Analytic approaches in conformal bootstrap

Apratim Kaviraj

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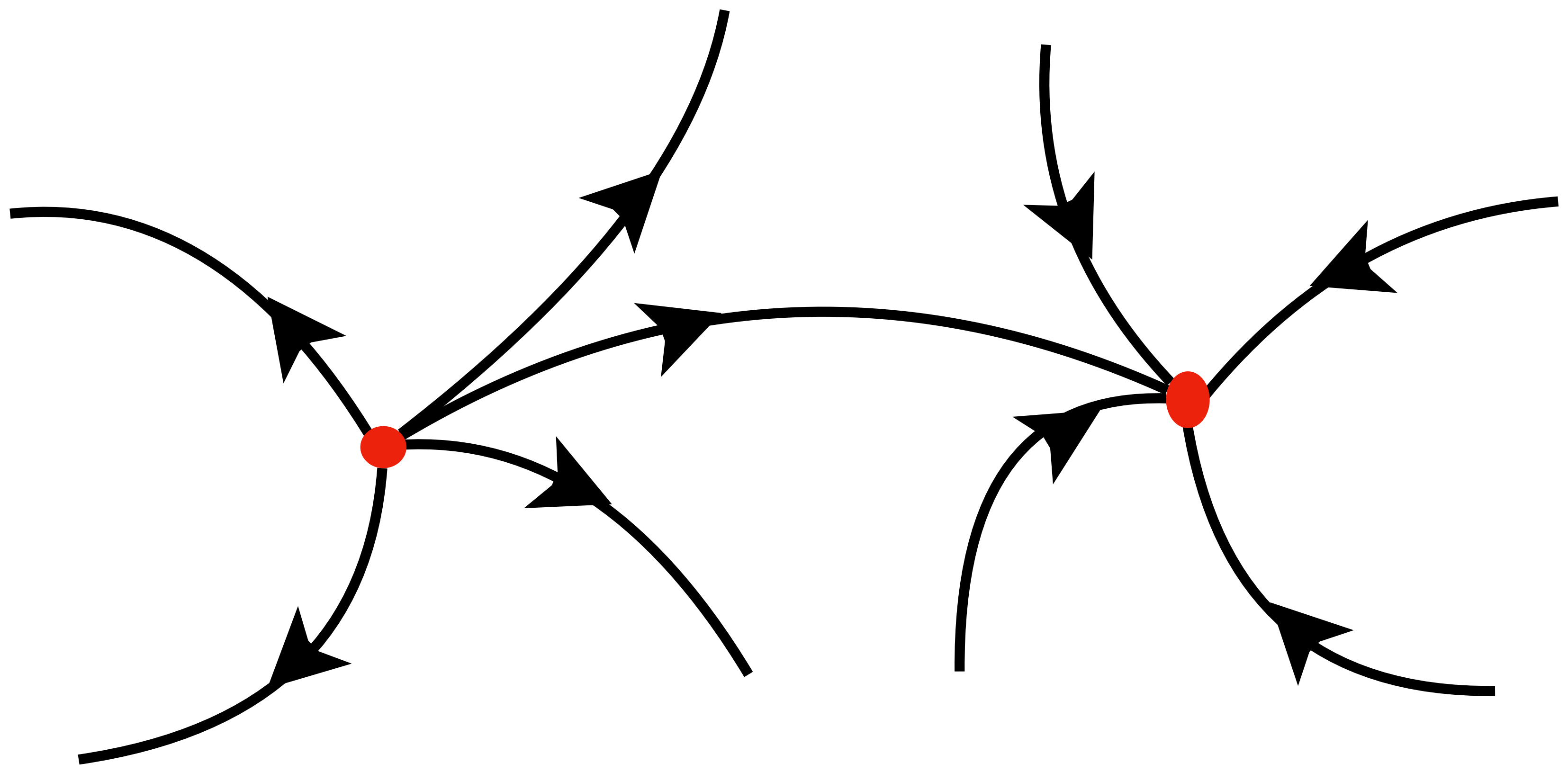
Outline

- Introduction and CFT fundamentals
- Conformal bootstrap - basics
- Numerical bootstrap and bounds
- Analytic (lightcone) bootstrap
- Outlook

