Field Theory & the EW Standard Model
Part I: QFT in a nutshell

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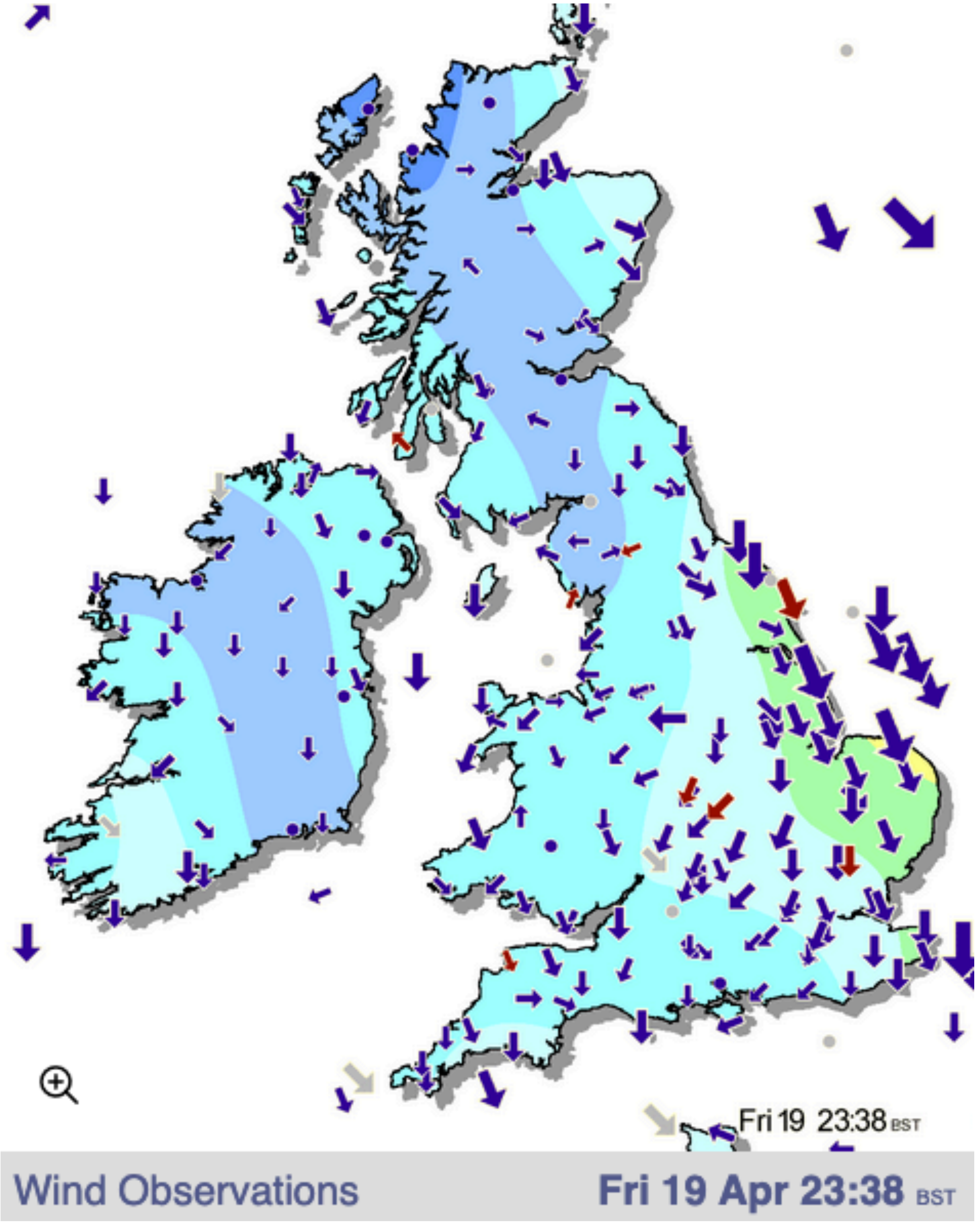
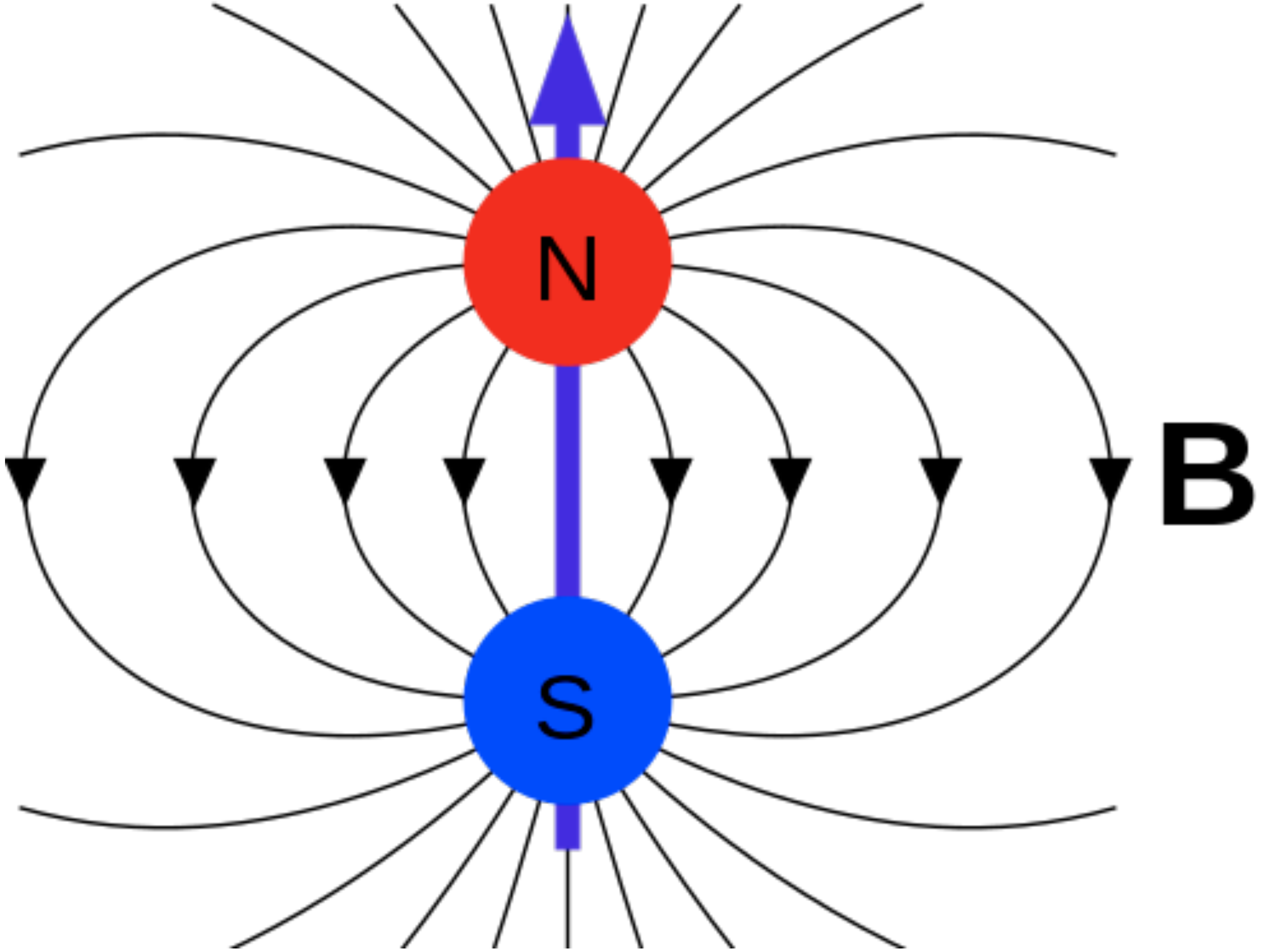
Field

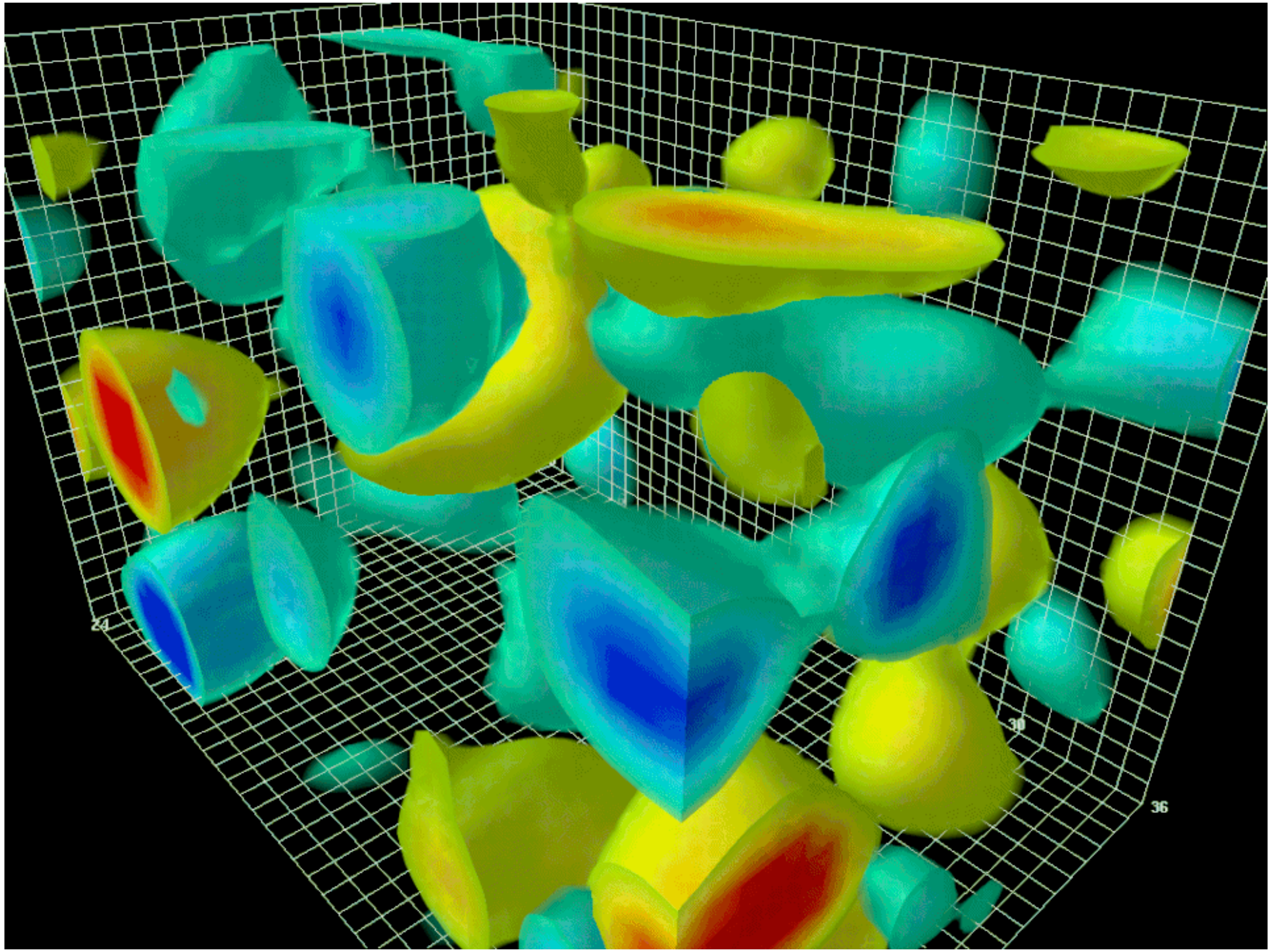
Field



Field Theory



Quantum Field Vacuum



Motivation for QFT

Key motivation: consistent combination of Quantum Mechanics + Special Relativity

Naive combination: relativistic quantum mechanics describes system of fixed number of particles

Schrödinger eq.:

$$i\partial_t \phi(t,\boldsymbol{x}) = \left(-\frac{1}{2m}\nabla^2 + V(\boldsymbol{x})\right)\phi(t,\boldsymbol{x}) = \hat{H}\,\phi(t,\boldsymbol{x})$$

plane wave solutions:

$$\phi(t,\boldsymbol{x}) \propto e^{-i(Et-\boldsymbol{p}\cdot\boldsymbol{x})} = e^{-ip\cdot x}$$

classical energy-momentum relation:

$$E = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})\,.$$

Klein-Gordon eq.:

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi(t,\boldsymbol{x}) = \left(\partial_\mu\partial^\mu + m^2\right)\phi(x) = (\Box + m^2)\phi(x) = 0\,,$$

plane wave solutions:

$$\phi(t,\boldsymbol{x}) \propto e^{-i(Et-\boldsymbol{p}\cdot\boldsymbol{x})} = e^{-ip\cdot x}$$

relativistic energy-momentum relation:

$$E^2 = m^2 + \boldsymbol{p}^2$$

Problem: **negative energy solutions** $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$

spectrum not bounded from below

Notation/Conventions/SR recap

$$c = \hbar = 1$$
$$\mu, \nu = 0,1,2,3 \text{ (greek indices)}$$
$$i, j = 1,2,3 \text{ (latin indices)}$$

Metric tensor is:

$$\eta_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Space-time coordinate **4-vector**: $x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}$

Einstein's summation convention

Lorentz transformation: $x^{'\mu} = \sum_{\nu=0}^{3} \Lambda^\mu_\nu \, x^\nu \equiv \Lambda^\mu_\nu \, x^\nu$

Lorentz boost in x-direction:

$$\underbrace{\begin{pmatrix} t' \\ x' \\ y' \\ z \end{pmatrix}}_{4-\text{vector } x'} = \underbrace{\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{4x4 \text{ matrix } \Lambda} \underbrace{\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}}_{4-\text{vector } x}$$

Use metric to raise/lower indices: $x_\mu = \eta_{\mu\nu} x^\nu \quad x^\mu = \eta^{\mu\nu} x_\nu$

Notation/Conventions/SR recap

Invariant space-time interval: $t^2 - \vec{x}^2 = x^\mu x^\nu \eta_{\mu\nu} = x^\mu \eta_{\mu\nu} x^\nu = x^T \eta x = x^\mu x_\mu \equiv x^2$

Covariant 4-vector: $x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$

Lorentz invariant scalar product: $a \cdot b = ab \equiv a^\mu b_\mu = a^\mu \eta_{\mu\nu} b^\nu$.

relativistic energy-momentum relation

Important examples of 4-vectors:

- **4-momentum** $p^\mu = (E, \vec{p})$, i.e. $\boxed{p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 \equiv m^2}$

  “Mass" is the "length" of the 4-momentum.

