QCD (for Colliders) Lecture 2

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Yesterday:

- ➤ QCD Lagrangian
- ➤ Running coupling
- ➤ Soft gluon emission & its divergences

Today

- ➤ Real–virtual cancellation
- ➤ Factorisation
- ➤ Parton Distribution Functions (PDFs)
- ➤ Total cross sections & their perturbative series

GLUON EMISSION FROM A QUARK

Consider an emission with

- ➤ energy **E** ≪ **√s** ("soft")
- \blacktriangleright angle $\theta \ll 1$

("collinear" wrt quark)

Examine correction to some hard process with cross section **σ⁰**

$$
d\sigma \simeq \sigma_0 \times \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\theta}
$$

This has a divergence when E→**0 or** θ→**0** [in some sense because of quark propagator going on-shell]

How come we get finite cross sections?

Divergences are present in both real and virtual diagrams.

If you are "**inclusive**", i.e. your measurement doesn't care whether a soft/collinear gluon has been emitted then the **real and virtual divergences cancel.**

Beyond inclusive cross sections: infrared and collinear (IRC) safety *^e*+*e* ! *qq*¯ DIIU IIILLUƏIVE LI U Infrared and collinear (IRC) safety

For an observable's distribution to be calculable in [fixed-order] perturbation theory, the observable should be infra-red safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if \vec{p}_i *is any momentum occurring in its definition, it must be invariant under the branching*

$$
\vec{\rho_i} \rightarrow \vec{\rho_j} + \vec{\rho_k}
$$

whenever \vec{p}_j and \vec{p}_k are parallel [collinear] or one of them is small [infrared]. [QCD and Collider Physics (Ellis, Stirling & Webber)]

Examples Examples

Multiplicity of gluons is not IRC safe

I Enodified by soft/collinear splitting

Energy of hardest particle is not IRC safe

[modified by collinear splitting]

Energy flow into a cone is IRC safe

[soft emissions don't change energy flow, collinear emissions don't change its direction]

proton proton

A proton-proton collision: FINAL STATE

(actual final-state multiplicity ~ several hundred hadrons)

A proton-proton collision: FILLING IN THE PICTURE

proton proton

A proton-proton collision: SIMPLIFYING IN THE PICTURE

$$
\sigma (h_1 h_2 \to ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} \left(x_1, \mu_F^2\right) f_{j/h_2} \left(x_2, \mu_F^2\right)
$$

$$
\times \hat{\sigma}_{ij \to ZH + X}^{(n)} \left(x_1 x_2 s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),
$$

the heavy-discrete included included included included included in the PDFs μ and μ and μ

11 + µ µ− **b _** σ **u Z _ u H b proton proton** ^σ (h1h² [→] ^W ⁺ ^X) = # ∞ n=0 αn s % µ2 R &# i,j \$ dx1dx² fi/h¹ % x1, µ² F & fj/h² % x2, µ² F & [×] ^σˆ(n) ij→W+X % x1x2s, µ² R, µ² F & + O) Λ2 M⁴ W * , (1.15) [∗] LO is generally taken to mean the lowest order at which a quantity is non-zero. This definition is nearly always unambiguous, the one major exception being for the case of the hadronic branching ratio of virtual photons, Z, τ , etc., for which two conventions exist: LO can either mean the lowest order that contributes to the hadronic branching fraction, i.e. the term "1" in Eq. (1.7); or it can mean the lowest order at which the hadronic branching ratio becomes sensitive to the coupling, n = 1 in Eq. (1.8), as is relevant when extracting the value of the coupling from a measurement of the branching ratio. Because of this ambiguity, we avoid use of the term "LO" in that context. *Parton distribution function (PDF): e.g. number of up antiquarks carrying fraction x2 of proton's momentum ZH ZH+X*

the heavy-quark contribution already included in the PDFs [46,47,48].

$$
\sigma(h_1h_2 \to ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2 \frac{f_{i/h_1}\left(x_1, \mu_F^2\right) f_{j/h_2}\left(x_2, \mu_F^2\right)}{\times \hat{\sigma}_{ij \to ZH + X}^{(n)}\left(x_1x_2s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right)},
$$
\n
$$
\mu^{-1}
$$
\n
$$
\mathbf{b}
$$
\nParton distribution function (PDF): e.g. number of up
\nquarks carrying fraction x₁ of
\nproton's momentum
\n
$$
\overline{\mathbf{a}}
$$

the heavy-quark contribution already included in the PDFs [46,47,48].

THE MASTER EQUATION — FACTORISATION heavy-quark contributions at high Q² scales. For scales near the threshold, it is instead necessary to appropriate the standard massive coefficient functions to account for a coefficient for a coefficient for \mathbf{r}

the heavy-definition alternation alternation allows and included in the PDFs [46,47,48]. In the PDFs [46,47,48

$$
\sigma(h_1h_2 \to ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} \left(x_1, \mu_F^2\right) f_{j/h_2} \left(x_2, \mu_F^2\right)
$$

\n
$$
\times \frac{\hat{\sigma}_{ij \to ZH + X}^{(n)} \left(x_1 x_2 s, \mu_R^2, \mu_F^2\right)}{\hat{\sigma}_{ij \to ZH + X}^{(n)} \left(x_1 x_2 s, \mu_R^2, \mu_F^2\right)} + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),
$$

\nAt each perturbative order n
\nwe have a specific "hard matrix element" (sometimes several for different subprocesses)
\n
$$
\overline{\sigma}
$$

\n
$$
\overline{\sigma}
$$

15 + µ µ− **b _** σ **u Z _ u H b proton proton** ^σ (h1h² [→] ^W ⁺ ^X) = # ∞ n=0 αn s % µ2 R &# i,j \$ dx1dx² fi/h¹ % x1, µ² F & fj/h² % x2, µ² F & [×] ^σˆ(n) ij→W+X % x1x2s, µ² R, µ² F & + O) Λ2 M⁴ W * , (1.15) [∗] LO is generally taken to mean the lowest order at which a quantity is non-zero. This definition is nearly always unambiguous, the one major exception being for the case of the hadronic branching ratio of virtual photons, Z, τ , etc., for which two conventions exist: LO can either mean the lowest order that contributes to the hadronic branching fraction, i.e. the term "1" in Eq. (1.7); or it can mean the lowest order at which the hadronic branching ratio becomes sensitive to the coupling, n = 1 in Eq. (1.8), as is relevant when extracting the value of the coupling from a measurement of the branching ratio. Because of this ambiguity, we avoid use of the term "LO" in that context. May 5, 2016 21:57 *Additional corrections from non-perturbative effects (higher "twist", suppressed by powers of QCD scale (Λ) / hard scale) ZH ZH+X*

the heavy-quark contribution already included in the PDFs [46,47,48].

PARTON DISTRIBUTION FUNCTIONS (PDFs)

DEEP INELASTIC SCATTERING

Hadron-hadron is complex because of two incoming partons — so start with simpler Deep Inelastic Scattering (DIS).

$$
x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xyz
$$

$$
\sqrt{s} = \text{c.o.m. energy}
$$

Kinematic relations:

- \triangleright Q^2 = photon virtuality \leftrightarrow *transverse resolution* at which it probes proton structure
- \triangleright $x =$ *longitudinal momentum fraction* of struck parton in proton
- \blacktriangleright $y =$ momentum fraction lost by electron (in proton rest frame)

DEEP INELASTIC SCATTERING

DEEP INELASTIC SCATTERING E.g.: extracting *u* & *d* distributions [PDFs] $\overline{}$

Write DIS X-section to zeroth order in α_{s} ('quark parton model'):

$$
\frac{d^2 \sigma^{em}}{dx dQ^2} \simeq \frac{4\pi \alpha^2}{xQ^4} \left(\frac{1 + (1 - y)^2}{2} F_2^{em} + \mathcal{O}(\alpha_s) \right)
$$

 $\propto F_2^{em}$ [structure function]

$$
F_2 = x(e_u^2 u(x) + e_d^2 d(x)) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)
$$

 $[u(x), d(x)]$: parton distribution functions (PDF)]

NB:

- \triangleright use perturbative language for interactions of up and down quarks
- ▶ but distributions themselves have a *non-perturbative* origin.

