QCD (for colliders)
Lecture 1: Introduction

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QUANTUM CHROMODYNAMICS

The theory of quarks, gluons and their interactions

It's central to all modern colliders.
(And QCD is what we're made of)

The ingredients of QCD

- Quarks (and anti-quarks): they come in 3 colours
- Gluons: a bit like photons in QED

  But there are 8 of them, and they're colour charged
- And a coupling, α_s, that's not so small and runs fast

  At LHC, in the range 0.08(@ 5 TeV) to $\mathcal{O}(1)$(@ 0.5 GeV)

Aims of this course

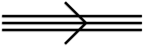
I'll try to give you a feel for:

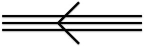
How QCD works

How theorists handle QCD at high-energy colliders

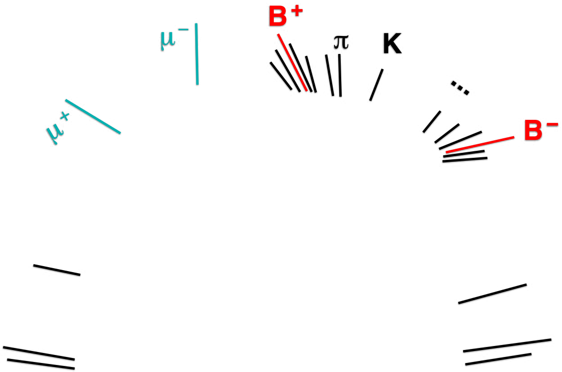
How *you* can work with QCD at high-energy colliders

A proton-proton collision: INITIAL STATE


proton


proton

A proton-proton collision: FINAL STATE



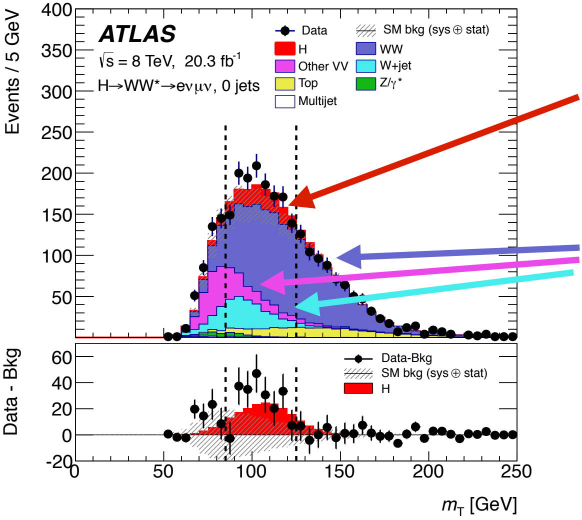
(actual final-state multiplicity ~ several hundred hadrons)

3 Signal and background models

The ggF and VBF production modes for $H \to WW^*$ are modelled at next-to-leading order (NLO) in the strong coupling α_S with the Powheg MC generator [22–25], interfaced with Pythia8 [26] (version 8.165) for the parton shower, hadronisation, and underlying event. The CT10 [27] PDF set is used and the parameters of the Pythia8 generator controlling the modelling of the parton shower and the underlying event are those corresponding to the AU2 set [28]. The Higgs boson mass set in the generation is 125.0 GeV, which is close to the measured value. The Powheg ggF model takes into account finite quark masses and a running-width Breit–Wigner distribution that includes electroweak corrections at NLO [29]. To improve the modelling of the Higgs boson p_T distribution, a reweighting scheme is applied to reproduce the prediction of the next-to-next-to-leading-order (NNLO) and next-to-next-to-leading-logarithm (NNLL) dynamic-scale calculation given by the HRes 2.1 program [30]. Events with ≥ 2 jets are further reweighted to reproduce the p_T^H spectrum predicted by the NLO Powheg simulation of Higgs boson production in association with two jets (H + 2 jets) [31]. Interference with continuum WW production [32, 33] has a negligible impact on this analysis due to the transverse-mass selection criteria described in Section 4 and is not included in the signal model.

Jets are reconstructed from topological clusters of calorimeter cells [50–52] using the anti-k_t algorithm with a radius parameter of $R = 0.4$ [53]. Jet energies are corrected for the effects of calorimeter non-

ATLAS H →WW* ANALYSIS [1604.02997]



(a) $N_{\rm jet} = 0$

That whole paragraph was just for the red part of this distribution (the Higgs signal).

Complexity of modelling each of the backgrounds is comparable

Lagrangian + colour

Quarks — 3 colours: $\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

Quark part of Lagrangian:

Let's write down QCD in full detail

(There's a lot to absorb here — but it should become more palatable as we return to individual elements later)

$SU(3$

corre

A representation is: $t^A = \frac{1}{2}\lambda^A$,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

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$$\mathcal{L}_q = \bar{\psi}_a (i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t^C_{ab} \mathcal{A}^C_\mu - m)\psi_b$$

$SU(3)$ local gauge symmetry $\leftrightarrow$ 8 $(= 3^2 - 1)$ generators $t^1_{ab} \ldots t^8_{ab}$ corresponding to 8 gluons $\mathcal{A}^1_\mu \ldots \mathcal{A}^8_\mu$.