  Note: $p \cdot x = p^\mu x_\mu = Et - \vec{p}\vec{x}$ is invariant (regularly used in QFT)

- **4-derivative** $\partial_\mu = (\partial_0, \partial_i)$ is a covariant 4-vector, i.e. $\partial^\mu = (\partial_t, -\partial_i)$,

  $\partial_\mu \partial^\mu = \partial_t^2 - \Delta = \Box$ and $\partial_\mu p^\mu(x) = \partial_0 p^0 + \partial_i p^i$

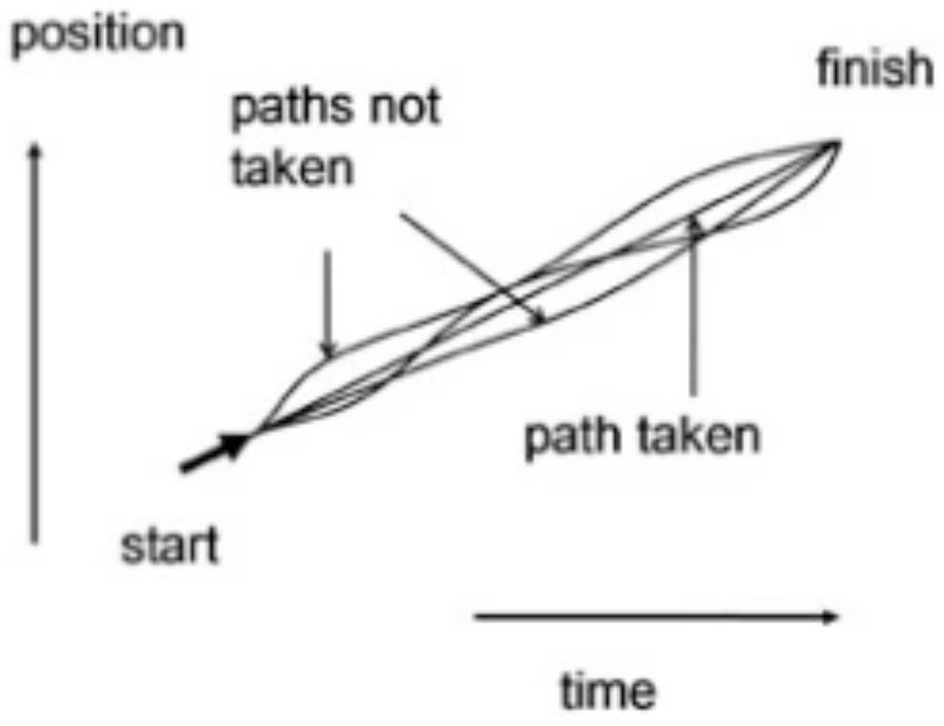
Lagrange formalism in classical mechanics

Classical mechanics can be formulated as a *least-action principle*.

Action: $S[x(t)] = \int_{t_A}^{t_B} L(x(t), \dot{x}(t), t)\, dt$

with Lagrange function $L = L(x(t), \dot{x}(t), t) = T(x, \dot{x}, t) - V(x, t) =$ kinetic energy − potential energy

Classical path such that: $\delta S[x(t)] = S[x(t) + \delta x(t)] - S[x(t)] = 0$



Equivalent with **Euler-Lagrange** (EL) equation:

$$\delta S[x(t)] = \int_{t_A}^{t_B} \delta L(x(t), \dot{x}(t), t) dt = \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} \delta x(t) + \frac{\partial L}{\partial \dot{x}} \delta \dot{x}(t) \right) dt$$

$$= \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x(t) dt + \frac{\partial L}{\partial \dot{x}} \delta x(t) \Big|_{t_A}^{t_B} \overset{!}{=} 0$$

IBP

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0}$$

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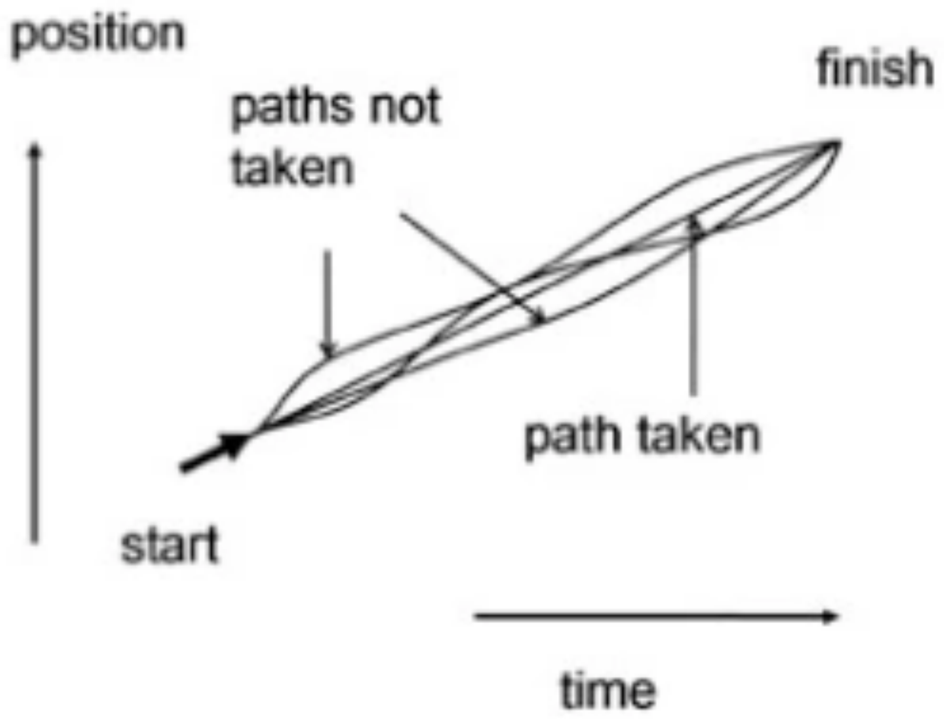

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IBP

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0}$$

Example:

$$L = \frac{m^2}{2} \dot{x}^2 - V(x) \rightarrow \frac{d}{dt} m\dot{x} + \frac{\partial V(x)}{\partial x} = m\ddot{x} + \frac{\partial V(x)}{\partial x} = 0\,.$$

$$\rightarrow m\ddot{x} = - \frac{\partial V(x)}{\partial x} = F(x)$$

Hamilton formalism in classical mechanics

Based on Lagrange function $L = L(q_i(t), \dot{q}_i(t), t)$ for a set of *generalised coordinates* q_i

define *generalised momenta:* $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$.

Aim to treat q_i, p_i as dynamical variables (instead of $q_i, \dot{q}_i$).

For this define transformation $H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

EL is equivalent to the Hamilton e.o.m:

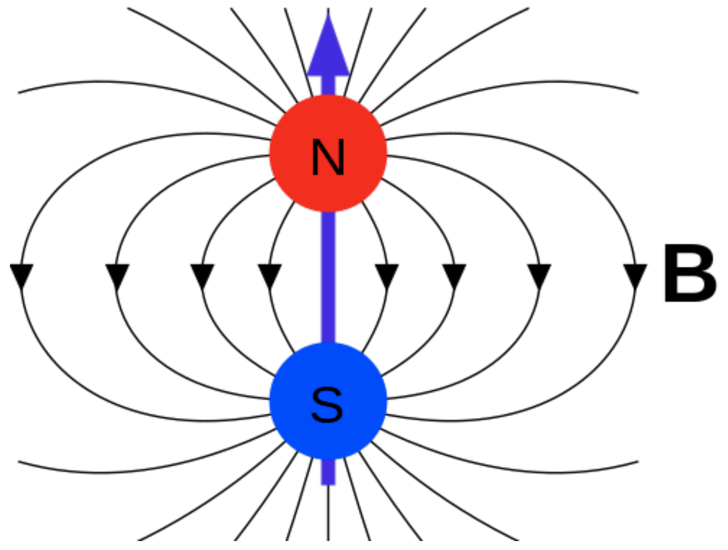
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial q_i},$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

The time-dependance of an observable $f = f(q_i, p_i, t)$ is then given by $\boxed{\dfrac{df}{dt} = \{f, H\} + \dfrac{\partial f}{\partial t}}$

Poisson bracket: $\{f, g\} = \sum_{i=1}^{N} \left(\dfrac{\partial f}{\partial q_i} \dfrac{\partial g}{\partial p_i} - \dfrac{\partial f}{\partial p_i} \dfrac{\partial g}{\partial q_i} \right)$

Least-action principle for classical fields



In classical field theory a field value is associated to every point in space. For a scalar field $\phi(\vec{x},t)$ this is a scalar value, while a vector field $A^\mu(\vec{x},t)$ associates a 4-vector to every point in space.

In order to formulate an action-principle for a field theory it is crucial to see the field itself as dynamical variable, while $\vec{x}$ plays the role of a label.

Least-action principle for field $\phi(\vec{x},t)$: $\delta S[\phi] = \int d^4x\, \mathscr{L}(\phi, \partial_\mu\phi, x) = 0$

Lagrange density

$$L = \int d^3x\, \mathscr{L}$$

is equivalent with EL for $\phi(\vec{x},t)$: $\boxed{\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0}$

As in classical mechanics we can define a *conjugated momentum field* $\pi(\vec{x},t) = \dfrac{\partial \mathscr{L}}{\partial(\partial_0\phi)}$

and a ***Hamilton density*** $\mathscr{H}(\phi,\pi,x) = \pi\partial_0\phi - \mathscr{L}(\phi,\partial_\mu\phi,x)$

Quantisation

Classical Mechanics

- observables: $q_i, p_i, f(x_i, p_i)$
- Poisson bracket
  $\{q_i, p_j\} = \delta_{ij}$
  $\{q_i, q_j\} = \{p_i, p_j\} = 0$

Quantum Mechanics

- operators: $\hat{x}_i, \hat{p}_i, \hat{f}(\hat{x}_i, \hat{p}_i)$
- Commutators
  $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$
  $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

remember: $\hbar = 1$

Field Quantisation

Classical Fields

- fields: $\phi, \pi, f(\phi, \pi)$
- Poisson bracket (at equal time)
  $\{\phi(t, \vec{x}), \pi(t, \vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})$
  $\{\phi(t, \vec{x}), \phi(t, \vec{y})\} = \{\pi(t, \vec{x}), \pi(t, \vec{y})\} = 0$

Quantum Fields

- Quantum fields: $\hat{\phi}(x), \hat{\pi}(x), \hat{f}(\hat{\phi}(x), \hat{\pi}(x))$
- Commutators (at equal time)
  $[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$
  $[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = [\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0$

Free scalar field

Lagrangian for a free real scalar field, describing neutral spin=0 particles with mass m: $\boxed{\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2}$

Euler-Lagrange for ϕ yields: $\boxed{(\partial_\mu\partial^\mu + m^2)\,\phi = (\Box + m^2)\,\phi = 0}$ ***Klein-Gordon equation***

This is a wave equation! → plane-wave solution as general ansatz

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \left[a(k)\,e^{-ikx} + a^*(k)\,e^{ikx}\right]$$

ensures Lorentz-invariance

momentum-space coefficients

This solves the Klein-Gordon equation for $k^0 = \sqrt{\vec{k}^2 + m^2}$ ← relativistic energy-momentum relation

Having both $a(k)$ and $a^*(k)$ ensures that $\phi(x)$ remains real.

Now we need to quantise this solution!

Free scalar field

General solution of KG equation: $\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \left[a(k)\, e^{-ikx} + a^*(k)\, e^{ikx}\right]$

Determine associated conjugate momentum field $\pi(\vec{x}, t)$, interpret $\hat{\phi}(x)$ and $\hat{\pi}(x)$ as operators demanding

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$
$$[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0$$

We thus have to promote $a(k) \rightarrow \hat{a}(k)$ and $a(k)^* \rightarrow \hat{a}^\dagger(k)$ to operators with commutators

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3\, \delta^{(3)}(\vec{k} - \vec{k}')$$
$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = 0$$

This is the algebra of a **simple harmonic oscillator** (SHO)!!

As for the SHO $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ can be interpreted as ladder operators that create and annihilate one-particle states:

$$a^\dagger(k)\,|0> \; = |k>$$ **creation operator**

$$a(k)\,|k'> \; = 2E_k\,\delta^{(3)}(\vec{k} - \vec{k}')\,|0>$$ **annihilation operator**

SHO in QM

Hamiltonian of SHO: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} x^2$

The Schrödinger eq. $i\hbar \frac{\mathrm{d}}{\mathrm{dt}} |\psi(t)> = \hat{H} |\psi(t)>$ can be solved algebraically introducing ladder operators

$\hat{a} = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{x} + \frac{i}{\sqrt{m\omega}} \hat{p} \right)$ creation operator

$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{x} - \frac{i}{\sqrt{m\omega}} \hat{p} \right)$ annihilation operator

In terms of these ladder operators the Hamiltonian reads $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \rightarrow \hat{H} |n> = E_n |n>$

And we have $[\hat{a}, \hat{a}^\dagger] = 1,\ [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$

$[\hat{H}, \hat{a}^\dagger] = \hbar\omega,\ [\hat{H}, \hat{a}] = -\hbar\omega$

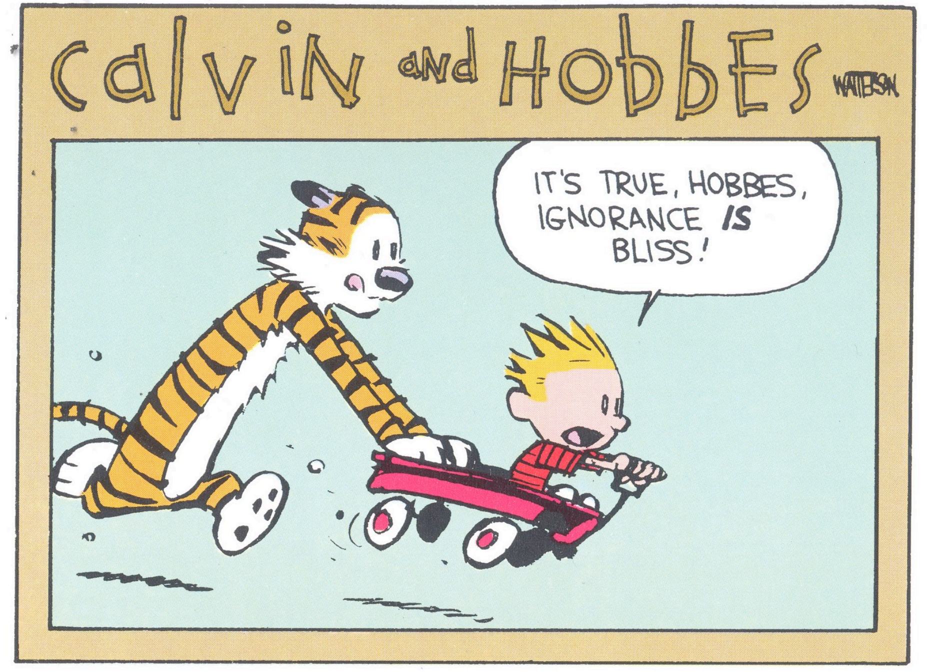
$(a^\dagger)^n |0> \sim |n>$ **creation operator**

$a |n> \sim |n-1>$ **annihilation operator**

Back to the scalar field

$a^\dagger(k)\,|\,0> \;= |\,k>$ **creation operator**

$a(k)\,|\,k'> \;= 2k^0\,\delta^{(3)}(\vec{k}-\vec{k}')\,|\,0>$ **annihilation operator**



In terms of these operators we find the Hamiltonian of the free scalar field as

$$H = \int d^3k\,\mathscr{H} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left(\hat{a}^\dagger(k)\hat{a}(k) + \frac{1}{2}\left[\hat{a}(k),\hat{a}^\dagger(k)\right]\right)$$

infinite number of SHOs

$\sim (2\pi)^3\delta^{(3)}(0) \rightarrow \infty$

infinite ground-state energy → ignore

formally: “normal ordering”

Vacuum state: $\hat{a}(k)\,|\,0> = 0$ and $<0\,|\,0> = 1$

As for SHO states $a^\dagger(\vec{k})\,|\,0>$ are eigenstates of the Hamiltonian $\hat{H}$: $\hat{H}\,\hat{a}^\dagger(\vec{k})\;|0> = E_k\,a^\dagger(\vec{k})\;|0>$

Generic n-particle state: $|\,\vec{k}_1\ldots\vec{k}_n> = (2E_{k_1})^{1/2}\ldots(2E_{k_n})^{1/2}\;\hat{a}^\dagger(\vec{k}_1)\ldots a^\dagger(\vec{k}_n)\;|0>$

Note: we have $|\,\vec{k}_1\vec{k}_2> \sim \hat{a}^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)\,|\,\vec{0}> = |\,\vec{k}_2\vec{k}_1>$ → Bose-Einstein statistics → scalar field is a **boson**

$[\hat{a}^\dagger(\vec{k}),\hat{a}^\dagger(\vec{k}')] = 0$

Particles ↔ Fields

The **field is a superposition of all possible momentum modes**.
Thus, the field contains all freedom to describe all possible configurations of one or more particles in a given momentum state.

particles = field excitations

Location of particles

we can define state $|\vec{x}> = \hat{\phi}(0,\vec{x})\,|\,0> \; = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int \dfrac{\mathrm{d}^3k}{2k^0}\,[\hat{a}(k)\,e^{+i\vec{k}\cdot\vec{x}} + \hat{a}^{\dagger}(k)\,e^{-i\vec{k}\cdot\vec{x}}\,]\,|\,0>$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} e^{-i\vec{k}\cdot\vec{x}}\,|\,\vec{k}> \qquad \text{where} \quad |\,\vec{k}> = \hat{a}^{\dagger}(\vec{k})\,|\,0>$$

i.e. $|\,\vec{x}>$ is a superposition of single-particle states that have well defined momentum and energy.