For initial state splitting, hard process occurs after splitting, and momentum entering hard process is modified: $p \rightarrow zp$.

$$
\sigma_{g+h}(p) \simeq \sigma_h(zp) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2} \qquad p \qquad \longrightarrow \qquad \frac{zp}{\gamma} \left(\frac{\sigma_h}{1-z} \right).
$$

For virtual terms, momentum entering hard process is unchanged

$$
\sigma_{V+h}(p) \simeq -\sigma_h(p) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2} \qquad \qquad \underbrace{p}_{\text{e.g.}} \qquad \qquad \underbrace{p}_{\text{e.g.}}.
$$

Total cross section gets contribution with two different hard X-sections

$$
\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \int \frac{dk_t^2}{k_t^2} \frac{dz}{1-z} [\sigma_h(zp) - \sigma_h(p)]
$$

NB: We assume σ_h involves momentum transfers $\sim Q \gg k_t$, so ignore extra transverse momentum in σ_h

Initial-state collinear divergence not in handout

<u>Higher order corrections from initial state splittings?</u>

$$
\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_0^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{infinite}} \underbrace{\int \frac{dz}{1-z} [\sigma_h(zp) - \sigma_h(p)]}_{\text{finite}}
$$

In soft limit $(z \to 1)$, $\sigma_h(zp) - \sigma_h(p) \to 0$: soft divergence cancels.

For $1 - z \neq 0$, $\sigma_h(zp) - \sigma_h(p) \neq 0$, so z integral is non-zero but finite.

BUT: k_t integral is just a factor, and is *infinite*

This is a collinear $(k_t \rightarrow 0)$ divergence. Cross section with incoming parton is not collinear safe!

This always happens with coloured initial-state particles So how do we do QCD calculations in such cases?

Parton distributions and DGLAP I Collinear divergence for incoming partons *not cancelled* by virtuals. AP

➤ Write up-quark distribution in proton as IV Situation angle distribution in proton as

$$
u(x,\mu_F^2)
$$

- ➤ Perturbative collinear (IR) divergence absorbed into the parton distribution (NB divergence not physical: non-perturbative physics provides a physical cutoff) $\sqrt{I/I}$ noncative commeas (in) arrespence asserbed mic the parton alsthsatic.
- \triangleright μ_F is the **factorisation scale** a bit like the renormalisation scale (μ_R) for the running coupling. Choice of factorization scale, *µ*², is and *Q*², is and *Q*², is and *Q*², is and *Q*² and *Q*² and *Q*²
	- ➤ As you vary the factorisation scale, the parton distributions evolve with a renormalisation-group type equation \blacktriangleright As vou vary the factorisation scale, the parton distributions evolve with a

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations

DGLAP EQUATION DGLAP equation (*q q*) [Initial-state splitting] [DGLAP]

take derivative wrt factorization scale μ^2

Change convention: (a) now *fix outgoing* longitudinal momentum *x*; (b)

Awkward to write real and virtual parts separately. Use more compact notation:

$$
\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi} \underbrace{\int_x^1 dz \, P_{qq}(z) \, \frac{q(x/z,\mu^2)}{z}}_{P_{qq} \otimes q}, \qquad P_{qq} = C_F \left(\frac{1+z^2}{1-z}\right)_+
$$

This involves the *plus prescription:*

$$
\int_0^1 dz \, [g(z)]_+ f(z) = \int_0^1 dz \, g(z) \, f(z) - \int_0^1 dz \, g(z) \, f(1)
$$

 $z = 1$ divergences of $g(z)$ cancelled if $f(z)$ sufficiently smooth at $z = 1$

DGLAP EQUATION DGLAP flavour structure [Initial-state splitting]

[DGLAP]

Proton contains both quarks and gluons — so DGLAP is a *matrix in flavour* space:

$$
\frac{d}{d\ln Q^2} \left(\begin{array}{c} q \\ g \end{array} \right) = \left(\begin{array}{cc} P_{q \leftarrow q} & P_{q \leftarrow g} \\ P_{g \leftarrow q} & P_{g \leftarrow g} \end{array} \right) \otimes \left(\begin{array}{c} q \\ g \end{array} \right) \right)
$$

[In general, matrix spanning all flavors, anti-flavors, $P_{qq'} = 0$ (LO), $P_{\bar{q}g} = P_{qg}$]

Splitting functions are:

$$
P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right], \qquad P_{gq}(z) = C_F \left[\frac{1 + (1-z)^2}{z} \right],
$$

\n
$$
P_{gg}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \delta(1-z) \frac{(11C_A - 4n_f T_R)}{6}.
$$

Have various symmetries / significant properties, e.g.

 \triangleright P_{qg} , P_{gg} : *symmetric* $z \leftrightarrow 1 - z$ (except virtuals) \triangleright P_{qq} , P_{gg} : *diverge for* $z \to 1$ soft gluon emission \triangleright P_{gg} , P_{gg} : *diverge for* $z \to 0$ Implies PDFs grow for $x \to 0$

2015 EPS HEP prize to Bjorken, Altarelli, Dokshitzer, Lipatov & Parisi

NLO DGLAP

NLO:

$$
P_{\rm ps}^{(1)}(x) = 4 C_F n_f \left(\frac{20}{9} \frac{1}{x} - 2 + 6x - 4H_0 + x^2 \left[\frac{8}{3} H_0 - \frac{56}{9} \right] + (1+x) \left[5H_0 - 2H_{0,0} \right] \right)
$$

$$
P_{\text{qg}}^{(1)}(x) = 4 C_A n_f \left(\frac{20}{9} \frac{1}{x} - 2 + 25x - 2p_{\text{qg}}(-x)H_{-1,0} - 2p_{\text{qg}}(x)H_{1,1} + x^2 \left[\frac{44}{3} H_0 - \frac{218}{9} \right] \right)
$$

+4(1-x)\left[H_{0,0} - 2H_0 + xH_1 \right] - 4\zeta_2 x - 6H_{0,0} + 9H_0 \right) + 4 C_F n_f \left(2p_{\text{qg}}(x) \left[H_{1,0} + H_{1,1} + H_2 - \zeta_2 \right] + 4x^2 \left[H_0 + H_{0,0} + \frac{5}{2} \right] + 2(1-x) \left[H_0 + H_{0,0} - 2xH_1 + \frac{29}{4} \right] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_0 \right)

$$
P_{\rm gq}^{(1)}(x) = 4 C_A C_F \left(\frac{1}{x} + 2p_{\rm gq}(x) \left[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6} H_1 \right] - x^2 \left[\frac{8}{3} H_0 - \frac{44}{9} \right] + 4 \zeta_2 - 2
$$

