} \log \Sigma_{\mathrm{N}^k\mathrm{LO}}(\mu) \right|$$

measure the scale dependence of Σ

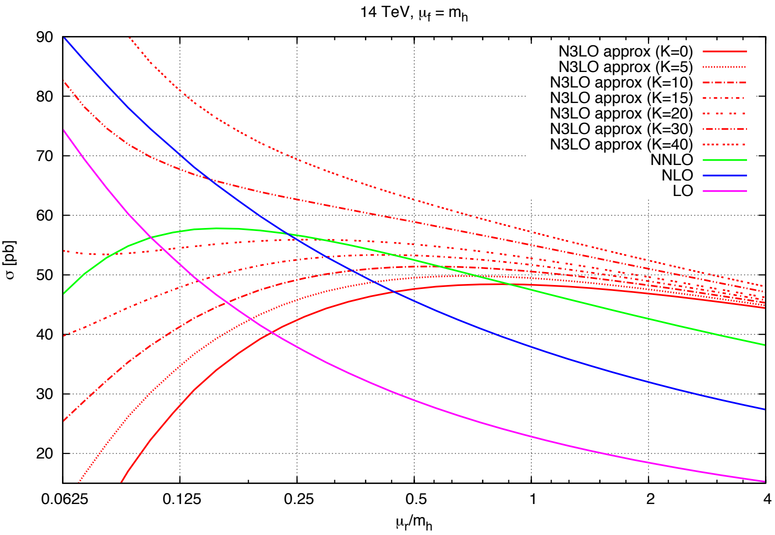
My proposal: scale variation model

$$|\delta_{k+1}(\mu)| \leq \lambda \, r_k(\mu)$$

depends on one hidden parameter λ

Canonical scale variation is approximately recovered for $\lambda = \log 2$

Not all higher orders are good...



[Buehler,Lazopoulos 1306.2223]

New model (3): Constrained scale dependence

Because $r_k(\mu) = \mathcal{O}(\alpha_s^{k+1})$, they should also behave perturbatively

Idea: require perturbativity of the $r_k(\mu)$ as a model condition!

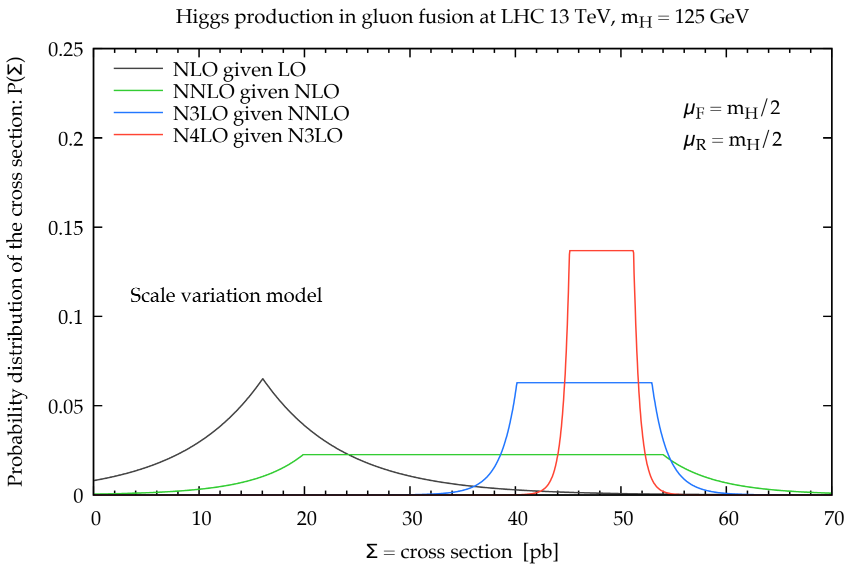
Two conditions:

$$|\delta_{k+1}(\mu)| \leq \lambda r_k(\mu)$$
$$|r_{k+1}(\mu)| \leq \eta r_k(\mu)$$