QFTs vs scale

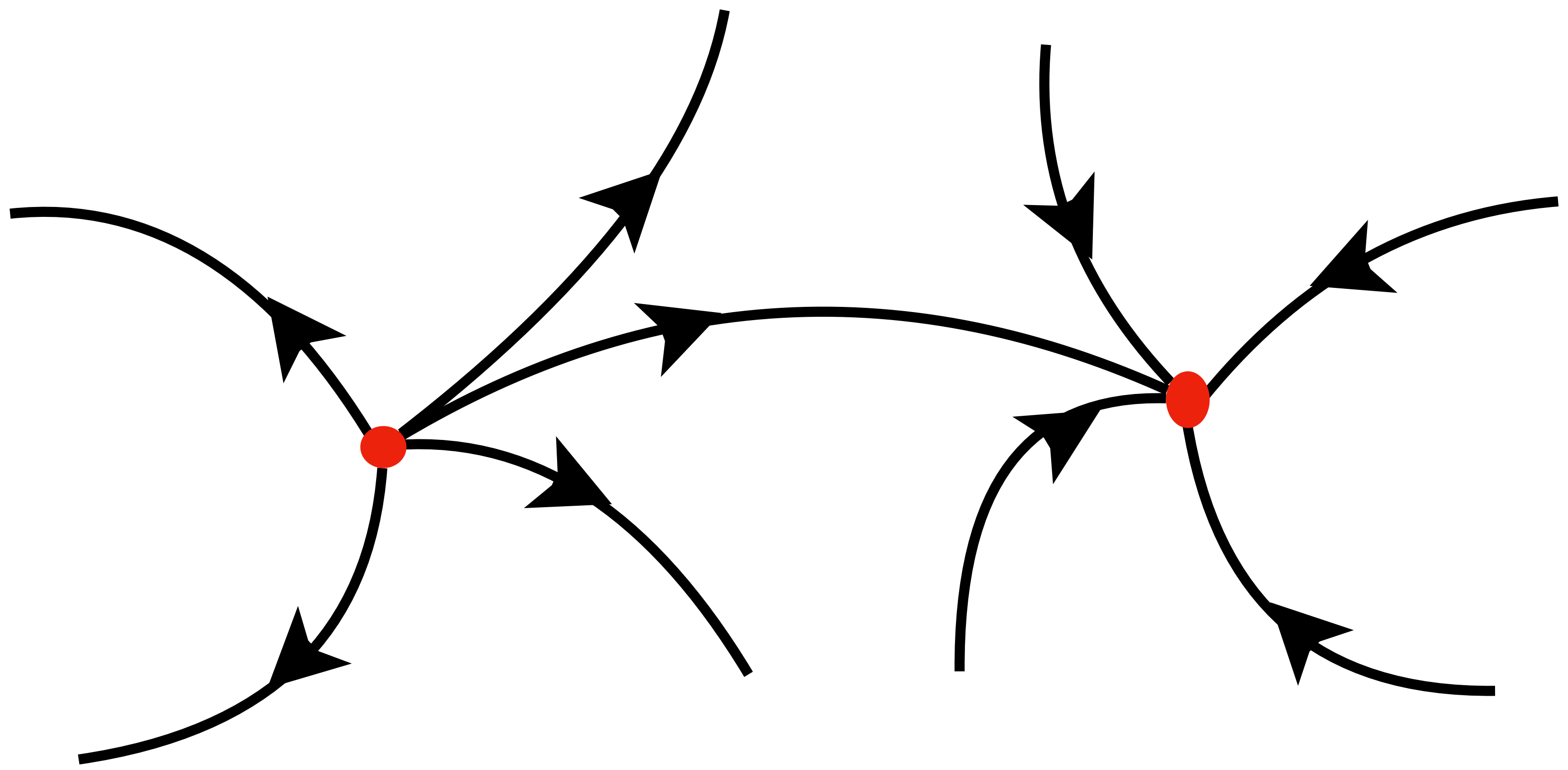
- QFTs at different scales are connected by RG flows.
- End points of an RG flow are usually the scale invariant fixed points
- Usually scale invariance also implies conformal invariance!



QFTs vs scale

- Conformal field theories (CFTs): are universal theories i.e. described w/o a Lagrangian.

- Knowing a CFT $\Leftrightarrow$ knowing RG flows around them.



QFTs vs scale

- Conformal field theories (CFTs): are universal theories i.e. described w/o a Lagrangian.
- Knowing a CFT $\Leftrightarrow$ knowing RG flows around them.
- **Applications:**

~ Analytic structure has parallels with scattering amplitudes!

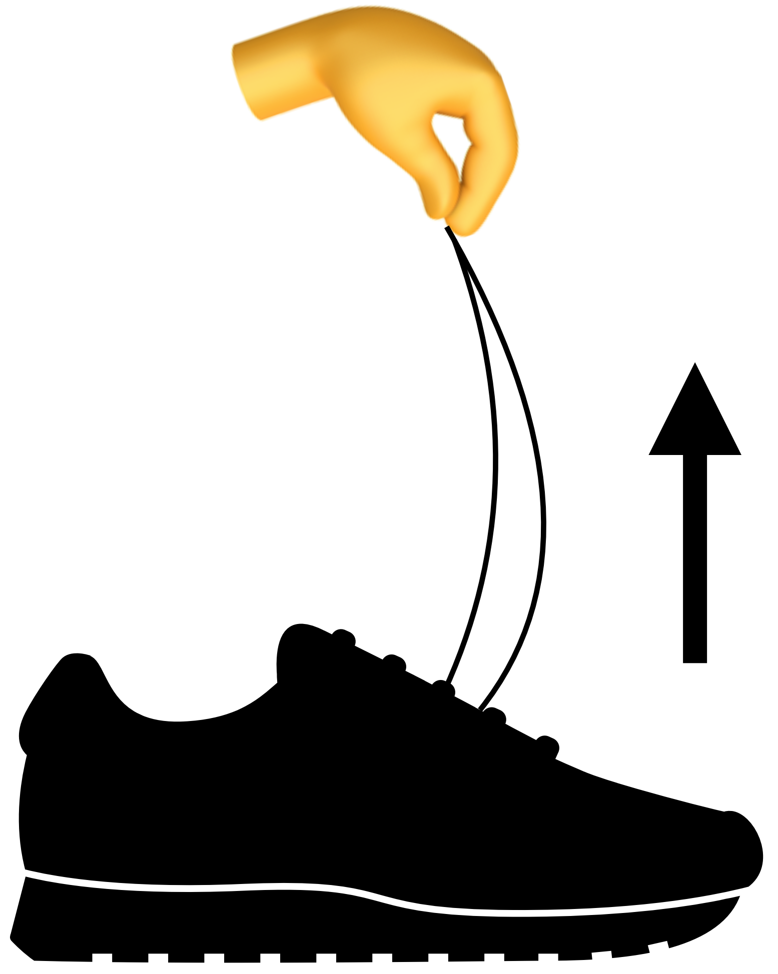
~ Describe critical phenomena in statistical systems.

