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Decomposition of Anomalous Diffusion in Variable Speed Generalized Lévy Walks

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Diffusive behavior

- Governed by the Central Limit Theorem (CLT)
- $t \to \infty$, the PDF of $x(t)$ is Gaussian
- Width of the distribution$\sim t^{\frac{1}{2}}$

Anomalous Diffusion

- First found by Lewis Richardson in 1926 (Relative motion of tracer particles in turbulence)
- $\langle x^2(t) \rangle \sim t^{2H}$ and $H \neq \frac{1}{2}$
- H is known as Hurst exponent.

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Examples:

- The diffusion of tracer microbeads in reconstituted, cross-linked actin networks [Wong et al. *Physical Review Letters* (2004)]
- The motion of functionalized colloidal particles along a complementarily functionalized surface [Xu et al. *Physical Review Letters* (2011)]
- The stochastic pathway of potassium channels diffusing in the plasma membrane of living human cells [Weigel et al. *Proceedings of the National Academy of Sciences* (2011)]
- Diffusion of tracer particles in the cytoplasm of mammalian cells [Sabri et al. *Physical Review Letters* (2020)]
- Blinking quantum dots [Stefani, Hoogenboom, Barkai. *Phys. Today* (2009)]
- Chaotic transport in zonal flows in analogous geophysical and plasma systems [Diego et al. *Physics of Plasmas* 7, 1702 (2000)]
- Scaling laws of marine predator search behaviour[Sims, D., Southall, E., Humphries, N. et al. *Nature* 451, 1098–1102 (2008)]

The root causes of anomalous diffusive behavior [1] [2] [3] .

- Joseph effect (J) caused by increment correlations. $\left\langle\left\langle\Delta x^2(\tau)\right\rangle_t\right\rangle_E \sim \tau^{2J}$
- Noah effect (L) caused by fat tails of increment probability. $\left\langle \mathrm{v}^2 \right\rangle \propto t^{2L+2M-2}$
- Moses effect (M) caused by non-stationary increments. $\left\langle \mathrm{v} \right\rangle \propto t^{M-\frac{1}{2}}$

It can be possible to show that $\mathrm{H} = \mathrm{J} + \mathrm{L} + \mathrm{M} - 1$

[1] Benoit B. Mandelbrot and James R. Wallis. "Noah, Joseph, and operational hydrology." Water resources research 4.5 (1968): 909-918.

[2] C. Lijian, K. E. Bassler, J. L. McCauley, and G. H. Gunaratne, Anomalous scaling of stochastic processes and the Moses effect, *Phys. Rev. E* 95, 042141 (2017)

[3] P. G. Meyer, V. Adlakha, H. Kantz and K. E. Bassler, Anomalous diffusion and the Moses effect in an aging deterministic model, *New J. Phys.* 20, 113033 (2018)

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In many cases, the dynamics in these systems can be modeled with non-linearly coupled Lévy Walks.

- Steps of random duration τ chosen from a probability distribution $p(\tau) \sim \tau^{-1-\gamma}$
- During each step, i, the direction (i.e. forward or backward) is random, but the magnitude of velocity $|\overrightarrow{\mathrm{v}_i}| \propto \tau_i^{\nu-1}$

An example of a Lévy walk path with parameter $\gamma = 0.52, \nu = 0.5$. [1]

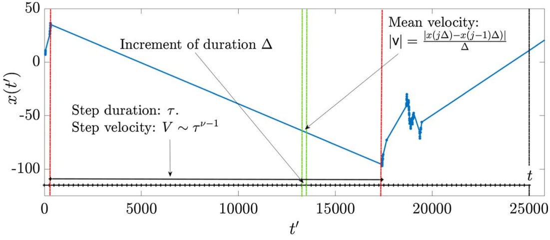

Figure: Example of Lévy Walks.

[1] E. Aghion, P. G. Meyer, V. Adlakha, H. Kantz and K. E. Bassler, Moses, Noah and Joseph effects in Lévy Walks, *New J. Phys.* 23, 023002 (2021)

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Generalized Lévy walks

- Velocity is a piecewise constant function.

