QCD (for Colliders)
Lecture 2

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in lecture 1 handout

1st lecture summary

- QCD at colliders mixes weak and strong coupling
- No calculation technique is rigorous over that whole domain

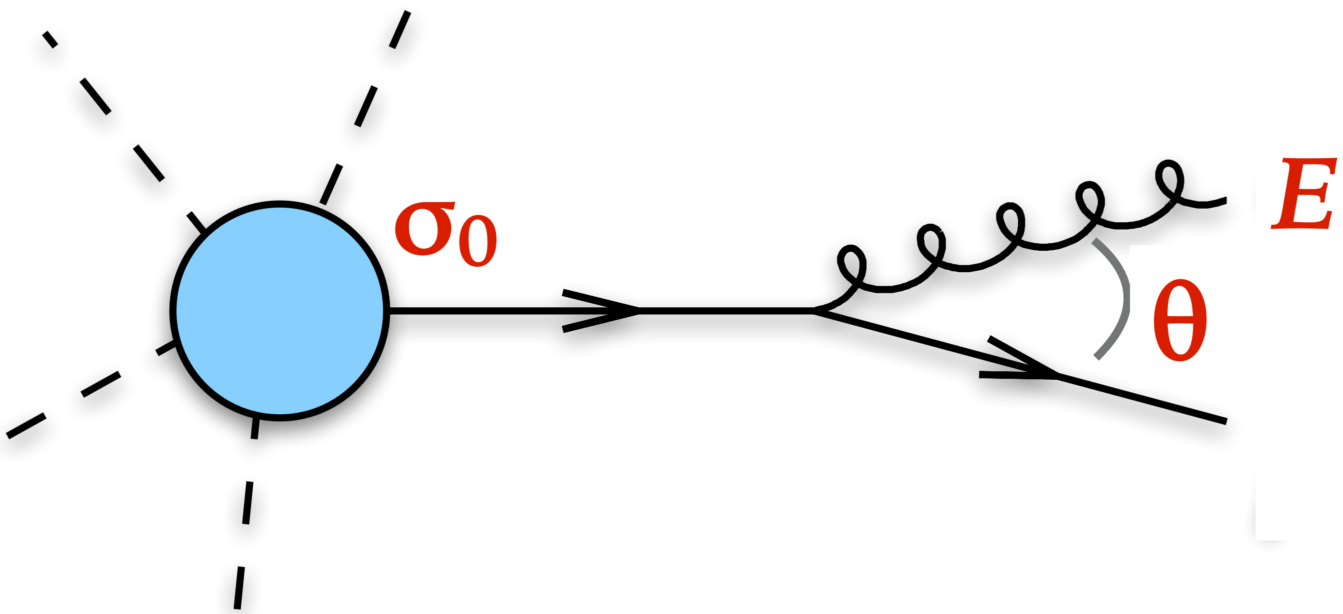
- Gluon emission repaints a quark's colour
- That implies that gluons carry colour too

- Quarks emit gluons, which emit other gluons: this gives characteristic "shower" structure of QCD events, and is the basis of *Monte Carlo simulations*
- ~~To use perturbation theory one must measure quantities that are insensitive to the (divergent) soft & collinear splittings, like *jets*.~~

Today

- Infrared and Collinear Safety
- Factorisation of cross sections into PDFs and “hard” parts
- Parton Distribution Functions (PDFs)
- Total cross sections & their perturbative series

GLUON EMISSION FROM A QUARK



Consider an emission with

- energy $E \ll \sqrt{s}$ ("soft")
- angle $\theta \ll 1$ ("collinear" wrt quark)

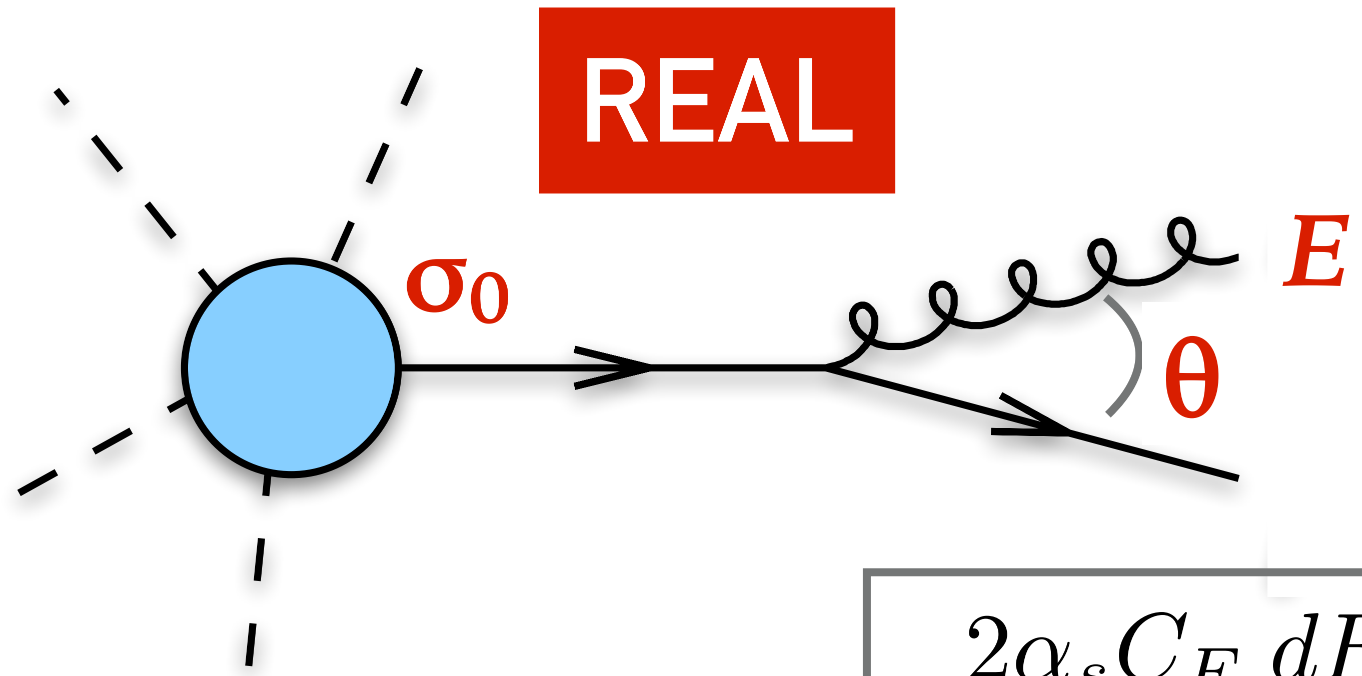
Examine correction to some hard process with cross section σ_0

$$d\sigma \simeq \sigma_0 \times \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\theta}$$

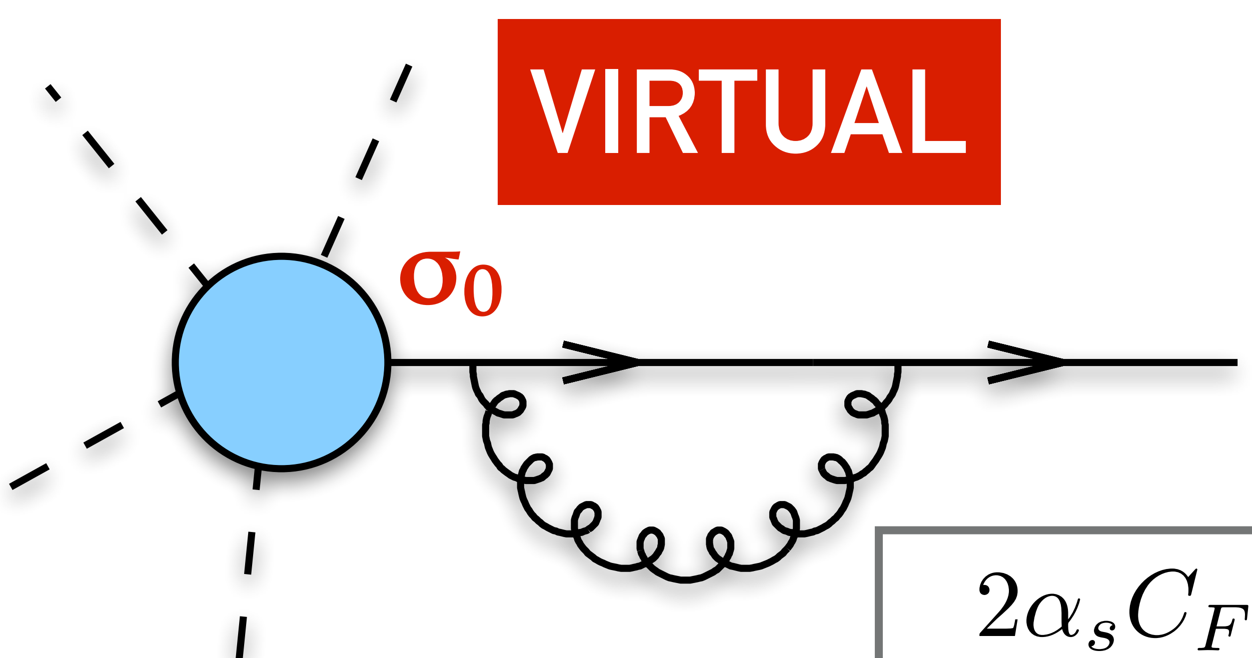
This has a divergence when E→0 or θ→0

[in some sense because of quark propagator going on-shell]

How come we get finite cross sections?



$$+\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$



$$-\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$

Divergences are present in both real and virtual diagrams.

If you are "**inclusive**", i.e. your measurement doesn't care whether a soft/collinear gluon has been emitted then the **real and virtual divergences cancel.**

Beyond inclusive cross sections: infrared and collinear (IRC) safety

For an observable's distribution to be calculable in [fixed-order] perturbation theory, the observable should be infra-red safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \to \vec{p}_j + \vec{p}_k$$

whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel [collinear] or one of them is small [infrared].

[QCD and Collider Physics (Ellis, Stirling & Webber)]

Examples

Multiplicity of gluons is not IRC safe

[modified by soft/collinear splitting]

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Examples

Multiplicity of gluons is not IRC safe

[modified by soft/collinear splitting]

Energy of hardest particle is not IRC safe

[modified by collinear splitting]

Beyond inclusive cross sections: infrared and collinear (IRC) safety

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[QCD and Collider Physics (Ellis, Stirling & Webber)]

Examples

Multiplicity of gluons is not IRC safe

[modified by soft/collinear splitting]

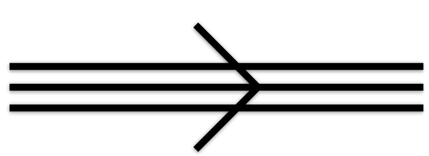
Energy of hardest particle is not IRC safe

[modified by collinear splitting]

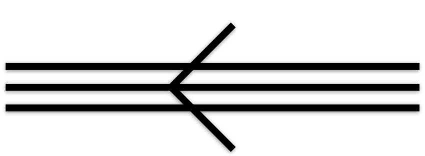
Energy flow into a cone is IRC safe

[soft emissions don't change energy flow,
collinear emissions don't change its direction]