~ Duality with gravity/string theory (AdS-CFT)

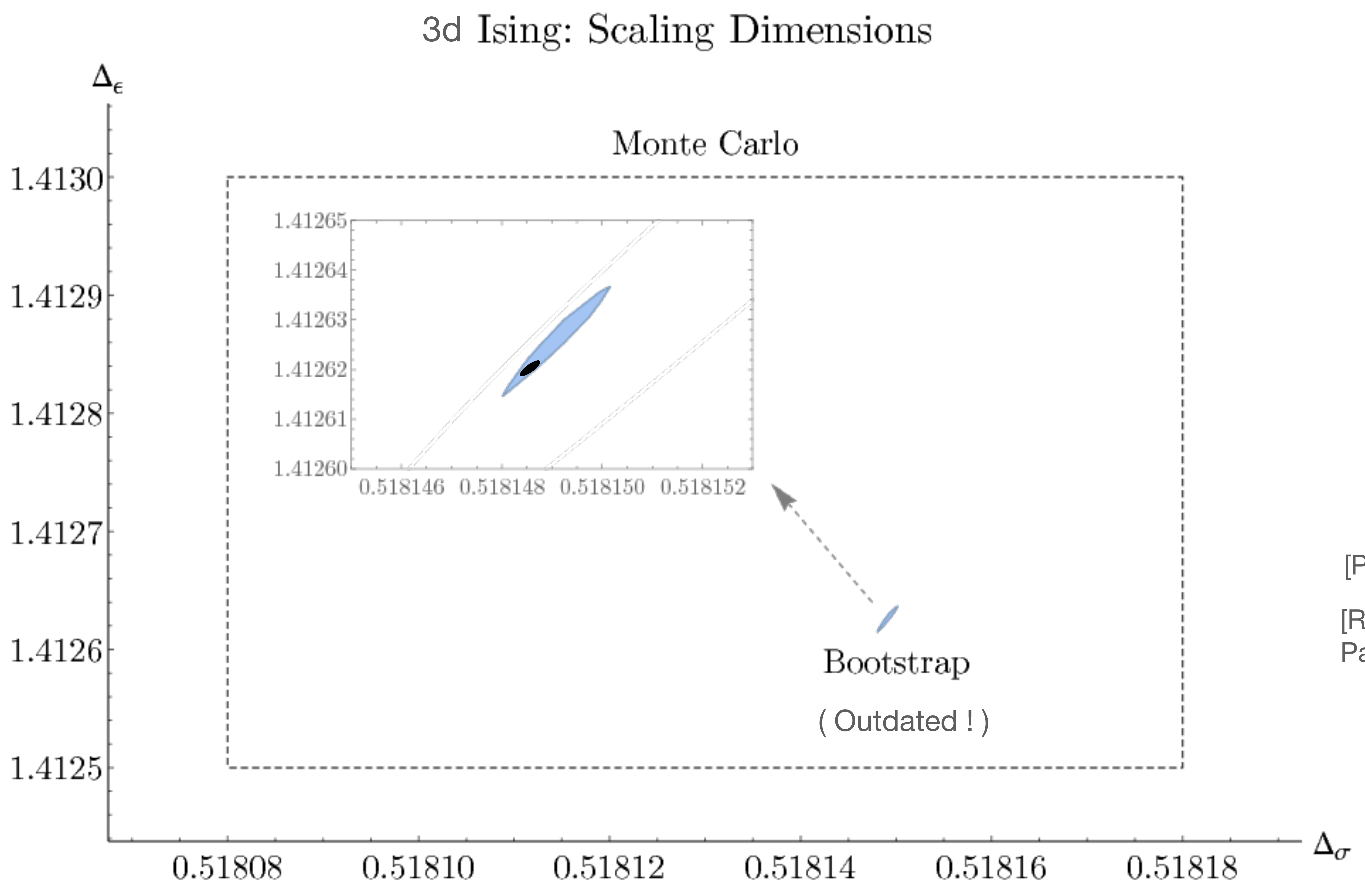
~ …….

What is a non-Lagrangian approach?

- All CFTs share similar kinematics (conformal invariance)
- Quantum states are described by local operators
- CFT observables obey the rules of QM (unitarity, OPE)
- Can these axioms alone fix a CFT?



The power of bootstrap !



[Polyakov]

[Rychkov, Simmons-Duffin, Paulos, Poland, van Rees, Su, ….]