-7H₀ + 2H_{0,0} - 2H₁x + (1 + x) \left[2H_{0,0} - 5H₀ + \frac{37}{9} \right] - 2p_{\rm gq}(-x)H_{-1,0} - 4 C_Fn_f \left(\frac{2}{3} x
-p_{gq}(x) \left[\frac{2}{3} H_1 - \frac{10}{9} \right] \right) + 4 C_F² \left(p_{\rm gq}(x) \left[3H_1 - 2H_{1,1} \right] + (1 + x) \left[H_{0,0} - \frac{7}{2} + \frac{7}{2} H_0 \right] - 3H_{0,0}
+ 1 - \frac{3}{2} H_0 + 2H_1 x \right)

$$
P_{gg}^{(1)}(x) = 4 C_A n_f \left(1 - x - \frac{10}{9} p_{gg}(x) - \frac{13}{9} \left(\frac{1}{x} - x^2 \right) - \frac{2}{3} (1 + x) H_0 - \frac{2}{3} \delta (1 - x) \right) + 4 C_A^2 \left(27 + (1 + x) \left[\frac{11}{3} H_0 + 8 H_{0,0} - \frac{27}{2} \right] + 2 p_{gg} (-x) \left[H_{0,0} - 2 H_{-1,0} - \zeta_2 \right] - \frac{67}{9} \left(\frac{1}{x} - x^2 \right) - 12 H_0
$$

$$
- \frac{44}{3} x^2 H_0 + 2 p_{gg}(x) \left[\frac{67}{18} - \zeta_2 + H_{0,0} + 2 H_{1,0} + 2 H_2 \right] + \delta (1 - x) \left[\frac{8}{3} + 3 \zeta_3 \right] + 4 C_F n_f \left(2 H_0
$$

$$
+ \frac{2}{3} \frac{1}{x} + \frac{10}{3} x^2 - 12 + (1 + x) \left[4 - 5 H_0 - 2 H_{0,0} \right] - \frac{1}{2} \delta (1 - x) \right) .
$$

$$
P_{ab} = \frac{\alpha_s}{2\pi} P^{(0)} + \frac{\alpha_s^2}{16\pi^2} P^{(1)}
$$

Curci, Furmanski & Petronzio '80

$NNLO$ DGLAP NNLO splitting functions [Initial-state splitting]

Divergences for *x* 1 are understood in the sense of -distributions. The third-order pure-singlet contribution to the quark-quark splitting function (2.4), corre-

sponding to the anomalous dimension (3.10), is given by $P_{\rm ps}^2$ *x* 16*C_AC_Fn_f* $\frac{4}{3}$ $\frac{1}{x}$ *x*² $\frac{13}{3}$ H₁₀ $\frac{14}{9}$ H₀ $\frac{1}{2}$ H₁₂ H₁₁₀ 2H₁₀₀ 4 *x*² 3 *x*² 3 ^x 3^x 3^x 3^x 3^x 3^x 3² 3*x*₂¹ 4 3²₂¹ 4 3²₂¹ 4 3²₂¹ 4 3²₃¹ 4 3²₃² 4 3³ 4 34³ 4 34466 3²¹ 4 34466 3²

3H₁₀ 2H₁₁ 1 3₄ 34² 5² 5² 5² 5² $\frac{13}{6}$ H₁₀ 3*x*H₁₀ H₃₀ H₂²₂ 2H₂ 2₁₀ 3H₂ 0 0 $\frac{1}{2}$ H₀ 6₂ $\frac{1}{2}$ H₂⁶₂ $\frac{7}{2}$ H₁²₃ $\frac{113}{18}$ ₁₂ $\frac{826}{18}$ ₂ $\frac{826}{27}$ H₁₀ 3 4 5 H_{20} 16 $\frac{16}{3}$ H₁₀ 6*x*H₁₀ $\frac{31}{6}$ H₀₀₀ $\frac{17}{6}$ H₂₁ $\frac{117}{20}$ ζ_2 ² 9H₀ ζ_3 $\frac{5}{2}$ H₁ ζ_2 2H₂₁₀ 1 $\text{H}_{100} \quad \text{2H}_{12} \quad \text{H}_2 \zeta_2 \quad \frac{7}{2} \text{H}_{200} \quad \text{H}_{110} \quad \text{2H}_{211} \quad \text{H}_{31} \quad \frac{1}{2} \text{H}_4 \quad \text{5H}_{20} \quad \text{H}_{21}$ $_{\rm H_{0\,0\,0\,0}}$ 1 ζ2 2 4H 30 4H₀ ζ_3 $\frac{32}{9}$ H₀ $\frac{29}{12}$ H₀ $\frac{235}{12}\zeta_2$ $\frac{511}{12}$ $\frac{97}{12}$ H₁ $\frac{33}{4}$ H₂ H₃ 11 $\frac{1}{2}H_0\zeta_2$ 11 $\frac{11}{2}\zeta_3$ $\frac{3}{2}H_{2\,0}$ 10H₀₀₀ $\frac{2}{3}x^2$ $\frac{83}{4}H_{0\,0}$ $\frac{243}{4}H_0$ 10 ζ_2 $\frac{511}{8}$ $\frac{97}{8}H_1$ $\frac{4}{3}H_2$ $4\zeta_3$ H₀₅² H₃ ζ_2 B₃ + 2₃ + 3₃ + 3₃ + 3₅ + 3₃ + 3³₄ + 3³₄ + 4³₄ + 5³₄ + 5³₄ + 5³₄ + 5⁴₄ $\frac{2}{2}x^2$ H₂ ζ₂ 3 19 $\frac{1}{6}$ H₀ 2 9 1 1 $\frac{1}{3}$ 1 *x* $\frac{4}{3}$ 3 $_{\rm H_2}$ $\begin{split} \frac{25}{4}\mathrm{H}_0\mathrm{O} &\quad \frac{158}{12}\mathrm{H}_0\quad \frac{17}{12}\mathrm{H}_0\quad \frac{17}{34}\mathrm{H}_2\quad \frac{7}{4}\mathrm{H}_2\quad \mathrm{H}_3\quad \mathrm{SH}_{2,0}\quad \mathrm{H}_{2,1}\quad \mathrm{H}_0\zeta_2\quad x^2\,\frac{55}{12}\\ \frac{85}{12}\mathrm{H}_1\quad \frac{2}{3}\mathrm{H}_0\mathrm{O} &\quad \frac{13}{6}\mathrm{H}_0\quad \frac{25}{34}\mathrm{H}_2\quad \frac{28}{3}\mathrm{$ $\frac{101}{54}$ $\frac{73}{4}$ ζ_2 $\frac{73}{4}$ H₂ H₃ 5 H₂ 0 H₂ 1 H₀ ζ_2 x^2 $\frac{55}{12}$ 7H0ζ³ 6H0 ⁰ζ² 4H0 ⁰ ⁰ ⁰ H2 ⁰ ⁰ 2H2 ¹ ⁰ 2H2 ¹ ¹ 4H3 ⁰ H3 ¹ 6H4 (4.12) Due to Eqs. (3.11) and (3.12) the three-loop gluon-quark and quark-gluon splitting functions read P_{qg}^2 *x* 16*C_AC_Fn_f* p_{qg} *x* $\frac{39}{2}$ H₁₍₃ 4H₁₁₁ 3H₂ 0 0 $\frac{15}{4}$ H₁₂ $\frac{9}{4}$ H₁₁₀ 3H₂₁₀

 $H_0 \zeta_3$ 2 $H_{2,1,1}$ 4 $H_2 \zeta_2$ $\frac{173}{12} H_0 \zeta_2$ $\frac{551}{72} H_{0,0}$ $\frac{64}{3} \zeta_3$ ζ_2 ² $\frac{49}{4} H_2$ $\frac{3}{2} H_{1,0,0,0}$ $\frac{1}{3} H_{1,0,0}$ 16