A CORE FORMULA FOR HADRON COLLIDERS

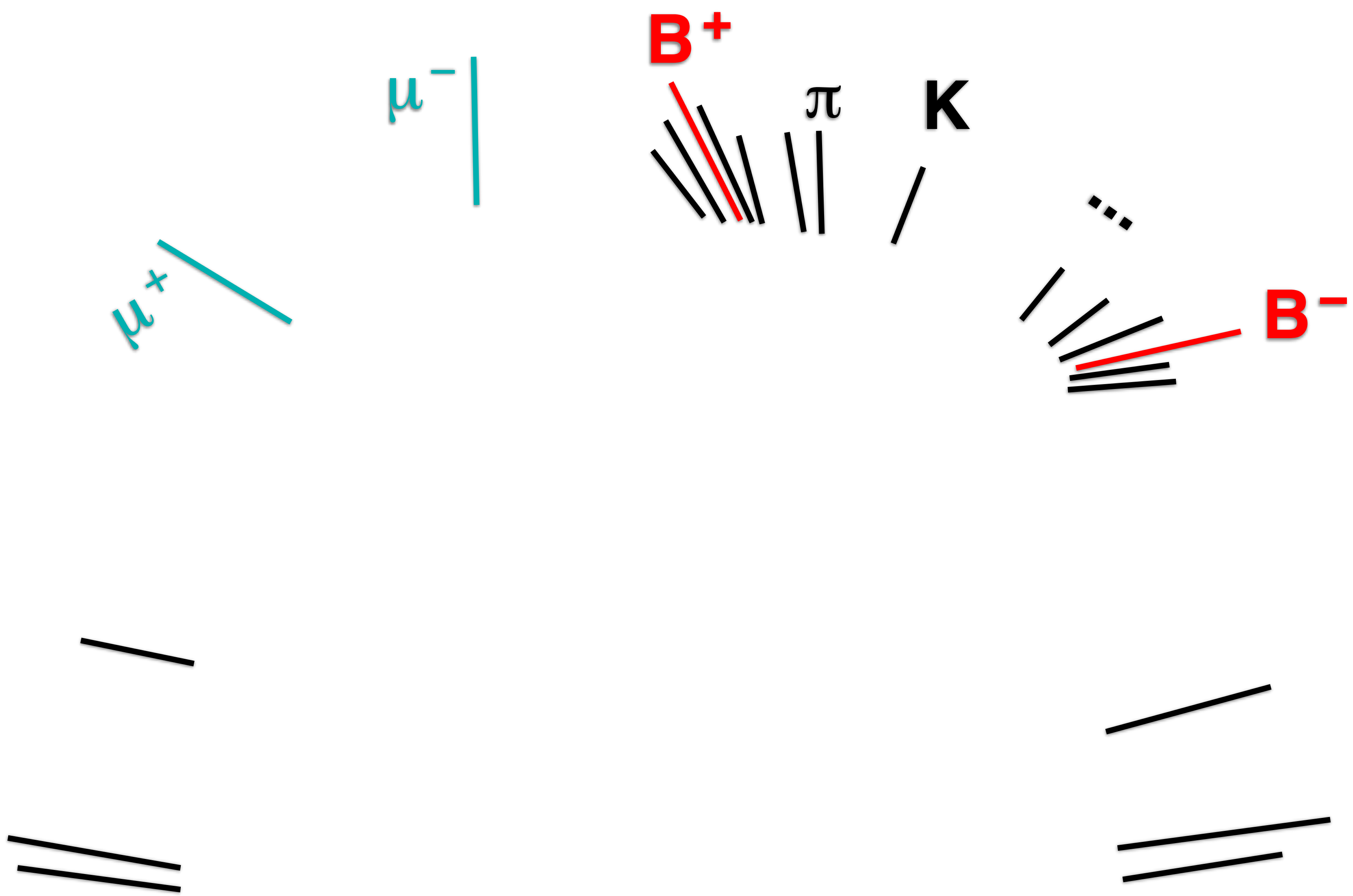


proton



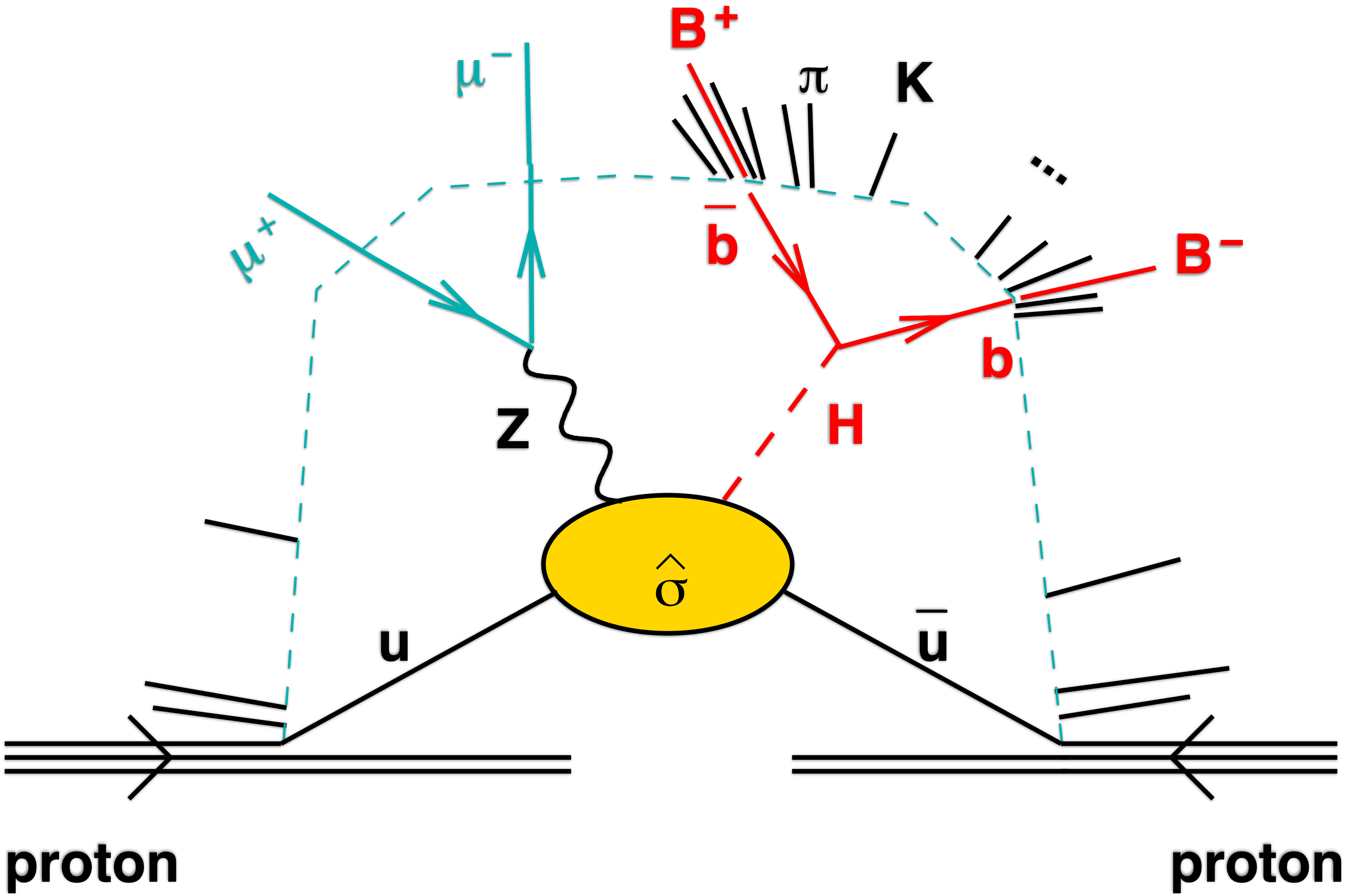
proton

A proton-proton collision: FINAL STATE

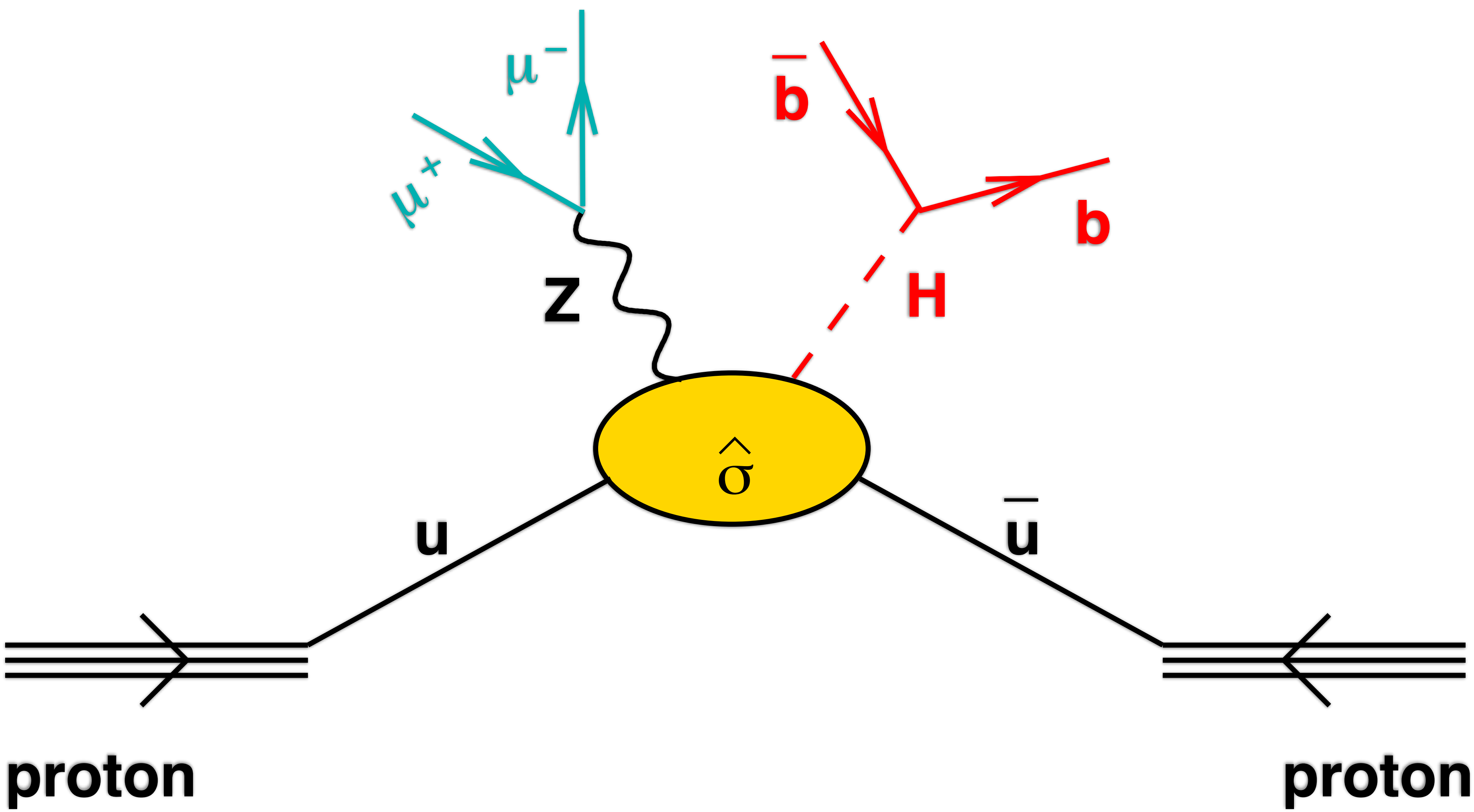


(actual final-state multiplicity ~ several hundred hadrons)

A proton-proton collision: FILLING IN THE PICTURE



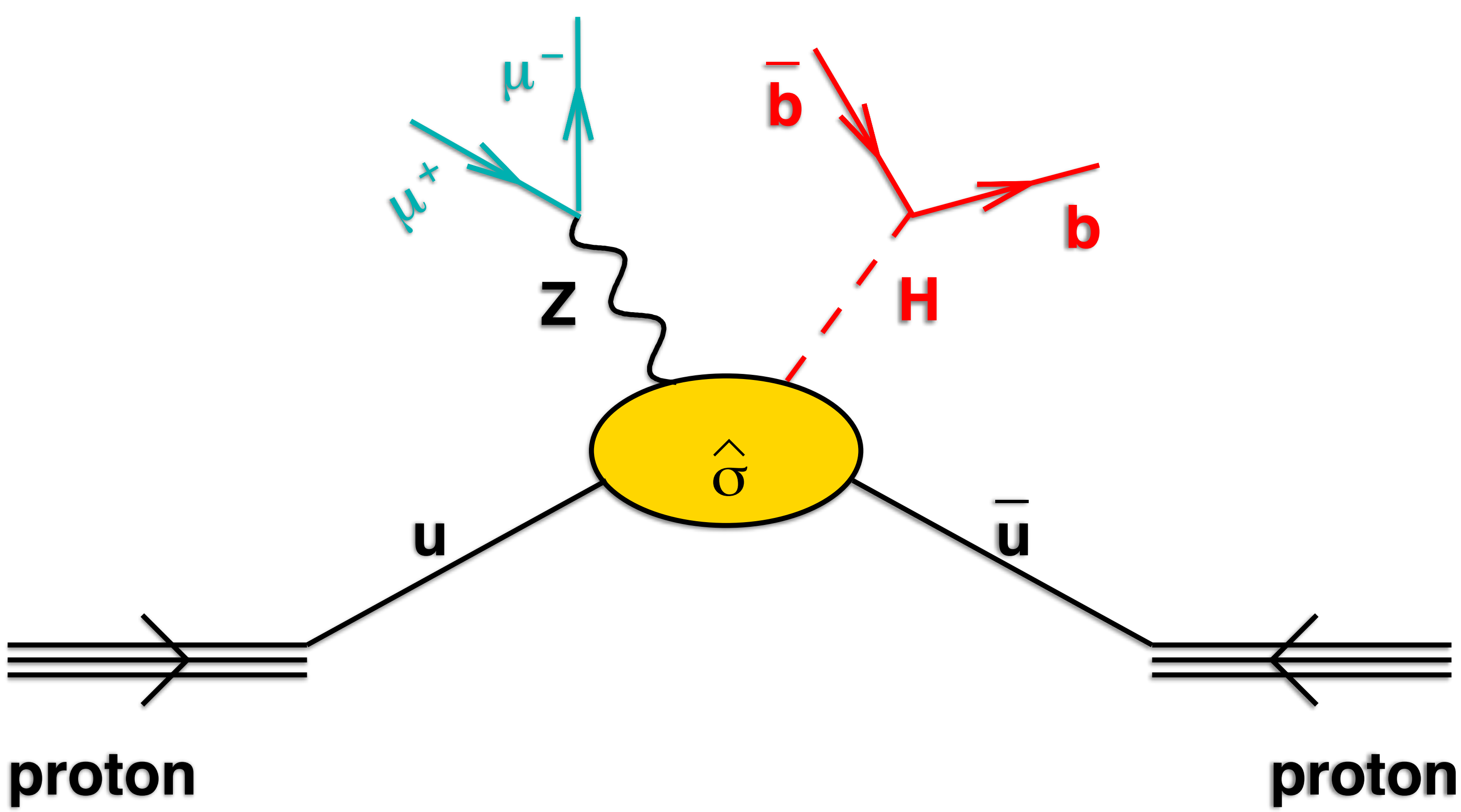
A proton-proton collision: SIMPLIFYING IN THE PICTURE



THE MASTER EQUATION — FACTORISATION

$$\sigma\left(h_1 h_2 \to ZH + X\right) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2\, f_{i/h_1}\left(x_1, \mu_F^2\right) f_{j/h_2}\left(x_2, \mu_F^2\right)$$

$$\times\, \hat{\sigma}^{(n)}_{ij \to ZH+X}\left(x_1 x_2 s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),$$



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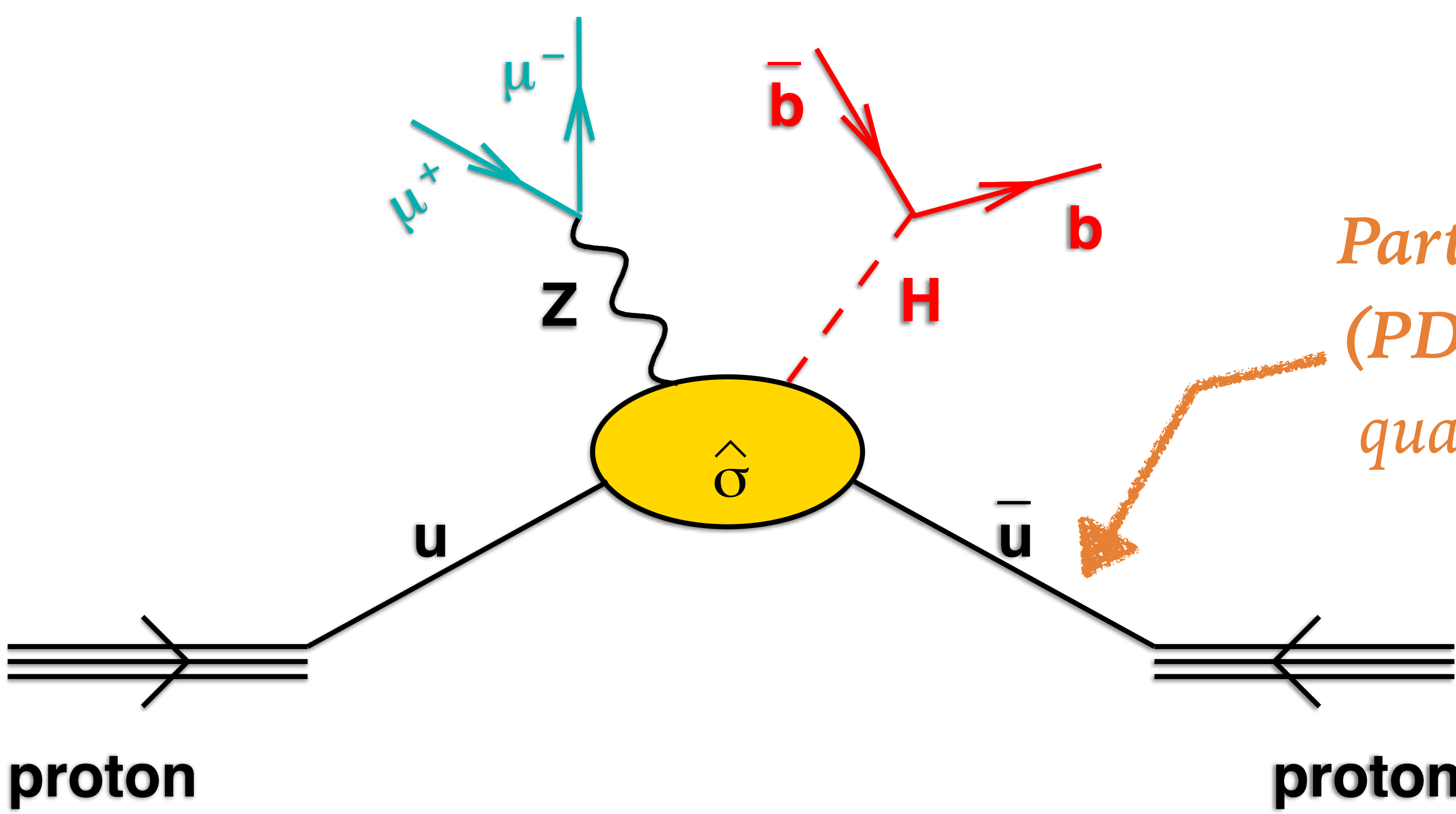

Parton distribution function (PDF): e.g. number of up anti-quarks carrying fraction x_2 of proton's momentum

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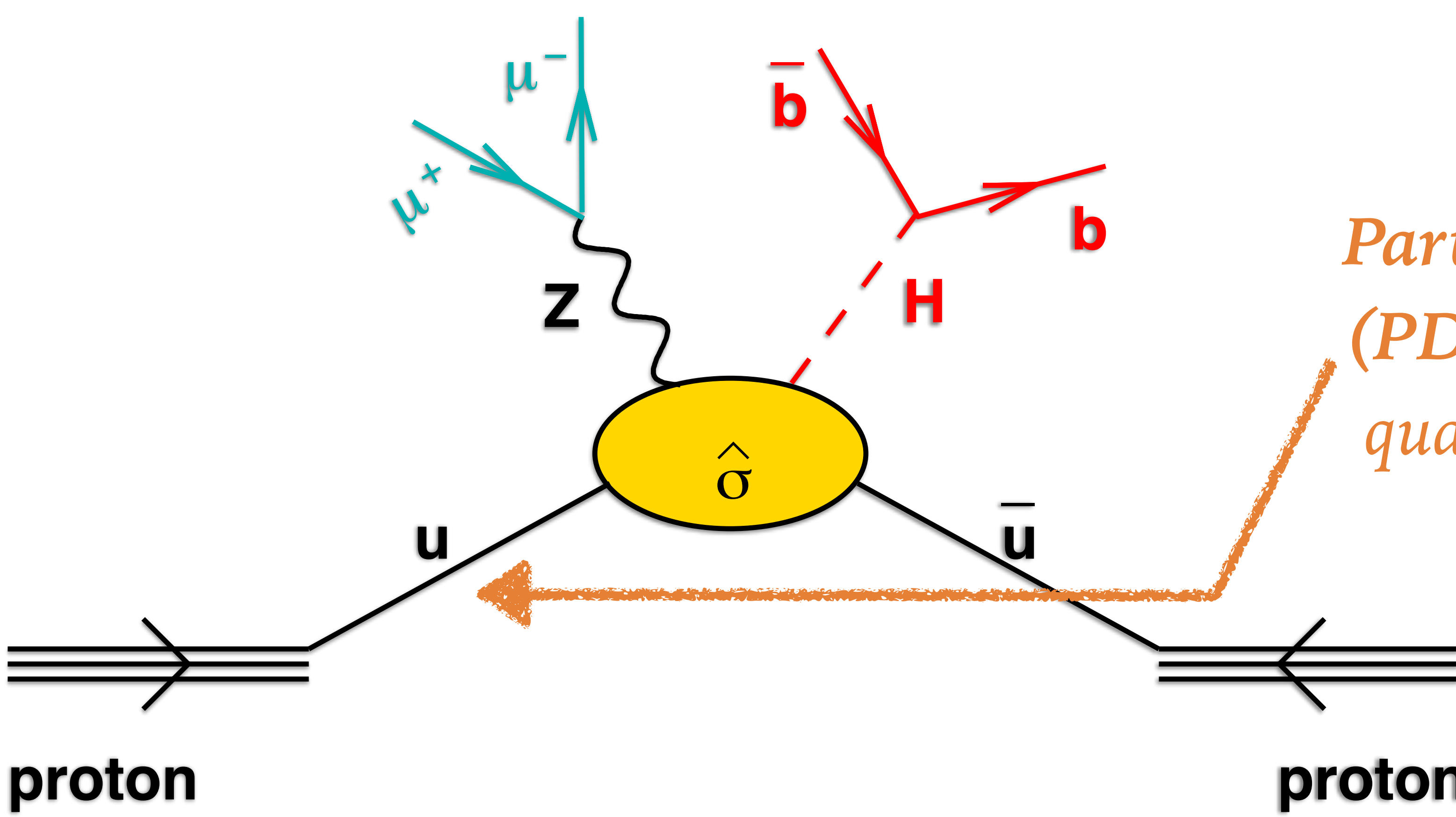
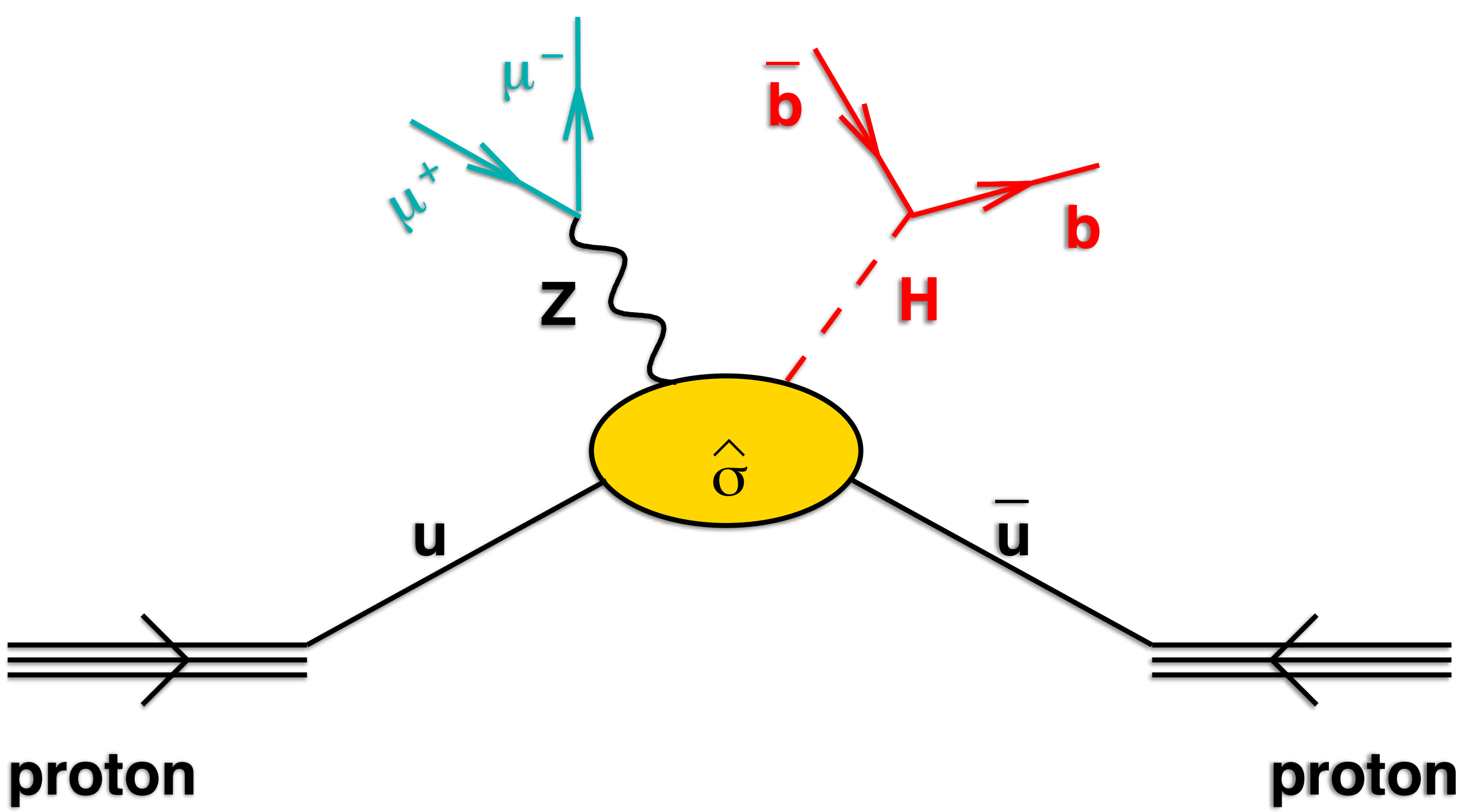