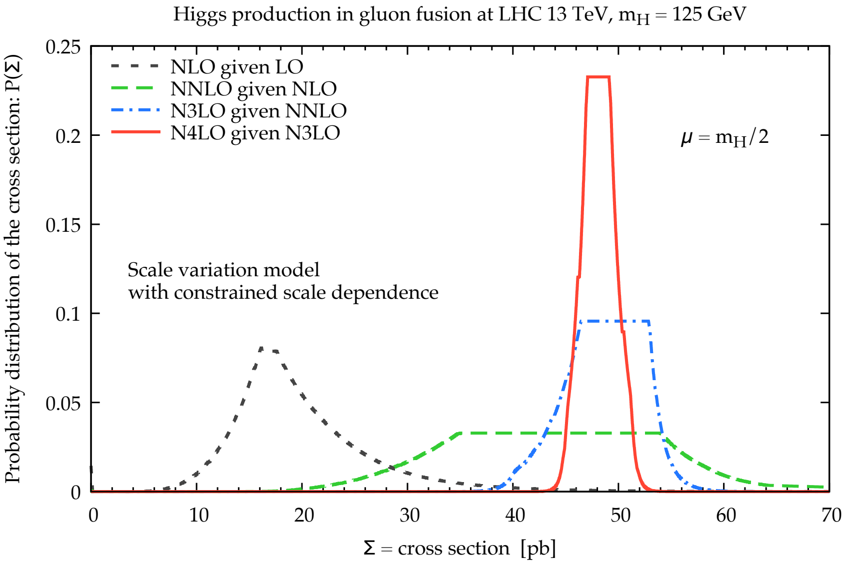
that depends on two hidden parameters λ, η

Leads to more stable and narrower results
(but the implementation is numerical, hence slow)

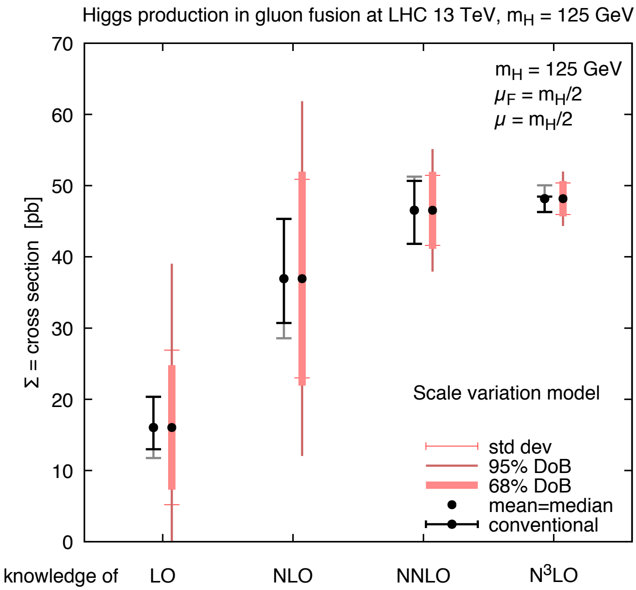
Higgs in gluon fusion at LHC: probability distributions



Higgs in gluon fusion at LHC: probability distributions

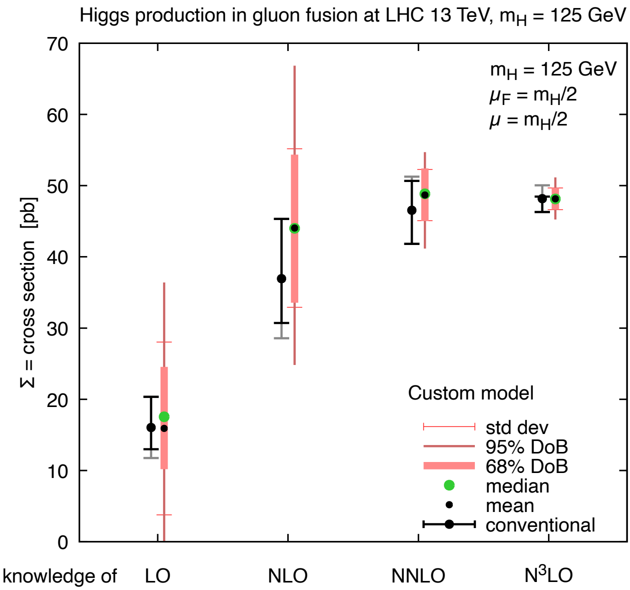


From distributions to statistical estimators



conventional result:
canonical scale
variation

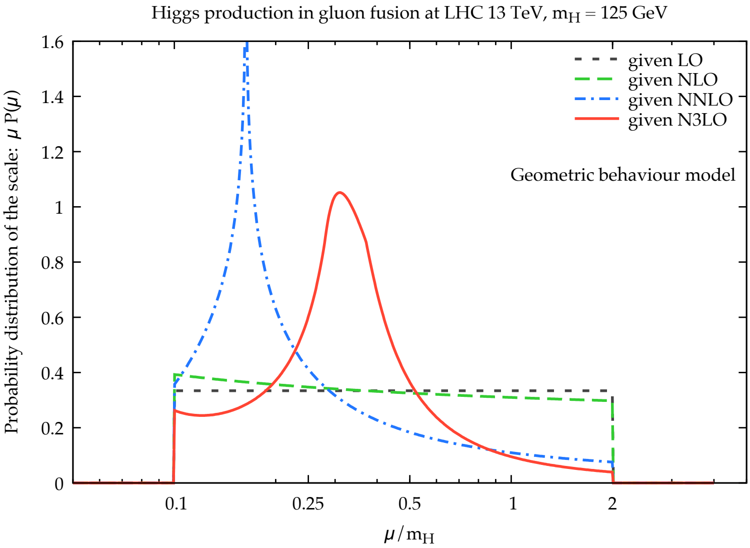
new result:
scale variation
inspired model