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Lagrangian: gluonic part

Field tensor: $F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\nu - g_s\, f_{ABC} \mathcal{A}^B_\mu \mathcal{A}^C_\nu$ $\qquad [t^A, t^B] = i f_{ABC} t^C$

f_{ABC} are structure constants of $SU(3)$ (antisymmetric in all indices — $SU(2)$ equivalent was ϵ^{ABC}). Needed for gauge invariance of gluon part of Lagrangian:

$$\mathcal{L}_G = -\frac{1}{4} F_A^{\mu\nu} F^{A\,\mu\nu}$$

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$$\mathcal{L}_G = -\frac{1}{4} F^{\mu\nu}_A F^{A\,\mu\nu}$$

Solving QCD

Two main approaches to solving it

- Numerical solution with discretized space time (lattice)
- Perturbation theory: assumption that coupling is small

Also: effective theories

Lattice QCD

- Put all the quark and gluon fields of QCD on a 4D-lattice

  NB: with imaginary time

- Figure out which field configurations are most likely (by Monte Carlo sampling).

- You've solved QCD

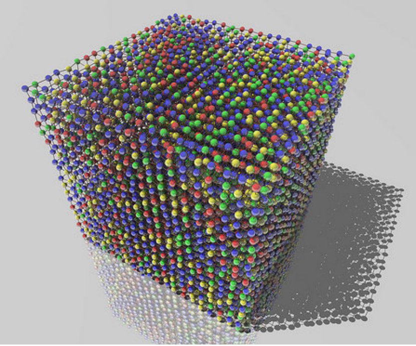
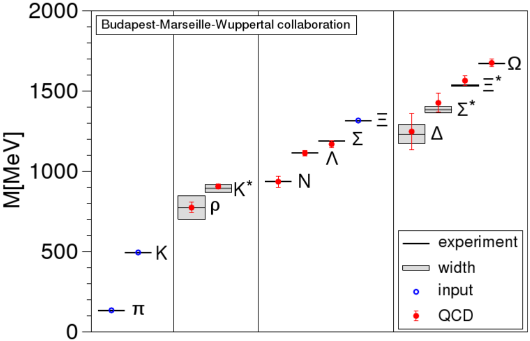


image credits: fdecomite [Flickr]

Lattice hadron masses

Lattice QCD is great at calculation static properties of a single hadron.

E.g. the hadron mass spectrum



Durr et al '08

Lattice for LHC?

How big a lattice do you need for an LHC collision @ 14 TeV?

Lattice spacing: $\dfrac{1}{14\ \text{TeV}} \sim 10^{-5}\,\text{fm}$

Lattice extent:

- non-perturbative dynamics for quark/hadron near rest takes place on timescale $t \sim \dfrac{1}{0.5\ \text{GeV}} \sim 0.4\ \text{fm}/c$
- But quarks at LHC have effective boost factor $\sim 10^4$
- So lattice extent should be ~ 4000 fm

Total: need $\sim 4 \times 10^8$ lattice units in each direction, or 3×10^{34} nodes total.

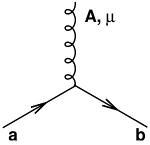
Plus clever tricks to deal with high particle multiplicity, imaginary v. real time, etc.

Perturbation theory

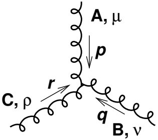
Relies on idea of order-by-order expansion small coupling, $\alpha_s \ll 1$

$$\alpha_s + \underbrace{\alpha_s^2}_{\text{small}} + \underbrace{\alpha_s^3}_{\text{smaller}} + \underbrace{\ldots}_{\text{negligible?}}$$

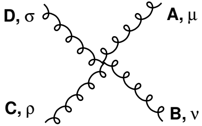
Interaction vertices of Feynman rules:



$-ig_s t^A_{ba} \gamma^\mu$



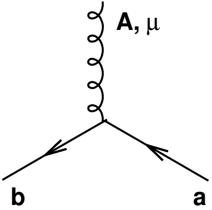
$$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$$



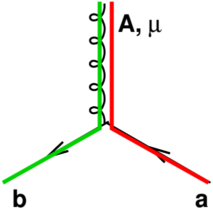
$$-ig_s^2 f^{XAC} f^{XBD}[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$$

These expressions are fairly complex, so you really don't want to have to deal with too many orders of them!
i.e. α_s had better be small. . .

What do Feynman rules mean physically?



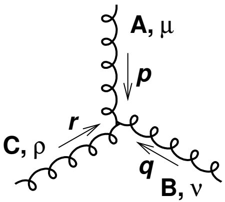
$$\bar{\psi}_b(-ig_s t^A_{ba}\gamma^\mu)\psi_a$$



$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{\bar{\psi}_b} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{t^1_{ab}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\psi_a}$$

A gluon emission **repaints** the quark colour.
A gluon itself carries colour and anti-colour.

What does "ggg" Feynman rule mean?



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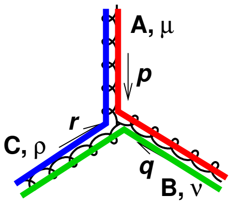
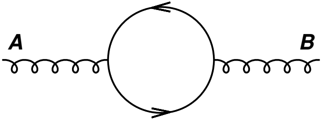


A gluon emission also repaints the gluon colours.