Some fundamentals of CFTs

- Conformal invariance: Poincare + scaling + special conformal invariance
- Operators are labeled by dimensions Δ and spins ℓ
  - Primaries $\mathcal{O}_{\Delta,\ell}$
  - Descendents $\partial_\mu \partial_\nu \cdots \mathcal{O}_{\Delta,\ell}$
- 2point and 3point functions

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim x^{-2\Delta}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle \sim C_{123}|x_{12}|^{\Delta_3-\Delta_1-\Delta_2}|x_{13}|^{\Delta_2-\Delta_1-\Delta_3}|x_{23}|^{\Delta_1-\Delta_3-\Delta_2}$$

Some fundamentals of CFTs

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- 4 and higher point functions determined up to cross ratios

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \ \sim \ G(u,v)$$

- Operator product expansion (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \sum_{\Delta,\ell} C^{ij}_{\Delta,\ell}\,\mathcal{O}_{\Delta,\ell}(0) \ + \ \text{descendants}$$

$$\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}\rangle \ = \ \sum_{\Delta,\ell} C^{12}_{\Delta,\ell} C^{34}_{\Delta,\ell}\, G_{\Delta,\ell}(u,v)$$

Crossing symmetry

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Associativity of taking OPE

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \;=\; \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

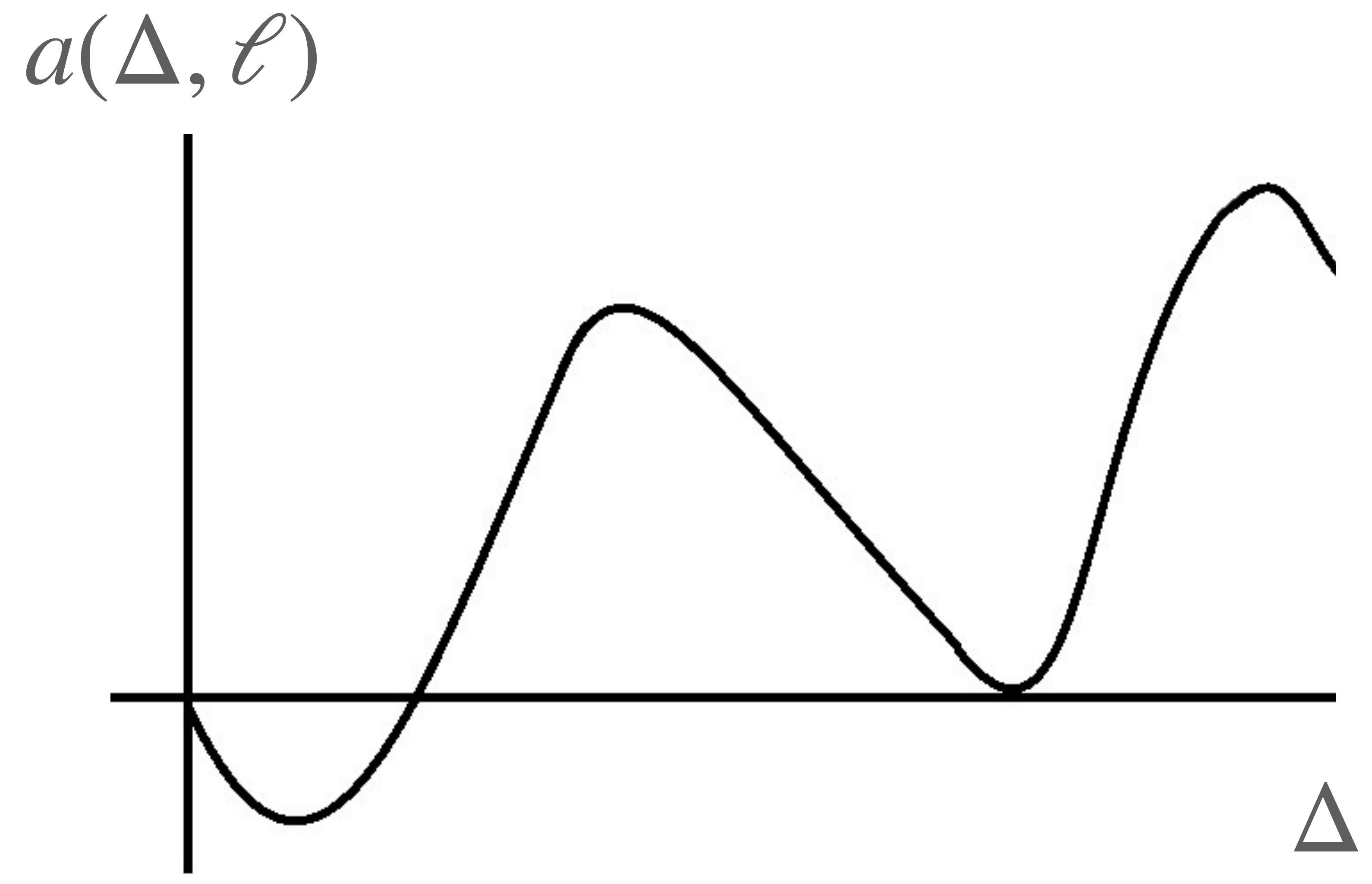
$$\sum_{\Delta,\ell} C_{\Delta,\ell}^2 G_{\Delta,\ell}(u,v) \;=\; \sum_{\Delta,\ell} C_{\Delta,\ell}^2 G_{\Delta,\ell}(v,u)$$