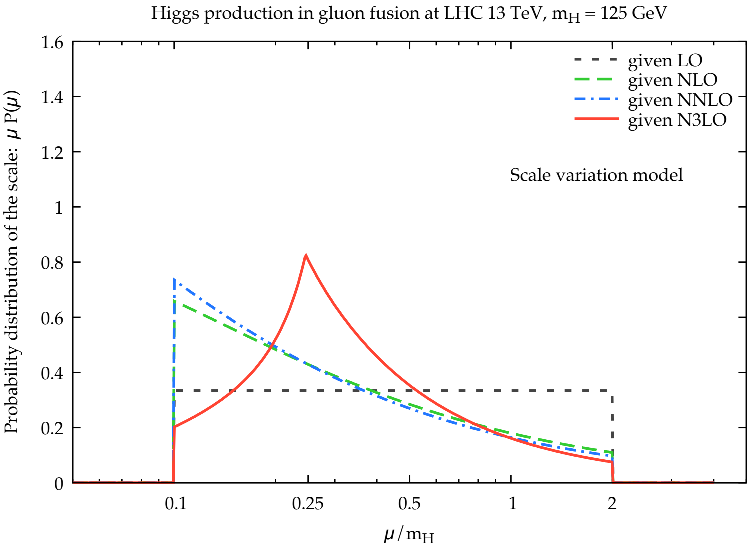
conventional result: canonical scale variation

new result: scale variation inspired model with contraints on higher order scale dependence