Interpretation: $\hat{\phi}(0,\vec{x})$ **field operator acts on the vacuum and creates a particle** at position $\vec{x}$. That particle does not have a unique momentum, but the probability to find it with momentum $\vec{k}$ is given by

$<0\,|\,\hat{\phi}(0,\vec{x})\,|\,\vec{k}> \sim e^{+i\vec{k}\cdot\vec{x}}$ **incoming state** – – – – – – – •

$<\vec{k}\,|\,\hat{\phi}(0,\vec{x})\,|\,0> \sim e^{-i\vec{k}\cdot\vec{x}}$ **outgoing state** • – – – – – – –

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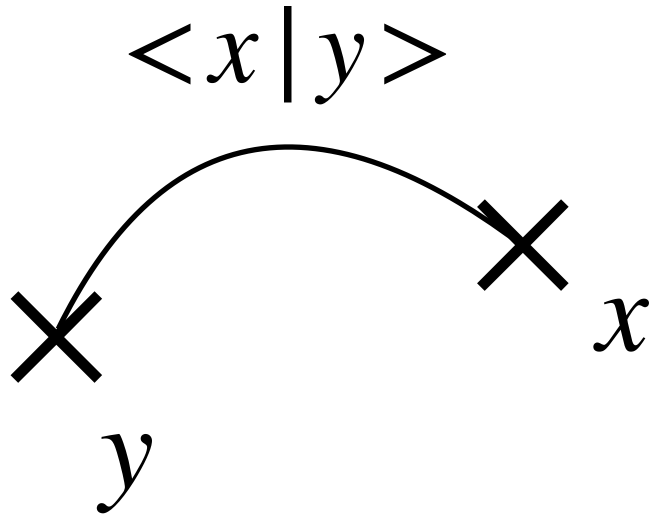
$|x> = \hat{\phi}(t,\vec{x})\,|\,0>$

$<0\,|\,\hat{\phi}(0,\vec{x})\,|\,\vec{k}> \sim e^{+i\vec{k}\cdot\vec{x}}$ **incoming state** – – – – – – – • $<0\,|\,\hat{\phi}(t,\vec{x})\,|\,k> \sim e^{-ik\cdot x}$

$<\vec{k}\,|\,\hat{\phi}(0,\vec{x})\,|\,0> \sim e^{-i\vec{k}\cdot\vec{x}}$ **outgoing state** • – – – – – – – $<k\,|\,\hat{\phi}(t,\vec{x})\,|\,0> \sim e^{+ik\cdot x}$

Feynman Propagator



Based on $|\vec{x}>$ we can also define a state $|x> = \hat{\phi}(x)|0> = \hat{\phi}(t,\vec{x})|0>$

Amplitude for the propagation from y to x: $<x|y>$

$$= <0|\hat{\phi}(x)\hat{\phi}(y)|0> \equiv D(x,y)$$

$$= D(x-y) = \ldots = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}e^{-ik\cdot(x-y)}$$

$D(x,y)$ only depends on $x-y$: only the distance matters.

In order to ensure **causality** we need to further refine this picture and define the **Feynman Propagator**

$$D_F(x-y) = \begin{cases} D(x-y) & \text{if} \quad x^0 > y^0 \\ D(y-x) & \text{if} \quad y^0 > x^0 \end{cases} = D(x-y)\Theta(x^0-y^0) + D(y-x)\Theta(x^0-y^0) = <0|\hat{T}\hat{\phi}(x)\,\hat{\phi}(y)|0>$$

where we make use of the time-ordering operator $\hat{T}$:

$$\hat{T}\hat{\phi}(x)\,\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & \text{if} \quad x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x) & \text{if} \quad y^0 > x^0 \end{cases}$$

The Feynman propagator is in essential ingredients of the Feynman rules needed to compute Feynman diagrams.

Momentum-space Feynman Propagator

The Feynman propagator is a Green's function of the inhomogeneous Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\, D_F(x-y) = -\,\delta^4(x-y)$$

Solutions to this differential equation can be obtained via Fourier transformation:

$$D_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, D(k)\, e^{-ik(x-y)}$$

In Fourier/momentum-space the inhomogeneous Klein-Gordon equation reads:

$$(k^2 - m^2)\, D(k) = 1$$

remember:

$$\delta^{(4)}(x-y) = \int \frac{d^4 k}{(2\pi)^4} e^{-i\,k\cdot(x-y)}$$

With the **momentum-space Feynman propagator** as solution:

convention

$$\boxed{i\, D(k) = \frac{i}{k^2 - m^2 + i\epsilon}}$$

ensures time-ordering i.e. causality

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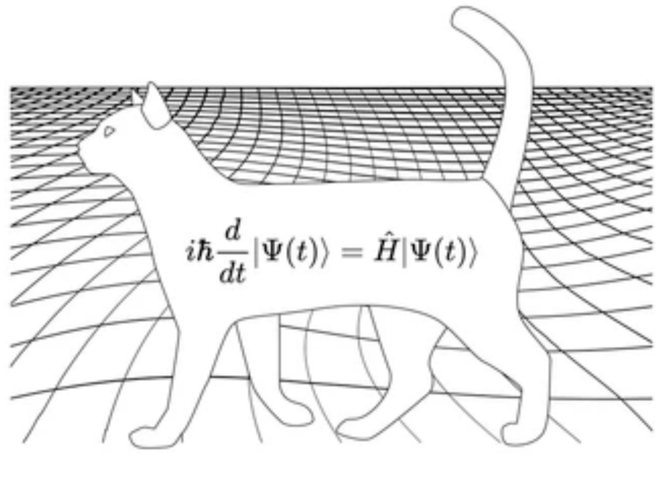
We can see this via

$$iD_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2k^0} \left[e^{-ik(x-y)}\, \Theta(x^0 - y^0) + e^{ik(x-y)}\, \Theta(y^0 - x^0) \right]_{k^0 = \sqrt{\vec{k}^2 + m^2}} = <0|\, \hat{T}\, \hat{\phi}(x)\, \hat{\phi}(y)\, |0>$$

$d^0 k$ integral via contour in lower/upper half-plane

Quantum Pictures



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Schrödinger picture:

- states $|\phi_S(t)>$ are **time-dependent**: $|\phi_S(t)> = e^{-i\hat{H}_S(t-t_0)}|\phi_S(t_0)> = U(t,t_0)|\phi_S(t_0)>$
- operators $\hat{A}_S$ are **time-independent**

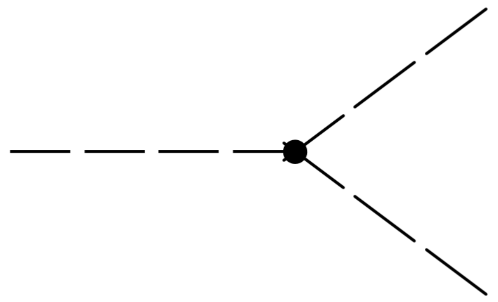
time-evolution operator

Heisenberg picture:

- states $|\phi_H> = |\phi_S(t_0)>$ are **time-independent**
- operators $\hat{A}_H(t)$ **time-dependent**: $\hat{A}_H = U^\dagger(t,t_0)\,\hat{A}_S\,U(t,t_0)$

Quantum Pictures

Interaction picture:



- separate $\boxed{\hat{H} = \hat{H}_0 + \hat{H}_I}$
- states $|\phi_I(t)>$ are **time-dependent**: $|\phi_I(t)> = e^{i\hat{H}_0(t-t_0)}|\phi_S(t)> = \hat{U}_0^\dagger(t,t_0)|\phi_S(t)>$
  $= e^{-i\hat{H}_I(t-t_0)}|\phi_S(t_0)> = \hat{U}_I(t,t_0)|\phi_S(t_0)>$

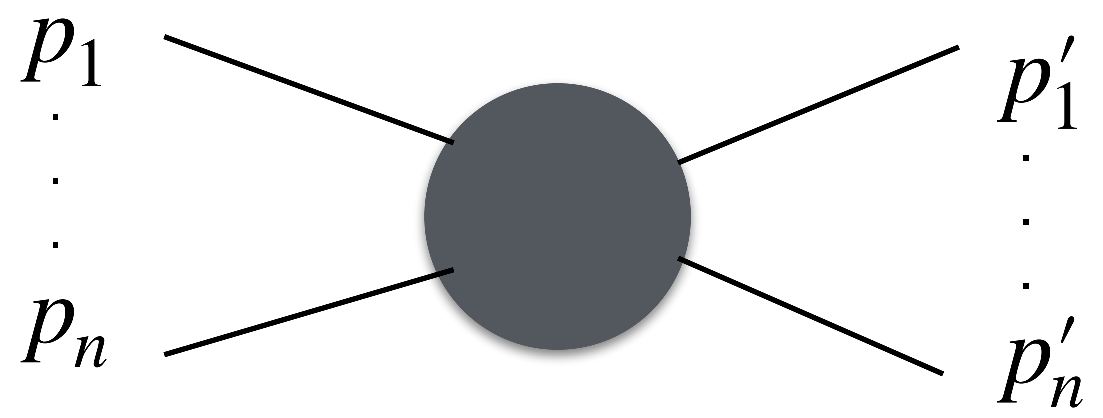
  → states evolve with interaction Hamiltonian $\hat{H}_I$
- operators $\hat{A}_I(t)$ **time-dependent**: $\hat{A}_I = \hat{U}_0^\dagger(t,t_0)\,\hat{A}_S\,\hat{U}_0(t,t_0)$

  → operators evolve with free Hamiltonian $\hat{H}_0$

To be precise: $\boxed{\hat{U}_I(t,t_0) = \hat{T}\,e^{-i\int_{t_0}^{t}\hat{H}_I(t')dt'}}$ as a solution of $i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\,\hat{U}(t,t_0)$

S-matrix

Ultimately we want to compute **cross sections for scattering processes**, i.e. probabilities for



free in-states interactions free out-states

$$|\mathrm{in}> = |\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{n}; \mathrm{in}> = |\phi(\mathrm{t} = -\infty)> \quad \longrightarrow \quad |\mathrm{out}> = = |\mathrm{p}'_1, \ldots, \mathrm{p}'_\mathrm{n}; \mathrm{out}> = |\phi(\mathrm{t} = +\infty)>$$

In interaction picture free in-state evolves in interaction region: $|\phi(t)> = U_I(t, -\infty)|\mathrm{in}>$

The projection of this state $|\phi(t)>$ onto the out-state defines the **S-matrix** element

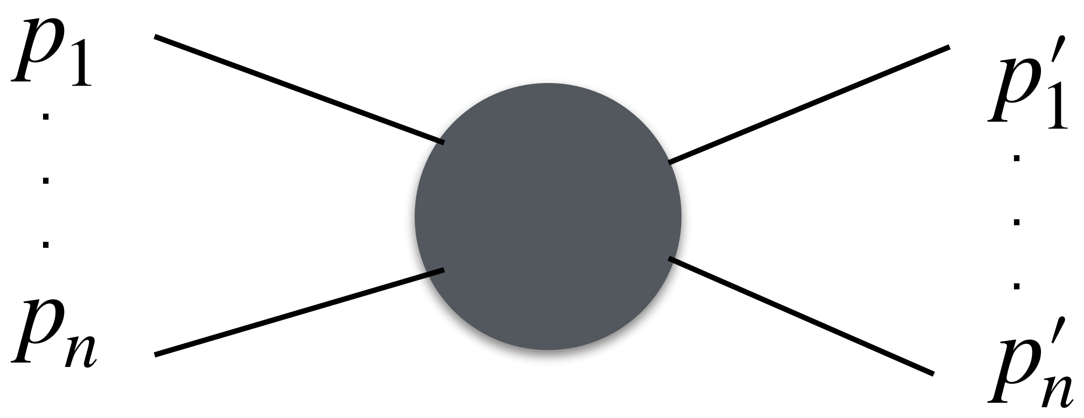
$$S_{\mathrm{fi}} = <\mathrm{f}|\hat{\mathrm{S}}|\mathrm{i}> = \lim_{\mathrm{t}\to+\infty} <\mathrm{f}|\phi(\mathrm{t})> = <\mathrm{out}|\mathrm{U_I}(+\infty, -\infty)|\mathrm{in}>$$

$$\to \hat{S} = \hat{U}(+\infty, -\infty) = \hat{T}\, e^{-i\int_{-\infty}^{\infty} \hat{H}_I(t')dt'}$$

Note: for $\hat{H}_I = 0 \to S = \mathbf{1}$

S-matrix

Ultimately we want to compute **cross sections for scattering processes**, i.e. probabilities for



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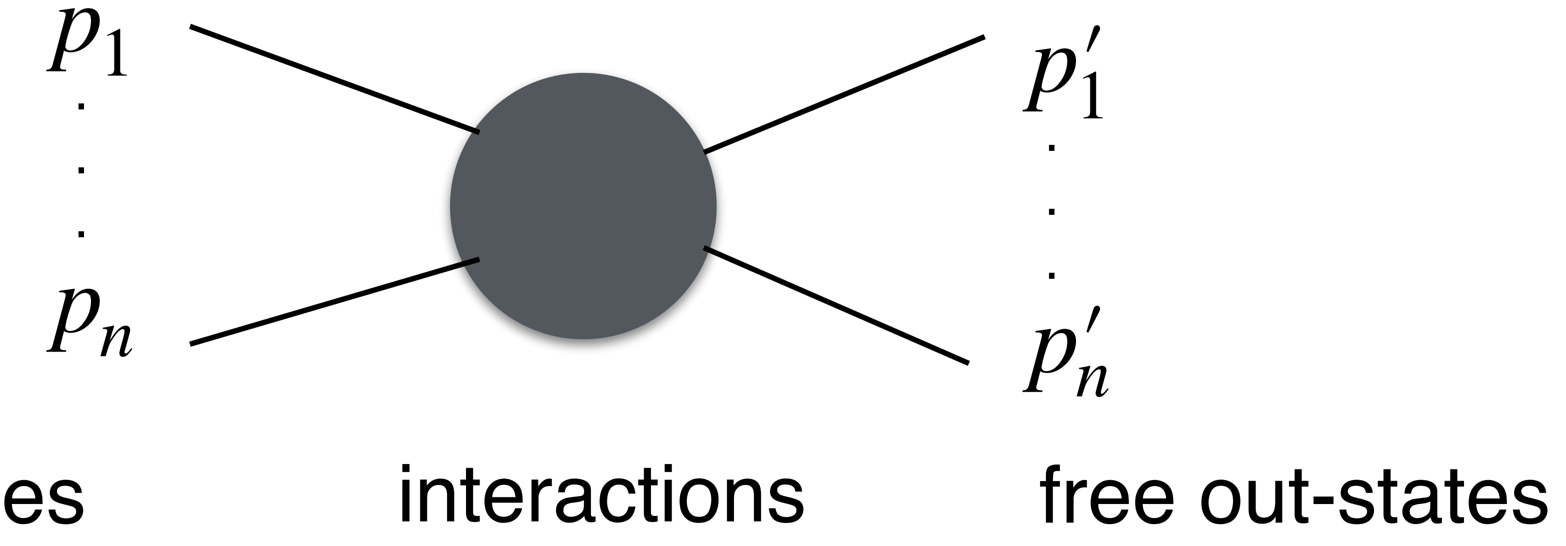
$$\to \hat{S} = \hat{U}(+\infty, -\infty) = \hat{T}e^{-i\int_{-\infty}^{\infty}\hat{H}_I(t')dt'} = \hat{T}\left(1 - i\int_{-\infty}^{\infty} H_I(t')\,dt' + \ldots\right)$$

perturbative expansion

S-matrix

Ultimately we want to compute **cross sections for scattering processes**, i.e. probabilities for