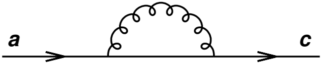
Because a gluon carries colour + anti-colour, it emits $\sim$ twice as strongly as a quark (just has colour)

Quick guide to colour algebra

$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$$



$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$



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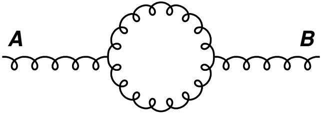
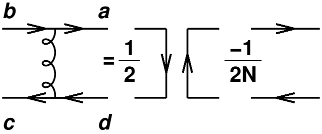


$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$



$N_c \equiv$ number of colours $= 3$ for QCD

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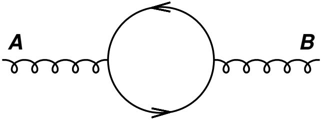

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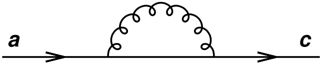

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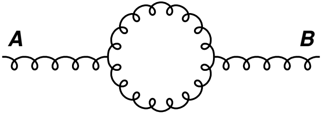

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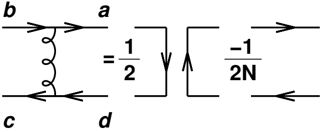

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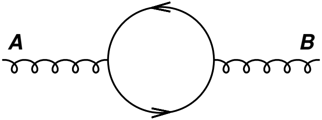

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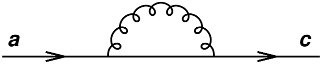

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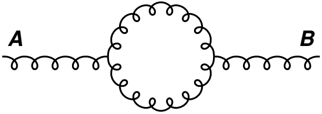

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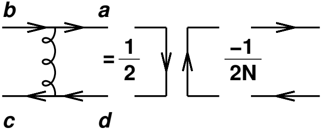

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QCD-colour-matrices.py

```python
#!/usr/bin/env python3
"""Small program to explore colour matrices and structure constants in QCD.

Written by Gavin Salam as an illustration for lecture 1 of the QCD
course at the 2024 European School of High-Energy Physics.
"""
import numpy as np

def main():

    print("Warning: the A,B,C indices here run from 0 to 7, not 1 to 8 as in the lecture notes.")
    print("Warning: the a,b,c indices here run from 0 to 2, not 1 to 3 as in the lecture notes.")

    nF = 3 # number of degrees of freedom in Fundamental representation (quarks)
    nA = 8 # number of degrees of freedom in Adjoint representation (gluos)
    lambdas = get_lambdas()
    ts = [0.5*l for l in lambdas]

    #---------------------------------------------
    header("check Tr(t^A t^B) = T_R delta^{AB}")
    for A in range(nA):
        for B in range(nA):
            trace = np.trace(np.matmul(ts[A],ts[B]))
            if abs(trace)>1e-10: print(f"{A=} {B=}: Trace(t^A t^B)={trace}")
```

https://gist.github.com/gavinsalam/6010007fd38f560d5424886c5b2f8649

How big is the coupling?

All couplings run (QED, QCD, EW), i.e. they depend on the momentum scale (Q^2) of your process.

The QCD coupling, $\alpha_s(Q^2)$, runs **fast:**

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s)\,, \qquad \beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + \dots)\,,$$

$$b_0 = \frac{11 C_A - 2n_f}{12\pi}\,, \qquad b_1 = \frac{17 C_A^2 - 5 C_A n_f - 3 C_F n_f}{24\pi^2} = \frac{153 - 19 n_f}{24\pi^2}$$

Note sign: Asymptotic Freedom, due to gluon to self-interaction

2004 Nobel prize: Gross, Politzer & Wilczek

- At high scales Q, coupling becomes small

  ➥quarks and gluons are almost free, interactions are weak

- At low scales, coupling becomes strong

  ➥quarks and gluons interact strongly — confined into hadrons

  Perturbation theory fails.

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Running coupling (cont.)

Solve $Q^2 \frac{\partial \alpha_s}{\partial Q^2} = -b_0 \alpha_s^2 \quad \Rightarrow \quad \alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 + b_0 \alpha_s(Q_0^2) \ln \frac{Q^2}{Q_0^2}} = \frac{1}{b_0 \ln \frac{Q^2}{\Lambda^2}}$

$\Lambda \simeq 0.2$ GeV (aka Λ_{QCD}) is the fundamental scale of QCD, at which coupling blows up.

- Λ sets the scale for hadron masses (NB: Λ not unambiguously defined wrt higher orders)
- Perturbative calculations valid for scales $Q \gg \Lambda$.

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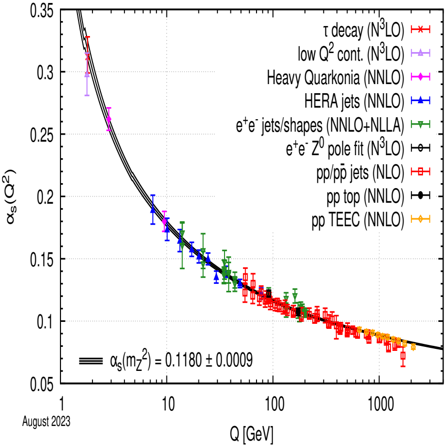
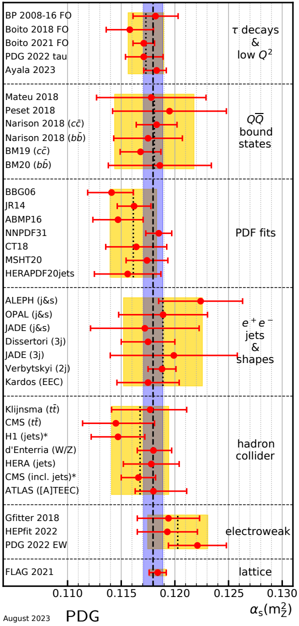