 $\frac{385}{72}H_{10} \quad \frac{31}{2}H_{11} \quad \frac{113}{12}H_1 \quad \frac{49}{4}H_{20} \quad \frac{5}{2}H_1\zeta_2 \quad \frac{79}{6}H_{000} \quad \frac{173}{12}H_3 \quad \frac{1259}{32} \quad \frac{2833}{216}H_0 \nonumber \\ \frac{6H_{21}}{32} \quad \frac{3H_{112}}{320} \quad \frac{1259}{216} \quad \frac{1237}{6}H_{121} \quad \frac{2833}{120} \quad$ $^{6}H_{13}$ $^{49}_{4}$ $^{42}_{4}$ $^{29}_{2}$ $^{17}_{29}$ $^{18}_{20}$ $^{17}_{21}$ $^{16}_{12}$ $^{5}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ $^{17}_{2}$ 6H 1 1 10 6H 1 100 6H 1 12 9H 1 0ζ² 9H 1 1ζ² 2H ¹²⁰ 11 ² ^H ¹⁰⁰⁰ 6H 1 3 1 *^x ^x*² ⁵⁵ ¹² ⁴ζ³ 23 ⁹ H1 0 4 3 H110 1 *^x ^x*² ² 3 H100 371 ¹⁰⁸H1 23 ⁹ H1 1 2 3 H111 1 *x* 6H210 3H211 5 6 H111 7H200 2H1 2 39H0ζ³ 4H2ζ² 16 ³ ^ζ³ H1 1 0 154 ³ H0ζ² 899 ²⁴ H0 ⁰ 121 ¹⁰ ^ζ² ² 607 ³⁶ H2 5 2 H1ζ² 65 ⁶ H1 0 0 29 ¹²H1 ⁰ 13 ¹⁸H1 ¹ 1189 ¹⁰⁸ H1 67 ³ H2 ¹ 29H2 ⁰ 949 ³⁶ ^ζ² 67 ² H0 0 0 142 ³ H3 215 32 3989 ⁴⁸ H0 2H 3 0 1 *x* H ₁₀₀ 10H 2ζ2 6H 200 2H₀ 0ζ2 9H ₁ 10 7H 12 9H 20 2H_{3 1} 4H 2 10 4H4 4H3 0 4H0000 ² ^H 1 0 ² ¹ *^x* ^H ¹ζ² 4H ²⁰⁰ 2H0 0ζ² H2ζ² 3H1 1 0 2H0 ⁰⁰⁰ H 3 0 9H2 1 0 9 2 H2 1 1 11 ³ H1 1 1 19 ² H2 0 0 9 2 H1 ² 91 ² H0ζ³ 8H ²ζ² 5 2 H 1 10 5 2 H 1 2 9 2 H 1 0 39 ² ^H 2 0 473 ¹² H0ζ² 1853 ⁴⁸ H0 0 217 ¹² ^ζ³ 59 ⁴ ^ζ² ² 169 ¹⁸ H2 13 ⁴ H1ζ² 2 3 H100 167 ²⁴ H1 0 191 ¹⁸ H1 1 1283 ¹⁰⁸ H1 185 ¹² H2 1 75 ⁴ H2 0 170 ⁹ ^ζ² 85 ⁴ H000 425 ¹² H3 7693 192 3659 ⁴⁸ H0 ²*x x*H2 2 4H3 0 4H 2 2 37 5 $\begin{array}{l} 16 C_A n_j^2-\frac{1}{6} P_{88} \times \begin{array}{l} H_{12} \times H_{12} \times H_{160} \times H_{110} \times H_{111} \times \frac{229}{18} H_{9} - \frac{4}{3} H_{00} \times \frac{11}{2} \times \frac{1}{6} H_{2} \\ \frac{53}{18} H_{9} - \frac{17}{6} H_{90} - \frac{1}{5} \times \frac{11}{18} \xi_2 - \frac{11}{108} - \frac{13}{3} P_{88} \times \begin{array}{l} H_{10} \times H_{2$ H_{1 1}1 2H₂ 20 3H₁ 2 3H₀ 2 3H₀ 0 H₁ (2 H₁₀ 0 H₁₁ 10 2H₁₁
2H₁₂ 2H₁₂ _{Pqg} *x* H₁₁² 2H₁₂ 6H ₁₀ H₁₁ 2H₂² 3H₂₀ 3H₁₀ 0 3H₁₁₂ 3H₂₀

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6H ₁ ₁ 1 0 2H ₁ 3 2H ₁₂₁ $\frac{1}{x}$ x^2 $\frac{2}{3}$ H₂₁ $\frac{32}{9}$ _{S₂ 2H₁₀₀ $\frac{4}{3}$ H₁₁₀ $\frac{10}{9}$ H₁₁} $\frac{8}{3}$ H₁₀ $\frac{3}{2}$ H₁₀ $6\zeta_3$ $\frac{161}{36}$ H₁ $\frac{2351}{108}$ $\frac{2}{3}$ $\frac{1}{x}$ x^2 $\frac{26}{3}$ H _{1 0} $\frac{28}{9}$ H₀ 2H _{1 10} 2H ₁₂ H₁ζ₂ H₁ζ₂ 10 $\frac{31}{6}$ H₂ H₁ 1 1 1 x 15H₀000 5H₂ ζ_2 $\frac{29}{6}$ ζ_3 $\frac{23}{6}$ H₁ 1₁
 $\frac{31}{6}$ H₂ 0 $\frac{17}{12}$ H₁ 0 $\frac{551}{20}$ ζ_2 $\frac{29}{4}$ H₁ 0 0 $\frac{113}{4}$ H₂ $\frac{18691}{72}$ H₀ 65 11 3 H4 5 H_0 0ζ2 H_1 10 31 $\frac{1}{6}H_2$ 0 2243 108 265 $\frac{6}{6}$ H₁₀₀ 33 $\frac{2}{2}H_{200}$ 19H₂₁ 31 $\frac{1}{12}H_{11}$ 23 $\frac{2}{2}$ H 20 497 $\frac{17}{36}$ ζ₂ 29 $\frac{6}{6}H_1\zeta_2$ 143 $\frac{12}{12}H_3$ 11 $\frac{1}{6}$ H₁₁₁ 19 $\frac{1}{12}H_0\zeta_2$ 1223 $\frac{1}{72}$ H₁ 43 $\frac{1}{6}H_{000}$ 3011 $\frac{36}{36}H_{00}$ 1 *x* 8H₂₁₀ 4H₁₂ 7H 1 10 35 $5H_2$ $\frac{6}{6}$ H_{111} $5H_2$ $\frac{7}{2}$ $11H_2$ 00 $\frac{1}{3}$ H_{10} $\frac{10}{2}$ H_1 $\frac{7}{2}$ $8H_3$ $10H_2$ 10
 $5H_2$ $\frac{7}{2}$ $4H_2$ 11 H_3 0 $36H_0$ $\frac{7}{2}$ $5H_2$ $\frac{7}{2}$ 2 H_1 12 $6H_1$ 10 1 H ₁₀ 15 11H0000 5H3 1 25 ⁴ H111 13 ² ^H ²ζ² 27 ² ^H ²⁰⁰ 11 ² ^H 3 0 13 ² H2ζ² 17 ⁴ H100 13H 2 10 17 ¹²H111 3 4 H4 1 4 H0 0ζ² H1 2 11 ² H110 79 ¹²H2 0 67 ⁸ H1 0 263 ⁸ ^ζ² 2 $\begin{split} \frac{119}{3}\zeta_3 & \frac{957}{24}H_2 - \frac{305}{12}H_{-10} - 24H_0 \zeta_3 & H_1 \zeta_2 - \frac{1375}{122}H_0 - \frac{1889}{18} - 38H_{-100} - \frac{21}{2}H_{21} \\ \frac{79}{4}\mathrm{H_{200}} & \frac{217}{24}H_{11} - \frac{7}{2}H_{-20} - \frac{79}{72}\zeta_2 - \frac{4}{3}H_1 \zeta_2 - \frac{17}{12}H_{111} - \frac{$ $\begin{aligned} &2H_{-3,0}-7H_1\xi_3-5H_{2,2}-6H_{3,0}-6H_{3,1}-H_{2,10}-4H_{2,00}-3H_{2,1}-2H_{2,11}-\frac{5}{2}H_{2,0}\\ &\frac{61}{8}H_2-\frac{61}{8}\zeta_2-\frac{87}{8}H_1-\frac{11}{2}H_{1,2}-\frac{61}{8}H_{1,1}-\frac{17}{2}H_{1,0}-7H_{0,0}\zeta_2-\frac{5}{2}H_{1,00}-\frac{5}{2}H_{1,10}-\frac{19}{2}\zeta_3\\ &\frac{$ $6H_{1,2,0}$ $6H_{1,2,1}$ $4p_{qq}$ *x* $H_{0,0,0,0}$ *H* $_{2,0}$ *H* $_{1,1,0}$ *H* $_{2,0,0}$ $\frac{1}{2}$ *H* $_{1,2,0}$ $\frac{5}{2}$ *H* $_{1,0}$ $\frac{5}{2}$
 $\frac{1}{4}$ H 100 $\frac{1}{2}$ H 30 $\frac{1}{2}$ H 1²₂ H 1 100 $\frac{1}{4}$ H 1000 2 1 *x* H₂ 10 H₂ 00 H₂2 $H_{3\,1}$ 2H₃₀ 2H ₁ ζ_2 H₁₂ H₁₀₀ H₁₁₀ H₂ ζ_2 ζ_2 ² $\frac{43}{8}$ H₂ $\frac{49}{8}\zeta_2$ $\frac{13}{8}$ H₁₁