Variable speed generalized Lévy walks

- Velocity is not fixed throughout a flight.
- $\mathrm{v}\left(t_f, t_B\right) = \eta c t_B^{\nu-\eta} t_f^{\eta-1}$
  Where t_f is intended stretch duration and t_B is on the actual time in motion[1]

[1] M. Bothe, F. Sagues, and I. M. Sokolov. Mean squared displacement in a generalized Lévy walk model. *Physical Review E* 100.1 (2019)

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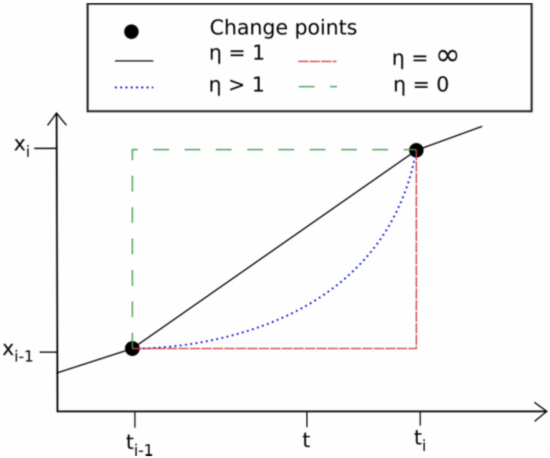

Figure: Example of Variable speed generalized Lévy walks[1]

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Hurst Exponent (H)

$$\left\langle \Delta \mathbf{x}^2(t) \right\rangle_{\mathrm{E}} = \left\langle [\mathbf{x}(t) - \mathbf{x}(0)]^2 \right\rangle_{\mathrm{E}} \sim t^{2H} \quad (1)$$

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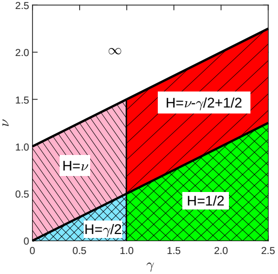
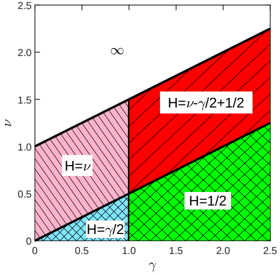


Figure: Phase diagram of H exponent for generalized Lévy walks [a]

[a] Albers, T. and Radons, G., 2022. Nonergodicity of d-dimensional generalized Lévy walks and their relation to other space-time coupled models. Physical Review E, 105(1), p.014113.



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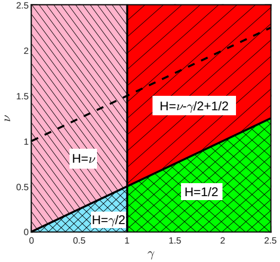

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Joseph Exponent (J)

$$\left\langle \Delta \mathbf{x}^2(\tau) \right\rangle_{\mathrm{T}} = \frac{1}{T-\tau} \int_0^{T-\tau} [\mathbf{x}(t+\tau) - \mathbf{x}(t)]^2 dt \sim \tau^{2J} \quad (2)$$

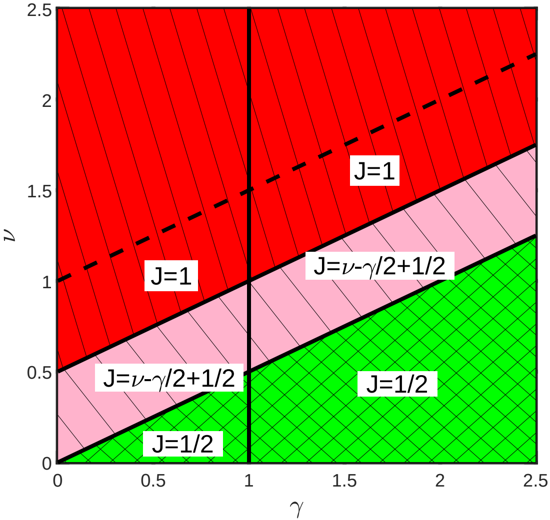

Figure: Phase diagram of J exponent for variable speed generalized Lévy walks [1]

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Moses exponent (M) and Noah exponent (L)

$$\langle|\mathrm{v}|\rangle \propto t^{M-\frac{1}{2}} \tag{3}$$

$$\left\langle \mathrm{v}^2 \right\rangle \propto t^{2L+2M-2} \tag{4}$$

$$\langle \mathrm{v}^2 \rangle = \int \mathrm{v}^2 p(\mathrm{v}, t) d\mathrm{v} \tag{5}$$

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Based on the value of the parameter ν and η, $p(\mathrm{v}, t)$ can be categorized as of 4 types for Variable speed generalized Lévy walks