2023 PDG, QCD chapter

Strong-coupling determinations

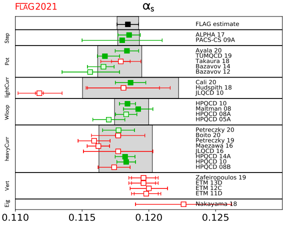


Current world average is

$$\alpha_s(m_Z) = 0.1180 \pm 0.0009$$

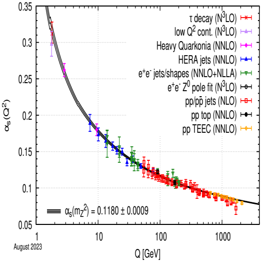
The world average has been stable over many years, but be aware of what goes into it:

- most determinations with small uncertainties ($\lesssim 0.001$) are systematics dominated
- some determinations (not shown on left) are in tension ($\sim 4\sigma$) with world average



Question of how best to determine α_s is an active research topic spanning many sub-fields of QCD!

QCD perturbation theory (PT) & LHC?



- Higgs, SM and searches at colliders probe scales $Q \sim p_t \sim 50$ GeV – 5 TeV
  The coupling certainly is small there!
- But we're colliding protons, $m_p \simeq 0.94$ GeV
  The coupling is large!

When we look at QCD events (this one is interpreted as $e^+e^- \to Z \to q\bar{q}$), we see:

- hadrons (PT doesn't hold for them)
- lots of them — so we can't say 1 quark/gluon $\sim$ 1 hadron, and we limit ourselves to 1 or 2 orders of PT.

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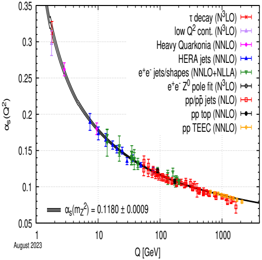

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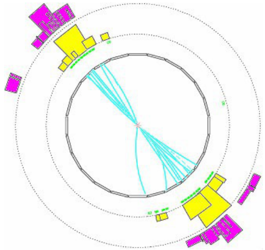


Neither lattice QCD nor perturbative QCD can offer a full solution to using QCD at colliders

What the community has settled on is perturbative QCD inputs + non-perturbative *modelling/factorisation*

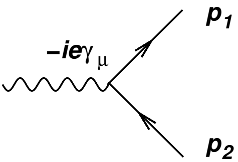
Rest of this lecture: take a simple environment ($e^+e^- \to$ hadrons) and see how PT allows us to understand why QCD events look the way they do.

Next lectures: dealing with incoming protons, jets, modern predictive tools

Soft gluon amplitude

Start with $\gamma^* \to q\bar{q}$:

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$



Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$

Make gluon *soft* $\equiv k \ll p_{1,2}$; ignore terms suppressed by powers of k:

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$\not{p}\not{k} + \not{k}\not{p} = 2p.k$

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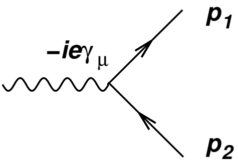

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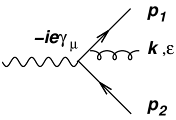

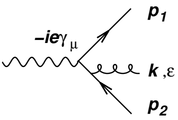

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Soft gluon amplitude

$$\bar{u}(p_1) i g_s \not{\epsilon}\, t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) = -i g_s \bar{u}(p_1) \not{\epsilon} \frac{\not{p}_1 + \not{k}}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\not{A}\not{B} = 2A.B - \not{B}\not{A}$:

$$= -i g_s \bar{u}(p_1) [2\epsilon.(p_1+k) - (\not{p}_1 + \not{k})\not{\epsilon}] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

Use $\bar{u}(p_1)\not{p}_1 = 0$ and $k \ll p_1$ (p_1, k massless)

$$\simeq -i g_s \bar{u}(p_1)[2\epsilon.p_1] \frac{1}{(p_1+k)^2} e_q \gamma_\mu t^A v(p_2)$$

$$= -i g_s \frac{p_1.\epsilon}{p_1.k} \underbrace{\bar{u}(p_1) e_q \gamma_\mu t^A v(p_2)}_{\text{pure QED spinor structure}}$$

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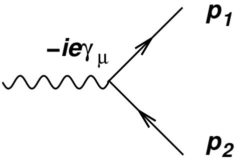

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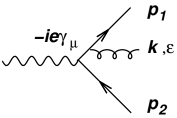

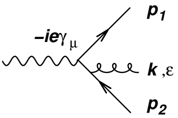

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Squared amplitude

$$|M^2_{q\bar{q}g}| \simeq \sum_{A,\text{pol}} \left| \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\; g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2 p_1.p_2}{(p_1.k)(p_2.k)}$$

Include phase space:

$$d\Phi_{q\bar{q}g} |M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}} |M^2_{q\bar{q}}|)\; \frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2 p_1.p_2}{(p_1.k)(p_2.k)}$$

Note property of factorisation into hard $q\bar{q}$ piece and soft-gluon emission piece, $d\mathcal{S}$.

$$d\mathcal{S} = E dE\, d\cos\theta\, \frac{d\phi}{2\pi} \cdot \frac{2\alpha_s C_F}{\pi} \frac{2 p_1.p_2}{(2p_1.k)(2p_2.k)} \qquad \theta \equiv \theta_{p_1 k}, \quad \phi = \text{azimuth}$$

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Soft & collinear gluon emission