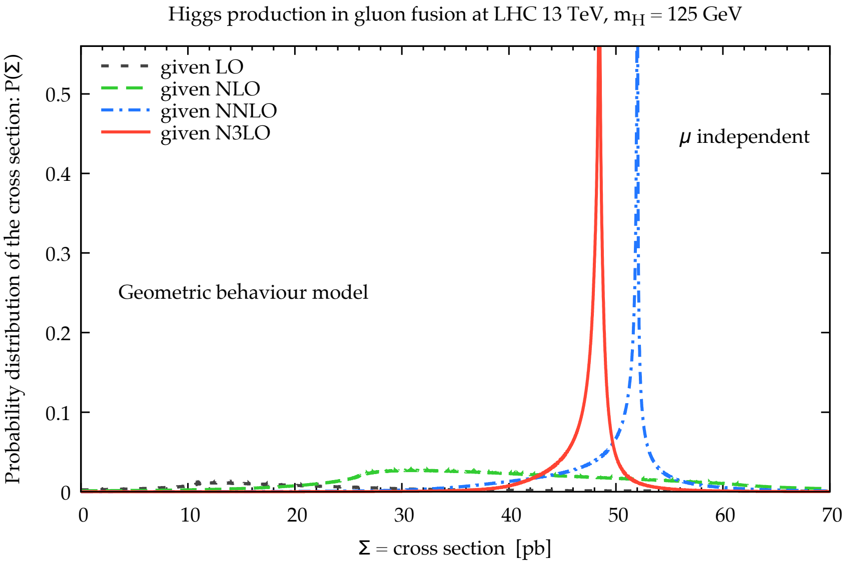
Posterior distribution for the scale μ



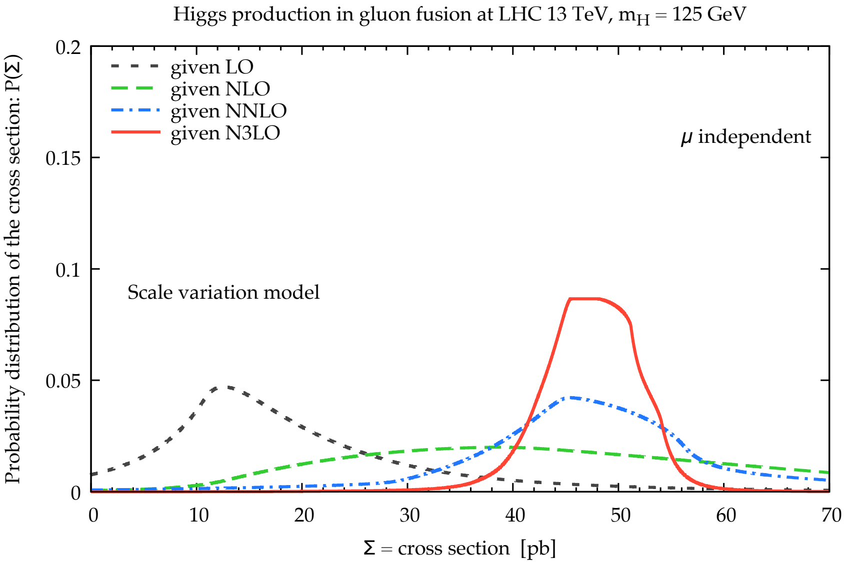
Posterior distribution for the scale μ



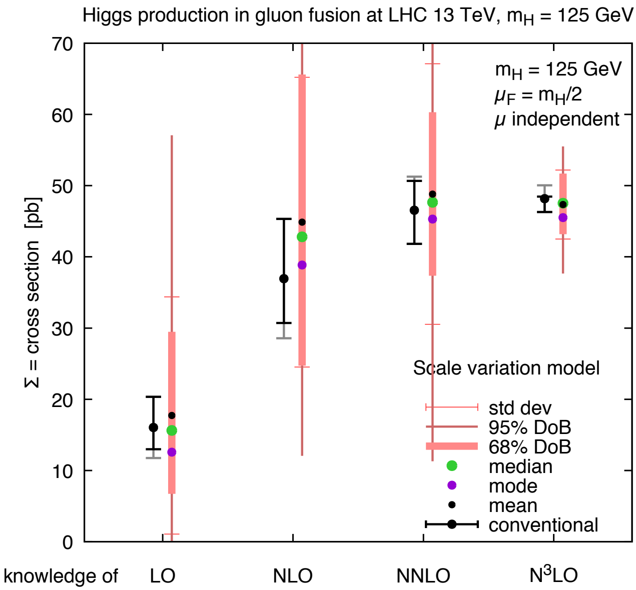
Higgs in gluon fusion at LHC: scale independent distributions



Higgs in gluon fusion at LHC: scale independent distributions



Higgs in gluon fusion at LHC: final results



conventional result: canonical scale variation by a factor of 2 about $\mu_R = m_H/2$ (best convergence properties)

new result: scale variation inspired model

How can a theory uncertainty from missing higher orders be probabilistic?

Frequentist approach to probability $\rightarrow$ requires repeatable events $\rightarrow$ no way...

Bayesian approach $\rightarrow$ probability defined as the **degree of belief** of an "event"

Initially no information $\rightarrow$ the probability of an event is given by a *prior* distribution, which encodes our subjective and arbitrary prejudices.

Acquiring information $\rightarrow$ changes the degree of belief through inference (Bayes theorem), making it less and less dependent on the prior.

see e.g. G.D'Agostini, Bayesian reasoning in data analysis

"Event" means something that can happen in different ways with different likelihoods.
In our case, the "event" is *"the observable takes the value* $\boldsymbol{\Sigma}$*"*, and its probability distribution will be a function of $\boldsymbol{\Sigma}$:

$$P(\boldsymbol{\Sigma}|\text{information}, \text{hypotheses})$$

Information = perturbative expansion of the observable.
Bayes theorem $\rightarrow$ improve the knowledge on the observable, namely update the distribution of $\boldsymbol{\Sigma}$.

Model 1: Geometric behaviour model (improved Cacciari-Houdeau)

Generalized condition that accounts for a possible power growth

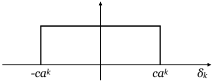
$$|\delta_k(\mu)| \le c a^k \qquad \forall k < k_{\text{asympt}} \qquad\qquad \text{CH: } \left|c_k \alpha_s^k\right| \le \bar{c}\alpha_s^k$$

depends on two hidden parameters c, a

It accounts for a possible power growth of the coefficients within the model!