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655 576 $\frac{655}{576}-\frac{151}{6}\begin{bmatrix} 53\\ 51\end{bmatrix} \text{H}_{11}+\frac{1}{6}\begin{bmatrix} 14\\ 11\end{bmatrix}+\frac{95}{9}\begin{bmatrix} 12\\ 2\\ 6\end{bmatrix} \text{H}_{21}-\frac{17}{4}\begin{bmatrix} 14\\ 1\end{bmatrix}+\frac{1}{14}\begin{bmatrix} 14\\ 1\end{bmatrix}+\frac{1}{14}\begin{bmatrix} 14\\ 1\end{bmatrix}+\frac{1}{16}\begin{bmatrix} 16\\ 16\end{bmatrix}+\frac{19}{16}\begin{b$ 2H3 ⁰ 3 $_{\rm H_2}$ 4H2 ⁰ H_2 ζ₂ 6H ₂ζ₂ 12H _{2 10} 6H ₂₀₀ *x* 3H₁₁₁ H_{0 0}ζ₂ 9 2 H 1 0 $\frac{35}{8}$ H₁ 0 2H₄ 1 3H₁ 1₂ ¹ 3_{H₂ ³ ³ ³_{H₄ ³_{H₄ ³H₁ 0 ³_{H₁} 3H₁ 3H₁ 3H₁ 3H₁ 3H₁ 1 0 6H₀² 3H₂ ³H₁ ³H₁ 6 ³_{H₁} 6 ³H₁ ³H₁ 6 ³H₁ ³H₁ 6 ⁴_M}}} $^{2}C_{F}$ x^{2} $\frac{2}{3}H_{1}\zeta_{2}$ $\frac{2105}{91}$ $\frac{77}{19}$ 6H3 4 3 4H ² ⁰ 584 ²⁷ H0 *^p*gq *^x* ⁷ 2 H1ζ³ 138305 2592 1 3 H2 ⁰ 13 ⁴ ^H ¹ζ² 2H2 ¹ ¹ 11 ² H1 ⁰ ⁰ 43 ⁶ H1 ¹ ¹ 109 ¹² ^ζ² 17 ³ H2 ¹ 71 ²⁴H1 ⁰ 11 ⁶ ^H ² ⁰ 21 ² ^ζ³ 3 2 H1 ⁰ ⁰ ⁰ H1 ² ⁰ 4H3 ¹ 395 $\frac{595}{54}H_0$ 2H₁₀ ζ_2 H₁₁ ζ_2 $\frac{55}{12}H_{110}$ 2H₁₁₀₀ 4H₁₁₁₀ 2H₁₁₁₁ 4H₁₁₂ $\frac{55}{12}H_{12}$ $6H_{1\,2\,0}$ $4H_{1\,2\,1}$ $4H_{1\,3}$ $3H_{2\,1\,0}$ $3H_{2\,2}$ p_{gq} x $\frac{23}{2}H$ $_1\zeta_3$ $5H$ $_2\zeta_2$ $2H$ $_2$ $_{1\,0}$ $\begin{array}{cccccc} \frac{109}{12} H & {}_{10} & {}_{10}H_{5} {}_{9} & \frac{17}{5} {}_{5} {}_{2} {}^{2} & \frac{1}{6} H_{1} {}_{5} {}_{2} & 2 H_{2} {}_{2} {}_{2} & \frac{65}{24} H_{11} & \frac{19}{2} H & {}_{1} & {}_{10} & 4 H_{3} {}_{0} & 3 H_{2} {}_{0} \\ \frac{3}{2} H & {}_{1} & {}_{2} & \frac{3379}{216} H_{1} & 4 H & {}_{2} & \frac{49}{6} H & {}_{10} {}_{0} & \$ 1.1 1 10 12H 1 10 10H 1 12 10 10H 1 0 2² 1 000 ²² 1 1 2 0 2H 1 2 1

1 1 1 1 1 0 1 2H 1 1 0 1 0H 1 1 1 2 1 0 1 1 1 2 0 2H 1 2 1 0

1 1 1 1 3 $\frac{1196}{6}$

1 x $\frac{41699}{2592}$ 3H 2 10 $\frac{3}{2}$ H $\frac{1}{25}$

2 $\frac{128}{$ $\begin{split} &\frac{7H_1\zeta_2}{\zeta_1}+\frac{2H_{100}}{\zeta_2}+\frac{3H_{100}}{\zeta_3}+\frac{2H_{200}}{\zeta_3}+3H_{0000}\qquad 1-x^{-4H_{31}}+\frac{H_{311}}{\zeta_1}+\frac{2H_{112}}{\zeta_1}+\frac{4H_{121}}{\zeta_2}+\frac{4H_{121}}{\zeta_3}+\frac{4H_{121}}{\zeta_3}+\frac{4H_{121}}{\zeta_3}+\frac{4H_{121}}{\zeta_3}+\frac{4H_{121}}{\zeta_3$ $\frac{67}{40}$ ζ₂² $\frac{67}{6}$ H₁₂ H₁₀ 8H₂₂ 25H₀ζ₂ $\frac{472}{9}$ H₁ $\frac{650}{9}$ H₀ $\frac{1}{4}$ H₄ 65H₃ 38H₀₀ 9H 30 $\frac{17}{3}$ H₀₀₀ x $\frac{27}{2}$ H 10 $\frac{1}{2}$ H₀₀₀₀ $\frac{3}{4}$ H₀₀₂ $\frac{1}{2}$ H 30 14H₀₀₀ $\frac{1}{12}$ H₁₁₁
 $\frac{43}{36}$ _C₂ $\frac{1}{2}$ H₂^C₂ $\frac{7}{72}$ H₀ $\frac{749}{54}$ H₁ $\frac{135}{4}$ _C $\frac{97}{24}$ H₁₀