Parton distribution function (PDF): e.g. number of up quarks carrying fraction x_1 of proton's momentum

THE MASTER EQUATION — FACTORISATION

Perturbative sum over powers of the strong coupling: typically we know first 2-4 orders

$$\sigma\left(h_1 h_2 \to ZH + X\right) = \sum_{n=0}^{\infty} \alpha_s^n\left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2\, f_{i/h_1}\left(x_1, \mu_F^2\right) f_{j/h_2}\left(x_2, \mu_F^2\right) \times \hat{\sigma}^{(n)}_{ij\to ZH+X}\left(x_1 x_2 s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),$$



THE MASTER EQUATION — FACTORISATION

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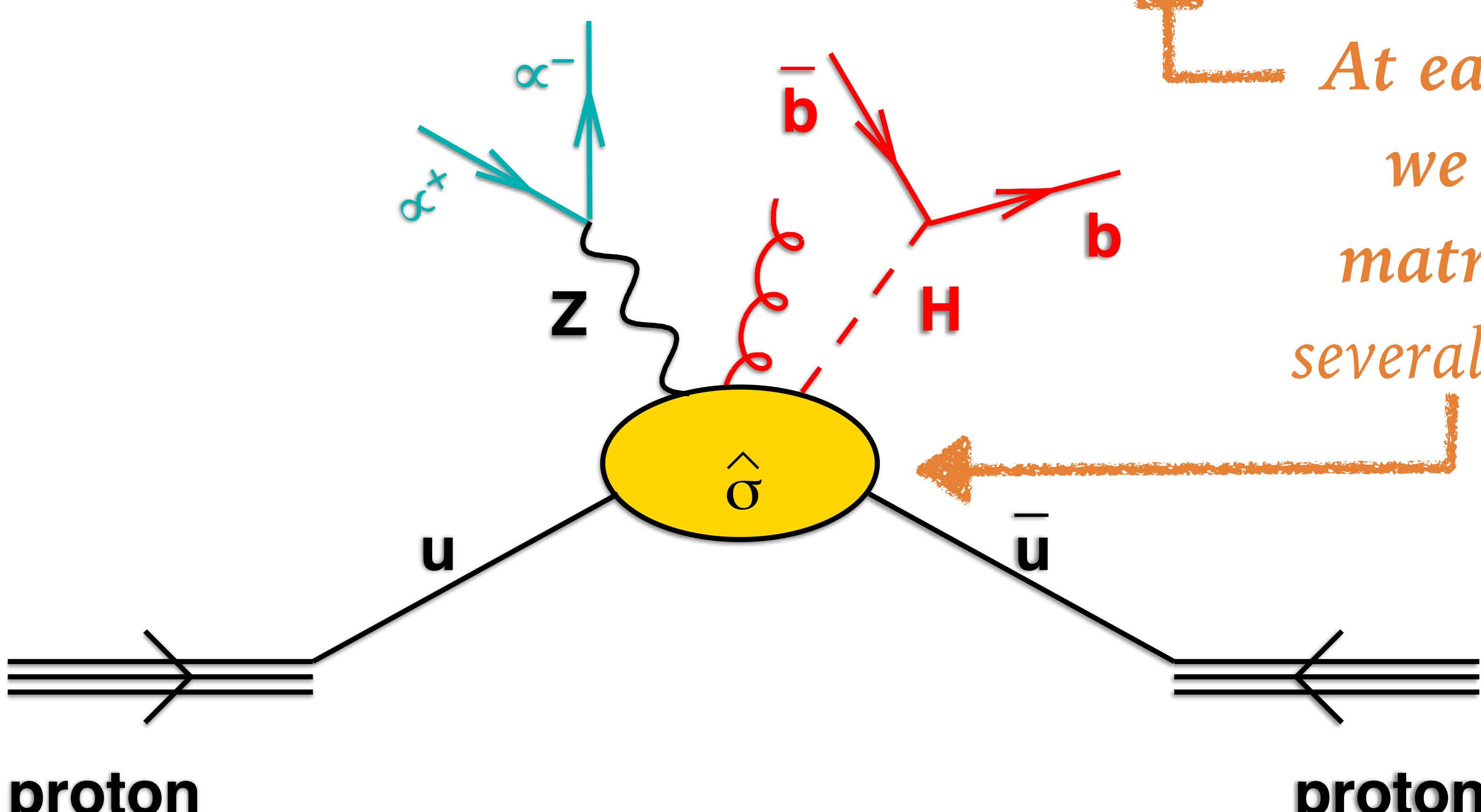
At each perturbative order n we have a specific "hard matrix element" (sometimes several for different subprocesses)



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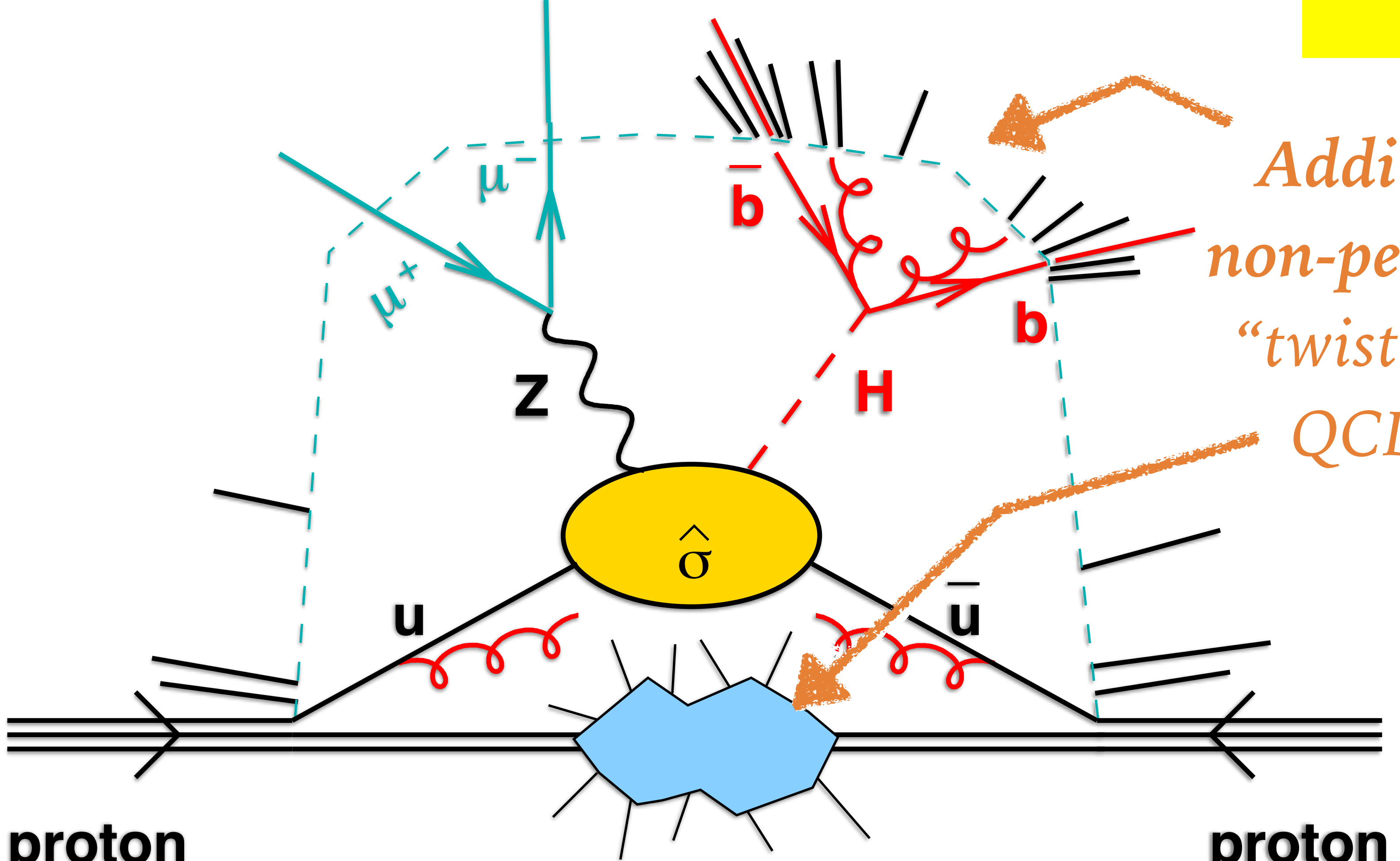
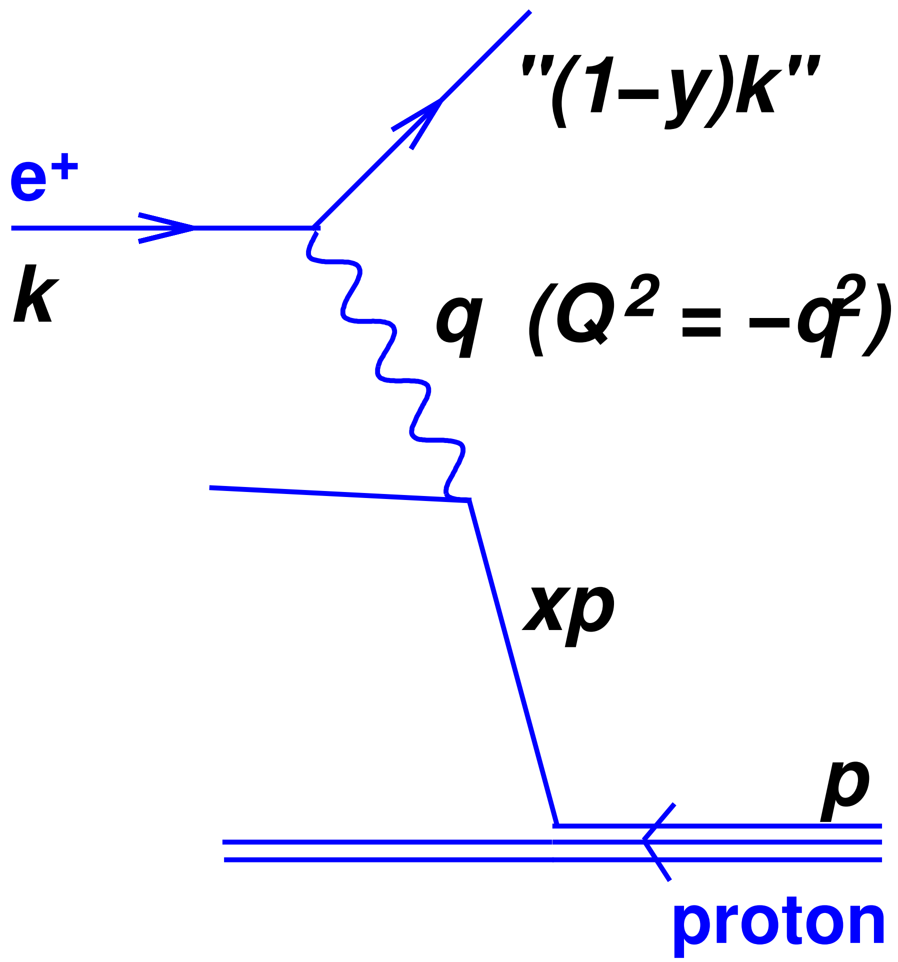


Additional corrections from non-perturbative effects (higher "twist", suppressed by powers of QCD scale (Λ) / hard scale)

PARTON DISTRIBUTION FUNCTIONS (PDFs)

Hadron-hadron is complex because of two incoming partons — so start with simpler Deep Inelastic Scattering (DIS).



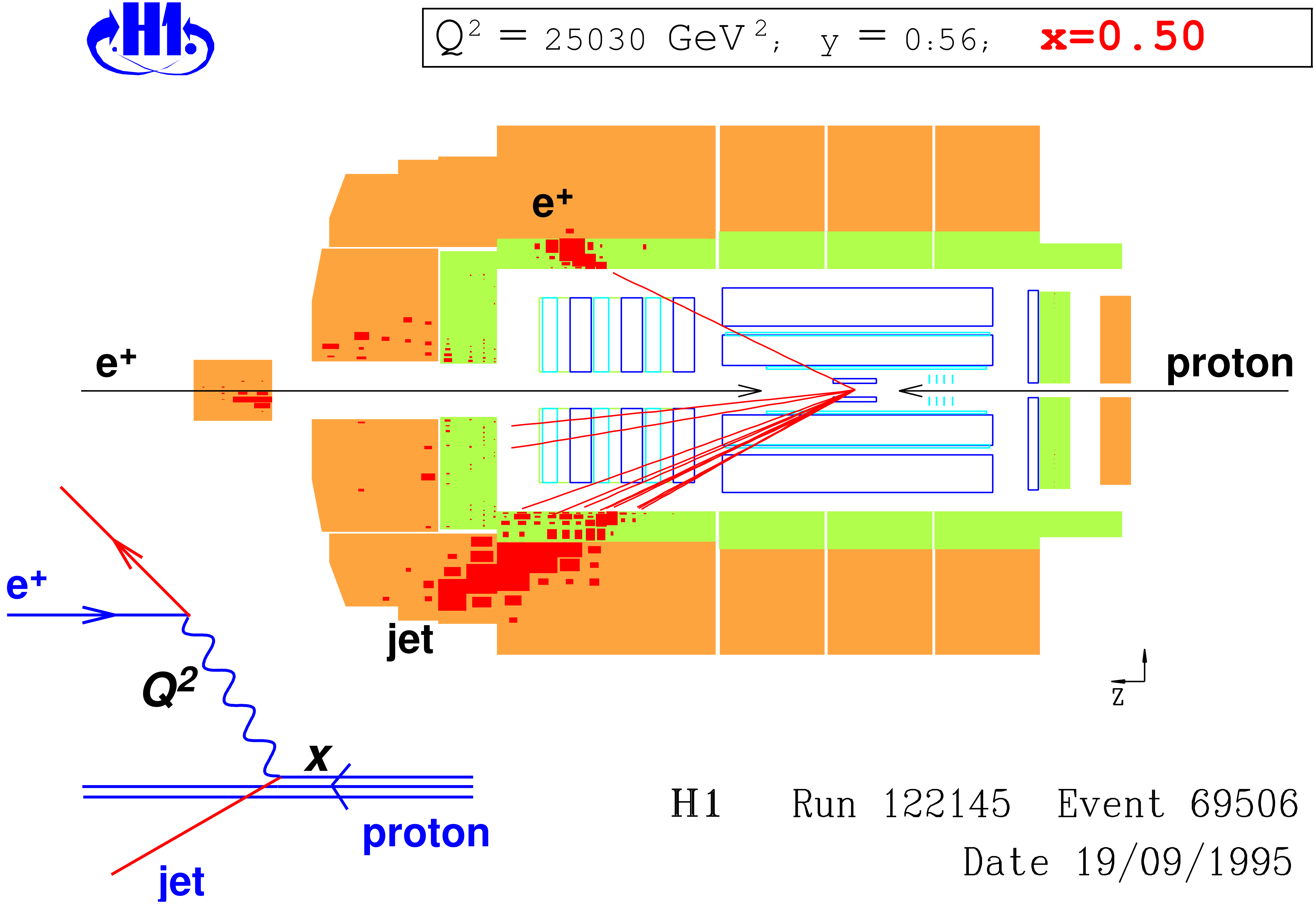
Kinematic relations:

$$x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys$$

$\sqrt{s}$ = c.o.m. energy

- Q^2 = photon virtuality $\leftrightarrow$ *transverse resolution* at which it probes proton structure
- x = *longitudinal momentum fraction* of struck parton in proton
- y = momentum fraction lost by electron (in proton rest frame)

Deep Inelastic Scattering (Past = HERA, future = EIC@Brookhaven ~ 2030)



Write DIS X-section to zeroth order in α_s ('quark parton model'):

$$\frac{d^2\sigma^{em}}{dxdQ^2} \simeq \frac{4\pi\alpha^2}{xQ^4}\left(\frac{1+(1-y)^2}{2}F_2^{em} + \mathcal{O}(\alpha_s)\right)$$

$\propto F_2^{em}$ [structure function]

$$F_2 = x(e_u^2 u(x) + e_d^2 d(x)) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

[$u(x)$, $d(x)$: parton distribution functions (PDF)]

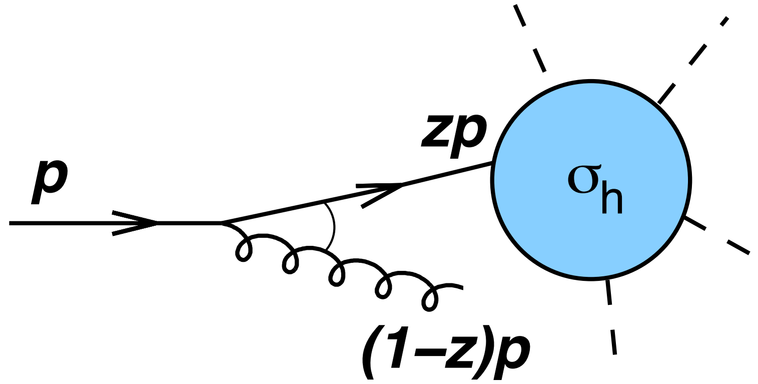
NB:

- use perturbative language for interactions of up and down quarks
- but distributions themselves have a *non-perturbative* origin.