s-channel

t-channel

Conformal bootstrap - numerical approach

$$\sum_{\Delta,\ell} C^2_{\Delta,\ell}\, a(\Delta,\ell) = 0$$



Some functional

e.g. $a(\Delta,\ell) :=$ combination of derivatives

$$\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v) \;=\; \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)$$

s-channel t-channel

Conformal bootstrap - numerical approach

- Unitarity + crossing symmetry = numerical bounds in OPE space
- Multiple correlators: island-like bounds
- Solving the spectrum: is there a more systematic approach?
- Let’s look at crossing equation in a different limit….

Lightcone analytic bootstrap

[Komargodski, Zhiboedov, Fitzpatrick, Kaplan, Simmons-Duffin, Poland, Alday, Sinha, Sen, AK…. (2013-2016)]

$$\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v) \;=\; \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)$$

$u \ll v \ll 1$

$$1 \;+\; C_\star u^{\frac{\Delta_\star - \ell_\star}{2}} + \cdots$$

$$G_{\Delta,\ell}(v,u) \sim u^{\Delta_{\mathcal{O}}} \log u$$

Identity $(\Delta, \ell = 0)$ $+$ other operators

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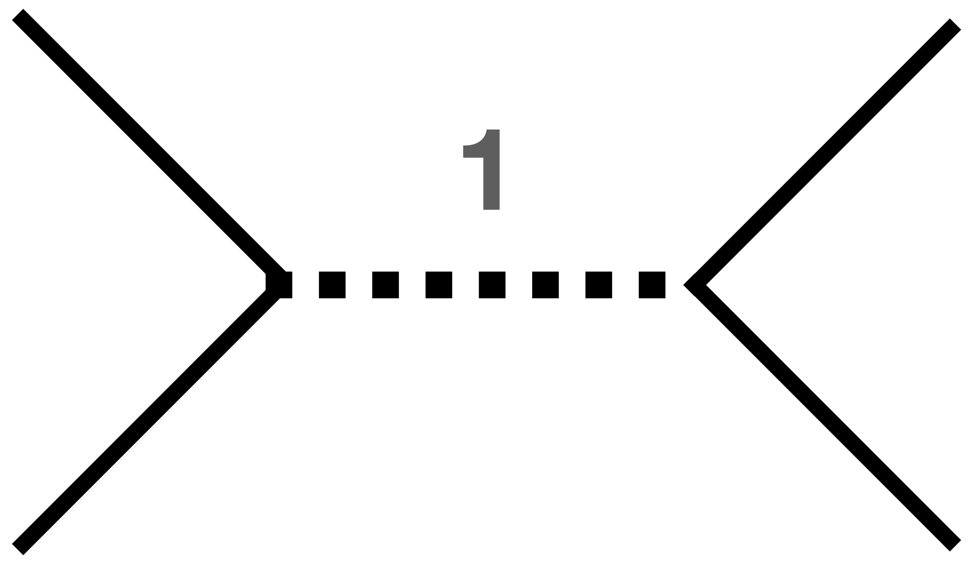
Identity $(\Delta, \ell = 0)$ + other operators

$$\sum_{\ell \gg 1} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u) \approx 1$$

$$\Delta \approx 2\Delta_{\mathcal{O}} + 2\ell$$

Lightcone analytic bootstrap (4pt vs 6pt)