Take squared matrix element and rewrite in terms of E, θ,

$$\frac{2p_1.p_2}{(2p_1.k)(2p_2.k)} = \frac{1}{E^2(1-\cos^2\theta)}$$

So final expression for soft gluon emission is

$$d\mathcal{S} = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

NB:

- It *diverges* for $E \to 0$ — *infrared (or soft) divergence*
- It *diverges* for $\theta \to 0$ and $\theta \to \pi$ — *collinear divergence*

Soft, collinear divergences derived here in specific context of $e^+e^- \to q\bar{q}$
But they are a very general property of QCD

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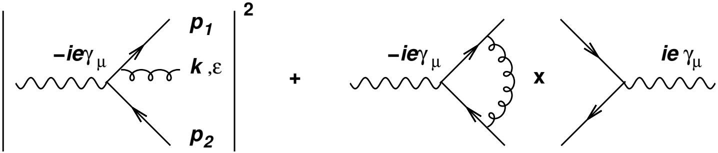
Real-virtual cancellations: total X-sctn

Total cross section: sum of all real and virtual diagrams



Total cross section must be *finite*. If real part has divergent integration, so must virtual part. (Unitarity, conservation of probability)

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} R(E/Q,\theta) - \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} V(E/Q,\theta)\right)$$

- $R(E/Q,\theta)$ parametrises real matrix element for hard emissions, $E \sim Q$.
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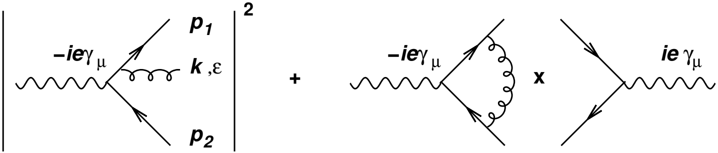

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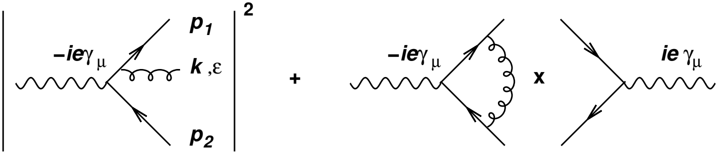

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Total X-section (cont.)

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- From calculation: $\lim_{E\to 0} R(E/Q,\theta) = 1$.
- For every divergence $R(E/Q,\theta)$ and $V(E/Q,\theta)$ should cancel:

$$\lim_{E\to 0} (R - V) = 0\,, \qquad \lim_{\theta\to 0,\pi} (R - V) = 0$$

Result:

- corrections to σ_{tot} come from hard ($E \sim Q$), large-angle gluons
- Soft gluons don't matter:

- Correct renorm. scale for α_s: $\mu \sim Q$ — perturbation theory valid

Total X-section (cont.)

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- corrections to σ_{tot} come from hard ($E \sim Q$), large-angle gluons
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  - Physics reason: soft gluons emitted on long timescale $\sim 1/(E\theta^2)$ relative to collision ($1/Q$) — cannot influence cross section.
  - Transition to hadrons also occurs on long time scale ($\sim 1/\Lambda$) — and can also be ignored.
- Correct renorm. scale for α_s: $\mu \sim Q$ — perturbation theory valid.

Total X-section (cont.)

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} \left(R(E/Q,\theta) - V(E/Q,\theta)\right)\right)$$

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total X-section (cont.)

Dependence of total cross section on only *hard* gluons is reflected in ‘good behaviour’ of perturbation series:

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + 1.045 \frac{\alpha_s(Q)}{\pi} + 0.94 \left(\frac{\alpha_s(Q)}{\pi} \right)^2 - 15 \left(\frac{\alpha_s(Q)}{\pi} \right)^3 + \cdots \right)$$

(Coefficients given for $Q = M_Z$)

Let's look at more "exclusive" quantities — structure of final state

Naive gluon multiplicity

Let's try and integrate emission probability to get the mean number of gluons emitted off a a quark with energy $\sim Q$:

$$\langle N_g \rangle \simeq \frac{2\alpha_s C_F}{\pi} \int^{Q} \frac{dE}{E} \int^{\pi/2} \frac{d\theta}{\theta}$$

This diverges unless we cut the integral off for transverse momenta ($k_t \simeq E\theta$) below some non-perturbative threshold, $Q_0 \sim \Lambda_{QCD}$.

On the grounds that perturbation no longer applies for $k_t \sim \Lambda_{QCD}$

Language of quarks and gluons becomes meaningless

With this cutoff, result is:

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Naive gluon multiplicity (cont.)

Suppose we take $Q_0 = \Lambda_{QCD}$, how big is the result?

Let's use $\alpha_s = \alpha_s(Q) = 1/(2b \ln Q/\Lambda)$

[Actually, over most of integration range this is optimistically small]

$$\langle N_g \rangle \simeq \frac{\alpha_s C_F}{\pi} \ln^2 \frac{Q}{\Lambda_{QCD}} \to \frac{C_F}{2b\pi} \ln \frac{Q}{\Lambda_{QCD}}$$

NB: given form for α_s, this is actually $\sim 1/\alpha_s$

Put in some numbers: $Q = 100$ GeV, $\Lambda_{QCD} \simeq 0.2$ GeV, $C_F = 4/3$, $b \simeq 0.6$,

$$\longrightarrow \langle N_g \rangle \simeq 2.2$$

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But the final result is $\sim 1/\alpha_s > 1\ldots$

Is perturbation theory completely useless?