Higher order corrections from initial state splittings?

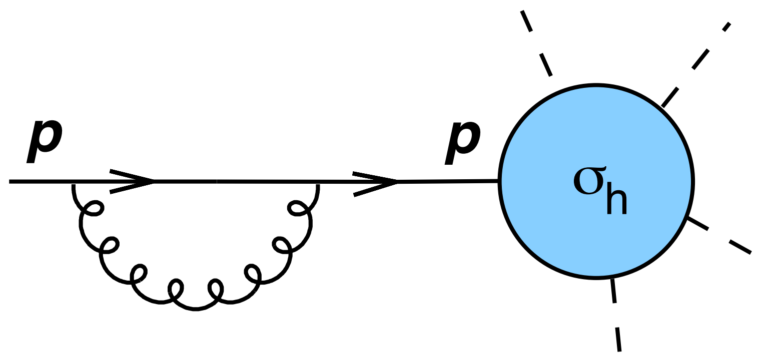
For initial state splitting, hard process occurs *after splitting*, and momentum entering hard process is modified: $p \to zp$.

$$\sigma_{g+h}(p) \simeq \sigma_h(zp)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$



For virtual terms, momentum entering hard process is unchanged

$$\sigma_{V+h}(p) \simeq -\sigma_h(p)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$



Total cross section gets contribution with *two different hard X-sections*

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi}\int \frac{dk_t^2}{k_t^2}\frac{dz}{1-z}[\sigma_h(zp) - \sigma_h(p)]$$

NB: We assume σ_h involves momentum transfers $\sim Q \gg k_t$, so ignore extra transverse momentum in σ_h

Higher order corrections from initial state splittings?

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_0^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{infinite}} \underbrace{\int \frac{dz}{1-z} [\sigma_h(zp) - \sigma_h(p)]}_{\text{finite}}$$

- In soft limit ($z \to 1$), $\sigma_h(zp) - \sigma_h(p) \to 0$: *soft divergence cancels*.
- For $1 - z \neq 0$, $\sigma_h(zp) - \sigma_h(p) \neq 0$, so z integral is non-zero but finite.

BUT: k_t integral is just a factor, and is *infinite*

This is a collinear ($k_t \to 0$) divergence.

Cross section with incoming parton is not collinear safe!

This always happens with coloured initial-state particles

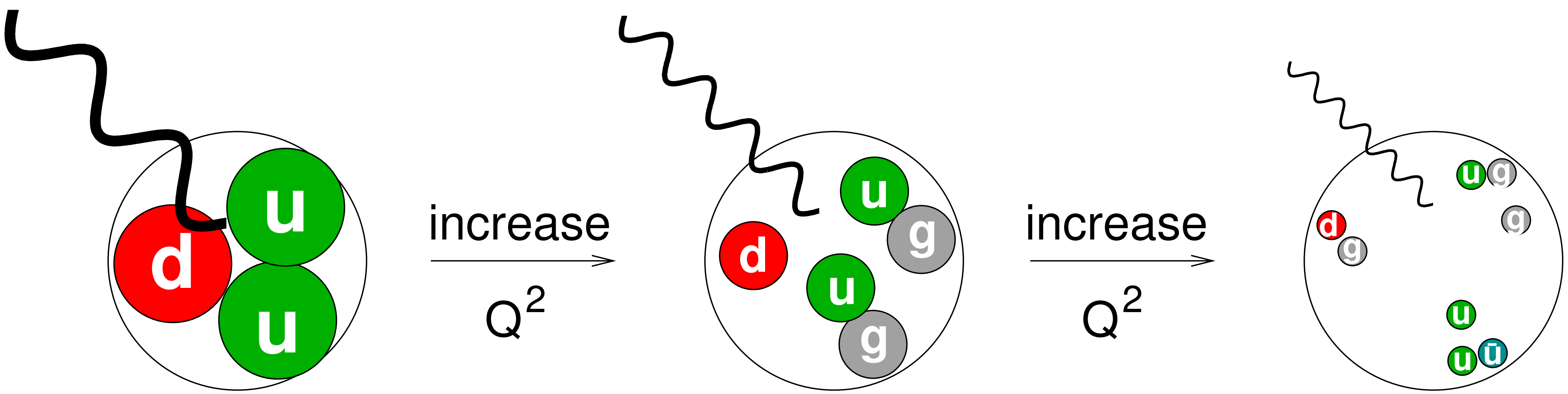
So how do we do QCD calculations in such cases?

Parton distributions and DGLAP

- Write up-quark distribution in proton as

$$u(x, \mu_F^2)$$

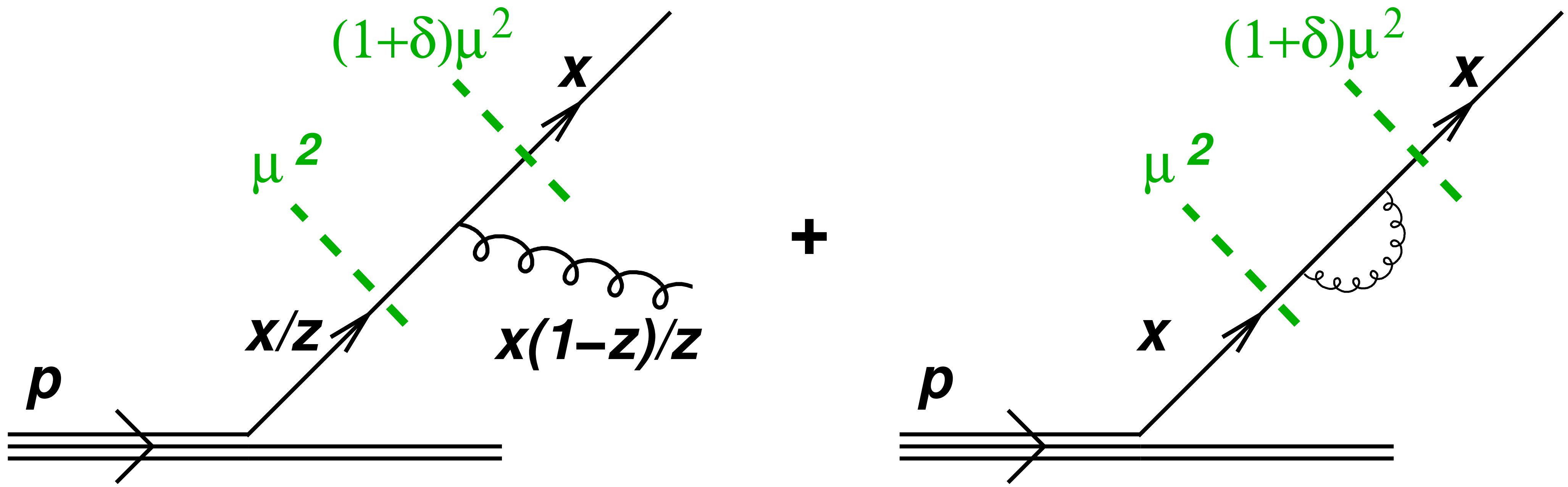
- Perturbative collinear (IR) divergence absorbed into the parton distribution (NB divergence not physical: non-perturbative physics provides a physical cutoff)
- μ_F is the **factorisation scale** — a bit like the renormalisation scale (μ_R) for the running coupling.
- As you vary the factorisation scale, the parton distributions evolve with a renormalisation-group type equation



Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations

DGLAP EQUATION

take derivative wrt factorization scale μ^2



$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 dz\, p_{qq}(z)\,\frac{q(x/z,\mu^2)}{z} - \frac{\alpha_s}{2\pi}\int_0^1 dz\, p_{qq}(z)\, q(x,\mu^2)$$

p_{qq} is real $q \leftarrow q$ **splitting kernel:** $p_{qq}(z) = C_F \frac{1+z^2}{1-z}$

Awkward to write real and virtual parts separately. Use more compact notation:

$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\underbrace{\int_x^1 dz\, P_{qq}(z)\,\frac{q(x/z,\mu^2)}{z}}_{P_{qq}\otimes q}, \qquad P_{qq} = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

This involves the *plus prescription:*

$$\int_0^1 dz\,[g(z)]_+\,f(z) = \int_0^1 dz\,g(z)\,f(z) - \int_0^1 dz\,g(z)\,f(1)$$

$z=1$ divergences of $g(z)$ cancelled if $f(z)$ sufficiently smooth at $z=1$

DGLAP EQUATION

Proton contains both quarks and gluons — so DGLAP is a *matrix in flavour space:*

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q \\ g \end{pmatrix} = \begin{pmatrix} P_{q\leftarrow q} & P_{q\leftarrow g} \\ P_{g\leftarrow q} & P_{g\leftarrow g} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

[In general, matrix spanning all flavors, anti-flavors, $P_{qq'} = 0$ (LO), $P_{\bar{q}g} = P_{qg}$]

Splitting functions are:

$$P_{qg}(z) = T_R\left[z^2 + (1-z)^2\right], \qquad P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right],$$

$$P_{gg}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \delta(1-z)\frac{(11C_A - 4n_f T_R)}{6}.$$

Have various symmetries / significant properties, e.g.

- P_{qg}, P_{gg}: *symmetric* $z \leftrightarrow 1-z$ (except virtuals)
- P_{qq}, P_{gg}: *diverge for* $z \to 1$ soft gluon emission
- P_{gg}, P_{gq}: *diverge for* $z \to 0$ Implies PDFs grow for $x \to 0$

2015 EPS HEP prize to Bjorken, Altarelli, Dokshitzer, Lipatov & Parisi

NLO DGLAP

$$P^{(1)}_{\rm ps}(x) = 4\,C_F n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4\mathrm{H}_0 + x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{56}{9}\right] + (1+x)\left[5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right]\right)$$

$$P^{(1)}_{\rm qg}(x) = 4\,C_A n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x)\mathrm{H}_{-1,0} - 2p_{\rm qg}(x)\mathrm{H}_{1,1} + x^2\left[\frac{44}{3}\mathrm{H}_0 - \frac{218}{9}\right] + 4(1-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_0 + x\mathrm{H}_1\right] - 4\zeta_2 x - 6\mathrm{H}_{0,0} + 9\mathrm{H}_0\right) + 4\,C_F n_f\left(2p_{\rm qg}(x)\left[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 - \zeta_2\right] + 4x^2\left[\mathrm{H}_0 + \mathrm{H}_{0,0} + \frac{5}{2}\right] + 2(1-x)\left[\mathrm{H}_0 + \mathrm{H}_{0,0} - 2x\mathrm{H}_1 + \frac{29}{4}\right] - \frac{15}{2} - \mathrm{H}_{0,0} - \frac{1}{2}\mathrm{H}_0\right)$$

$$P^{(1)}_{\rm gq}(x) = 4\,C_A C_F\left(\frac{1}{x} + 2p_{\rm gq}(x)\left[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 - \frac{11}{6}\mathrm{H}_1\right] - x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{44}{9}\right] + 4\zeta_2 - 2 - 7\mathrm{H}_0 + 2\mathrm{H}_{0,0} - 2\mathrm{H}_1 x + (1+x)\left[2\mathrm{H}_{0,0} - 5\mathrm{H}_0 + \frac{37}{9}\right] - 2p_{\rm gq}(-x)\mathrm{H}_{-1,0}\right) - 4\,C_F n_f\left(\frac{2}{3}x - p_{\rm gq}(x)\left[\frac{2}{3}\mathrm{H}_1 - \frac{10}{9}\right]\right) + 4\,{C_F}^2\left(p_{\rm gq}(x)\left[3\mathrm{H}_1 - 2\mathrm{H}_{1,1}\right] + (1+x)\left[\mathrm{H}_{0,0} - \frac{7}{2} + \frac{7}{2}\mathrm{H}_0\right] - 3\mathrm{H}_{0,0} + 1 - \frac{3}{2}\mathrm{H}_0 + 2\mathrm{H}_1 x\right)$$

$$P^{(1)}_{\rm gg}(x) = 4\,C_A n_f\left(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\left(\frac{1}{x} - x^2\right) - \frac{2}{3}(1+x)\mathrm{H}_0 - \frac{2}{3}\delta(1-x)\right) + 4\,{C_A}^2\left(27 + (1+x)\left[\frac{11}{3}\mathrm{H}_0 + 8\mathrm{H}_{0,0} - \frac{27}{2}\right] + 2p_{\rm gg}(-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_{-1,0} - \zeta_2\right] - \frac{67}{9}\left(\frac{1}{x} - x^2\right) - 12\mathrm{H}_0 - \frac{44}{3}x^2\mathrm{H}_0 + 2p_{\rm gg}(x)\left[\frac{67}{18} - \zeta_2 + \mathrm{H}_{0,0} + 2\mathrm{H}_{1,0} + 2\mathrm{H}_2\right] + \delta(1-x)\left[\frac{8}{3} + 3\zeta_3\right]\right) + 4\,C_F n_f\left(2\mathrm{H}_0 + \frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\left[4 - 5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right] - \frac{1}{2}\delta(1-x)\right)\,.$$

NLO:

$$P_{ab} = \frac{\alpha_s}{2\pi}P^{(0)} + \frac{\alpha_s^2}{16\pi^2}P^{(1)}$$

Curci, Furmanski & Petronzio '80

NNLO DGLAP

$$P_{ab} = \frac{\alpha_s}{2\pi} P_{ab}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{(2)}$$

NNLO, $P_{ab}^{(2)}$: Moch, Vermaseren & Vogt '04

N3LO DGLAP [in progress]

Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond

S. Moch[a], B. Ruijl[b,c], T. Ueda[b], J.A.M. Vermaseren[b] and A. Vogt[d]

arXiv:1707.08315v2 [hep-ph] 5 Oct 2017

$$P_{ab} = \frac{\alpha_s}{2\pi} P_{ab}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{(2)} + \left(\frac{\alpha_s}{2\pi}\right)^4 P_{ab}^{(3)}$$

+ *continued work over the past 7 years*

Additional moments and x-space approximations of four-loop splitting functions in QCD

S. Moch (Hamburg U., Inst. Theor. Phys. II), B. Ruijl (ETH, Zurich (main)), T. Ueda (Juntendo U.), J. Vermaseren (Nikhef, Amsterdam), A. Vogt (Liverpool U.) (Oct 9, 2023)