Likelihood:

$$P(\delta_k|c,a,\mu) \propto \theta(ca^k - |\delta_k(\mu)|) =$$



namely all values of δ_k within the allowed range are equally likely

Prior:

$$P(c,a|\mu) \propto \frac{\theta(c-1)}{c^{1+\epsilon}} \times (1-a)^\omega \theta(a)\theta(1-a), \qquad \epsilon = 0.1, \quad \omega = 1$$

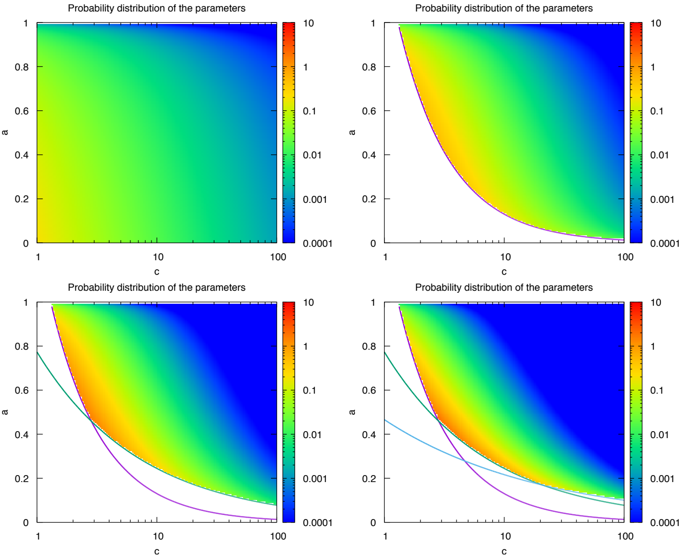
Inference scheme:

$$\underbrace{\delta_0, \ldots, \delta_n}_{\text{known}} \xrightarrow{\text{inference}} c, a \xrightarrow{\text{inference}} \underbrace{\delta_{n+1}, \delta_{n+2}, \ldots}_{\text{unknown}} \xrightarrow{\text{sum}} \Sigma$$

Final output:

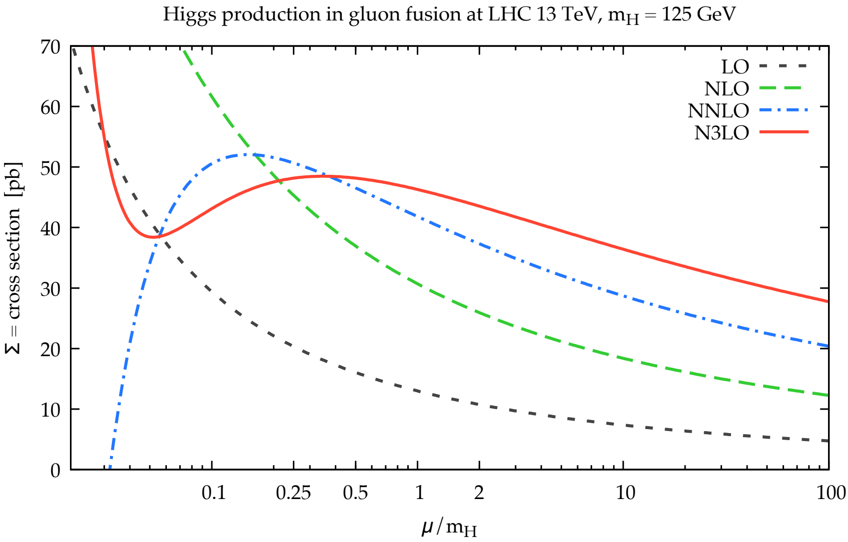
$$P(\Sigma|\delta_0, \ldots, \delta_n, \mu, \text{model}_1)$$

Posterior of c, a for Higgs production in gluon fusion



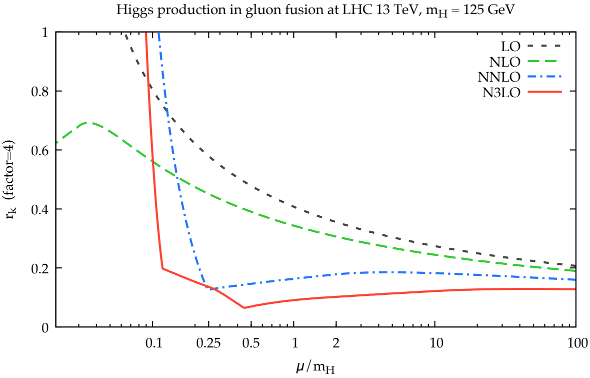
$$\boldsymbol{\delta_0 + \delta_1 + \delta_2 + \delta_3 + \ldots = 1 + 1.36 + 0.85 + 0.35 + \ldots}$$

Higgs production in gluon fusion at LHC



I want to define a model that uses scale variation.
I need a dimensionless number (to be compared to δ_k) that probes higher orders:

$$r_k(\mu) \simeq \left| \mu \frac{d}{d\mu} \log \Sigma_{\mathrm{N}^k\mathrm{LO}}(\mu) \right| = \mathcal{O}(\alpha_s^{k+1}) = \mathcal{O}(\delta_{k+1}(\mu))$$