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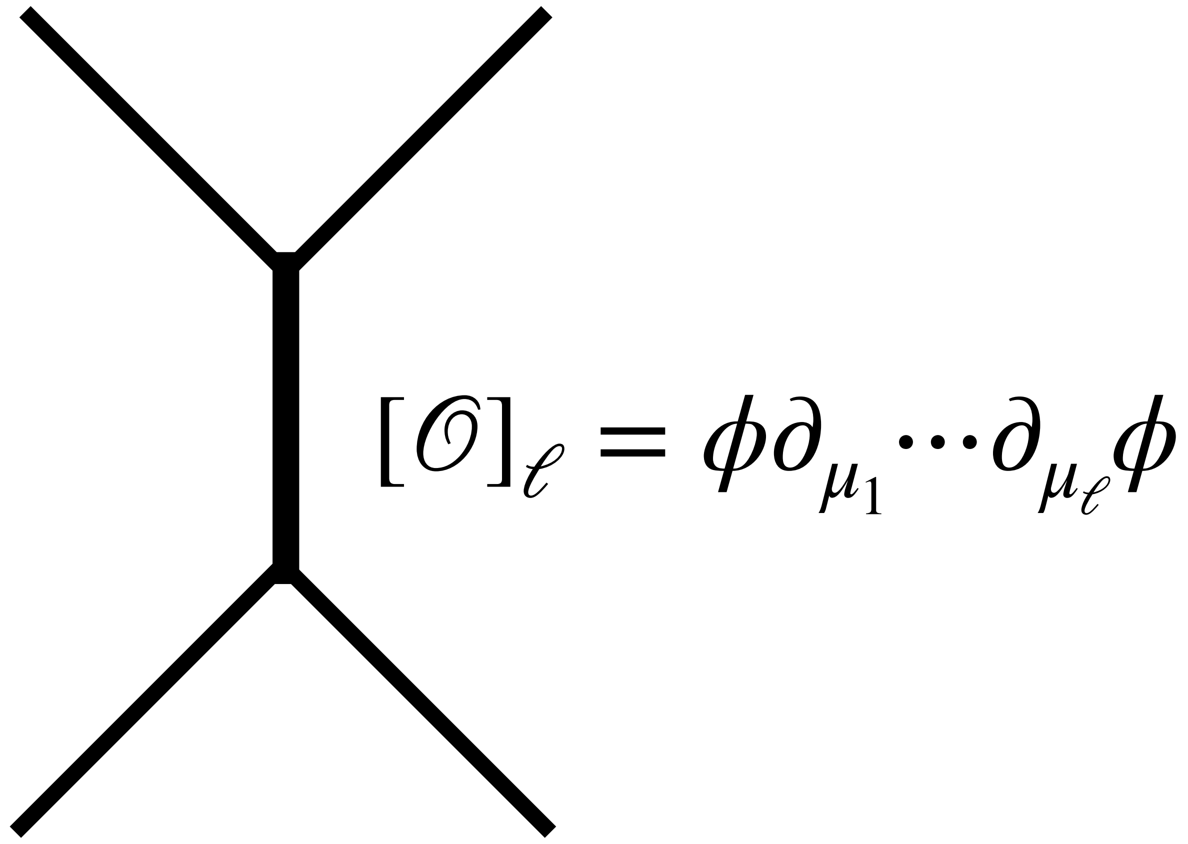