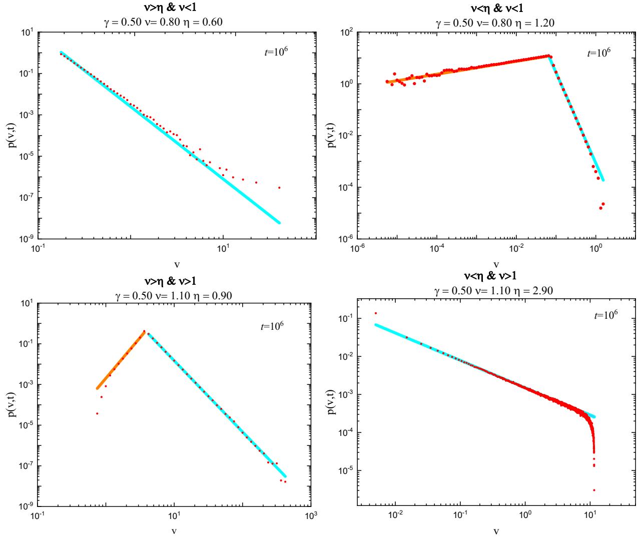

Figure: Numerical and Analytical estimation of $p(\mathrm{v}, t)$ for $\gamma < 1$. Symbols are the results of numerical simulations. Solid lines are the theories,

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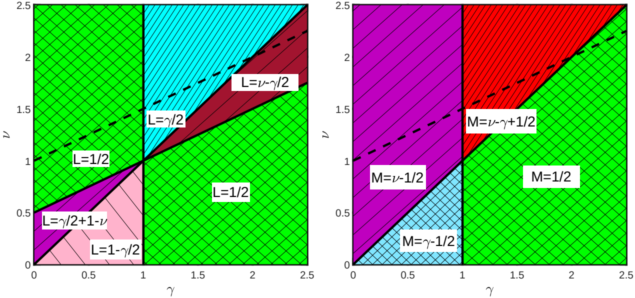

Figure: L and M exponent value for Variable speed generalized Lévy walks

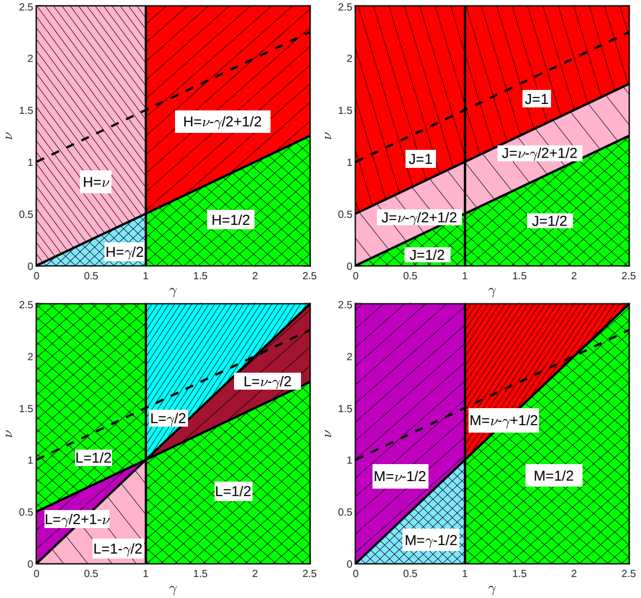

Figure: H,J,L and M exponent value for Variable speed generalized Lévy walks

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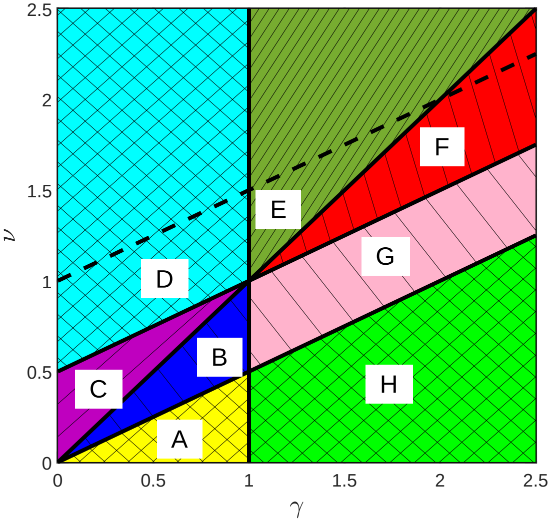

Figure: Phase decomposition for Variable speed generalized Lévy walks

References:

- C. Lijian, K. E. Bassler, J. L. McCauley, and G. H. Gunaratne,"Anomalous scaling of stochastic processes and the Moses effect", *Phys. Rev. E* 95, 042141 (2017)
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- A. Bera and K. E. Bassler, in preparartion.

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THANK YOU