Model 2: Scale variation inspired model

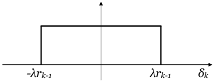
I propose the condition

$$|\delta_{k+1}(\mu)| \leq \lambda r_k(\mu) \qquad \forall k < k_{\text{asympt}}$$

that depends on one hidden parameter λ
Canonical scale variation is approximately recovered for $\lambda = \log 2$

Likelihood:

$$P(\delta_k|r_{k-1}, \lambda, \mu) \propto \theta(\lambda r_{k-1} - |\delta_k(\mu)|) =$$



namely all values of δ_k within the allowed range are equally likely

Prior:

$$P(\lambda|\mu) \propto \lambda^{\gamma} e^{-\lambda} \theta(\lambda), \qquad \gamma = 1$$

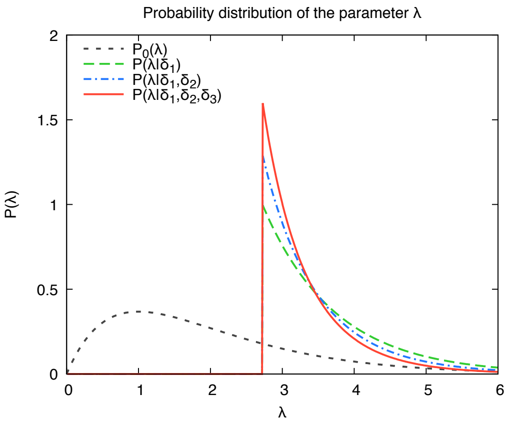
Inference scheme:

$$\underbrace{\delta_0, \dots, \delta_n, r_0, \dots, r_{n-1}}_{\text{known}} \xrightarrow{\text{inference}} \lambda \xrightarrow{\text{inference}+r_n} \underbrace{\delta_{n+1}}_{\text{unknown}} \xrightarrow{\text{sum}} \Sigma_{\mathrm{N}^{n+1}\mathrm{LO}}$$

in this case only the first missing higher order can be predicted:

$$P(\Sigma_{\mathrm{N}^{n+1}\mathrm{LO}}|\delta_0, \dots, \delta_n, r_0, \dots, r_n, \mu, \text{model}_2)$$

Posterior of λ for Higgs production in gluon fusion



The first non-trivial order $(\boldsymbol{\delta_1})$ sets the lower limit of $\boldsymbol{\lambda}$

$\rightarrow$ stable but possibly non optimal (overestimating uncertainty)

Improvable allowing violation of the bound (see appendix B.3)

Combining models and inventing new ones

Models can be combined together, requiring two or more conditions at the same time

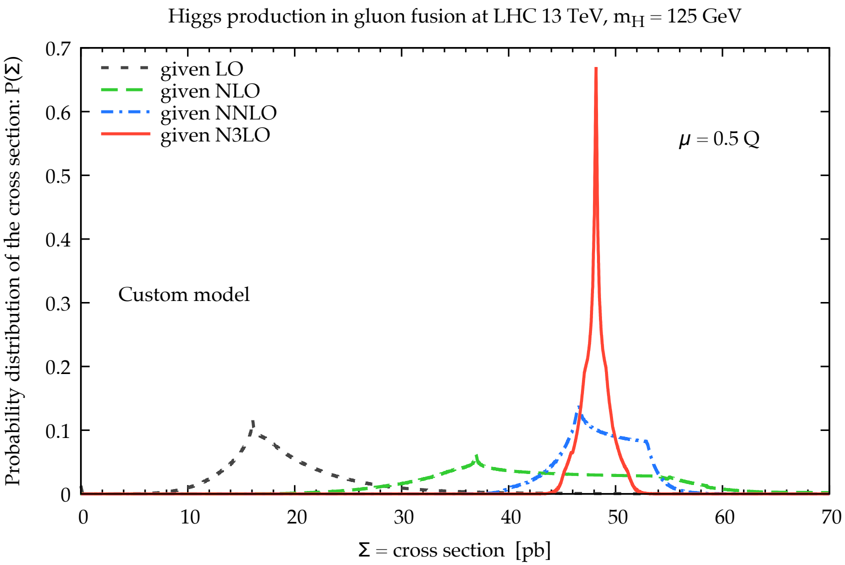
So far we have seen three conditions

$$
\begin{aligned}
|\delta_k(\mu)| &\le c a^k \\
|\delta_k(\mu)| &\le \lambda r_{k-1}(\mu) \\
|r_k(\mu)| &\le \eta r_{k-1}(\mu)
\end{aligned}
$$