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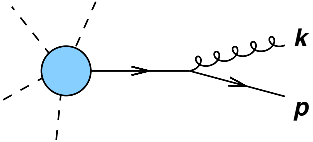
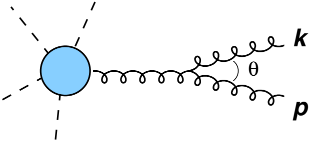
Is perturbation theory completely useless?

Given this failure of first-order perturbation theory, two possible avenues.

1. Continue calculating the next order(s) and see what happens

2. Try to see if there exist other observables for which perturbation theory is better behaved





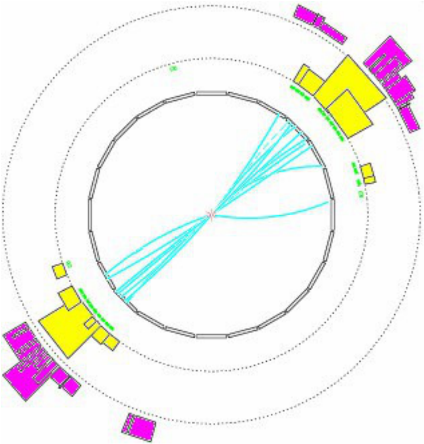
Gluon emission from quark: $\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$

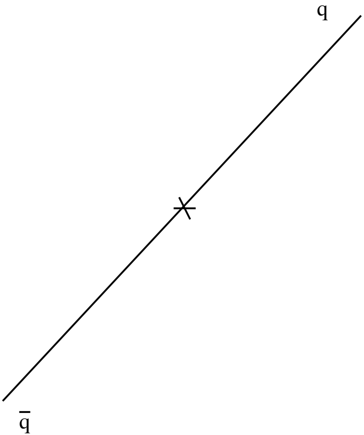
Gluon emission from gluon: $\frac{2\alpha_s C_A}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$

Both expressions valid only if $\theta \ll 1$ and energy soft relative to parent

- Same divergence structures, regardless of where gluon is emitted from
- All that changes is the colour factor ($C_F = 4/3$ v. $C_A = 3$)
- Expect low-order structure ($\alpha_s \ln^2 Q$) to be replicated at each new order