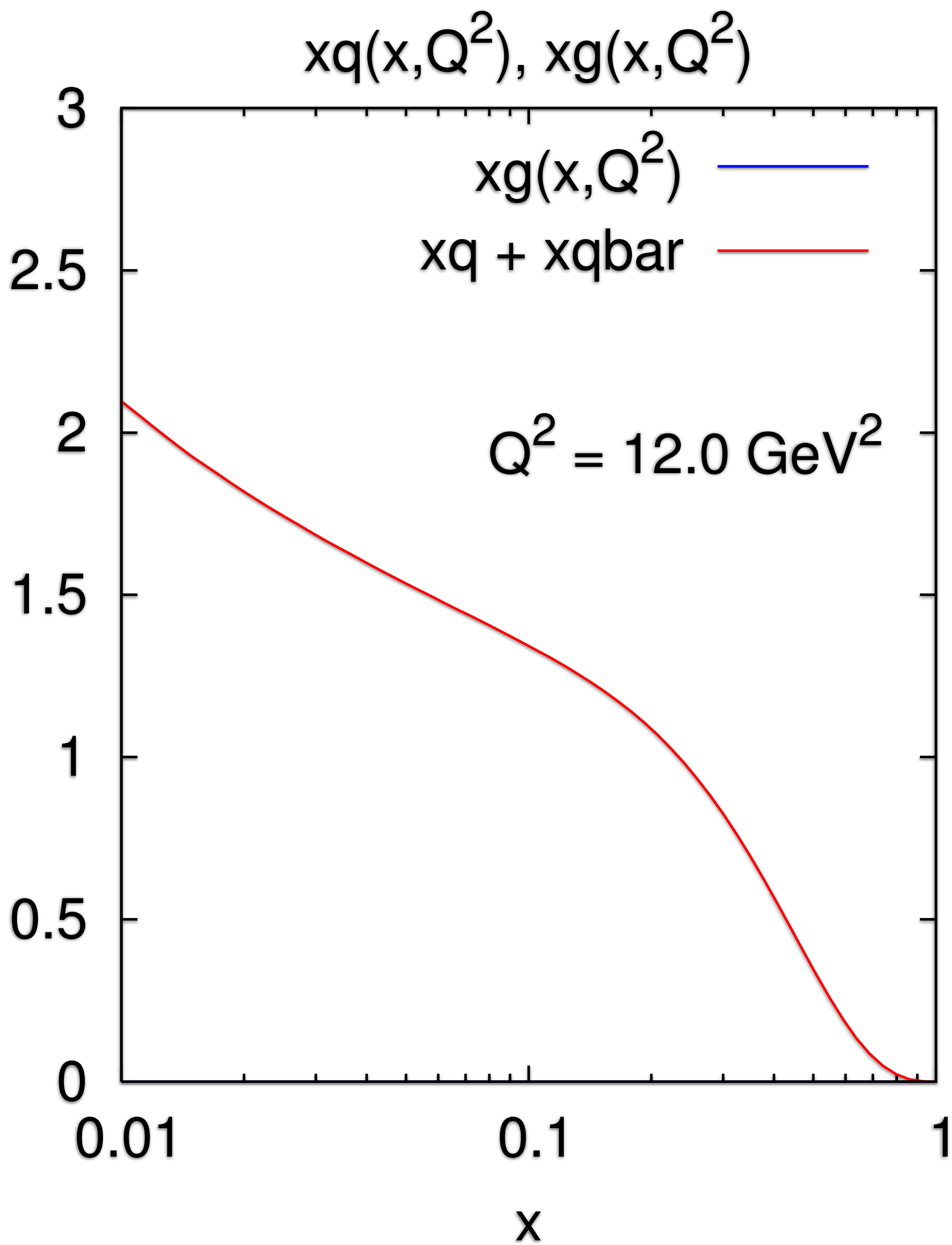
Published in: *Phys.Lett.B* 849 (2024) 138468 • e-Print: 2310.05744 [hep-ph]

The $N_f\, C_F^3$ contribution to the non-singlet splitting function at four-loop order #10

Thomas Gehrmann (Zurich U.), Andreas von Manteuffel (Regensburg U. and Michigan State U.), Vasily Sotnikov (Zurich U.), Tong-Zhi Yang (Zurich U. and Michigan State U.) (Oct 18, 2023)

Published in: *Phys.Lett.B* 849 (2024) 138427 • e-Print: 2310.12240 [hep-ph]

DGLAP evolution (initial quarks only)



Take example evolution starting with just quarks:

$$\partial_{\ln Q^2} q = P_{q \leftarrow q} \otimes q$$
$$\partial_{\ln Q^2} g = P_{g \leftarrow q} \otimes q$$

- quark is depleted at large x
- gluon grows at small x

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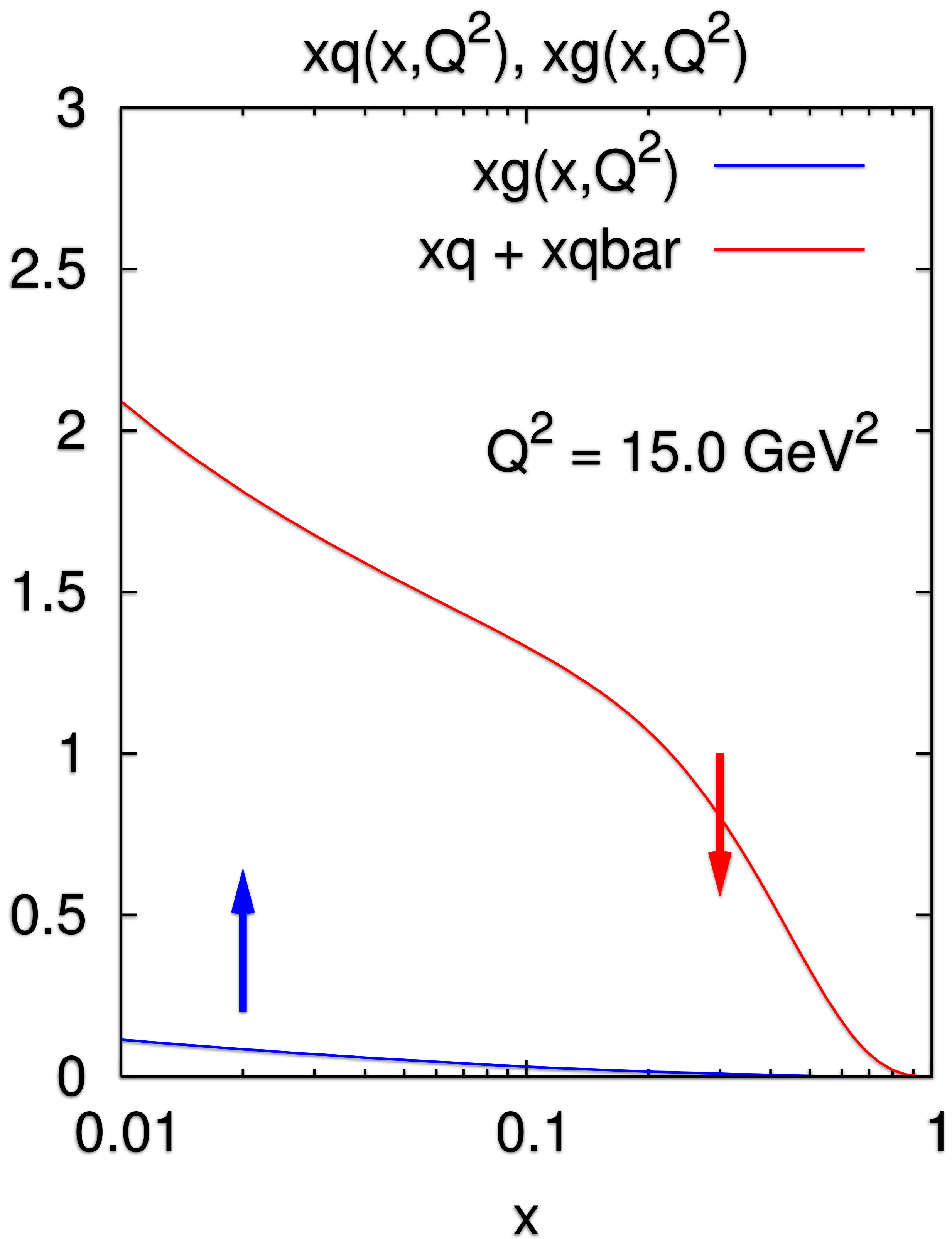

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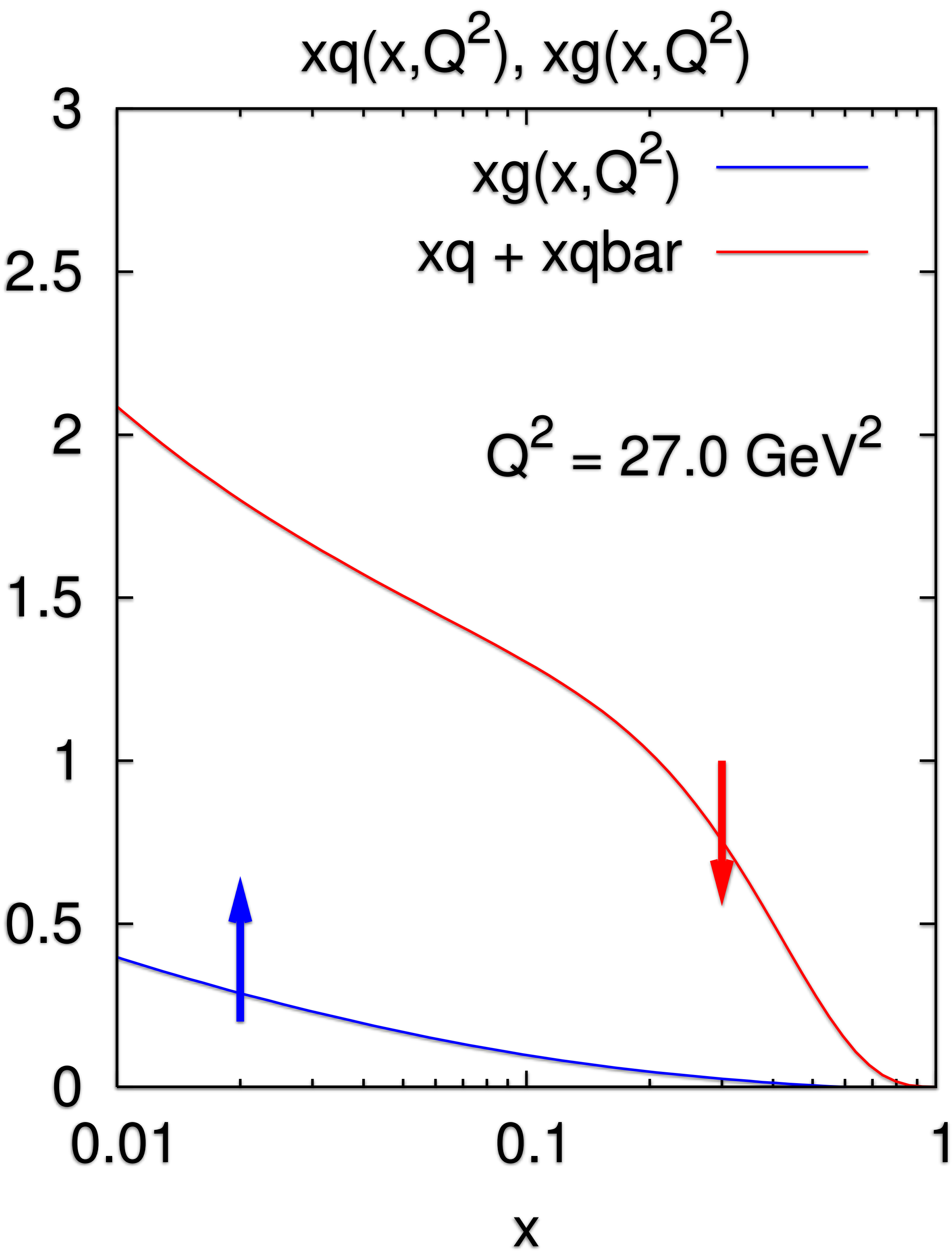

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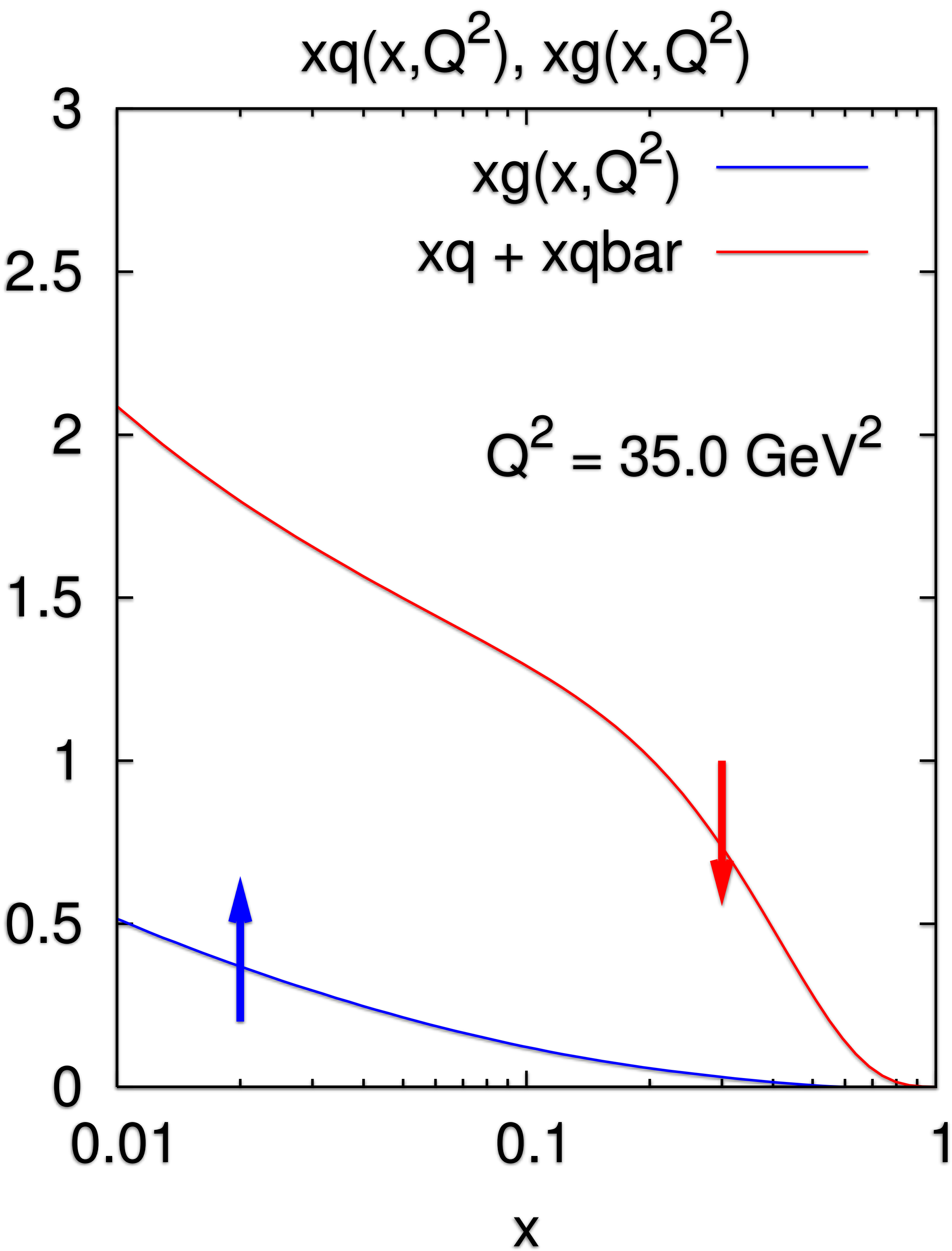

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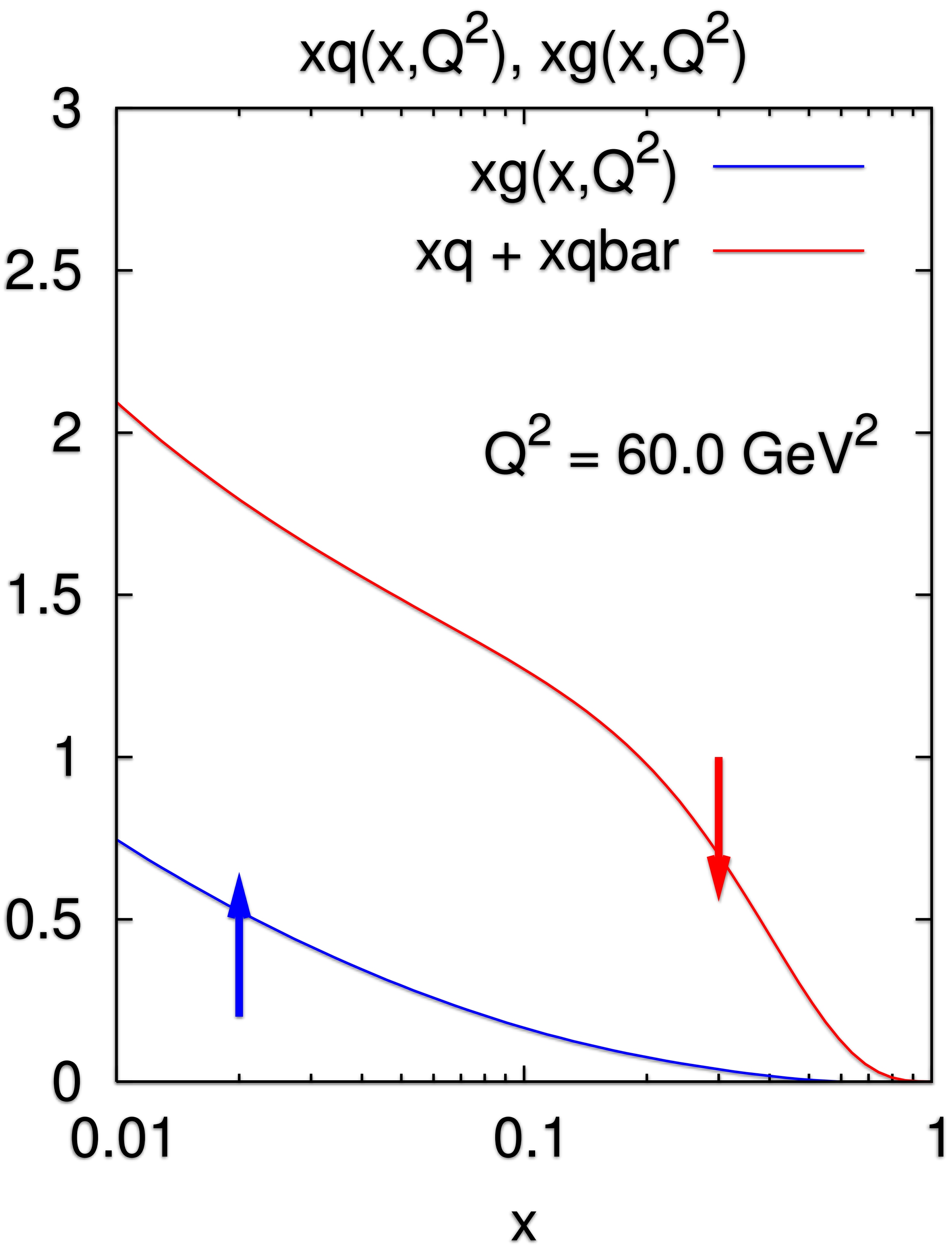