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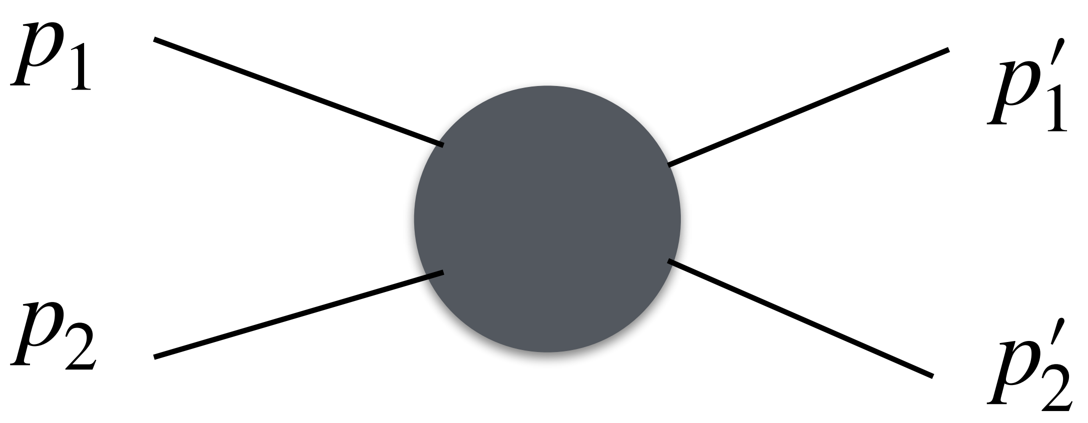
perturbative expansion

Scattering amplitude in ϕ^4-theory



$$\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\phi^4$$
$$= \mathscr{L}_0 + \mathscr{L}_I$$

$$A_{2\to2} = S_{\mathrm{fi}} = <\mathrm{f}\,|\,\hat{\mathrm{S}}\,|\,\mathrm{i}> = <0\,|\,\hat{\mathrm{a}}_{\vec{\mathrm{p}}_1'}\hat{\mathrm{a}}_{\vec{\mathrm{p}}_2'}\,\hat{\mathrm{S}}\,\hat{\mathrm{a}}^\dagger_{\vec{\mathrm{p}}_1}\,\hat{\mathrm{a}}^\dagger_{\vec{\mathrm{p}}_2}\,|0>$$

(assume: vacuum identical for in- and out-states)

where $$\hat{S} \approx \hat{T}\left(1 - i\int_{-\infty}^{\infty}\mathscr{H}_I(x')\,d^4x'\right) = \hat{T}\left(1 - \frac{i\,\lambda}{4!}\int_{-\infty}^{\infty}\hat{\phi}^4\,d^4x'\right)$$

leading order

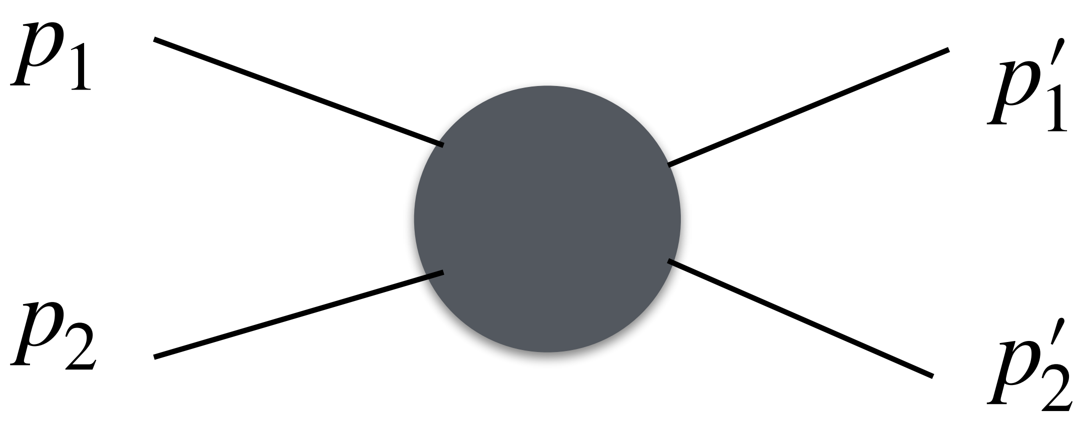
$$\mathscr{H}_I = \frac{\lambda}{4!}\phi^4$$

$$\hookrightarrow\; A_{2\to2} = <0\,|\,\hat{a}_{\vec{p}_1'}\hat{a}_{\vec{p}_2'}\,\hat{a}^\dagger_{\vec{p}_1}\,\hat{a}^\dagger_{\vec{p}_2}\,|0> - \frac{i\,\lambda}{4!}<0\,|\,\hat{T}(\hat{a}_{\vec{p}_1'}\hat{a}_{\vec{p}_2'}\,\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}\,\hat{a}^\dagger_{\vec{p}_1}\,\hat{a}^\dagger_{\vec{p}_2})\,|0>$$

= 0

$<f\,|\,i> = 0$

Scattering amplitude in ϕ^4-theory



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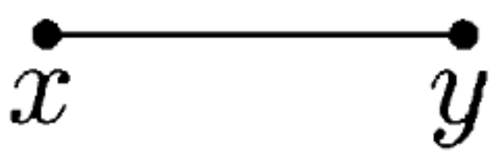
Wick's theorem:
such expectations values of multiple field operators can be decomposed into products of two-point function = propagators

$$A_{2\to2} = <0\,|\,\hat{a}_{\vec{p}_1'}\hat{a}_{\vec{p}_2'}\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}\,|0> - \frac{i\lambda}{4!}<0\,|\,\hat{T}(\hat{a}_{\vec{p}_1'}\hat{a}_{\vec{p}_2'}\,\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}\,\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2})\,|0>$$

= 0

$<f\,|\,i> = 0$

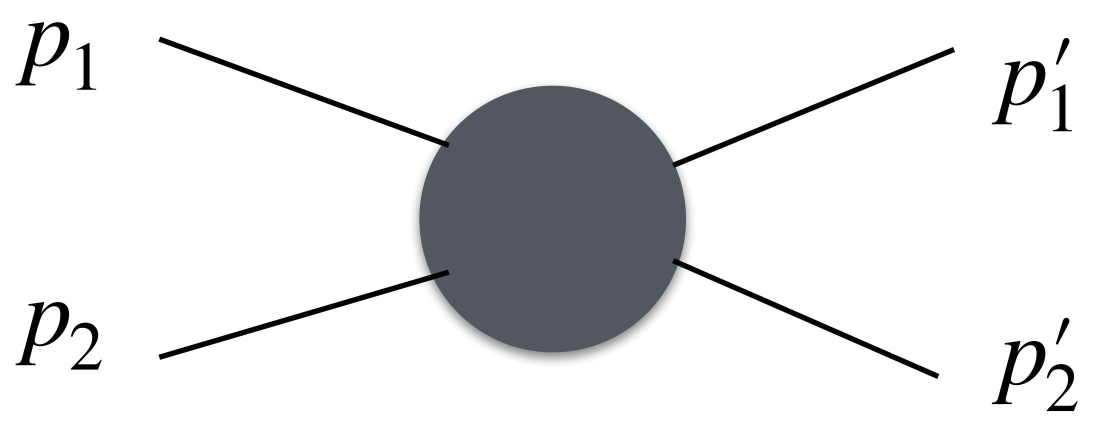
$$<0\,|\,\hat{\phi}(x)\hat{\phi}(y)\,|0> = D_{\mathrm{F}}(x-y) =$$



e.g.:

$$<0\,|\,\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}\,|0> =$$

Scattering amplitude in ϕ^4-theory



$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$
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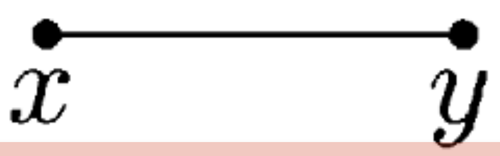
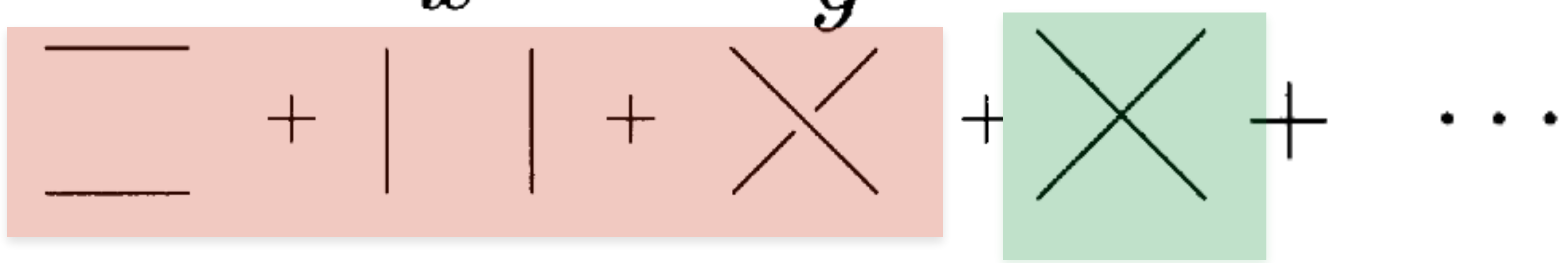
= 0

$<f|i> = 0$

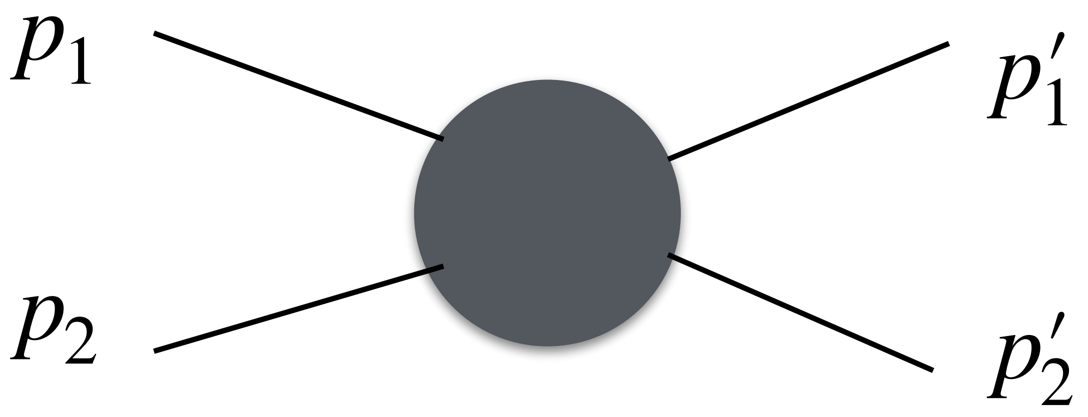
e.g.: $<0|\hat\phi(x)\hat\phi(y)|0> = D_{\mathrm{F}}(x-y) =$



$<0|\hat\phi\hat\phi\hat\phi\hat\phi|0> =$



Scattering amplitude in ϕ^4-theory



$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\phi^4 \\ &= \mathcal{L}_0 + \mathcal{L}_I\end{aligned}$$

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= 0

$<f\,|\,i> = 0$

use:

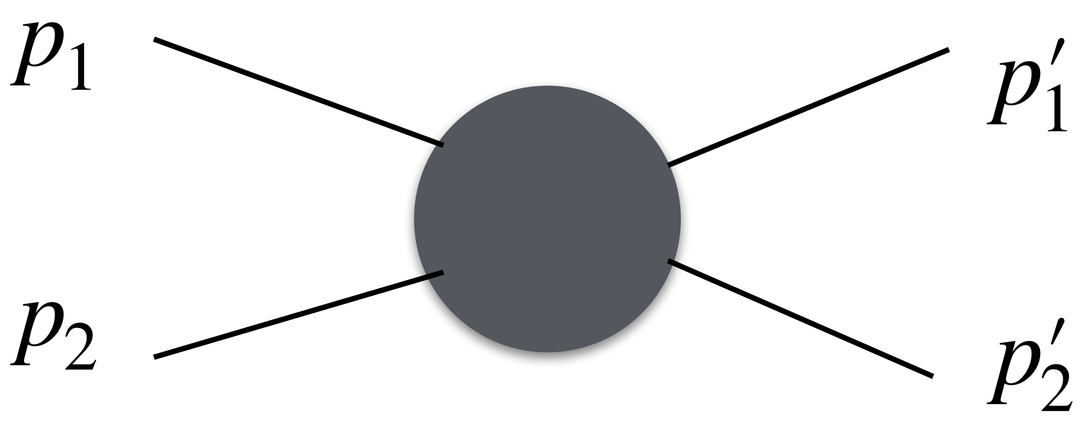
$$<0\,|\,\hat{T}(\hat{\phi}a^\dagger_{\vec{p}})\,|\,0> = <0\,|\,\hat{\phi}\,|\,\vec{p}> = 1\cdot e^{-ip\cdot x}$$

$$<0\,|\,\hat{T}(a_{\vec{p}}\hat{\phi})\,|\,0> = <\vec{p}\,|\,\hat{\phi}\,|\,0> = 1\cdot e^{ip\cdot x}$$

→ external lines

external momentum-space wf

Scattering amplitude in ϕ^4-theory



$$\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\phi^4$$
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$$A_{2\to 2} = S_{\rm fi} = <{\rm f}\,|\,\hat{\rm S}\,|\,{\rm i}> = <0\,|\,\hat{\rm a}_{\vec{\rm p}_1'}\hat{\rm a}_{\vec{\rm p}_2'}\,\hat{\rm S}\,\hat{\rm a}^\dagger_{\vec{\rm p}_1}\hat{\rm a}^\dagger_{\vec{\rm p}_2}\,|0>$$

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$$= \ldots = -i\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$ = sum of all connected diagrams at given order

