Reflected Entropy of Conformal Fields in Black Hole Background

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Outline

Entanglement Entropy

- Consider a bipartite quantum system $A \cup B$ described by the density matrix ρ_{AB}.
- ***Entanglement entropy (EE)*** : von Neumann entropy for the reduced density matrix $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$

$$S_{vN}(A) = -\,\mathrm{Tr}\,\left(\rho_A\ \log \rho_A\right).$$

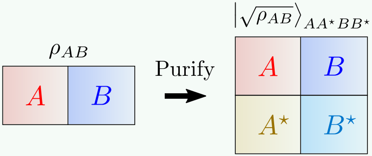
- In QFTs, EE may be obtained through the n-th Rényi entropy $S_n(A)$ as,

$$S_{vN}(A) = \lim_{n\to 1} S_n(A) = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}\,(\rho_A)^n\,.$$

- $A \cup B$ **in pure state :** EE is a good measure of entanglement.
- $A \cup B$ **in mixed state :** EE receives irrelevant classical as well as quantum correlations.
- We need another measure to properly quantify entanglement in a mixed state.

Reflected Entropy [Dutta, Faulkner: 19]

- Consider a bipartite mixed state $A \cup B$ described by the density matrix ρ_{AB}.
- The canonical purification of ρ_{AB} is given by the state $\left|\sqrt{\rho_{AB}}\right\rangle$ defined on the doubled Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{A^\star} \otimes \mathcal{H}_{B^\star}$.



- ***Reflected entropy*** : the von Neumann entropy of the reduced density matrix $\rho_{AA^*} = \mathrm{Tr}_{BB^*} \left|\sqrt{\rho_{AB}}\right\rangle \left\langle\sqrt{\rho_{AB}}\right|$,

$$S_R(A:B) \equiv S_{vN}(\rho_{AA^*})_{\sqrt{\rho_{AB}}}\,.$$

- For a pure state $A \cup B$: $S_R(A:B) = 2S(A)$.

Reflected Entropy in a CFT_2

- Replica technique : Construct a state $|\psi_m\rangle \equiv \left|\rho_{AB}^{m/2}\right\rangle$ on an m-replicated manifold. This state $|\psi_m\rangle$ could be understood as a purification of ρ_{AB}^m.
- Rényi reflected entropy $S_n(AA^*)_{\psi_m}$ is then defined as the Rényi entropy for the reduced density matrix $\rho_{AA^*}^{(m)} = \text{Tr}_{BB^*} |\psi_m\rangle \langle\psi_m|$.
- The Rényi reflected entropy may be obtained in terms the correlation functions of twist operators σ_{g_A} and σ_{g_B} inserted at the endpoints of the subsystems $A \equiv [z_1, z_2]$ and $B \equiv [z_3, z_4]$ as

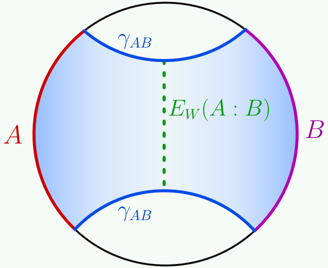
$$S_n\left(AA^\star\right)_{\psi_m} = \frac{1}{1-n} \log \frac{\left\langle \sigma_{g_A}(z_1)\sigma_{g_A^{-1}}(z_2)\sigma_{g_B}(z_3)\sigma_{g_B^{-1}}(z_4)\right\rangle_{\text{CFT}^{\otimes mn}}}{\left(\left\langle \sigma_{g_m}(z_1)\sigma_{g_m^{-1}}(z_2)\sigma_{g_m}(z_3)\sigma_{g_m^{-1}}(z_4)\right\rangle_{\text{CFT}^{\otimes m}}\right)^n}.$$

- Reflected entropy is obtained in the replica limit,

$$S_R(A:B) = \lim_{n,m\to 1} S_n(AA^*)_{\psi_m}.$$

Entanglement Wedge Cross Section

- ***Entanglement wedge*** : codimension-one bulk region bounded by the subsystem A and its RT surface γ_A.
- ***Entanglement wedge cross section (EWCS)*** : codimension-two surface with minimal area dividing the entanglement wedge for $A \cup B$.