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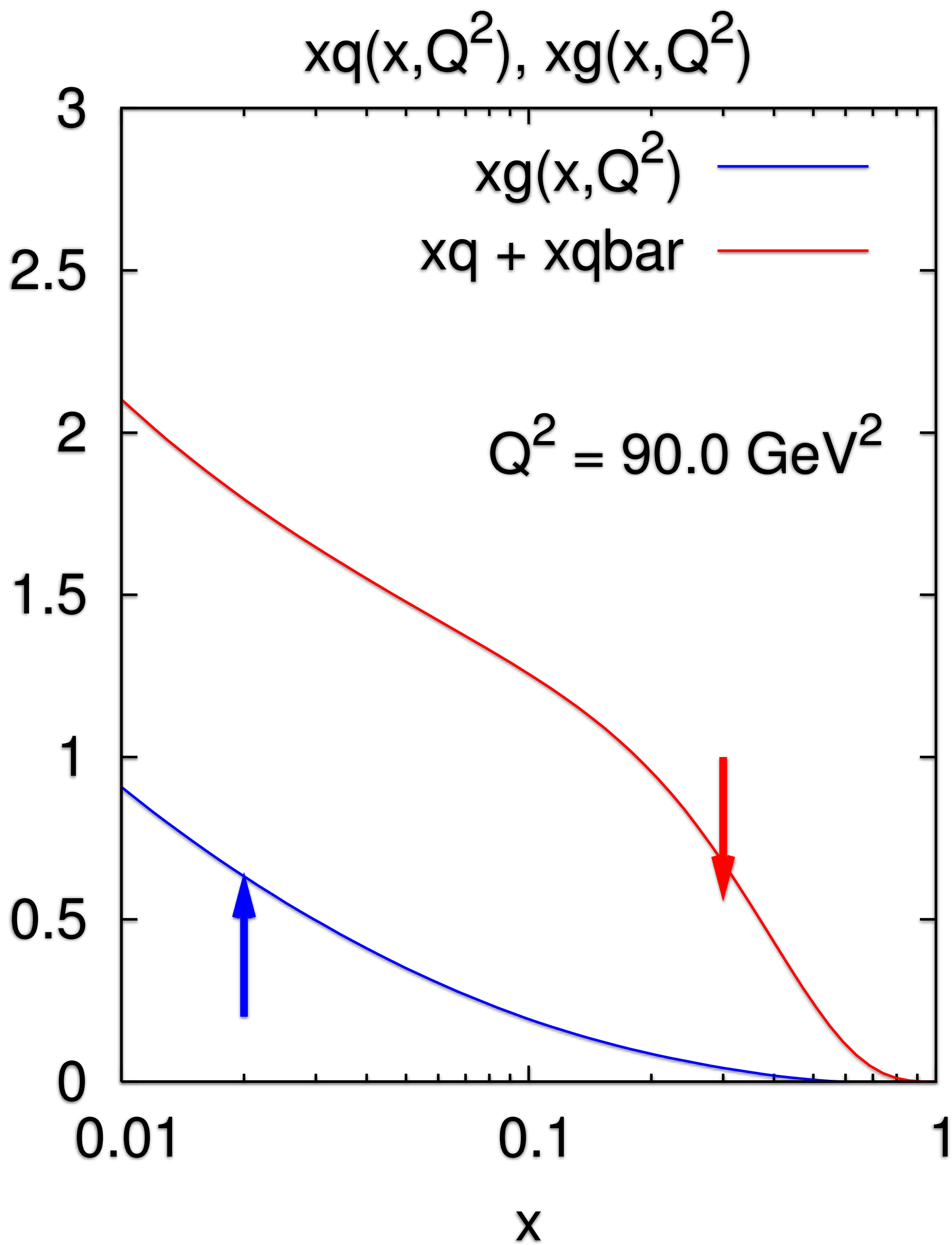

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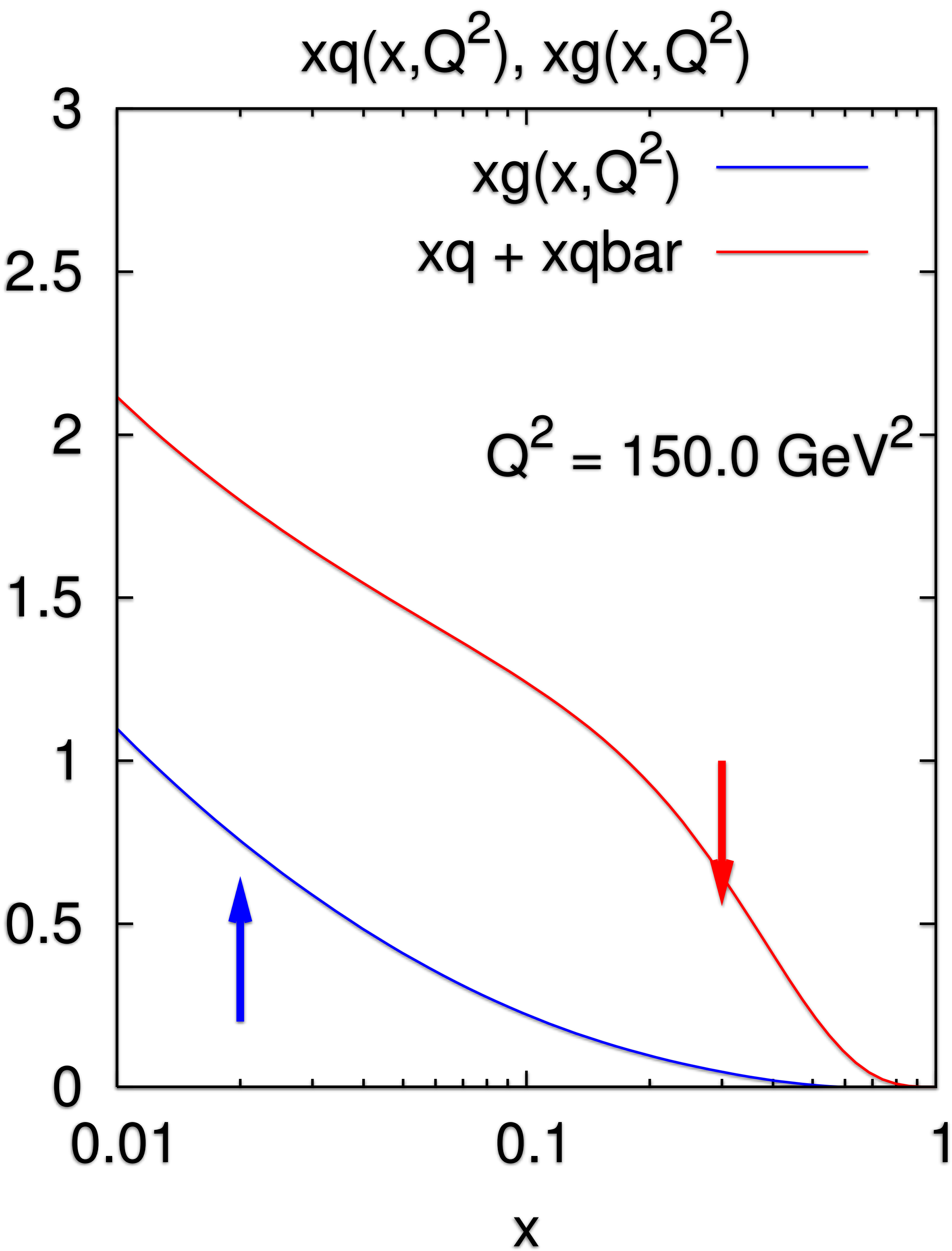

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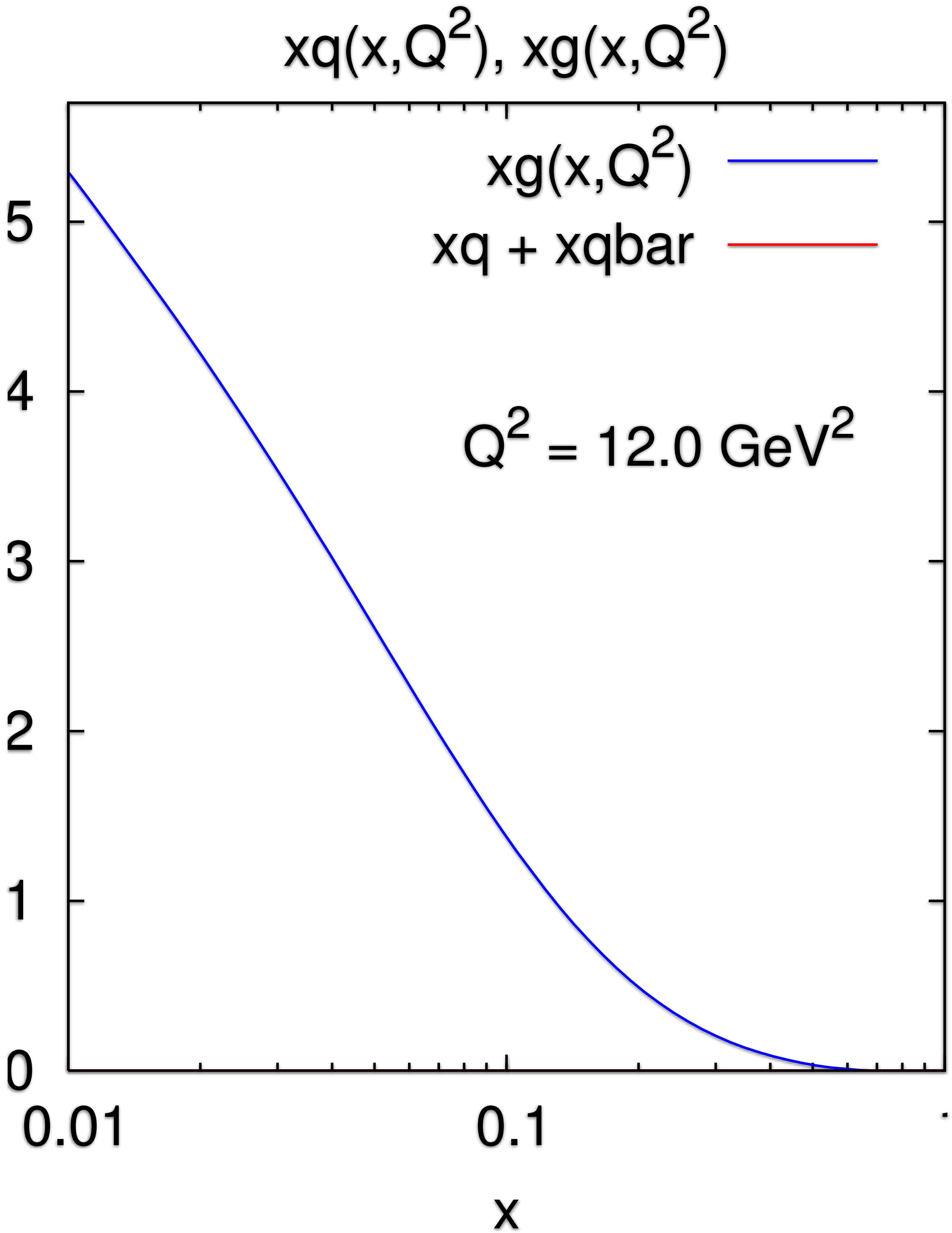
DGLAP evolution (initial gluons only)



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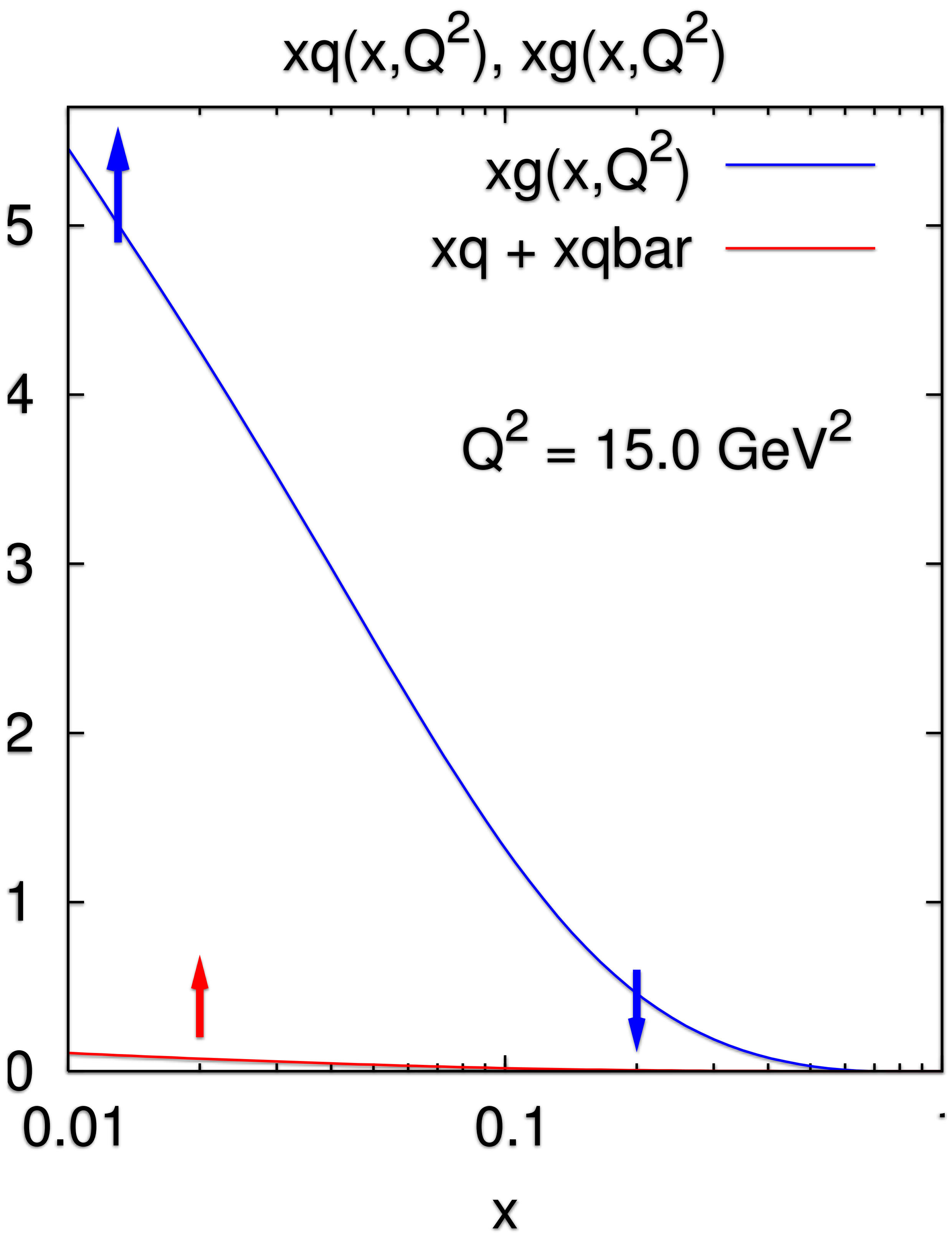

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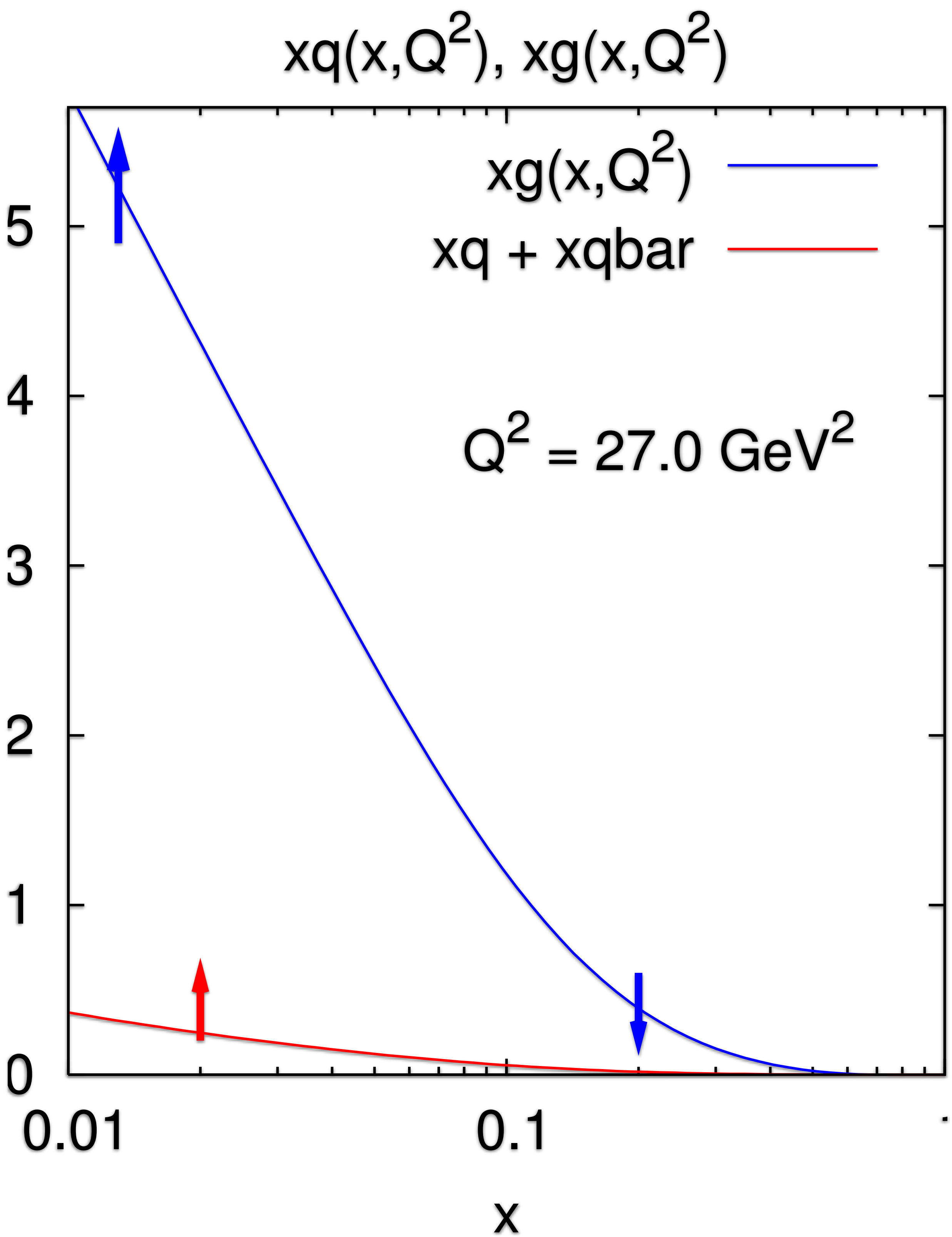