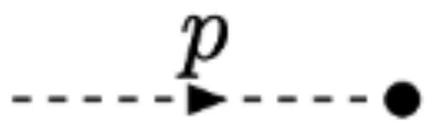
drop disconnected diagrams

vertex rule

$$\int d^4x e^{-ip_1x}\,e^{-ip_2x}\,e^{+ip_1'x}\,e^{ip_2x}$$ from external lines

Feynman rules for ϕ^4 theory

$$\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\phi^4$$



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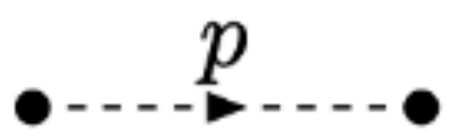
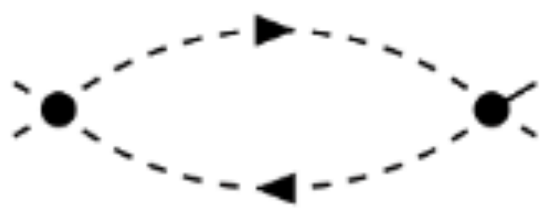


$$\frac{i}{k^2 - m^2 + i\epsilon}$$

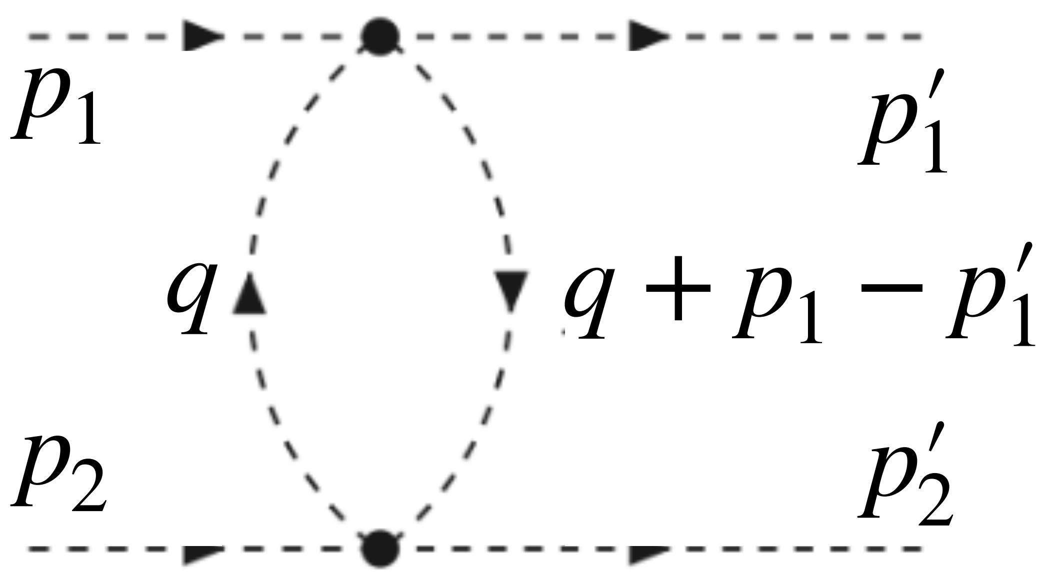
$$-i\lambda$$



$$\int \frac{d^4q}{(4\pi)^4}$$

+ symmetry factors

Example



=

$$\frac{1}{2}(-i\,\lambda)^2 \int \frac{d^4q}{(4\pi)^4}\,\frac{i}{q^2 - m^2}\,\frac{i}{(q + p_1 - p_1')^2 - m^2}$$

$$\times\,(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

Cross-sections

scattering process $a + b \to b_1 + b_2 + \cdots + b_n$ with momenta $P_i = p_a + p_b = P_f = p_1 + \cdots + p_n$

initial state: $|i> = |a(p_a), b(p_b)>$ final state: $|f> = |b_1(p_1), \cdots b_n(p_n)>$

Amplitude for transition from $|i>$ into $|f>$ given by S-matrix element

$$S_{fi} = <f|\hat{S}|i> = (2\pi)^4 \, \delta^{(4)}(P_i - P_f) \, \mathscr{M}_{fi} \, (2\pi)^{-3(n+2)/2}$$

total momentum conservation

matrix-element from Feynman rules

cross section: $\sigma = \frac{1}{N} \cdot \text{probability of interactions}$

Phase-space integral for final-state particles

$$= \frac{1}{\mathscr{F}} \cdot \prod_f \int d\Phi_f \, (2\pi)^4 \, \delta^{(4)}(P_i - P_f) \, |\mathscr{M}_{fi}|^2$$

Flux factor
in massless limit
$\mathscr{F} \simeq 2s$
$s = (p_a + p_b)^2$

$$= \frac{1}{64\pi^2 s} \frac{|\vec{p}_1|}{|\vec{p}_a|} \int |\mathscr{M}_{fi}|^2 \mathrm{d}\Omega$$

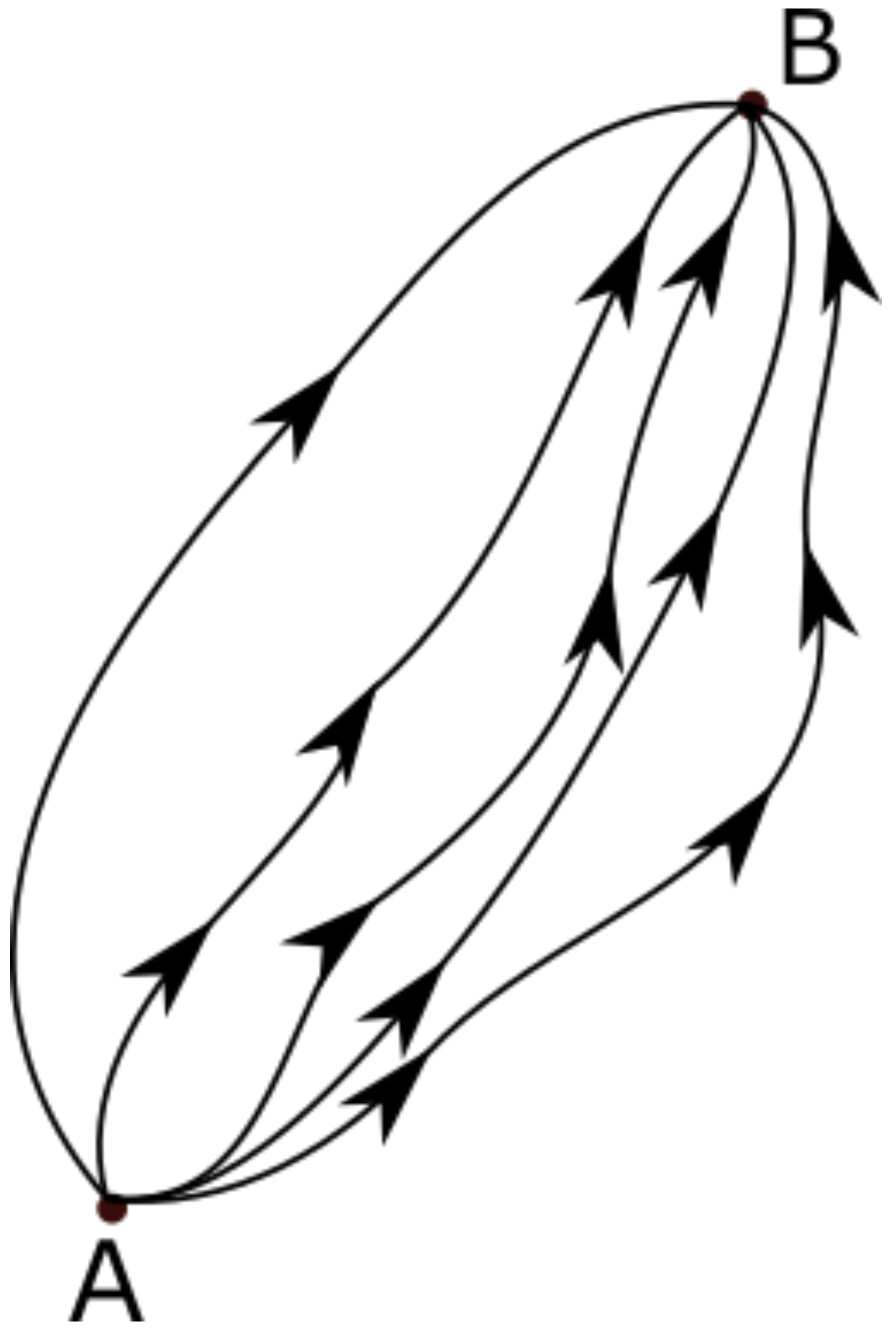
$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$

for $2 \to 2$ scattering

Alternative field quantisation: Path integral

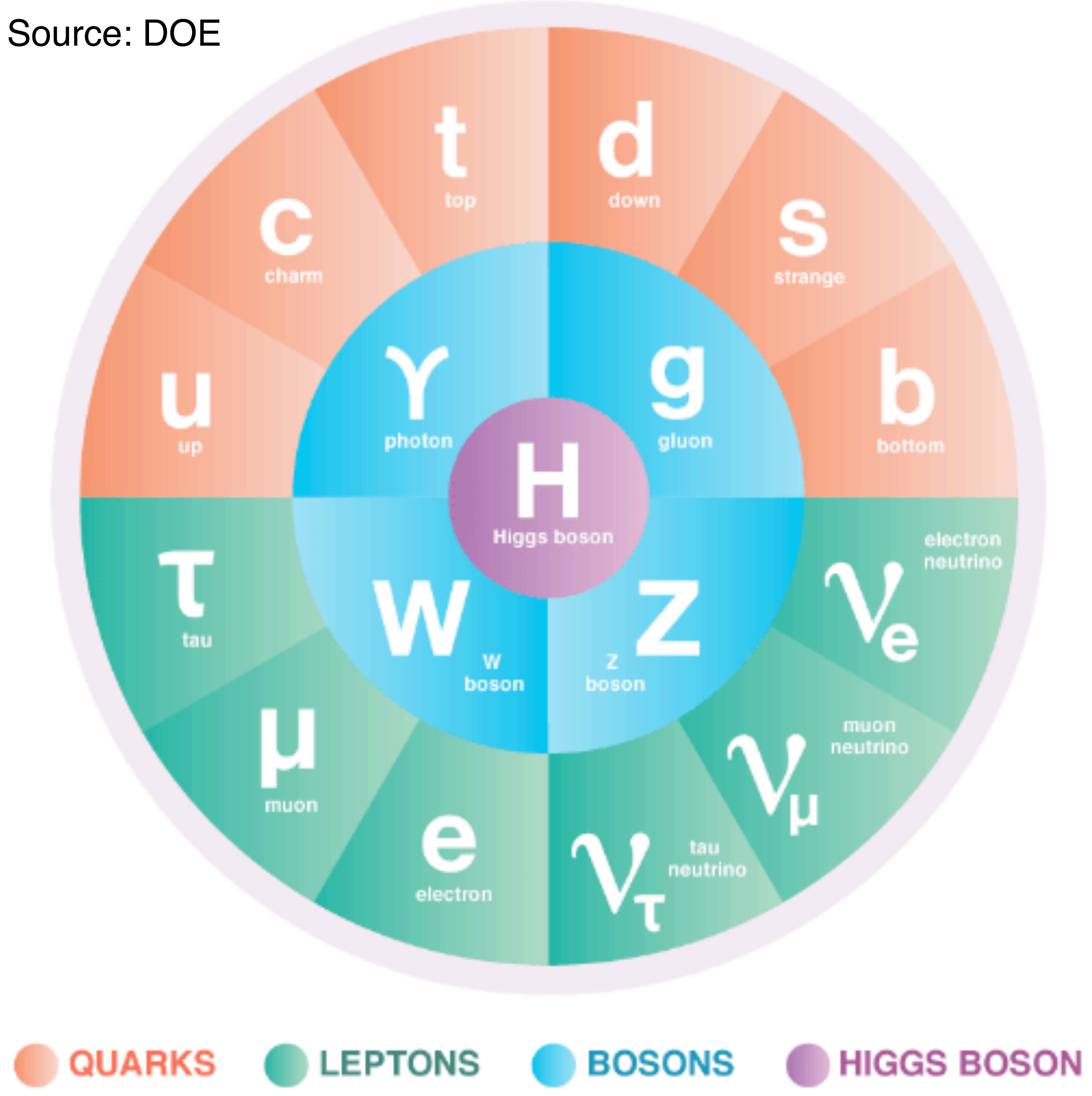
Wikipedia:
The path integral formulation of quantum field theory represents the transition amplitude (corresponding to the classical correlation function) as a weighted sum of all possible histories of the system from the initial to the final state.

$$\langle F \rangle = \frac{\int \mathcal{D}\varphi F[\varphi] e^{i\mathcal{S}[\varphi]}}{\int \mathcal{D}\varphi e^{i\mathcal{S}[\varphi]}}$$