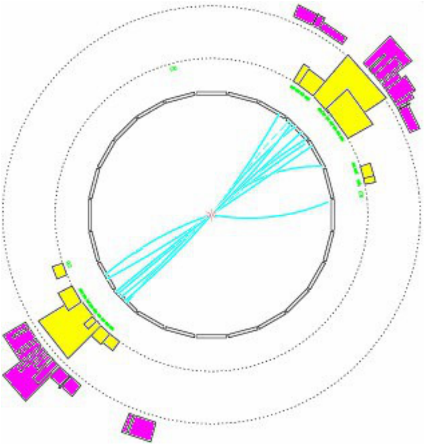
Picturing a QCD event

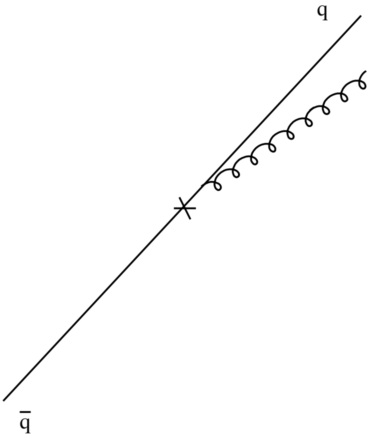




Start of with $q\bar{q}$

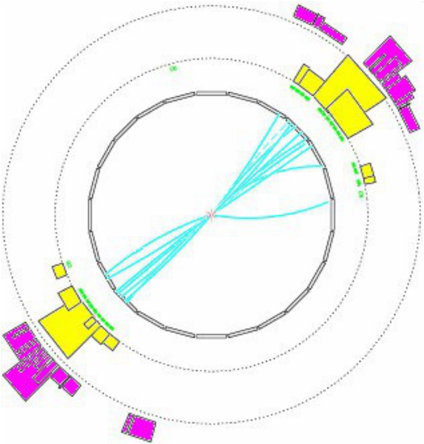
Picturing a QCD event

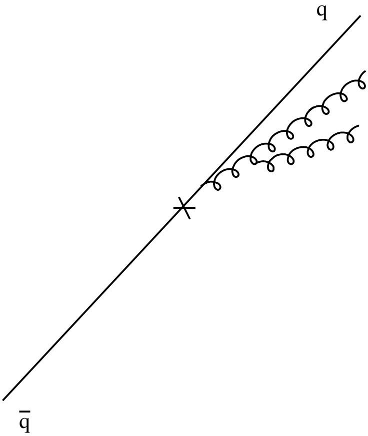




A gluon gets emitted at small angles

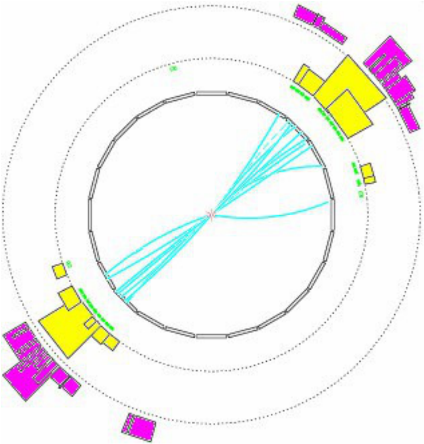
Picturing a QCD event

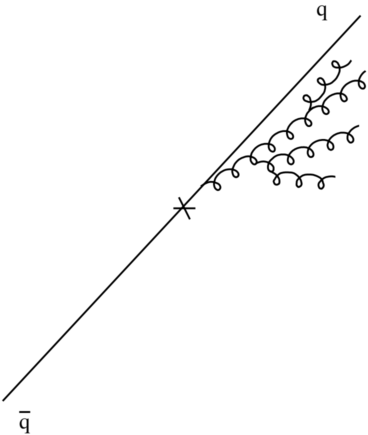




It radiates a further gluon

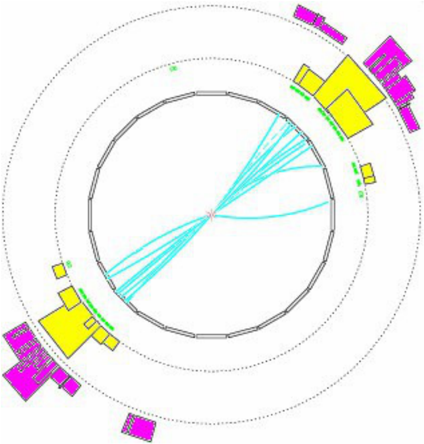
Picturing a QCD event

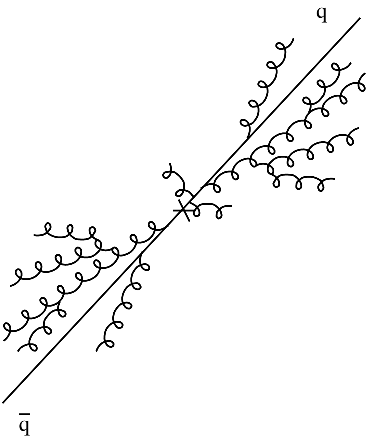




And so forth

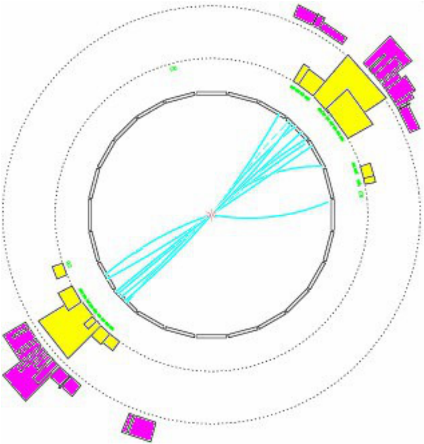
Picturing a QCD event

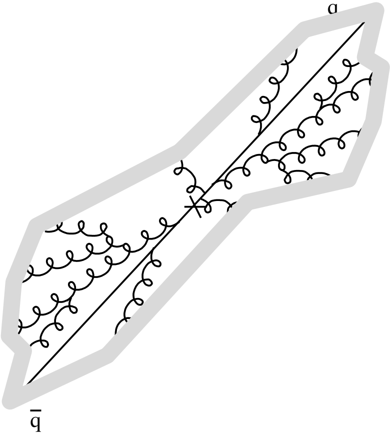




Meanwhile the same happened on other side of event

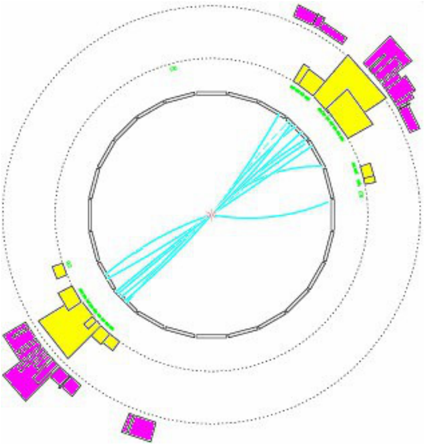
Picturing a QCD event

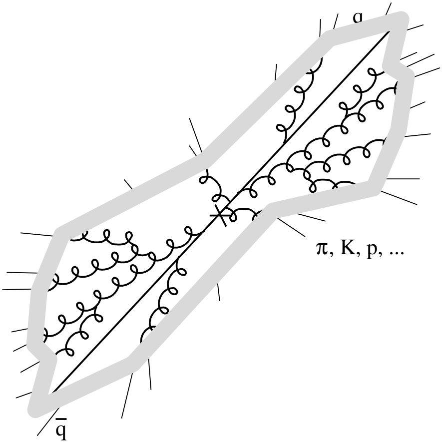




And then a non-perturbative transition occurs

Picturing a QCD event





Giving a pattern of hadrons that "remembers" the gluon branching

Hadrons mostly produced at small angle wrt $q\bar{q}$ directions or with low energy

Gluon v. hadron multiplicity

It turns out you can calculate the gluon multiplicity analytically, by summing all orders (n) of perturbation theory:

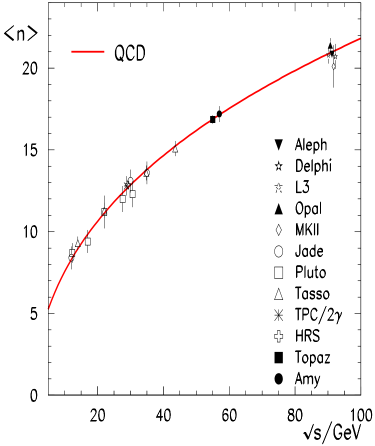
$$\langle N_g \rangle \sim \sum_n \frac{1}{(n!)^2} \left(\frac{C_A}{\pi b} \ln \frac{Q}{\Lambda} \right)^n$$

$$\sim \exp \sqrt{\frac{4C_A}{\pi b} \ln \frac{Q}{\Lambda}}$$

Compare to data for **hadron** multiplicity ($Q \equiv \sqrt{s}$)

Including some other higher-order terms and fitting overall normalisation

Agreement is amazing!



charged hadron multiplicity in e^+e^- events adapted from ESW

It's great that putting together all orders of gluon emission works so well!