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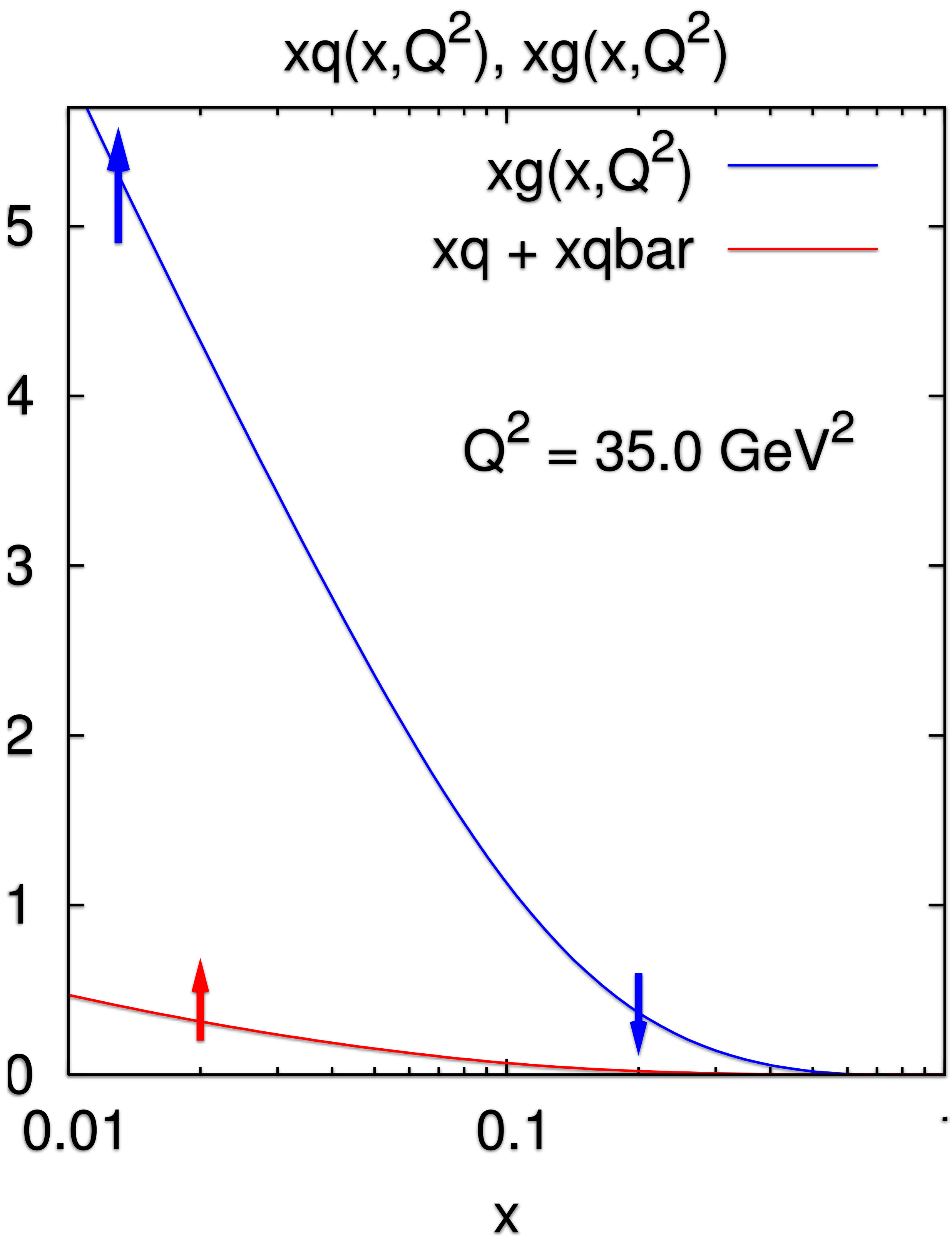

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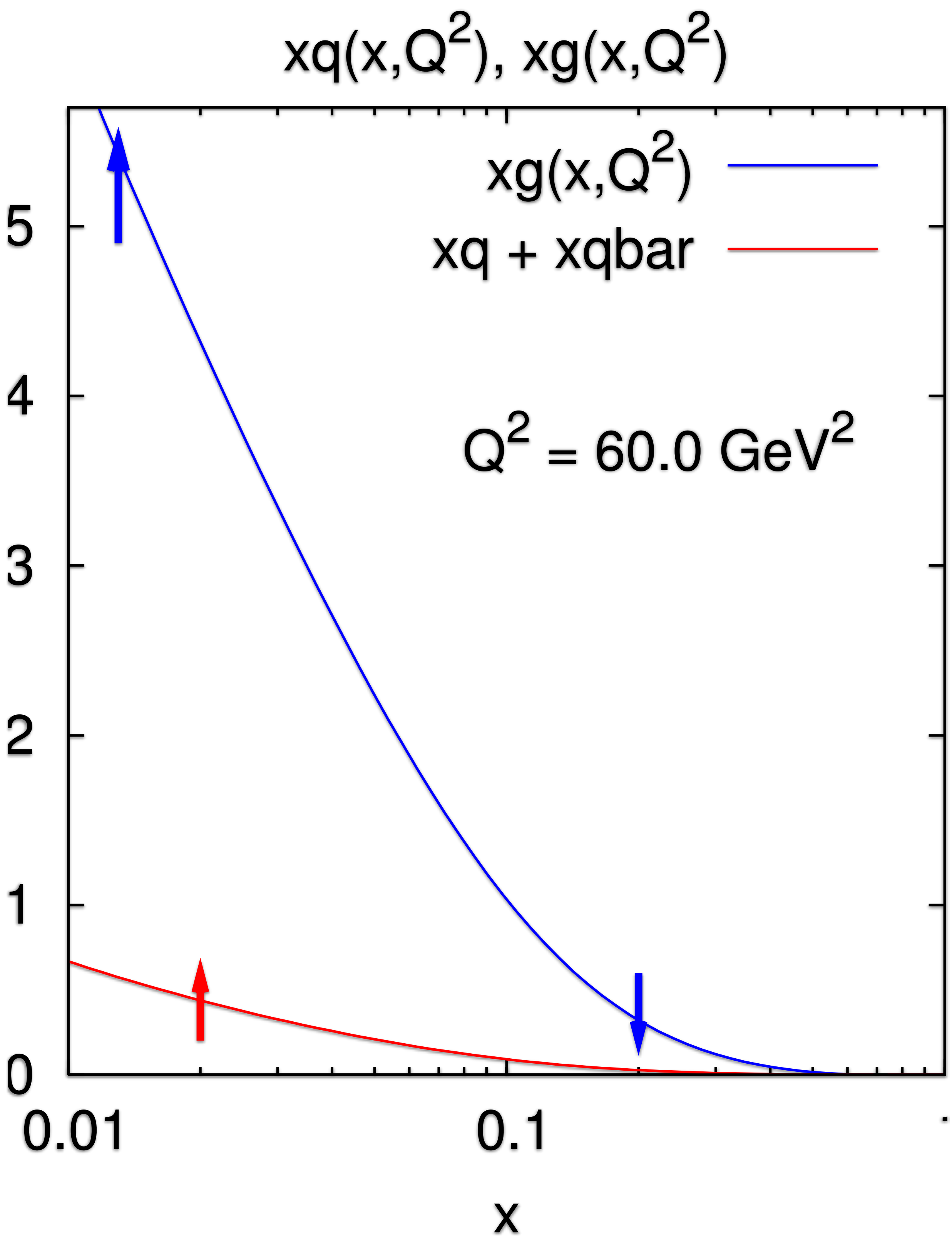

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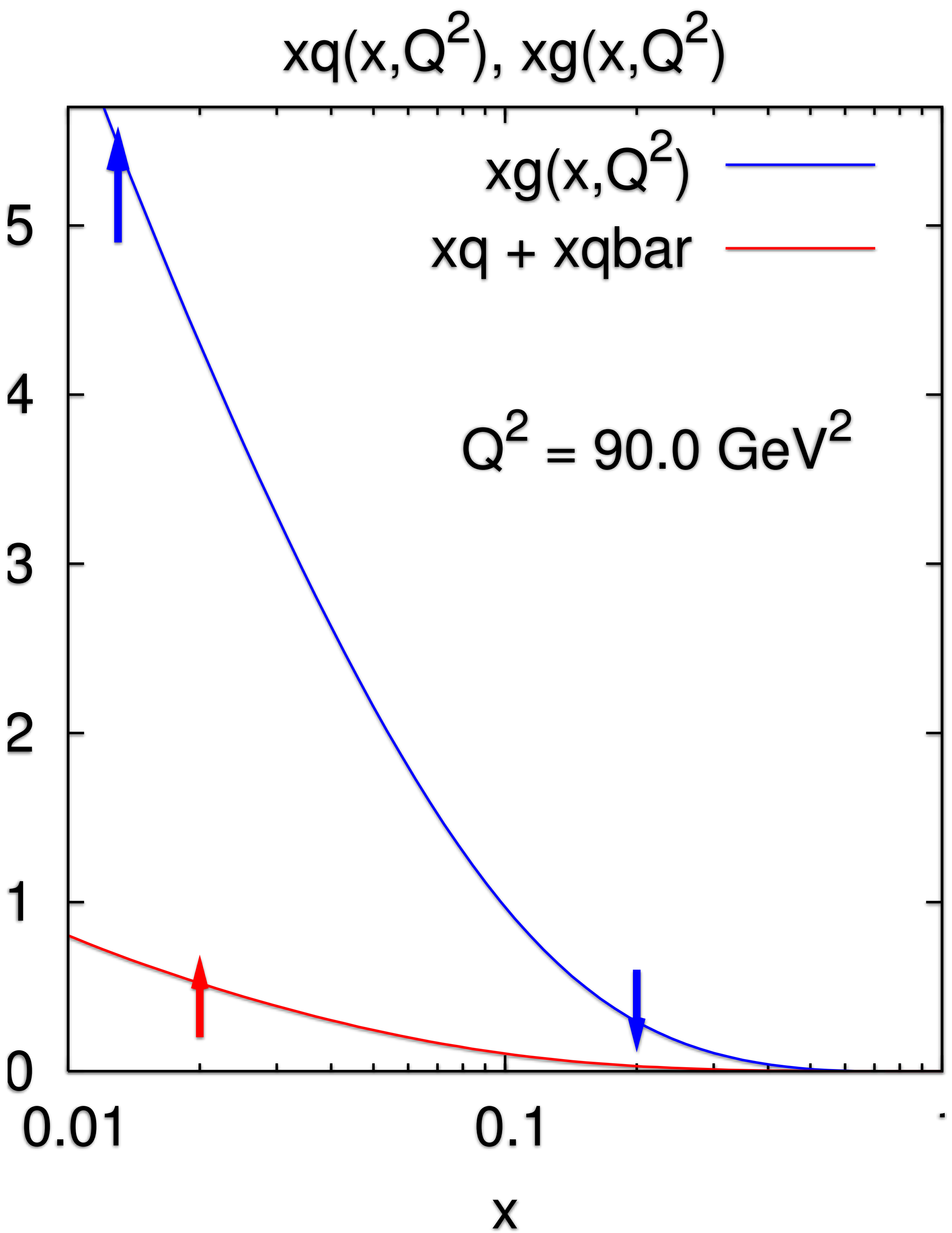

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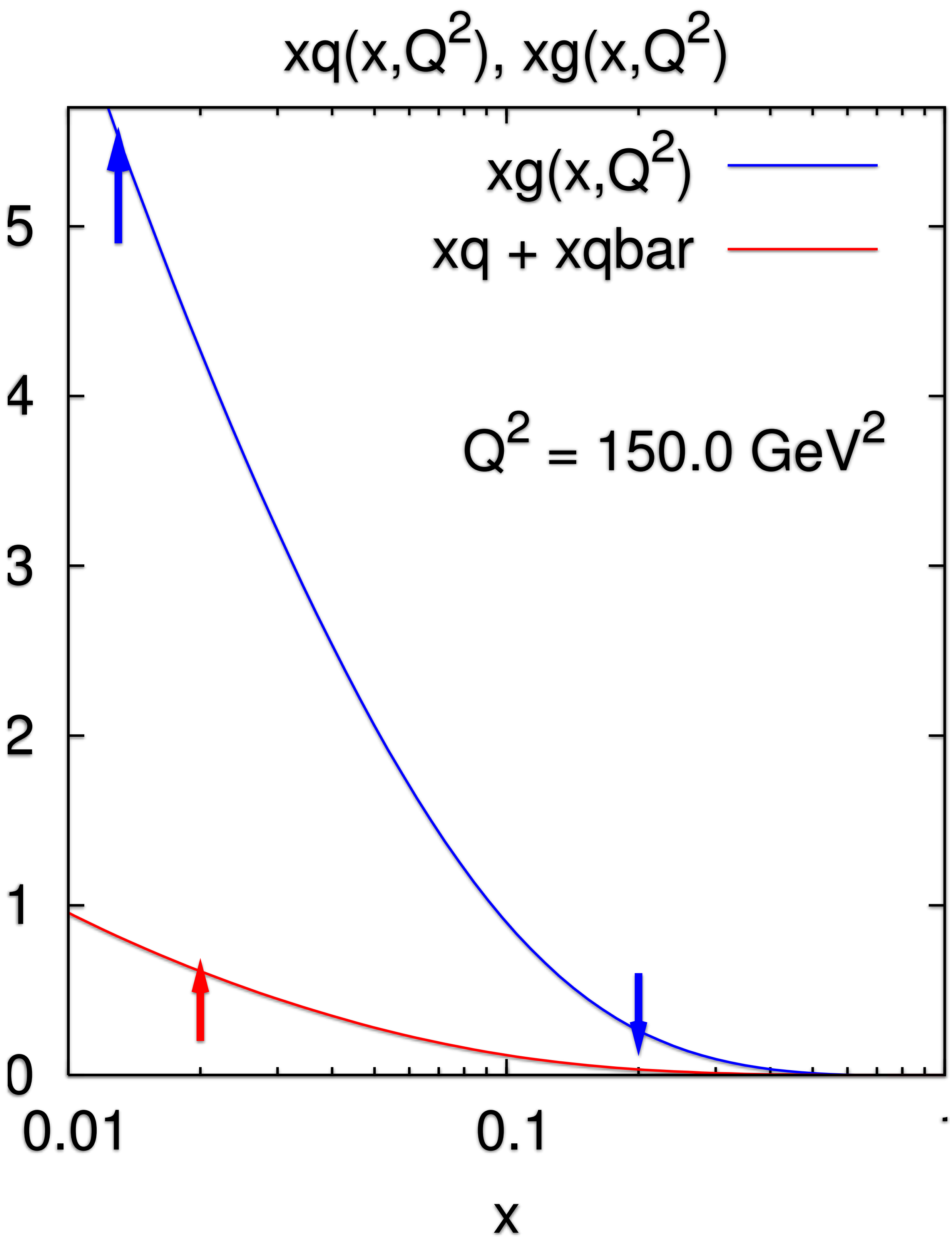

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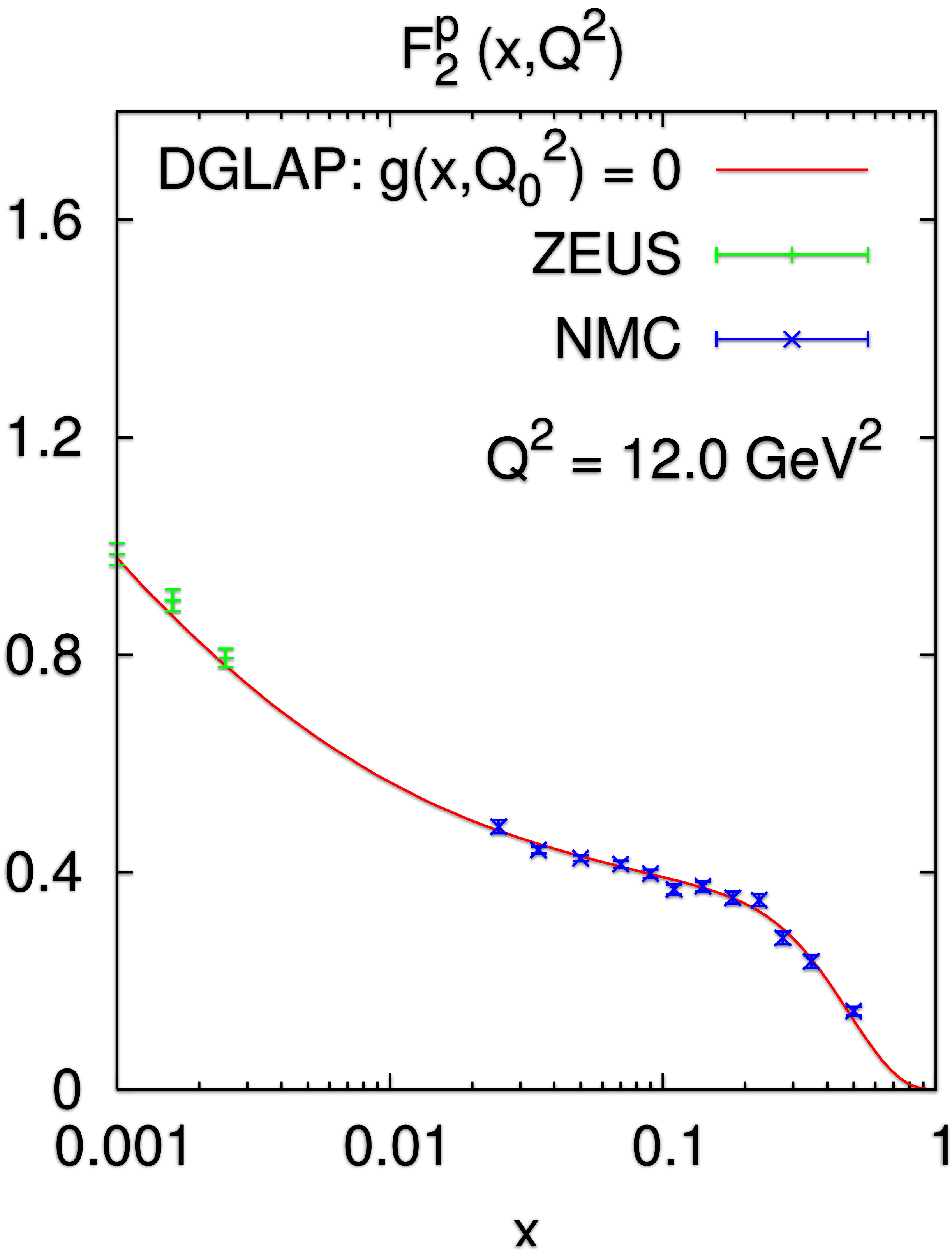
DGLAP evolution:
- partons lose momentum and shift towards smaller x
- high-x partons drive growth of low-x gluon

determining the gluon

which is critical at hadron colliders (e.g. ttbar, Higgs dominantly produced by gluon-gluon fusion), but not directly probed in Deep-Inelastic-Scattering

Consider DIS data — $F_2(x,Q^2)$ — in a world where the proton just had quarks



Fit quark distributions to $F_2(x, Q_0^2)$, at *initial scale* $Q_0^2 = 12\ \text{GeV}^2$.