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 $\frac{53}{12}$ H₂ $\frac{39}{4}$ H₁₁ $2H_{31}$ $\frac{13}{6}$ H₁₁₀ $\frac{7}{4}$ H₂₀₀ $4H_{110}$ $4H_{12}$ $16C_F n_f^2$ $\frac{1}{9}$ $\frac{11}{9x}$
 $\frac{2}{5}$ $\frac{1}{2}$ $\frac{1}{x^2}$ $\frac{1}{x^3}$ $\frac{1}{x^4}$ $\frac{5}{x^4}$ $\frac{1}{x^2}$ $16C_F^2 n_f \frac{4}{$ $\frac{1}{9}x$ $\frac{1}{6}xH_1$ $\frac{1}{6}P_{B91}x$ H_{11} $\frac{1}{3}H_1$ $16C_F^2n_f$ $\frac{1}{9}x^2$ H_{00} $\frac{1}{6}$ H_0 $\frac{1}{2}$ H_{10} $\frac{1}{3}P_{B91}x$ H_{12} H_{10} $H_{1}\zeta_2$ $9\zeta_3$ $\frac{83}{12}$ H_{11} $2H_{12}$ $\frac{7}{36}$ $\frac{13051}{288}$
 $\frac{1187}{216}$ H₀
 $\frac{23}{18}$ H ₁₀ $\begin{split} 2H_{110} = \frac{5}{2}H_{111} - \frac{31}{18}\rho_{B0} &\times \frac{95}{23}H_0 - \xi_2 - H_{10} - \frac{1}{3} \cdot 2 - \times 6H_{0000} - H_3 - \frac{1365}{238} \\ \frac{13}{2}\xi_3 - 4H_{120} - H_{20} - \frac{1}{2}H_{10} - \frac{1}{2}H_{21} - 2H_{000} - \frac{653}{24}H_{00} - 1 - \times - H_0\xi_2 - \frac{1187}{216}H_0 \\ \frac$ 2H₁₂₀ 2H₁₂₁ $\frac{9}{2}$ H₁₁₁ $\frac{3}{2}$ H₁₀₀ $\frac{47}{16}$ $\frac{47}{16}$ H₁ $\frac{15}{2}$ _{S3} *p*₈₉ *x* 2H₁₂₀
6H₁₁₀ 3H₁ ζ_2 $\frac{7}{4}$ H₁₀ $\frac{16}{5}$ ζ_2 ² 6H₁₀₀ $\frac{7}{2}$ H₁₀ 4H₁₁₀₀ 2H₁₀ ζ_2 H_{1000} 1 *x* 9H₁₀₀ H₁₁₁ $10H_1\zeta_2$ 3H₀ ζ_3 H₂2 H₂ ζ_2 H₀₀₀ 5H₂₀₀ $4H_3$ H_{211} $3H_{0.0}\zeta_2$ $3H_{3.1}$ $3H_4$ $\frac{211}{16}H_1$ $\frac{49}{20}\zeta_2^2$ 1 *x* $11\zeta_3$ $\frac{1}{4}H_{1.1}$ $\frac{1}{4}H_{1.0}$ $\frac{91}{16}H_9$ 36H $_{10}$ 8H $_{100}$ 14H $_{1}$ 10 7H $_{15}$ $\frac{20}{12}$ 2H₀ $_{20}$ $\frac{1}{2}$ $_{200}$ $\frac{2}{36}$ $_{202}$ $\frac{2}{32}$ $_{200}$ $\frac{2}{32}$ $\frac{287}{12}$ $\frac{11}{16}$ $_{200}$
 $\frac{11}{2}$ $_{21}$ $\frac{11}{2}$ $_{2200$ 4H 100 16H 30 4H 2⁵₂ 8H 2 10 5H₂⁵₂ $\frac{1}{4}$ H₂ H₂ $\frac{1}{8}$ H₂ 0 9H₀⁵₃
25H 20 6H 200 $\frac{3}{2}$ ^x $\frac{56}{3}$ ⁵₂ $\frac{7}{3}$ H₁⁵₂ 4H₁₁ $\frac{3}{2}$ H₁₁₁ $\frac{5}{2}$ H₁₀ 0 $\frac{175}{96}$ H₃₁ $\frac{1$ Finally the Mellin inversion of Eq. (3.13) yields the NNLO glu

P ² gg *^x* ¹⁶*CACFnf ^x*² ⁴ 9 H2 3H1 0 97 ¹²H1 8 3 H 2 0 2 3 H0ζ² 103 ²⁷ H0 16 ³ ^ζ² 2H3 6H 1 0 2H2 0 127 ¹⁸ H0 0 511 ¹² *^p*gg *^x* ²ζ³ 55 24 4 3 1 *^x ^x*² ¹⁷ ²⁴H1 0 43 ¹⁸H0 521 ¹⁴⁴H1 6923 432 1 2 H2 1 2H1ζ² H0ζ² 2H100 1 ¹²H1 1 H110 H111 175 ¹² H2 6H 1 0 8H0ζ³ 6H 2 0 53 ⁶ H0ζ² 49 ² H0 185 ⁴ ^ζ² 511 12 1 2 H2 ⁰ 3H1 ⁰ 4H0 ⁰⁰⁰

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 $\frac{67}{12}H_{0.0}$ $\frac{43}{2}\zeta_3$ $H_{2.1}$ $\frac{97}{12}H_1$ $4\zeta_2$ $\frac{9}{2}H_3$ $8H_{3.0}$ $\frac{33}{2}H_{0.00}$ $\frac{4}{3}$ $\frac{1}{x}$ x^2 $\frac{1}{2}H_2$ $H_{2.0}$ 11 $\begin{aligned} \frac{13}{5} \text{H} &+10 &+25 &-\frac{7}{6} \text{C}_2 &+2 \text{C}_3 &+14 \text{C}_2 &+14 \text{C}_1 &+2 \text{C}_1 &+14 \text{C}_1 &+14 \text{C}_2 &+14 \text{C}_1 &+14 \text{C}_2 &+$ $\frac{19}{6}$ ζ_2 2 ζ_3 H ₁ ζ_2 4H _{1 10} $\frac{1}{2}$ H ₁₀₀ H ₁₂ 1 *x* 9H₁ ζ_2 9H ¹⁰⁰ 241 ²⁸⁸^δ ¹ *^x* ¹⁶*CAnf* ² 19 ⁵⁴H0 1 ²⁴*x*H0 1 ²⁷ *^p*gg *^x* ¹³ 54 1 *^x ^x*² ⁵ ¹ *^x* ¹¹ ⁷²H1 71 216 2 ⁹ ¹ *^x* ^ζ² 13 ¹²*x*H0 1 2 H0 ⁰ H2 29 ²⁸⁸^δ ¹ *^x* 16*CA* ²*nf x*² ζ³ 11 ⁹ ^ζ² 11 ⁹ H0 0 2 3 H3 2 3 H0ζ² 1639 ¹⁰⁸ H0 2H 2 0 1 ³ *^p*gg *^x* ¹⁰ 209 ³⁶ ⁸ζ³ 2H 2 0 1 2 H0 10 ³ H0 0 20 ³ H1 0 H100 20 ³ H2 H3 10 $\frac{3}{3}$ H₁ $\frac{12}{3}$ ζ_2 $\frac{241}{36}$ 853 2H 2 0 $\frac{1}{2}$ H₀ $\frac{2}{3}$ H₀ $\frac{2}{3}$ H₀ $\frac{2}{3}$ H₁ 0 $\frac{1}{3}$ _{H2} H₃ $\frac{1}{9}$