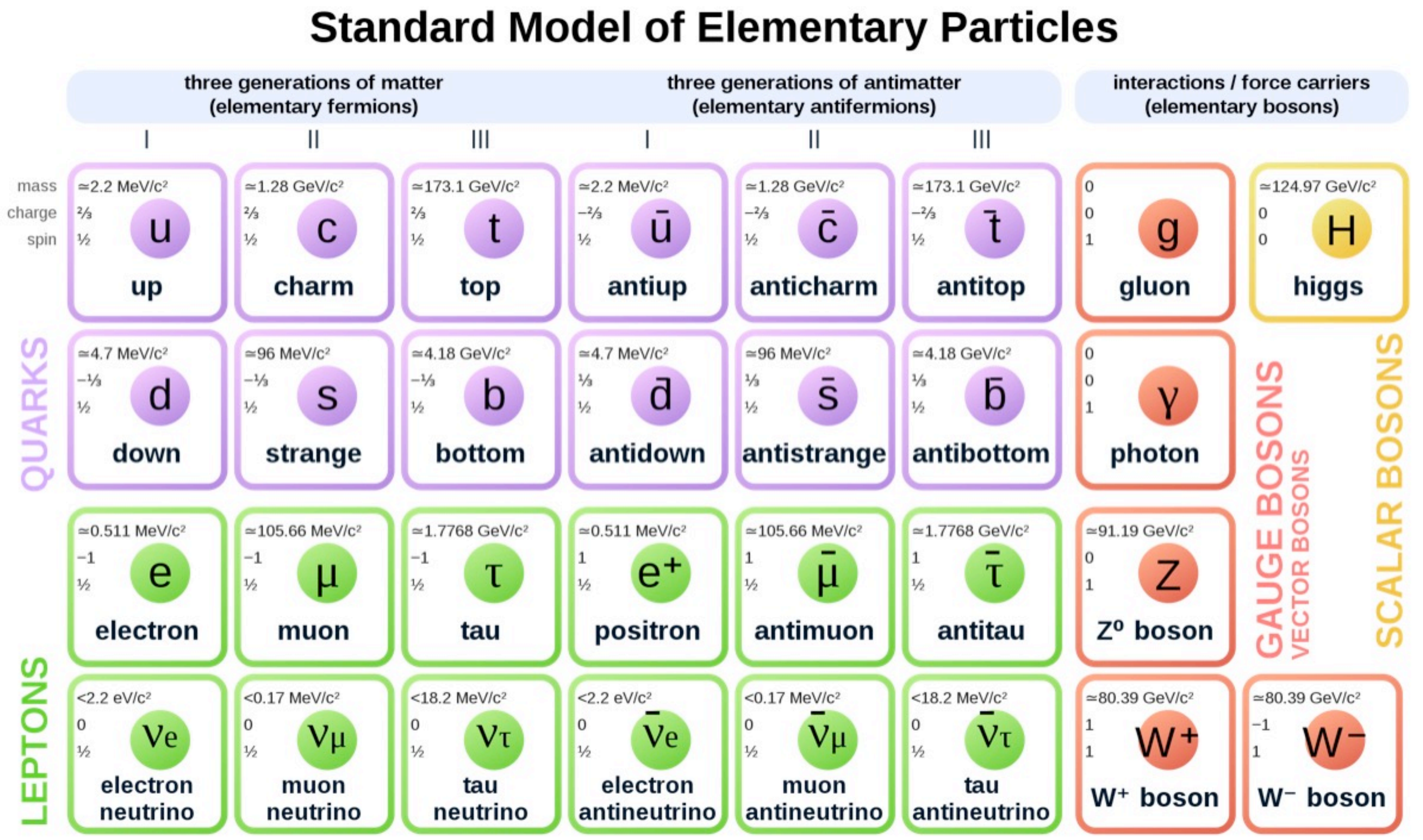


Field content of the SM

Source: DOE



Source: Ars Technika

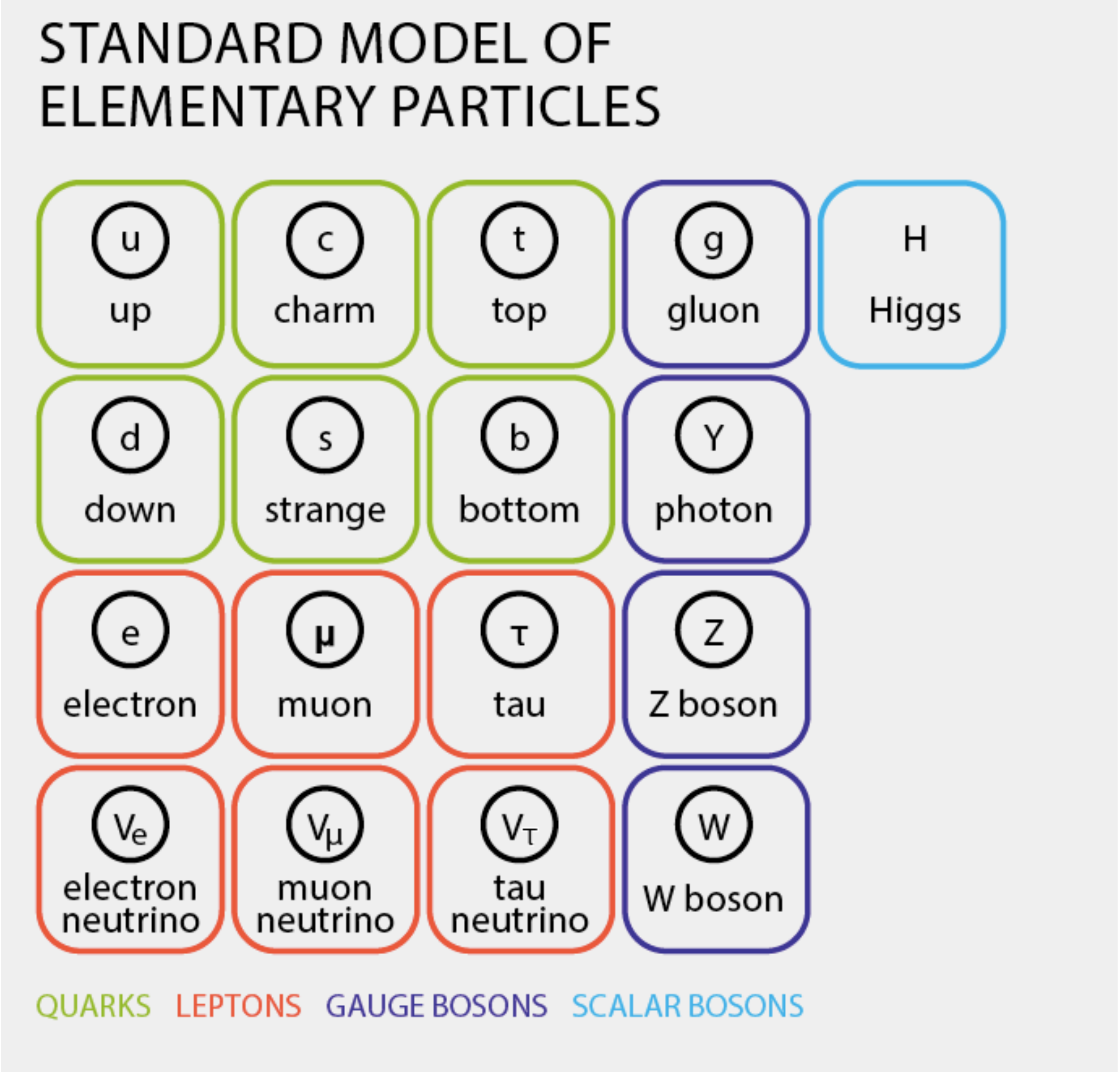


Standard Model of Elementary Particles

	three generations of matter (elementary fermions)			three generations of antimatter (elementary antifermions)			interactions / force carriers (elementary bosons)	
	I	II	III	I	II	III		
mass charge spin	≃2.2 MeV/c^2 ⅔ ½ u **up**	≃1.28 GeV/c^2 ⅔ ½ c **charm**	≃173.1 GeV/c^2 ⅔ ½ t **top**	≃2.2 MeV/c^2 −⅔ ½ $\bar{u}$ **antiup**	≃1.28 GeV/c^2 −⅔ ½ $\bar{c}$ **anticharm**	≃173.1 GeV/c^2 −⅔ ½ $\bar{t}$ **antitop**	0 0 1 g **gluon**	≃124.97 GeV/c^2 0 0 H **higgs**
QUARKS	≃4.7 MeV/c^2 −⅓ ½ d **down**	≃96 MeV/c^2 −⅓ ½ s **strange**	≃4.18 GeV/c^2 −⅓ ½ b **bottom**	≃4.7 MeV/c^2 ⅓ ½ $\bar{d}$ **antidown**	≃96 MeV/c^2 ⅓ ½ $\bar{s}$ **antistrange**	≃4.18 GeV/c^2 ⅓ ½ $\bar{b}$ **antibottom**	0 0 1 γ **photon**	
	≃0.511 MeV/c^2 −1 ½ e **electron**	≃105.66 MeV/c^2 −1 ½ μ **muon**	≃1.7768 GeV/c^2 −1 ½ τ **tau**	≃0.511 MeV/c^2 1 ½ e^+ **positron**	≃105.66 MeV/c^2 1 ½ $\bar{\mu}$ **antimuon**	≃1.7768 GeV/c^2 1 ½ $\bar{\tau}$ **antitau**	≃91.19 GeV/c^2 0 1 Z **Z^0 boson**	
LEPTONS	<2.2 eV/c^2 0 ½ ν_e **electron neutrino**	<0.17 MeV/c^2 0 ½ ν_μ **muon neutrino**	<18.2 MeV/c^2 0 ½ ν_τ **tau neutrino**	<2.2 eV/c^2 0 ½ $\bar{\nu}_e$ **electron antineutrino**	<0.17 MeV/c^2 0 ½ $\bar{\nu}_\mu$ **muon antineutrino**	<18.2 MeV/c^2 0 ½ $\bar{\nu}_\tau$ **tau antineutrino**	≃80.39 GeV/c^2 1 1 W^+ **W^+ boson**	≃80.39 GeV/c^2 −1 1 W^- **W^- boson**

GAUGE BOSONS
VECTOR BOSONS
SCALAR BOSONS

Source: BBC

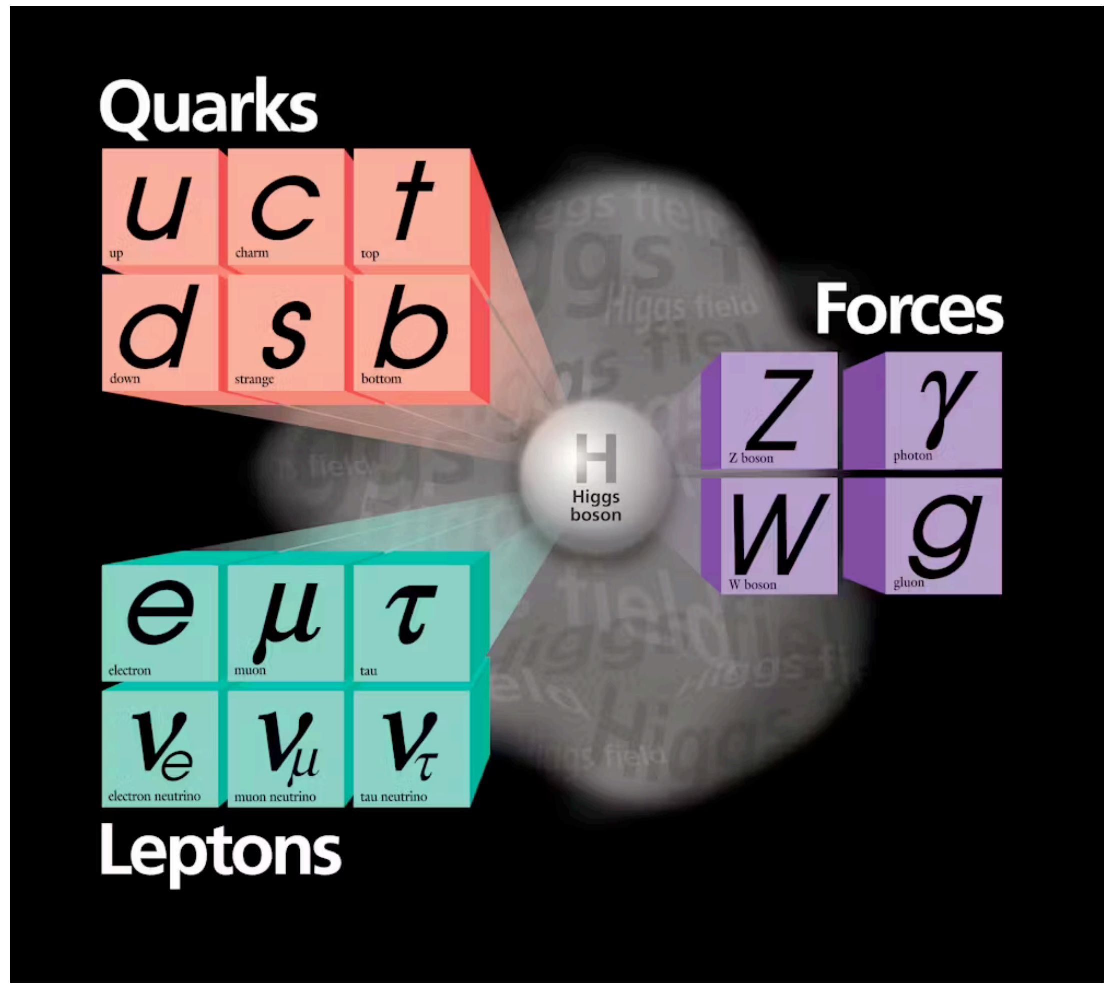


STANDARD MODEL OF
ELEMENTARY PARTICLES

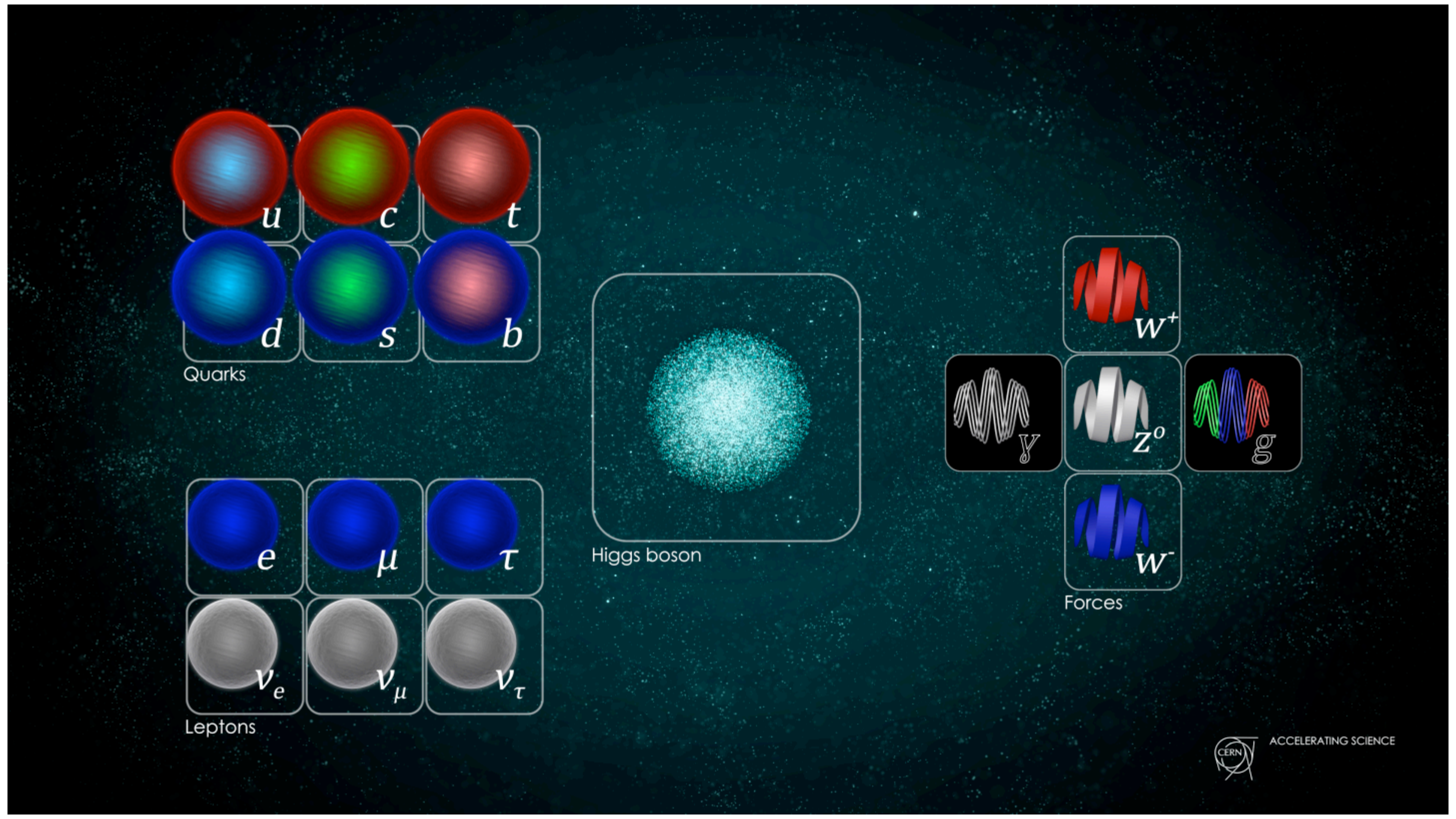
u up	c charm	t top	g gluon	H Higgs
d down	s strange	b bottom	γ photon	
e electron	μ muon	τ tau	Z Z boson	
ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson	