- Holographic duality : [Dutta, Faulkner: 19]

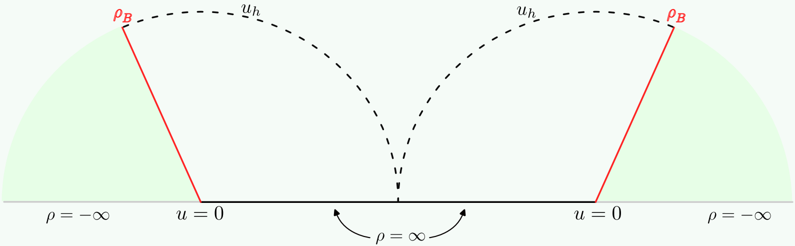
$$S_R(A:B) = 2E_W(A:B)\,.$$

- Begin with an eternal AdS$_3$ black string geometry truncated by an EOW brane:

$$\mathrm{d}s^2 = \cosh^2\rho \left[-\frac{\left(1-\frac{u}{u_h}\right)}{u^2}\mathrm{d}t^2 + \frac{\mathrm{d}u^2}{u^2\left(1-\frac{u}{u_h}\right)} \right] + \mathrm{d}\rho^2$$

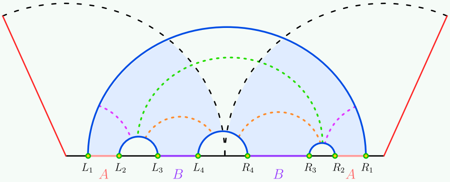
  with $\rho \in [\rho_B, \infty)$. The geometry on each constant ρ slice is an eternal AdS$_2$ black hole.

- The dual BCFT$_2$ is on one such eternal AdS$_2$ slice at the asymptotic boundary $\rho = \infty$ with conformal boundary conditions at $u = 0$.

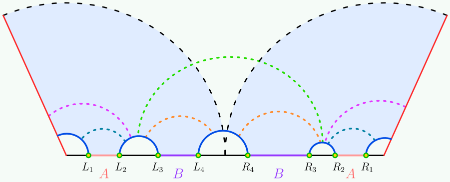


Reflected Entropy for BCFT on a Black Hole Background

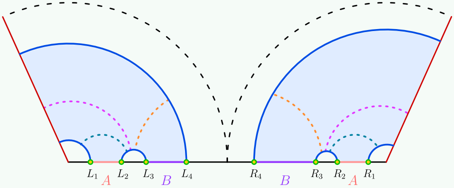
- Consider the generic configuration of two disjoint subsystems A and B.
- Four different EE phases for $A \cup B$ based on sizes and locations of the subsystems are investigated. Within each EE phase, different EWCS phases are depicted by dashed lines.



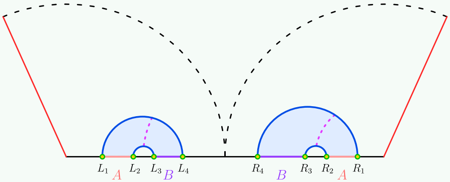
Phase - 1



Phase - 2



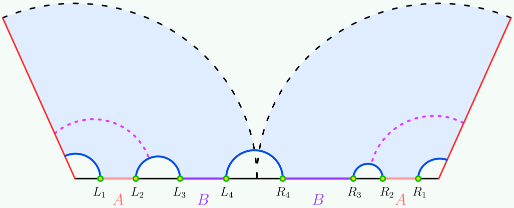
Phase - 3



Phase - 4

Reflected Entropy for BCFT on a Black Hole Background

Reflected entropy : Consider the subsystem A is close to the boundary while B is far away.



$$\langle \sigma_{g_A}(u_{L_1})\sigma_{g_A^{-1}}(u_{L_2})\sigma_{g_B}(u_{L_3})\sigma_{g_B^{-1}}(u_{L_4})\sigma_{g_B}(u_{R_4})\sigma_{g_B^{-1}}(u_{R_3})\sigma_{g_A}(u_{R_2})\sigma_{g_A^{-1}}(u_{R_1})\rangle_{\mathrm{BCFT}^{\otimes mn}}$$
$$= \langle \sigma_{g_A}(u_{L_1})\rangle\langle \sigma_{g_A^{-1}}(u_{R_1})\rangle\langle \sigma_{g_B^{-1}}(u_{L_4})\sigma_{g_B}(u_{R_4})\rangle\langle \sigma_{g_A^{-1}}(u_{L_2})\sigma_{g_B}(u_{L_3})\rangle\langle \sigma_{g_A}(u_{R_2})\sigma_{g_B^{-1}}(u_{R_3}).$$

By utilizing doubling trick and the form of the four point conformal block in the large central charge limit, the reflected entropy may be obtained as

$$S_R(A:B) = \frac{2c}{3}\log\left(\frac{1+\sqrt{1-\eta}}{\sqrt{\eta}}\right) + \frac{2c}{3}\log\left(\frac{1+\sqrt{1-\xi}}{\sqrt{\xi}}\right) + 4S_{\mathrm{bdy}},$$

where η and ξ are the cross ratios defined as

$$\eta = \frac{u_h\left(\sqrt{\Delta_{L_2}}-\sqrt{\Delta_{L_3}}\right)^2}{\left(u_h-\sqrt{\Delta_{L_2}}\sqrt{\Delta_{L_3}}\right)^2}, \qquad \xi = \frac{u_h\left(\sqrt{\Delta_{R_2}}-\sqrt{\Delta_{R_3}}\right)^2}{\left(u_h-\sqrt{\Delta_{R_2}}\sqrt{\Delta_{R_3}}\right)^2}.$$