2H₁₀ $\frac{3}{10}$ H₂ H₃ $\frac{1}{9}$ H₂ $\frac{1}{3}$

2H₁₀ $\frac{3}{10}$ H₂ $\frac{1}{3}$

12H₁₀ $\frac{3}{10}$

13 $\begin{split} 2H_{-1,2} &= 3H_2\xi_2 - \frac{2}{3}H_{2,0} - \frac{3}{2}H_{2,00} - \frac{3}{2}H_4 - \frac{1}{9}\xi_2 - 7H_{-2,0} - 2H_2 - \frac{438}{27}H_0 - H_{0,0}\xi_2 \\ \frac{3}{2}\xi_2^2 - 4H_{-3,0} &= x\,\frac{131}{12}H_{0,0} - \frac{8}{3}H_0\xi_2 - \frac{7}{2}H_3 - H_{0,0,00} - \frac{7}{6}H_{0,00} - \frac{1943}{216$ 8H₀₅2 4H₀ 0₂ 6H₁ 0₃ 6H₁ 20 10H₂₀ 6H₁ 0²₂ 8H₁ 000 8H₁ 0²₃ H₁ 0²₃ H₁ 0²₃ H₂⁵ H₁ 0²₃ H₂⁵ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶ H₂⁶

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NNLO, *P*(2) *ab* : Moch, Vermaseren & Vogt '04

The large-

Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond

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aII. Institute for Theoretical Physics, Hamburg University arXiv:1707.08315v2 [hep-ph] 5 Oct 2017

$$
\partial_{\ln Q^2} q = P_{q \leftarrow q} \otimes q
$$

$$
\partial_{\ln Q^2} g = P_{g \leftarrow q} \otimes q
$$

- **P** quark is depleted at large x
- \blacktriangleright gluon grows at small x

$$
\partial_{\ln Q^2} q = P_{q \leftarrow q} \otimes q
$$

$$
\partial_{\ln Q^2} g = P_{g \leftarrow q} \otimes q
$$

- **P** quark is depleted at large x
- \blacktriangleright gluon grows at small x

$$
\partial_{\ln Q^2} q = P_{q \leftarrow q} \otimes q
$$

$$
\partial_{\ln Q^2} g = P_{g \leftarrow q} \otimes q
$$

- **P** quark is depleted at large x
- \blacktriangleright gluon grows at small x

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$$

$$
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DGLAP evolution (initial gluons only) E↵ect of (LO) DGLAP: initial gluons [Initial-state splitting] [Example evolution]

2nd example: start with just gluons.

$$
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$$

$$
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$$

- ▶ gluon is depleted at large *x*.
- ▶ high-*x* gluon feeds growth of small *x* gluon & quark.

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- ▶ gluon is depleted at large *x*.
- \blacktriangleright high- x gluon feeds growth of small *x* gluon & quark.

DGLAP evolution:

- ➤ **partons lose momentum and shift towards smaller x**
- ➤ **high-x partons drive growth of low-x gluon**

determining the gluon

which is critical at hadron colliders (e.g. ttbar, Higgs dominantly produced by gluon-gluon fusion), but not directly probed in Deep-Inelastic-Scattering

Fit quark distributions to $F_2(x, Q_0^2)$, at *initial scale* $Q_0^2 = 12$ GeV². NB: *Q*⁰ often chosen lower

Assume there is no gluon at Q_0^2 :

$$
g(x,Q_0^2)=0
$$

Use DGLAP equations to evolve to higher *Q*2; compare with data.

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Use DGLAP equations to evolve to higher Q^2 ; compare with data.

Complete failure!

COMPLETE FAILURE to reproduce data evolution

$$
g\to q\bar{q}
$$

generates extra quarks at large Q2 WH faster rise of F2

$$
g\to q\bar{q}
$$

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generates extra quarks at large Q2 WH faster rise of F2

Global PDF fits (CT, MMHT, NNPDF, etc.) choose gluon distribution that leads to the correct Q2 evolution.

If gluon \neq 0, splitting $q \rightarrow q\bar{q}$

generates extra quarks at large Q2 WH faster rise of F2

If gluon \neq 0, splitting $g \rightarrow q\bar{q}$

generates extra quarks at large Q2 WH faster rise of F2

Global PDF fits (CT, MMHT, NNPDF, etc.) choose gluon distribution that leads to the correct Q2 evolution.

SUCCESS

Resulting gluon distribution is **HUGE!** Carries 47% of proton's momentum (at scale of 100 GeV) Crucial in order to satisfy momentum sum rule. Large value of gluon has big

impact on phenomenology

Today's PDF fits: huge array of data (and choices about which data to use) DUAY'S FUF 1115: HUYE ALLAY UL UALA (ANU CHUICES ADUUL WINCH UALA LU USE)

Table 6: The values of χ^2/N pts. for the non-LHC data sets included in the global fit at NLO and NNLO.

MSHT20 data sets & χ2

Table 7: The values of χ^2/N pts. for the LHC data sets included in the global fit and the overall global fit χ^2/N at NLO and NNLO. The corresponding values for the non-LHC data sets are shown in Table 6, and the total value corresponds to the sum over both tables.

data is precise, correlations between systematics are crucial

e.g. from MSHT20 (2012.04684)

today's PDF fits: fitting functions

⁰ = 1 GeV² is the input scale, and *T* Ch

and its low *x* power were tied to those of the light sea, *S*(*x*) = 2(¯*u*(*x*)+ ¯

A generic function $f(x)$ involves an infinite number of degrees of freedom. How can you fit this with a finite number of data points? Figure 2: As in Figure 2: As in Fig. 1, but at NLO. The state \mathcal{L}

CT / MSHT use parameterisations with hand-picked number of terms, e.g. up to $n = 6$ in Chebyshev series:

$$
xf(x, Q_0^2) = A(1-x)^{\eta} x^{\delta} \left(1 + \sum_{i=1}^{n} a_i T_i^{\text{Ch}}(y(x))\right)
$$

NNPDF use a *neural network* as a generic fit function, and In the Matter of the MMHT₀, the MMHT14 is to ache went parameterization of the summer parameterization of the studies of the studi LIE training subset, and stops when χ on training **d**
closure tests) were less well constrained by data, whilst for similar reasons two of the *s* + ¯*s* ('*s*+') Chebyshevs separate data into training / validation. Fit is done using just the training subset, and stops when χ^2 on training + validation starts to increase. (Supplemented with closure tests)

ⁱ (*y*) are Chebyshev polynomials in *y*, with

today's PDF fits: uncertainty estimation

With fits to $O(60)$ data sets, chances are they won't all be consistent (plainly inconsistent data sets may simply be excluded, but that can be biased)

CT / MSHT do a Hessian fit, with error eigenfunctions, scaled by a tolerance T that is like replacing $\Delta \chi^2 = 1$ with $\Delta \chi^2 = T$.

Squared error on a cross section is obtained by summing squared variations from each of the eigenfuntions.

NNPDF fits *Monte Carlo replica data sets*

i.e. fluctuate the data according to errors, and fit the fluctuated data; repeat over and over, to get $O(100)$ replica fits; prediction for any cross section is then average and std.dev. across the replicas

Charm-quark mass is around 1.5 GeV. Is this perturbative enough to treat it as purely perturbative generated? Or should one fit the charm as a light flavour?

CT / MSHT treat charm perturbatively, turning on its evolution from (almost zero) at the charm mass.