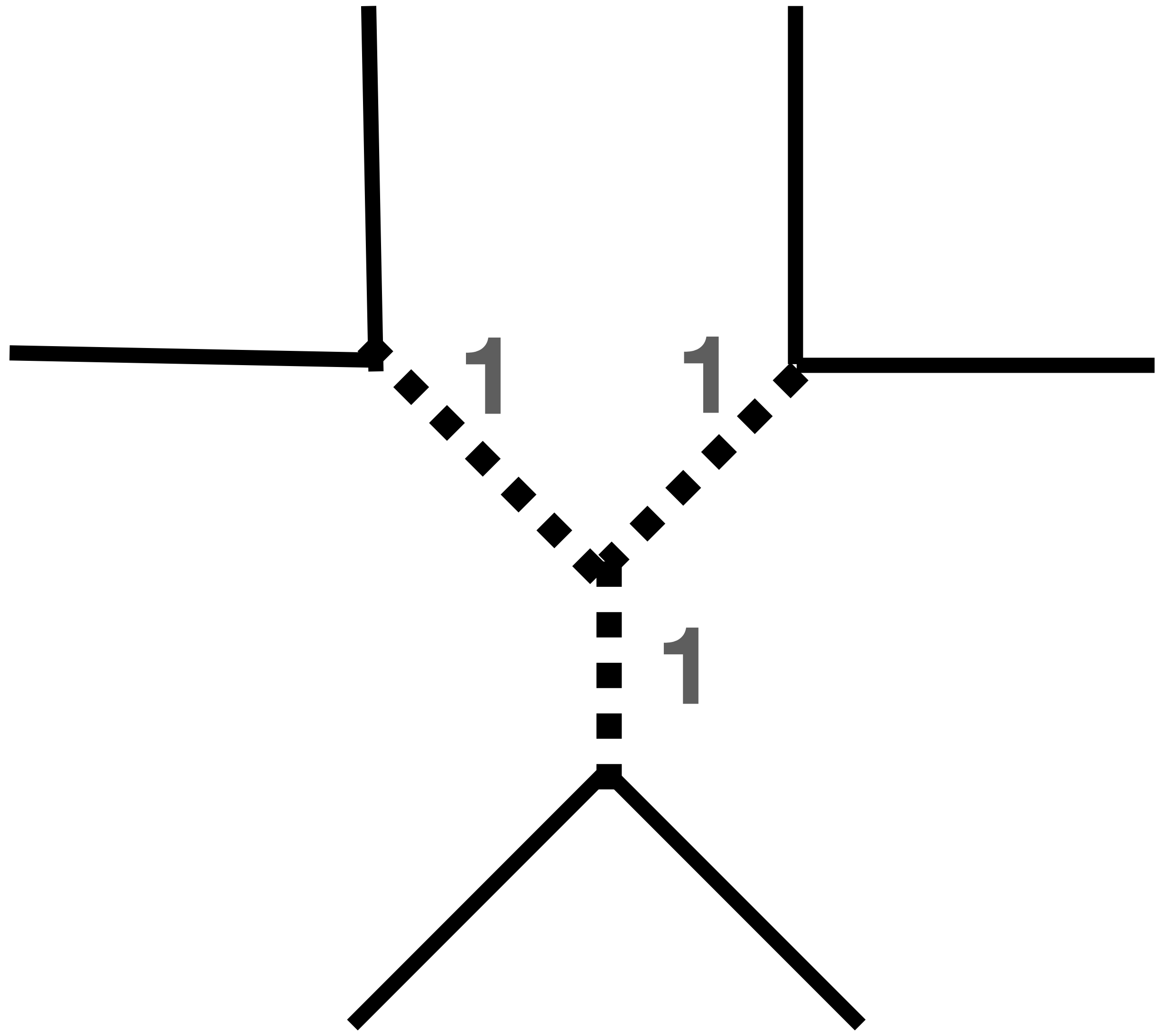

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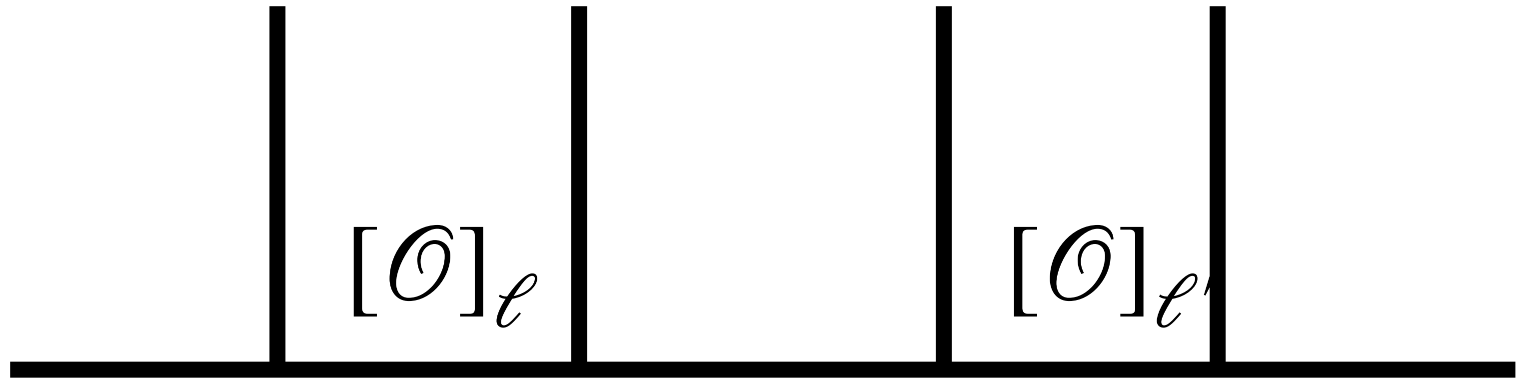

$$[\phi\mathcal{O}]_\ell^{(m)}$$
$$= \phi\partial_{\mu_1}\partial_{\mu_m}[\mathcal{O}]_{\ell-m}$$

[Harris, AK, Mann, Schomerus, Quintavalle (2024)]

Lightcone analytic bootstrap

- Identity operators correspond to Wick contractions (free theory)
- So crossing symmetry predicts: a sector of the spectrum approaches a **generalized free theory**
- This sector is “universal” (theory independent)
- A generalized free CFT_d $=$ a free massive QFT in AdS_{d+1}

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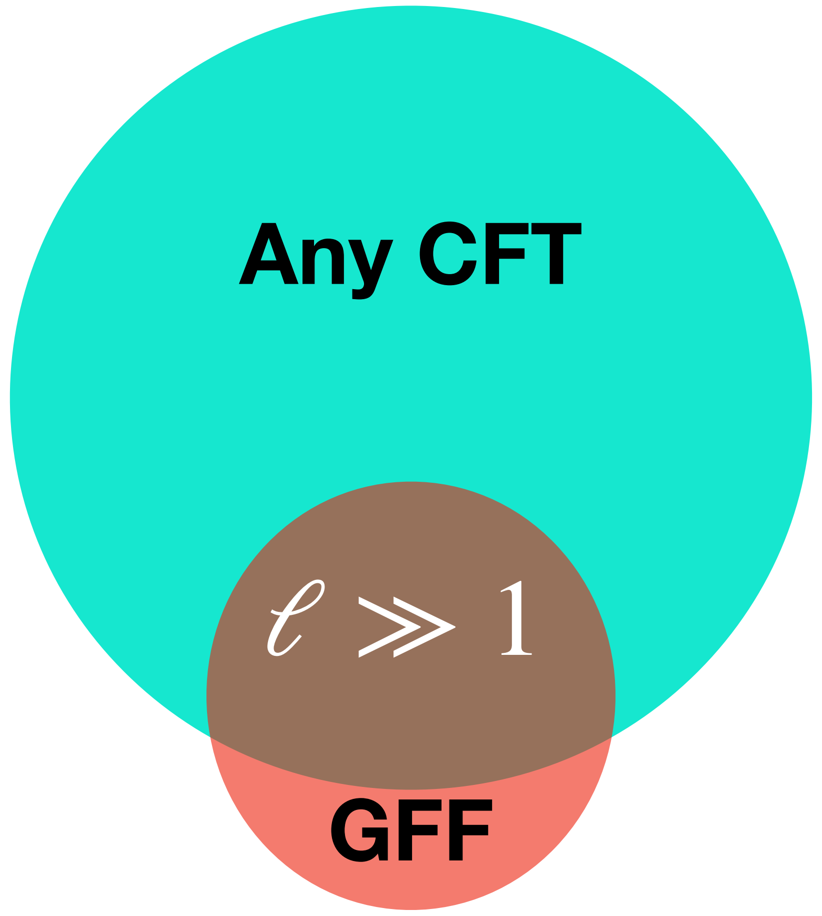

What is the most optimised approach?