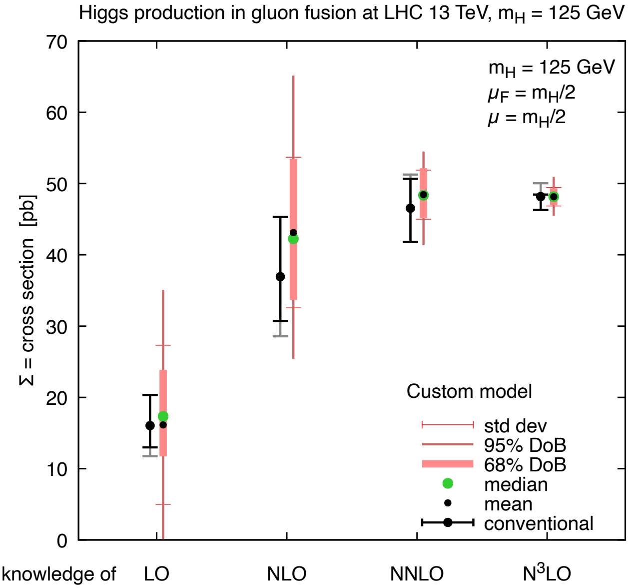
that are not contradictory and can thus hold at the same time

The models are implemented in a code named `THunc`, that provides a *custom model* feature to implement any customized model

Putting all conditions together....


Higgs production in gluon fusion at LHC 13 TeV, $m_H = 125$ GeV
given LO
given NLO
given NNLO
given N3LO
$\mu = 0.5$ Q
Custom model
Probability distribution of the cross section: P(Σ)
Σ = cross section [pb]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
10
20
30
40
50
60
70

go to slide **??**


Higgs production in gluon fusion at LHC 13 TeV, m_H = 125 GeV
m_H = 125 GeV
$\mu_F = m_H/2$
$\mu = m_H/2$
Σ = cross section [pb]
70
60
50
40
30
20
10
0
Custom model
std dev
95% DoB
68% DoB
median
mean
conventional
knowledge of
LO
NLO
NNLO
N^3LO

It's a generalisation of the geometric behaviour model,

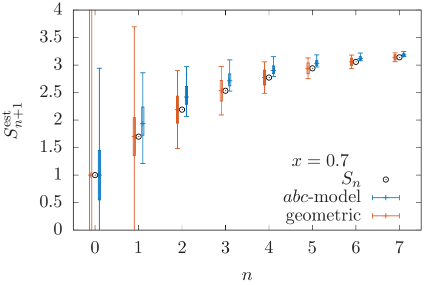
$$\text{geo:}\quad |\delta_k(\mu)| \le c a^k \qquad\qquad abc: \quad -c+b \le \frac{\delta_k(\mu)}{a^k} \le c+b$$

depends on three hidden parameters a, b, c

They keep requiring $|a| \le 1$, but the sign can be negative (to describe alternating sign series)

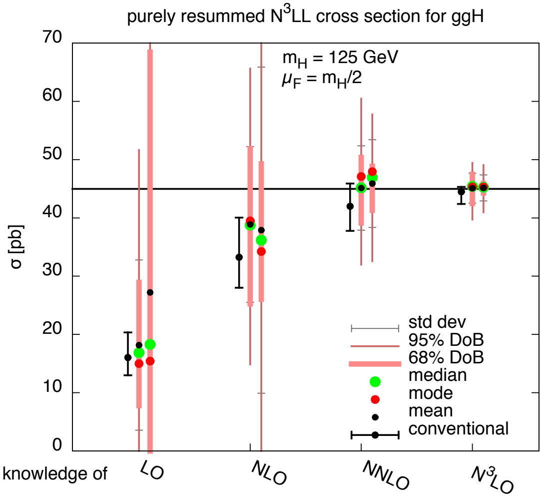
Moreover the b parameter accounts for asymmetric behaviour

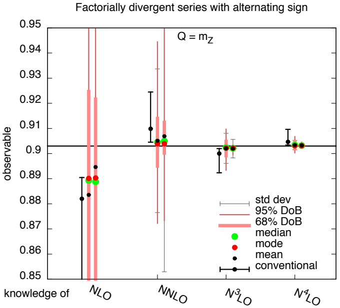
Comparison for $\sum_{k\ge 0} x^k$, $x = 0.7$