NB: CT also explores "fitted" charm

NNPDF fits by default treat the charm as light, but also provide PDF sets with perturbative charm

In much of region relevant to LHC, uncertainty is in the 1-2% range

- ➤ strange (anti-)quark is least well known PDF (small charge, few good experimental handles)
- ➤ charm: current debate about intrinsic charm
- ➤ bottom: mostly driven by gluon

"Think" at Leading Order (LO) in QCD:

- \blacktriangleright collide protons at CoM energy \sqrt{s} ,
- \triangleright take momentum fractions x_1 and x_2 from the two protons
- riangled producing a system of mass m requires $x_1 x_2 s = m^2$

Number of parton-parton collisions with flavours *i* and *j* is proportional to **partonic luminosity** $\mathscr{L}_{ij}(m^2)$

$$
\mathcal{L}_{ij}(m^2) = \int dx_1 dx_2 f_{i,p}(x_1, \mu_F^2) f_{j,p}(x_2, \mu_F^2) \delta(x_1 x_2 s - m^2)
$$

comparing PDF "luminosities"

Amazing that MSHT20 & NNPDF40 are reaching %-level precision

At this level, QED effects probably no longer optional (MSHT20QED: 0.9870)

Example: W mass

6.4 Combination LHCb

W mass is one area where LHC **FIGURE CDF**

Pred. unc.

[arXiv:2109.01113](https://arxiv.org/abs/2109.01113) MSHT20 $m_W = 80351 \pm 25$ $T20$ $m_W = 80351 \pm 23_{\text{stat}} \pm 10_{\text{exp}} \pm 17_{\text{theory}} \pm 7_{\text{PDF}} \text{MeV},$ considered. The correlation between the final \sim T-for and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\$ MSHT20 T- and σ and σ approach. These two probach. These two probach total uncertainties and only differ in the splitter in the spl $\text{DF31} \quad m_W = 80362 \pm 23_\text{stat} \pm 10_\text{exp} \pm 17_\text{theory} \pm 9_\text{PDF}\,\text{MeV},$ $t_{\text{max}} = 80350 + 23 + 10 + 17$ and $t_{\text{max}} = 10^{10}$ $m_W = 80350 \pm 23_{\text{stat}} \pm 10_{\text{exp}} \pm 17_{\text{theory}} \pm 12_{\text{PDF}} \text{MeV}$, NNPDF31 CT₁₈

- \triangleright In range $10^{-3} < x < 0.1$, core PDFs (up, down, gluon) known to \sim 1–3% accuracy
- ➤ For many LHC applications, you can use PDF4LHC21 set, which merges CT18, MSHT20, NNPDF31, all available at NNLO
- ➤ first N3LO sets also indicate
- ➤ Situation is not full consensus:
	- ➤ differences in errors (e.g. NNPDF40 v. CT18),
	- ➤ differences in central values (ABMP; approx N3LO v. NNLO)

SO FAR en ead adapt to appropriately adapt to appropriately adapt the standard massive coefficient functions to account for a coefficient for a coefficient functions to account for a coefficient functions to account for a coeffic $\frac{1}{20}$

➤ We discussed the "Master" formula H_{α} disconsecol the $(h_{\alpha}$ integrated h_{α} z we discussed the wiaster formula

$$
\sigma(h_1 h_2 \to W + X) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2 \frac{f_{i/h_1} \left(x_1, \mu_F^2\right) f_{j/h_2} \left(x_2, \mu_F^2\right)}{\times \hat{\sigma}_{ij \to W+X}^{(n)} \left(x_1 x_2 s, \mu_R^2, \mu_F^2\right)} + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),
$$

- ➤ and its main inputs λ and ite main inpute definition is nearly always unambiguous, the one major exception being for \mathbb{Z}_2 for \mathbb{Z}_2
	- \blacktriangleright the strong coupling a_s
- \bullet the strong coupling \circ L_{max} can be contributed that contributes to the hadronic branching fraction, L_{max} $\hat{\sigma}$ **u Z _ u H**
- ➤ Parton Distribution Functions (PDFs) $\sum_{i=1}^n$ $\sum_{i=1}^n$ $\sum_{i=1}^n$ in Equation s (PDFs) branching ratio becomes sensitive to the coupling, n $\mathcal{N} = 1$ in \mathcal{N} , as is relevant when $\mathcal{N} = 1$ in Eq. (1.8), as is relevant when $\mathcal{N} = 1$ in Eq. (1.8), as is relevant when $\mathcal{N} = 1$ is relevant when $\$
- ▶ Next: we discuss the actual scattering cross section of this ambiguity, we avoid use of the term C in the term C **proton proton**

the hard cross section

 $\sigma \sim \sigma_2 \alpha_s^2 + \sigma_3 \alpha_s^3 + \sigma_4 \alpha_s^4 + \sigma_5 \alpha_s^5 + \cdots$ **LO NLO NNLO N3LO**
INGREDIENTS FOR A CALCULATION (generic 2→2 process)

INGREDIENTS FOR A CALCULATION (generic 2→2 process)

NNLO

$$
\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} =
$$
\n
$$
= R_0 \left(1 + 0.32\alpha_s + 0.14\alpha_s^2 - 0.47\alpha_s^3 - 0.59316\alpha_s^4 + \cdots\right)
$$
\n
$$
Baikov et al., 1206.1288
$$
\n(numbers for γ -exchange only)

This is one of the few quantities calculated to N4LO Good convergence of the series at every order (at least for α**s(MZ) = 0.118)**

$$
\sigma(pp \to H) = (961 \,\text{pb}) \times (\alpha_s^2 + 10.4\alpha_s^3 + 38\alpha_s^4 + 48\alpha_s^5 + \cdots)
$$

$$
\alpha_s \equiv \alpha_s (M_H/2)
$$

$$
\sqrt{s_{pp}} = 13 \,\text{TeV}
$$

Anastasiou et al., 1602.00695 (ggF, hEFT)

pp→**H (via gluon fusion) is one of a few hadron-collider processes known at N3LO** (others are pp→H via weak-boson fusion, Drell-Yan production)

> Series convergence is poor until last term (explanations for why are only moderately convincing)

- ► On previous page, we wrote the series in terms of powers of $a_s(M_H/2)$
- But we are free to rewrite it in terms of $a_s(\mu)$ for any choice of "renormalisation scale" µ. **Higgs cross section**

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scale dependence (an intrinsic uncertainty) gets reduced as you go to higher order

Scale dependence as the "THEORY UNCERTAINTY"

Convention: "theory uncertainty" (i.e. from missing higher orders) is estimated by change of cross section when varying μ in range 1/2 \rightarrow 2 around central value 82

Scale dependence as the "THEORY UNCERTAINTY"

Here, only the renorm. scale μ (= μ _R) has been varied. In real life you need to change renorm. and factorisation (μ_F) scales.

Convention: "theory uncertainty" (i.e. from missing higher orders) is estimated by change of cross section when varying μ in range 1/2 \rightarrow 2 around central value 83

NLO/NNLO/N3LO for Drell-Yan process

Convergence from NNLO to N3LO is not so good.

(Apparent good convergence from NLO to NNLO & small NNLO uncertainty were perhaps accident of cancellation between flavour channels)

- ➤ LO: almost any process *(with MadGraph, Comix, ALPGEN, etc.)*
- ➤ NLO: most processes *(with MCFM, NLOJet++, MG5_aMC@NLO, POWHEG, OpenLoops/Blackhat/NJet/Gosam/etc.+Sherpa)*
- \triangleright NNLO: all 2→1, most 2→2, and a few 2→3 (some approx) *(top++, DY/HNNLO, FEWZ, MATRIX, MCFM, NNLOJet, MINNLO, Geneva etc.)*
- \triangleright N3LO: pp \rightarrow Higgs and Drell Yan *some with approximations (EFT, QCD₁×QCD₂)*
- \triangleright NLO EW corrections, i.e. relative a_{EW} rather than a_{s} : most $2\rightarrow1$, $2\rightarrow2$ and $2\rightarrow3$
- ➤ mixed NNLO (EW×QCD) for 2→1