This, together with a "hadronisation model", is part of what's contained in Monte Carlo event generators like Pythia, Herwig & Sherpa.

But are there things that we can calculate about the final state using just one or two orders perturbation theory?

Infrared and Collinear Safety (definition)

For an observable's distribution to be calculable in [fixed-order] perturbation theory, the observable should be infra-red safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \to \vec{p}_j + \vec{p}_k$$

whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel [collinear] or one of them is small [infrared]. [QCD and Collider Physics (Ellis, Stirling & Webber)]

Examples

- Multiplicity of gluons is *not* IRC safe [modified by soft/collinear splitting]
- Energy of hardest particle is *not* IRC safe [modified by collinear splitting]
- Energy flow into a cone *is* IRC safe [soft emissions don't change energy flow collinear emissions don't change its direction]

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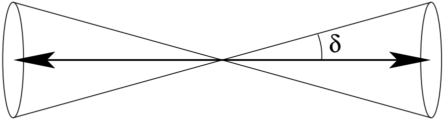
Sterman-Weinberg jets

The *original* (finite) jet definition

An event has 2 jets if at least a fraction $(1-\epsilon)$ of event energy is contained in two cones of half-angle δ.



$$\sigma_{2-jet} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \frac{d\theta}{\sin\theta} \left(R\left(\frac{E}{Q},\theta\right) \times \right.\right.$$
$$\left.\left. \times \left(1 - \Theta\left(\frac{E}{Q}-\epsilon\right)\Theta(\theta-\delta)\right) - V\left(\frac{E}{Q},\theta\right)\right)\right)$$

- For small E or small θ this is just like total cross section — full cancellation of divergences between real and virtual terms.
- For large E and large θ a *finite piece* of real emission cross section is *cut out.*
- Overall final contribution dominated by scales $\sim Q$ — cross section is perturbatively calculation.

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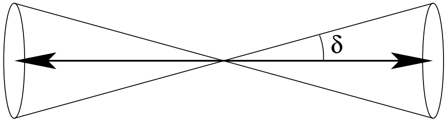

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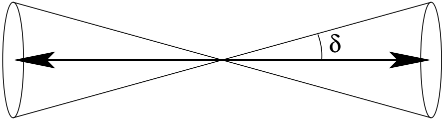

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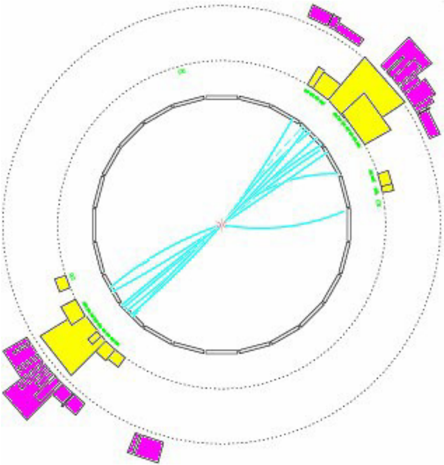
Real 2-jet event



Near 'perfect' 2-jet event

2 well-collimated jets of particles.

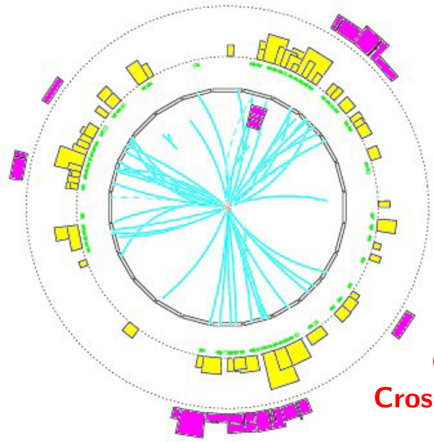
Nearly all energy contained in two cones.

Cross section for this to occur is

$$\sigma_{2-\text{jet}} = \sigma_{q\bar{q}}(1 - c_1\alpha_s + c_2\alpha_s^2 + \ldots)$$

where c_1, c_2 all ~ 1.

3-jet event



How many jets?

- Most of energy contained in 3 (fairly) collimated cones
- Cross section for this to happen is

$$\sigma_{3-\mathrm{jet}} = \sigma_{q\bar{q}}(c_1'\alpha_s + c_2'\alpha_s^2 + \ldots)$$

where the coefficients are all $\mathcal{O}(1)$

Cross section for extra gluon diverges
Cross section for extra jet is small, $\mathcal{O}(\alpha_s)$

NB: Sterman-Weinberg procedure gets complex for multi-jet events. 4th lecture will discuss modern approaches for defining jets.

1st lecture summary

- QCD at colliders mixes weak and strong coupling
- No calculation technique is rigorous over that whole domain

- Gluon emission repaints a quark's colour
- That implies that gluons carry colour too

- Quarks emit gluons, which emit other gluons: this gives characteristic "shower" structure of QCD events, and is the basis of *Monte Carlo simulations*
- To use perturbation theory one must measure quantities that are insensitive to the (divergent) soft & collinear splittings, like *jets*.