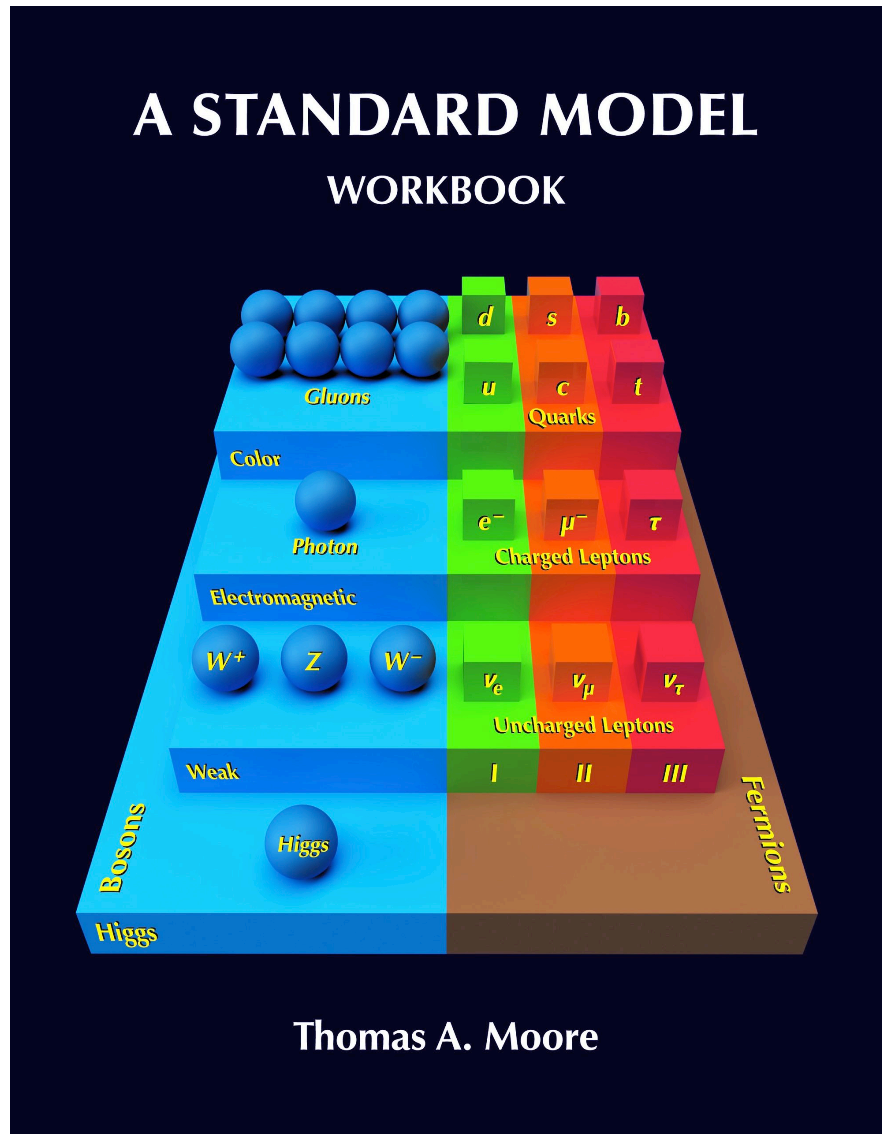
QUARKS LEPTONS GAUGE BOSONS SCALAR BOSONS

Source: unkown



Source: CERN





Field content of the SM

Source: The Particle Zoo

Field content of the SM

	field			spin
quarks	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	1/2
	u_R	c_R	t_R	1/2
	d_R	s_R	b_R	1/2
leptons	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	1/2
	e_R	μ_R	τ_R	1/2
Higgs-doublet		$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}_L$		0
gauge bosons		G_μ^a		1
		W_μ^i		1
		B_μ		1

spin-1/2 fermion fields $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

spin-0 (complex) scalar field $\phi = \mathrm{Re}\phi + i\,\mathrm{Im}\phi$

spin-1 vector fields $A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$
massless and massive

Free massive vector fields

Maxwell term   mass term

The dynamics of a free **massive vector field** is described by: $\boxed{\mathscr{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}Z_\mu Z^\mu}$

with the **field-strength tensor** $\boxed{F^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu}$ and the 4-potential $Z^\mu = (\phi_Z, \vec{Z})$

scalar potential   vector potential

EL eq. with respect to Z^ν gives free **Proca equations**: $\boxed{\left[(\Box + m^2)\, g^{\mu\nu} - \partial^\mu \partial^\nu\right] Z_\nu = 0}$

Plane wave solutions of Proca equations: $\sim \epsilon_\nu^{(\lambda)}(\vec{k})\, e^{-ikx}$

polarisation vectors with $\lambda = 1{,}2{,}3$ (2 x transverse, 1 x longitudinal)

Chosen such that $\epsilon^{(\lambda)} \cdot k = 0\,,\quad \epsilon^{(\lambda)*} \cdot \epsilon^{(\lambda')} = -\,\delta_{\lambda\lambda'}$ and we have $\sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} = -\,g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}$

orthonormal   completness

Free massive vector fields

General solution of Proca equation is given by superposition of plane waves:

$$\boxed{Z_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \int \frac{\mathrm{d}^3k}{2k^0} \left[a_\lambda(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{-ikx} \;+\; a_\lambda^\dagger(k)\, \epsilon_\mu^{(\lambda)}(k)^*\, e^{ikx} \right]}$$

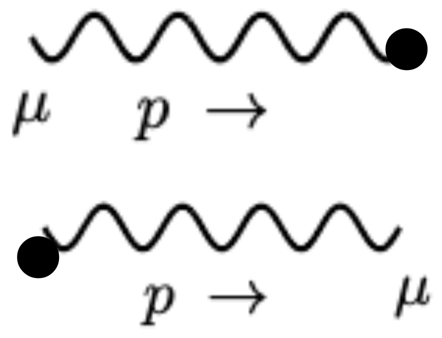
$a_\lambda^\dagger(k)\,|\,0\!>\; = |\,k\lambda\!>$ ***creation operator***

$a_\lambda(k)\,|\,k'\lambda'\!>\; = 2k^0\,\delta^3(\vec{k} - \vec{k}\,')\,\delta_{\lambda\lambda'}\,|\,0\!>$ ***annihilation operator***

wave-functions
for external state

$<\!0\,|\,A_\mu(x)\,|\,k\lambda\!> \;\sim \epsilon_\mu^{(\lambda)}(k)\, e^{-ikx}$ ***incoming massive vector***

$<\!k\lambda\,|\,A_\mu(x)\,|\,0\!> \;\sim \epsilon_\mu^{(\lambda)}(k)^*\, e^{ikx}$ ***outgoing massive vector***



propagator:
Green's function of
inhomogeneous Proca eq.

$$\left[(\Box + m^2)\, g^{\mu\rho} - \partial^\mu\partial^\rho\right] D_{\rho\nu}(x-y) = g^\mu{}_\nu\, \delta^4(x-y)$$

momentum space

$$\left[(-k^2 + m^2)\, g^{\mu\rho} + k^\mu k^\rho\right]\; D_{\rho\nu}(k) = g^\mu{}_\nu$$

$$\boxed{i\,D_{\rho\nu}(k) = \frac{i}{k^2 - m^2 + i\epsilon}\left(-g_{\nu\rho} + \frac{k_\nu k_\rho}{m^2}\right)}$$

momentum-space propagator

Free massless vector fields

Maxwell term

The dynamics of a free **massless vector field** is described by:

$$\boxed{\mathscr{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

EL equation with respect to A^ν gives free **Maxwell equations**:

$$\boxed{\Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0 = \partial_\mu F^{\mu\nu}}$$

propagator:
Green’s function of
inhomogeneous Maxwell eq.

$$\left(-k^2\, g^{\mu\rho} + k^\mu k^\rho\right)\, D_{\rho\nu}(k) = g^\mu{}_\nu$$

⚠ not invertible=degeneracy

Free massless vector fields

Maxwell term

The dynamics of a free **massless vector field** is described by: $$\boxed{\mathscr{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$ $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

EL equation with respect to A^ν gives free **Maxwell equations**: $$\boxed{\Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0 = \partial_\mu F^{\mu\nu}}$$

propagator:
Green's function of
inhomogeneous Maxwell eq.

$$\left(-k^2 g^{\mu\rho} + k^\mu k^\rho\right) D_{\rho\nu}(k) = g^\mu{}_\nu$$

⚠ not invertible=degeneracy

$F^{\mu\nu}$ is invariant under **gauge transformations** $A^\mu(x) \to A^\mu(x) - \partial^\mu \chi(x)$

This freedom is related to unphysical degrees of freedom: 2 d.o.f. for massless vector field vs 4 components of A^ν

Add **gauge-fixing** term to the Maxwell Lagrangian: $\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$

(ξ arbitrary,
no physical impact)

$$\boxed{i\, D_{\rho\nu}(k) = \frac{i}{k^2 + i\epsilon} \left[-g_{\nu\rho} + (1-\xi) \frac{k_\nu k_\rho}{k^2} \right]}$$

momentum-space propagator

(can e.g. choose $\xi = 1$
to simplify computations
→Feynman gauge)

Free Fermion field

The dynamics of a free fermion field is described by the **Dirac Lagrangian**: $\boxed{\mathscr{L}_{\text{Dirac}} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi}$

i) ψ is a 4-component **spinor** field: $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$

Pauli matrices

ii) Dirac γ^μ-matrices are 4x4 matrices in spinor space with $\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$

anti-commutator

γ^μ-matrices fulfil $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}$ (defining property)

iii) $\overline{\psi} = \psi^\dagger\,\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$ needed such that $\overline{\psi}\psi$ is Lorentz invariant

EL eq. for $\overline{\psi}$ yields: $\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi = 0}$ ***Dirac equation***

Two types of plane-wave solutions of Dirac equation:

with $E(p) = \sqrt{\vec{p}^2 + m^2}$

$\psi_+ = u(p)\,e^{-ipx}$ ***incoming fermion***

$\psi_- = v(p)\,e^{ipx}$ ***outgoing anti-fermion***

Free Fermion field

$$\not{a} = a_\mu \gamma^\mu$$

Spinors $u(p), v(p)$ fulfil the algebraic Dirac equations: $\left(\not{p} - m\right) u(p) = 0\,, \qquad \left(\not{p} + m\right) v(p) = 0$

Can be classified according to eigenvalues with respect to **helicity** operator $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$\sigma = \pm 1/2$

$$\frac{1}{2}\left(\vec{\Sigma}\cdot\vec{n}\right) u_\sigma(p) = \sigma\, u_\sigma(p)\,, \qquad -\frac{1}{2}\left(\vec{\Sigma}\cdot\vec{n}\right) v_\sigma(p) = \sigma\, v_\sigma(p) \qquad \vec{n} = \frac{\vec{p}}{|\vec{p}\,|}$$

direction of travel

General solution of Dirac equation is given by superposition of plane waves $\psi_\pm$:

$$\boxed{\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3k}{2k^0} \left[a_\sigma(\vec{k})\, u_\sigma(k)\, e^{-ikx} + b^\dagger_\sigma(\vec{k})\, v_\sigma(k)\, e^{ikx}\right]}$$

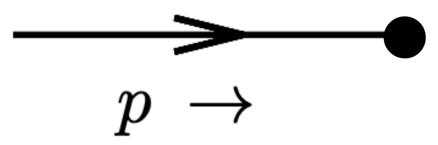
(similar for $\overline{\psi}$)

annihilation operators for particles

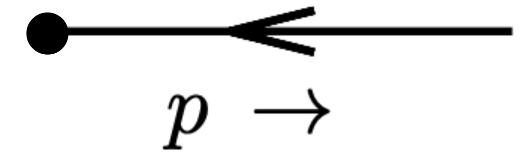
creation operators for anti-particles

wave-functions
for external state

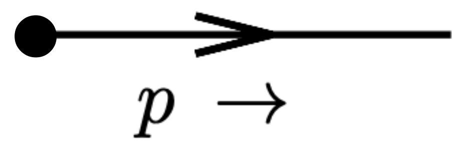
$< 0\,|\,\psi(x)\,|\,f, k\sigma > \ \sim u_\sigma(k)\, e^{-ikx}$



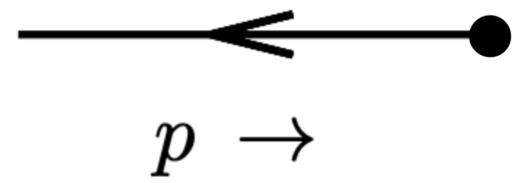
$<\bar{f}, k\sigma\,|\,\psi(x)\,|\,0 > \ \sim v_\sigma(k)\, e^{ikx}$



$< f, k\sigma\,|\,\overline{\psi}(x)\,|\,0 > \ \sim \bar{u}_\sigma(k)\, e^{ikx}$



$< 0\,|\,\overline{\psi}(x)\,|\,\bar{f}, k\sigma > \ \sim \bar{v}_\sigma(k)\, e^{-ikx}$



Dirac propagator

Determine Green's function of inhomogeneous Dirac equation: $(i\gamma^\mu \partial_\mu - m)\, S_F(x-y) = \mathbf{1}\, \delta^{(4)}(x-y)$

Fourier

Solution via Fourier ansatz: $S_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, S(k)\, e^{-ik(x-y)}$ with $(\not{k} - m)\, S(k) = \mathbf{1}\,.$

Dirac propagator is 4x4 matrix

$$\boxed{i\,S(k) = \frac{i}{\not{k} - m + i\epsilon} = \frac{i\,(\not{k} + m)}{k^2 - m^2 + i\epsilon}}$$

QED interaction

Maxwell equation sourced by 4-current: $\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$ where $\partial_\nu J^\nu = 0$ (current conservation)

Corresponding Lagrangian: $\mathscr{L}_{\text{MW}} = \mathscr{L}_{\text{EM}} + \mathscr{L}_{\text{int}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu$

Lorentz vector

A suitable 4-current in terms of a fermion (electron) field can be constructed as: $J^\mu \sim \bar{\psi}\gamma^\mu\psi$

This is indeed a conserved current iff ψ is a solution of the Dirac eq:

$$\begin{aligned}\partial_\mu J^\mu &= \bar{\psi}\,\overleftarrow{\not{\partial}}\psi + \bar{\psi}\,(\not{\partial}\psi) \\ &= (-m\bar{\psi})\psi + \bar{\psi}\,(m\psi) = 0\end{aligned}$$

Fixing the proportionality factor in J^μ to $-e$ (charge of electron) yields the QED Lagrangian:

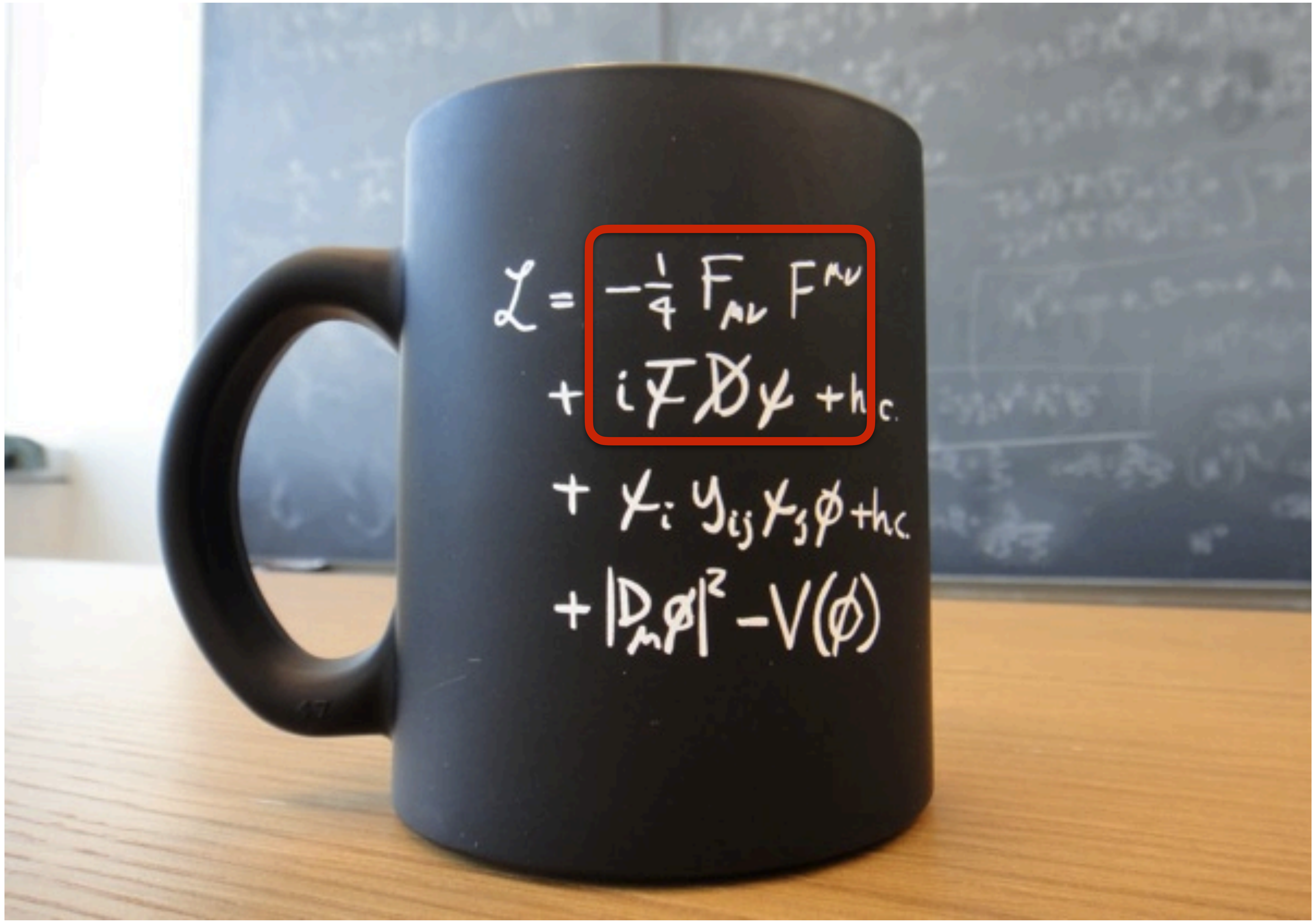
$$\boxed{\mathscr{L}_{\text{QED}} = \mathscr{L}_{\text{EM}} + \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{int}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\,(i\not{\partial} - m)\psi + e\bar{\psi}\,\gamma^\mu\psi A_\mu}$$