Analytic bootstrap

- Expansion around lightcone
  $u \ll v \ll 1$
- Lorentzian signature

Numerical bootstrap

- Expansion around a “good convergence” point
  $$u = v = \frac{1}{2}$$
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How do you manifest analytic properties, yet obtain numerical bounds?

$$\int\int \mathrm{d}u\, \mathrm{d}v\, h(u,v)$$

Singular regions in u,v complex plane

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[Paulos, Mazac, Rastelli, Zhou, Penedones, Zhiboedov, Silva, Carmi, Gopakumar, Sinha, Zahed, Ghosh, AK,....]

- Such integrals functionals are called analytic functionals
- They account for all analytic properties of blocks.
- **Related to block expansion coefficients of Witten diagrams (= Feynman diagrams in AdS)**

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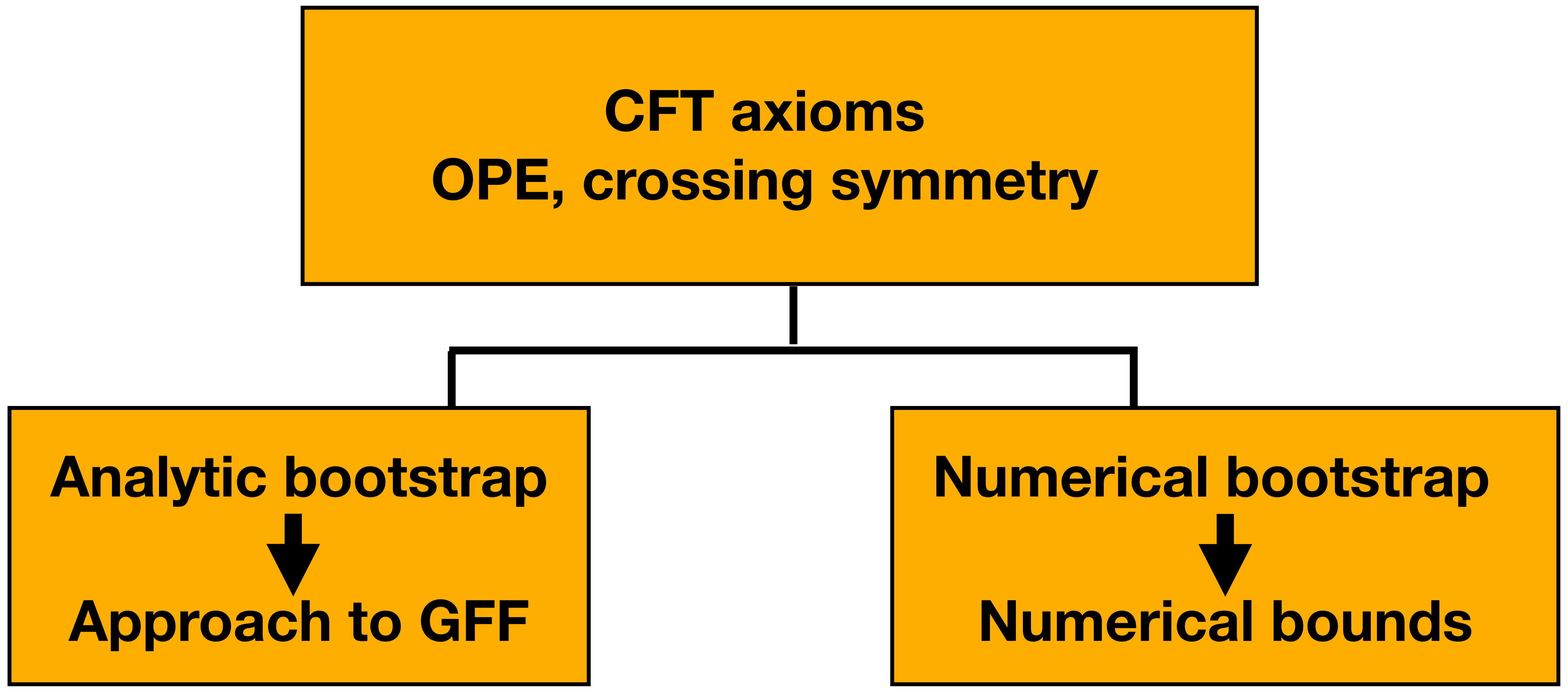
- Such integrals functionals are called analytic functionals
- They account for all analytic properties of blocks.
- **Related to block expansion coefficients of Witten diagrams (= Feynman diagrams in AdS)**
- Further established by dispersion relations for CFT
- An equivalence exists:
  CFT (in Mellin space) ↔ Scattering amplitudes (in momentum space)

Ongoing works + goals

- Bootstrapping CFTs using Witten diagrams
- Efficient implementation for multiple correlator + higher point correlators
- Systematic shrinking of allowed space of OPE data
- Can we solve theories completely without Lagrangian? ……

Summary

References:
EPFL lectures, Rychkov,
TASI lecture, Simmons-Duffin
Poland, Rychkov, Vichi, 2018



Ongoing works + goals

Thanks!

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