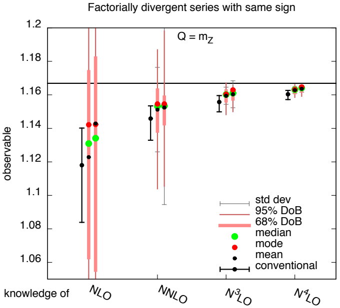


Note: I have proposed a different way to account for a sign pattern, which can be applied to any symmetric model (app. B.5)

Validation using known sums

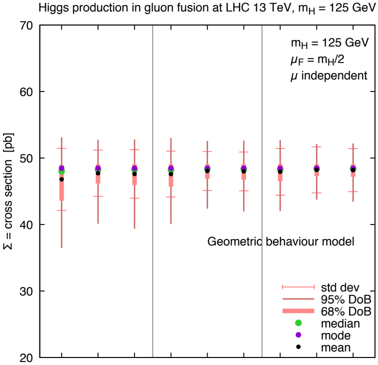


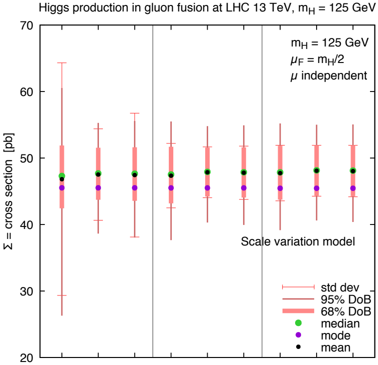




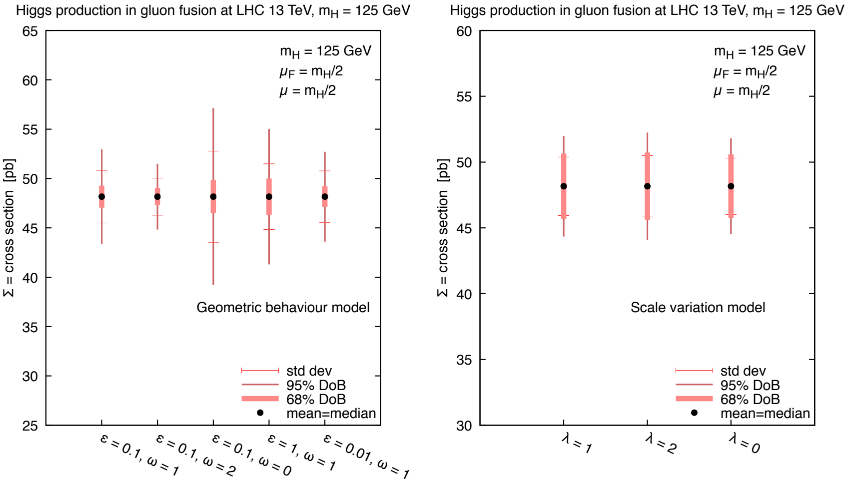
- purely resummed ggH at N^{3}LL, expanded in α_s
- factorially divergent series $\sum_k (-1)^k k! \alpha_s^k(m_Z)$
- factorially divergent series $\sum_k k! \alpha_s^k(m_Z)$

Scan of priors for the scale μ





Scan of priors for the model parameters



Explicit inference procedure in Cacciari-Houdeau

Probability of a missing higher order coefficient c_k given the knowledge of the first $c_0, ..., c_n$ orders

$$
\begin{aligned}
P(c_k|c_0,...,c_n) &= \frac{P(c_k,c_0,...,c_n)}{P(c_0,...,c_n)} \qquad\qquad (k>n)\\
&= \frac{\int d\bar{c}\, P(c_k,c_0,...,c_n,\bar{c})}{\int d\bar{c}\, P(c_0,...,c_n,\bar{c})}\\
&= \frac{\int d\bar{c}\, P(c_k,c_0,...,c_n|\bar{c})P_0(\bar{c})}{\int d\bar{c}\, P(c_0,...,c_n|\bar{c})P_0(\bar{c})}\\
&= \frac{\int d\bar{c}\, P(c_k|\bar{c})P(c_0|\bar{c})\cdots P(c_n|\bar{c})P_0(\bar{c})}{\int d\bar{c}\, P(c_0|\bar{c})\cdots P(c_n|\bar{c})P_0(\bar{c})}
\end{aligned}
$$

having used

$$P(A,B) = P(A|B)P(B), \qquad P(A) = \int dB\, P(A,B)$$

The probability for the full observable is given by

$$P(\Sigma|c_0,...,c_n) = \int dc_{n+1}dc_{n+2}\cdots\, P(c_{n+1},c_{n+2},...|c_0,...,c_n)\,\delta\left(\Sigma - \sum_{k=0}^{\infty} c_k\alpha_s^k\right)$$