(+ gauge-fixing)

$$\partial_\mu \;\rightarrow\; D_\mu = \partial_\mu - ieA_\mu \quad \hookrightarrow \quad = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\,(i\not{D} - m)\psi$$

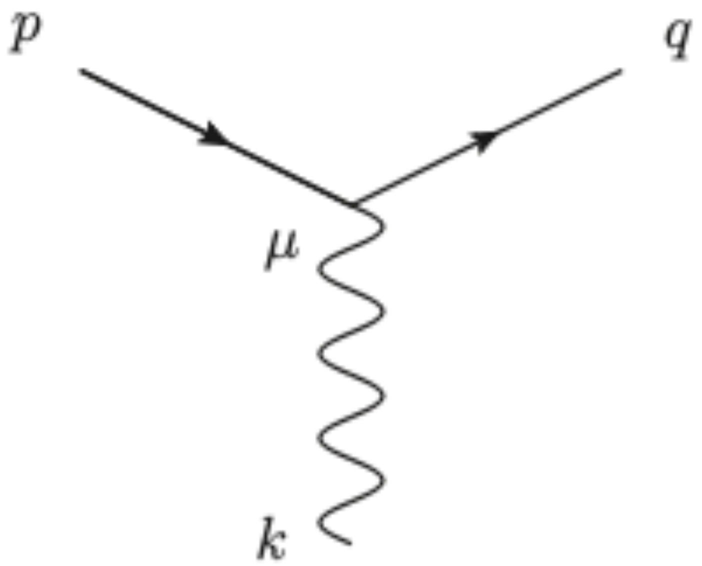
covariant derivative

The Standard Model coffee mug



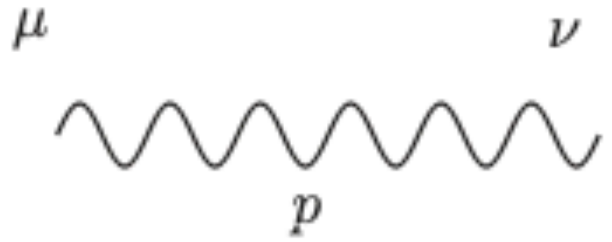
Not a coffee mug

Summary: QED Feynman rules



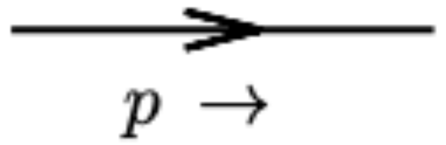
$$ie\gamma^\mu$$

$$\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

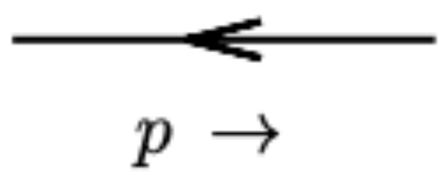


$$\frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

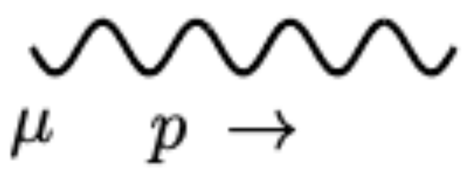
Incoming



$u(p)$

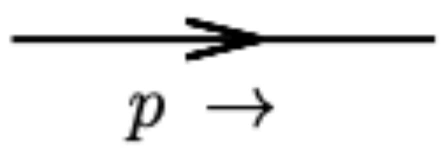


$\overline{v}(p)$

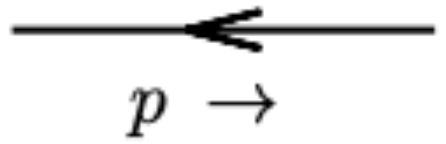


$\varepsilon_\mu(p)$

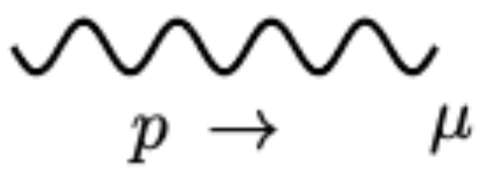
Outgoing



$\overline{u}(p)$



$v(p)$

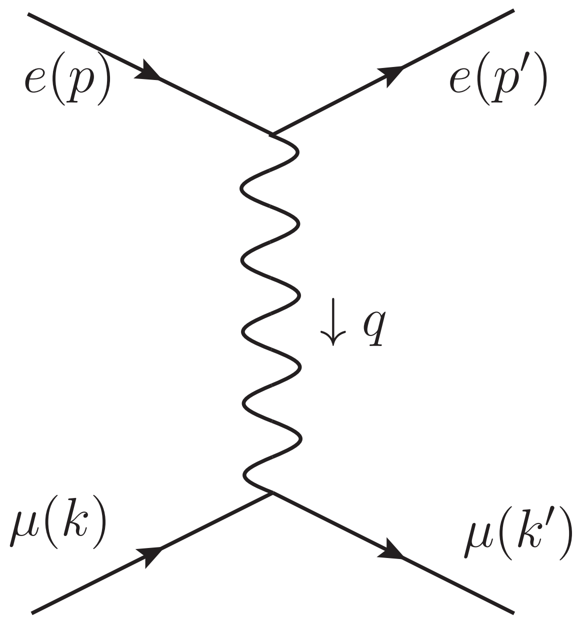


$\varepsilon^*_\mu(p)$

+ symmetry factors

The Feynman van

Example: Coulomb scattering



$$e(p)\,\mu(k) \;\to\; e(p')\,\mu(k')$$

Feynman rules

amplitude

$$i\mathscr{M} = ie^2\,[\bar{u}(p')\,\gamma^\mu\,u(p)]\,\frac{g_{\mu\nu}}{q^2}\,[\bar{u}(k')\,\gamma^\nu\,u(k)]$$

unpolarised squared amplitude

sum over final-state
and average over initial-state
polarisations

$$\overline{|\mathscr{M}|}^2 = \frac{1}{2}\sum_{r=1}^{2}\frac{1}{2}\sum_{s=1}^{2}\sum_{r'=1}^{2}\sum_{s'=1}^{2}|\mathscr{M}|^2$$

$$= \frac{1}{4}\frac{e^4}{(q^2)^2}\sum_{r,r'}[\bar{u}_{r'}(p')\,\gamma^\mu\,u_r(p)][\bar{u}_{r'}(p')\,\gamma^\rho\,u_r(p)]^*$$

$$\times\sum_{s,s'}[\bar{u}_{s'}(k')\,\gamma_\mu\,u_s(k)][\bar{u}_{s'}(k')\,\gamma_\rho\,u_s(k)]^*\,,$$

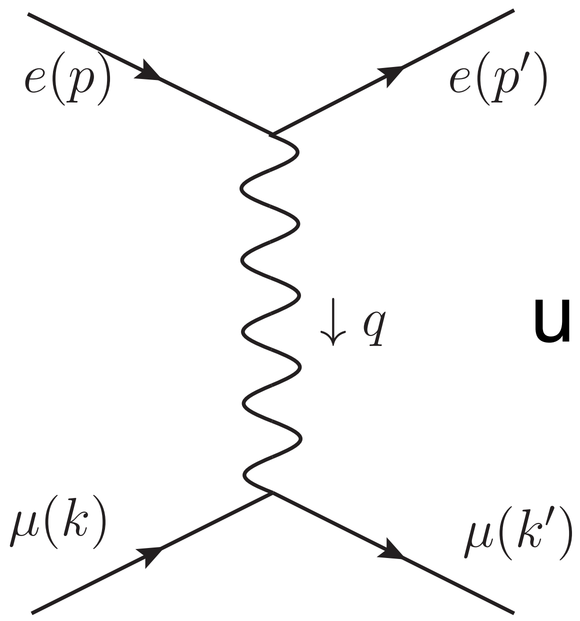
completeness relations,
Dirac equations for spinors
γ^μ trace simplifications,
…

$$= \frac{8e^4}{(q^2)^2}\Big((pk)\;(p'k') + (pk')(p'k) + 2m_e^2 m_\mu^2 - m_\mu^2(pp') - m_e^2(kk')\Big)$$

Mandelstam invariants

$$= \frac{2e^4}{t^2}\Big((s - m_e^2 - m_\mu^2)^2 + (u - m_e^2 - m_\mu^2)^2 + 2t\,(m_e^2 + m_\mu^2)\Big)$$

Example: Coulomb scattering



$$e(p)\,\mu(k) \;\to\; e(p')\,\mu(k')$$

unpolarised squared amplitude: $\overline{|\mathscr{M}|}^2 = \frac{2e^4}{t^2}\Big((s-m_e^2-m_\mu^2)^2 + (u-m_e^2-m_\mu^2)^2 + 2t\,(m_e^2+m_\mu^2)\Big)$

differential cross section: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}\,|}{|\overrightarrow{p'}|}\int |\mathscr{M}|^2$

$s \gg m_e^2, m_\mu^2$

↳ $s = 4\boldsymbol{p}^2\,, \qquad t = -\,4\boldsymbol{p}^2\sin^2(\theta/2)\,, \qquad u = -\,4\boldsymbol{p}^2\cos^2(\theta/2)$

↳ $\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{2s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}$

Crucial observation: gauge symmetry in QED

The QED Lagrangian $$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\,(i\not{\partial} - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\,(i\not{D} - m)\psi$$ is **invariant** under

a local (=x-dependent) **gauge transformation** $$\psi(x) \to \psi'(x) = e^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$$

Notes:

- $F_{\mu\nu}$ is invariant by construction but the shift of A_μ in interaction term cancels exactly the additional Dirac term
- we can demand this $U(1)$ gauge invariance to construct the QED interaction term
- a term $\sim A^\mu A_\mu$ (see $\mathscr{L}_{\text{Proca}}$) is NOT gauge-invariant → **massless photon**
- $\partial_\mu \;\to\; D_\mu = \partial_\mu - ieA_\mu$ ensures gauge invariance by construction → "**minimal coupling**"

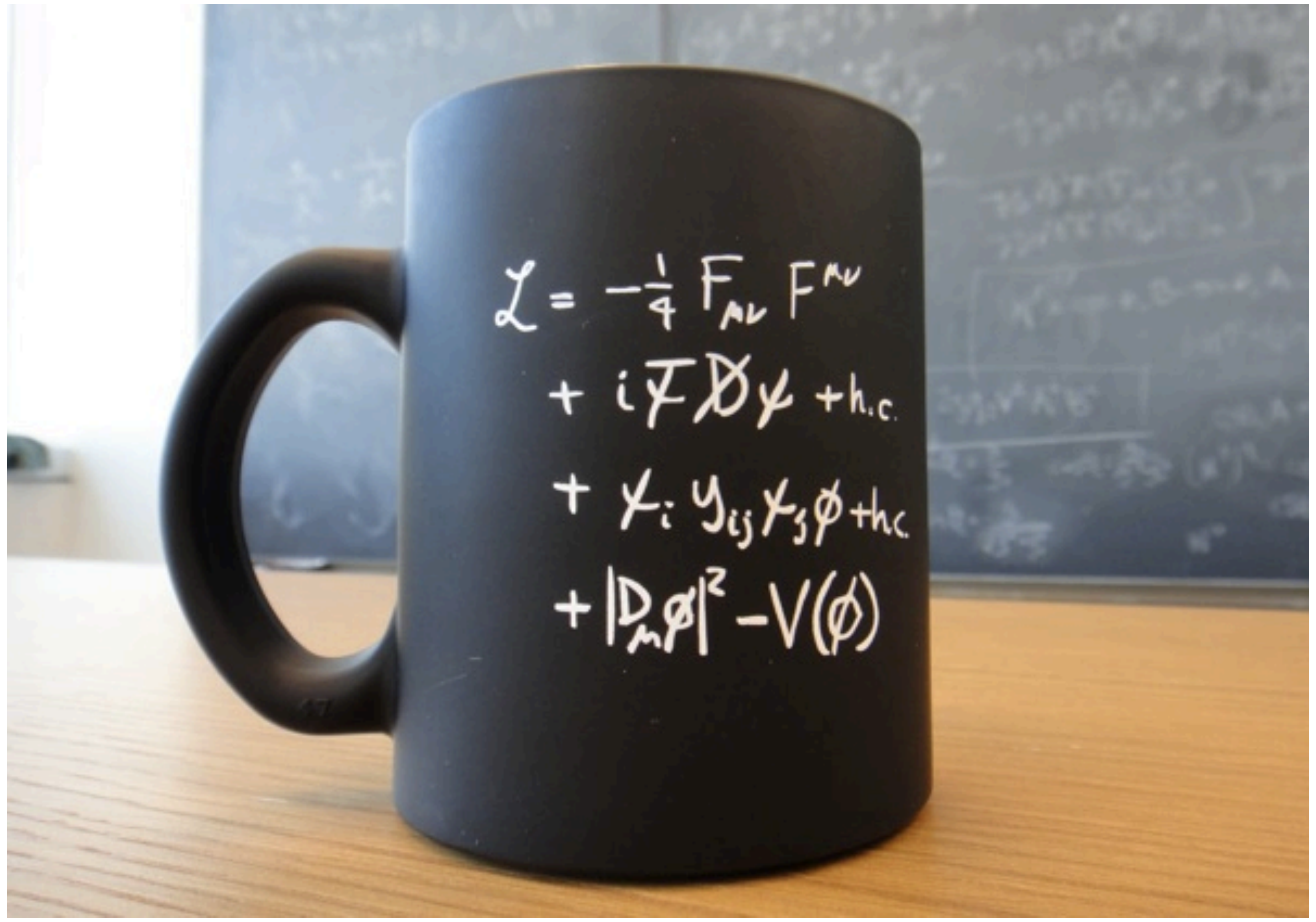
Guiding principles

- Causality
- Unitarity (conservation of probability)

- **Symmetry**
  - space-time: Lorentz invariance
  - internal: gauge invariance

- Renormalisability

- Minimality / Occam's razor



Conclusions

- QFT = QM + SR
- Every quantum field is superposition of quantised SHOs
- $S_{\mathrm{fi}} = <\mathrm{f}|\hat{\mathrm{S}}|\mathrm{i}> = <\mathrm{out}|\mathrm{U_I}(+\infty, -\infty)|\mathrm{in}>$
- Feynman diagrams: graphical representation of Wick's theorem
- Guiding principle to construct consistent theories: symmetries
- Local U(1) symmetry → QED interactions

Questions?