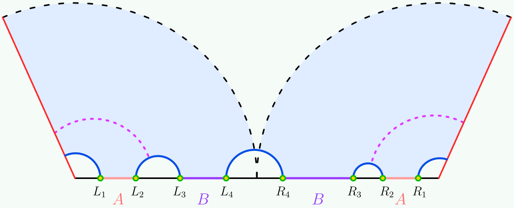
EWCS for AdS$_3$ Black String Geometry

EWCS : In the embedding space formalism for AdS$_3$ geometries, the EWCS for two disjoint intervals $A = [X_1, X_2]$ and $B = [X_3, X_4]$ is given by [Kusuki, Tamaoka: 2019]

$$E_W = \frac{1}{4G_N} \cosh^{-1}\left(\frac{1+\sqrt{u}}{\sqrt{v}}\right),$$

where u and v are defined in terms of $\xi_{ij} = -X_i \cdot X_j$ as,

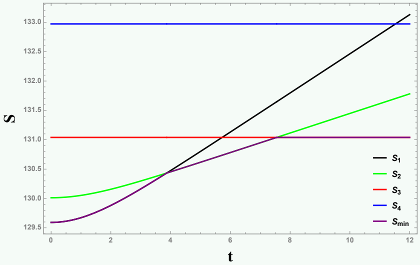
$$u = \frac{\xi_{12}\xi_{34}}{\xi_{13}\xi_{24}},\ v = \frac{\xi_{14}\xi_{23}}{\xi_{13}\xi_{24}}.$$

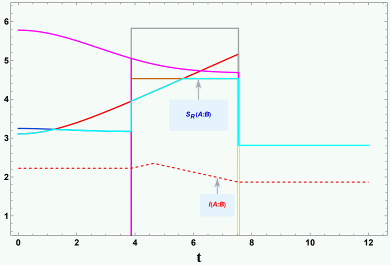


For the the configuration under consideration, we can compute the lengths of the left and right geodesic separately through the above prescription to obtain the EWCS.

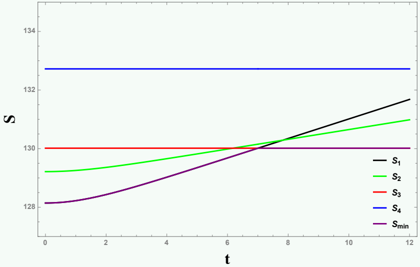
Page Curves

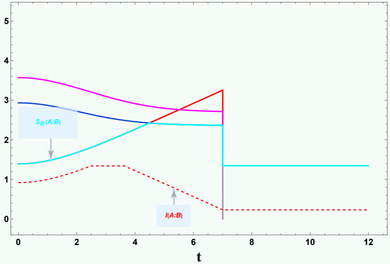
Case I : Subsystem A close to the boundary





Case II : Subsystem A far away from the boundary





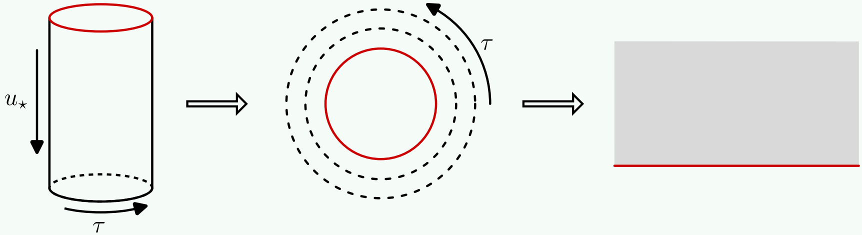
Summary

- We have investigated the mixed state entanglement of disjoint subsystems through the reflected entropy, in a KR braneworld model with the BCFT_2 located in a gravitational background.
- We have computed the reflected entropy and the bulk EWCS, and verified their holographic duality in this model.
- We plotted the Page curves for the reflected entropy for different values of the parameters and observed phase transitions.

Thank you

Backup: Holographic $BCFT_2$ on a Black Hole Background

- The computation of entanglement entropy (EE) in such $BCFT_2$s require transforming the twist field correlators to conformally flat geometry. To achieve this, we first map the eternal AdS_2 BH to a semi-infinite thermal cylinder described by the coordinates $(u_\star, \tau)$ with $\tau \sim \tau + \beta$. Subsequently through a series of transformations, it can be mapped to UHP.



- The dual bulk geometry described by the AdS_3 black string may be embedded as a codimension-one surface in $\mathbb{R}^{2,2}$ through a given parametrization.