NB: Q_0 often chosen lower

Assume there is no gluon at Q_0^2:

$$g(x, Q_0^2) = 0$$

Use DGLAP equations to evolve to higher Q^2; compare with data.

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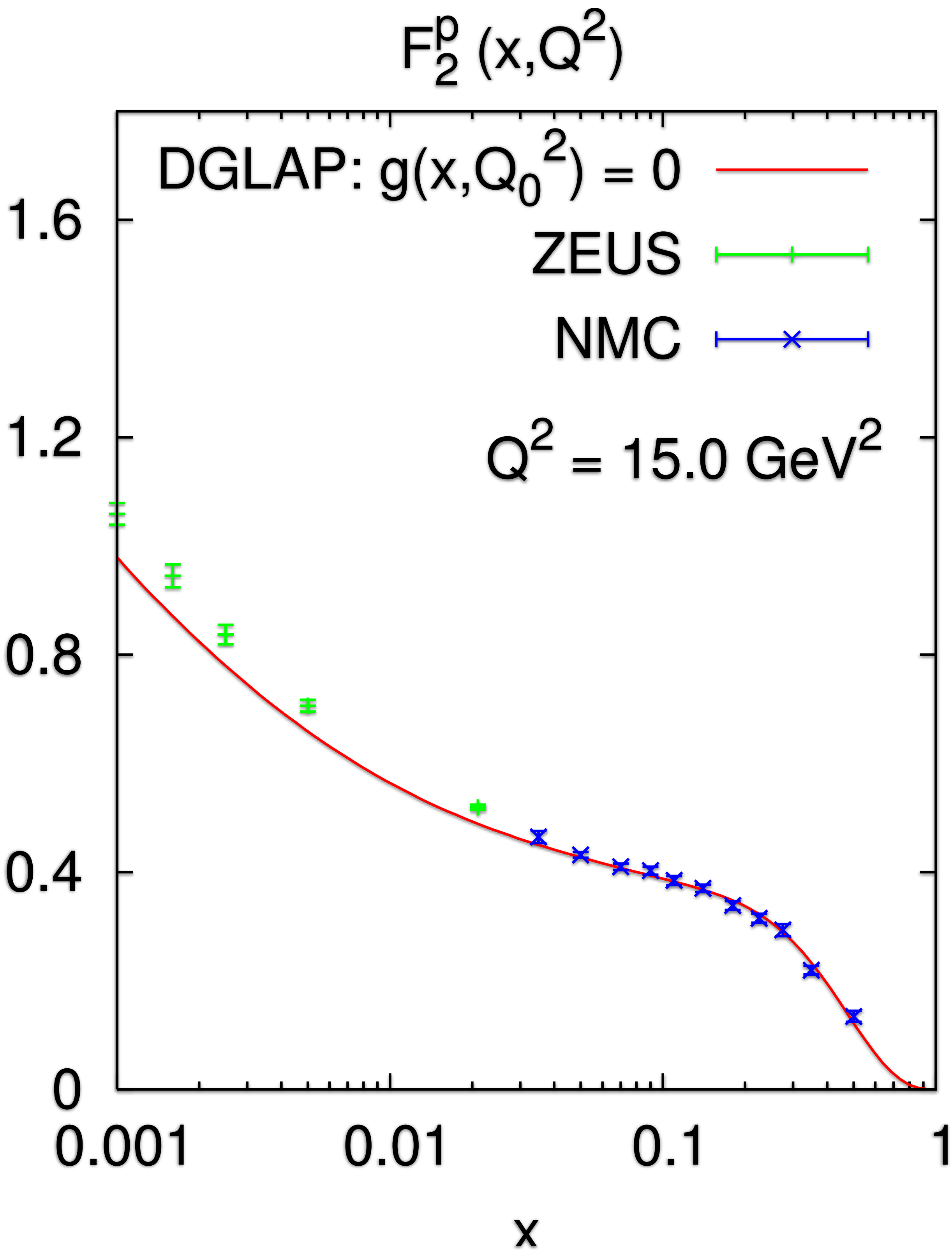

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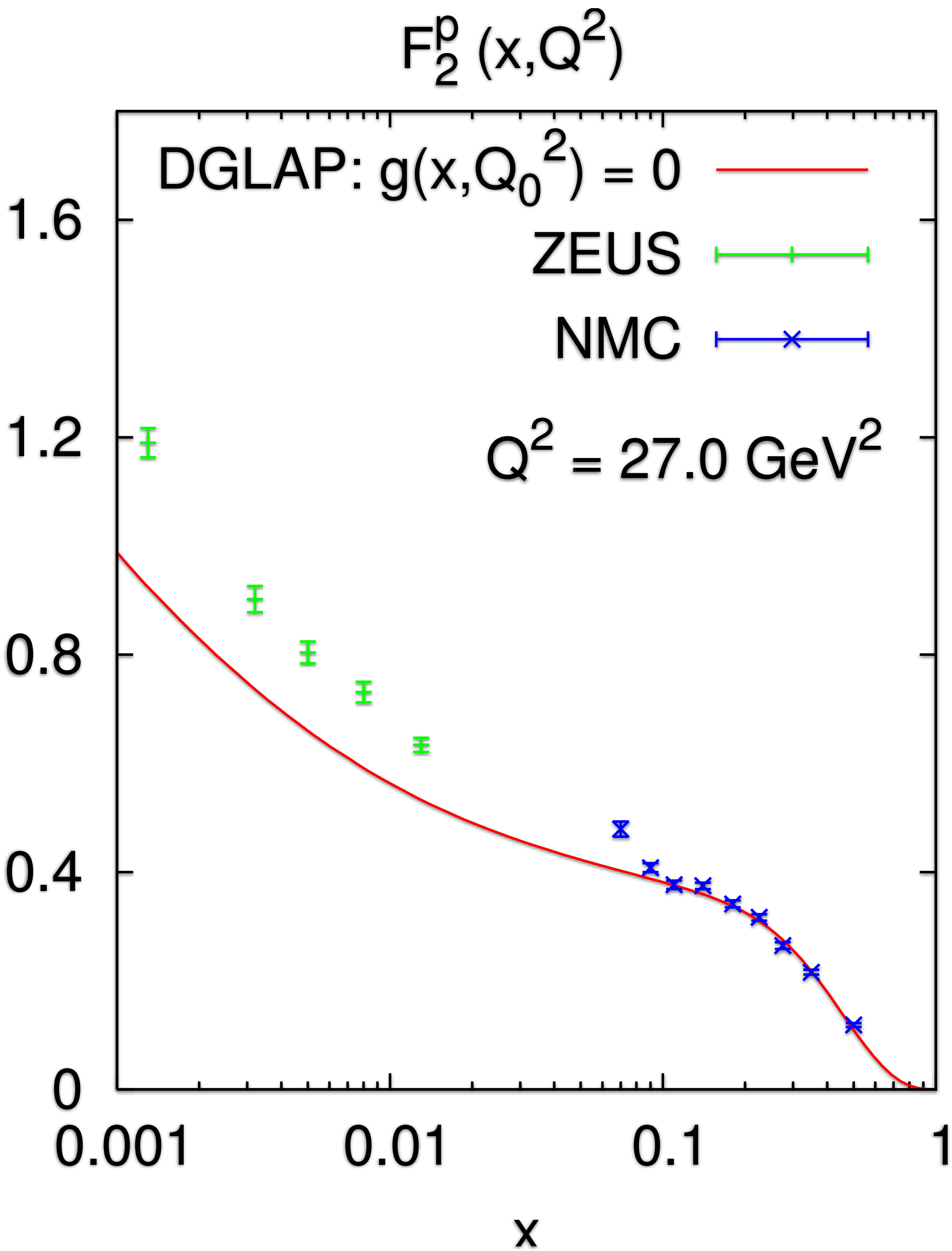

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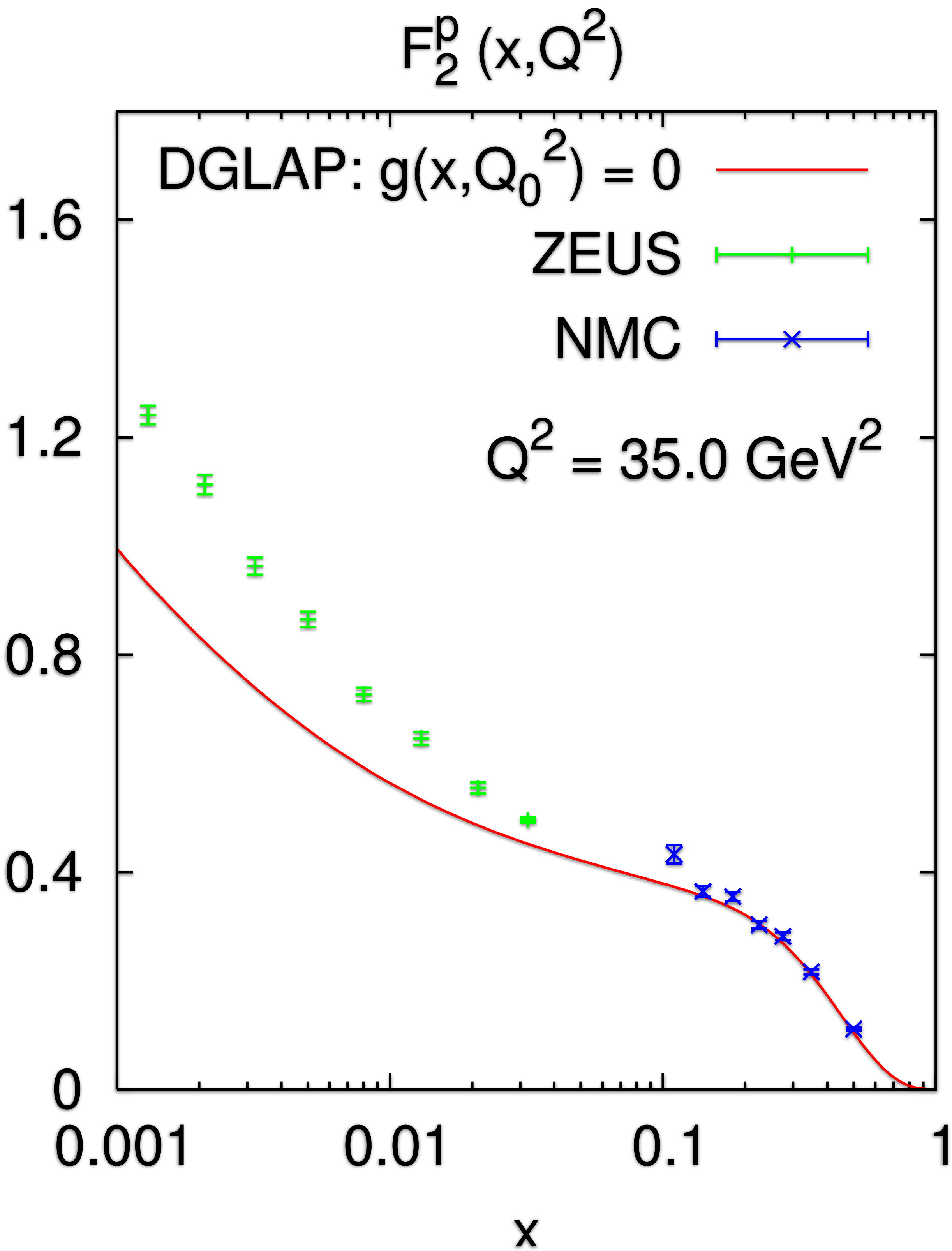

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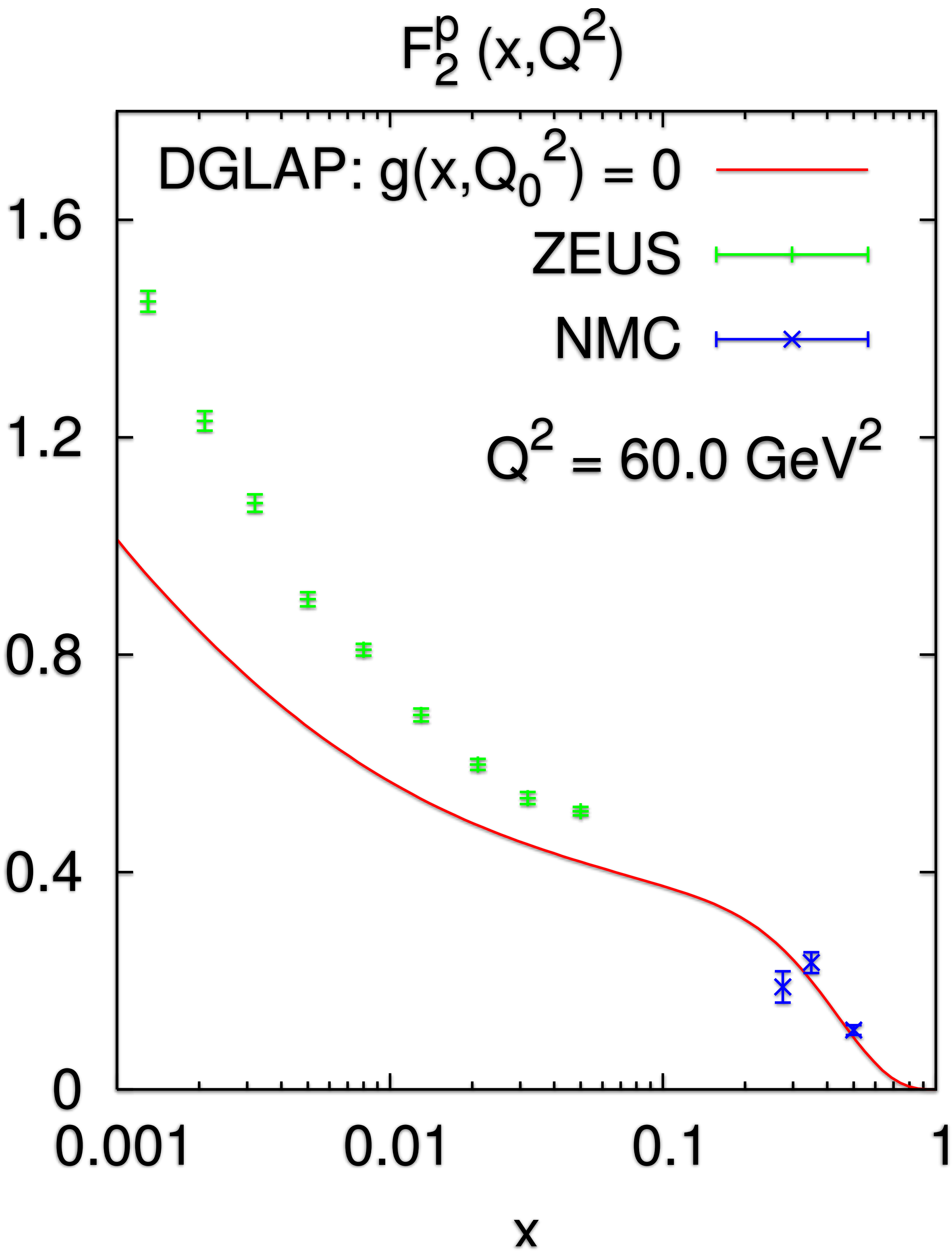

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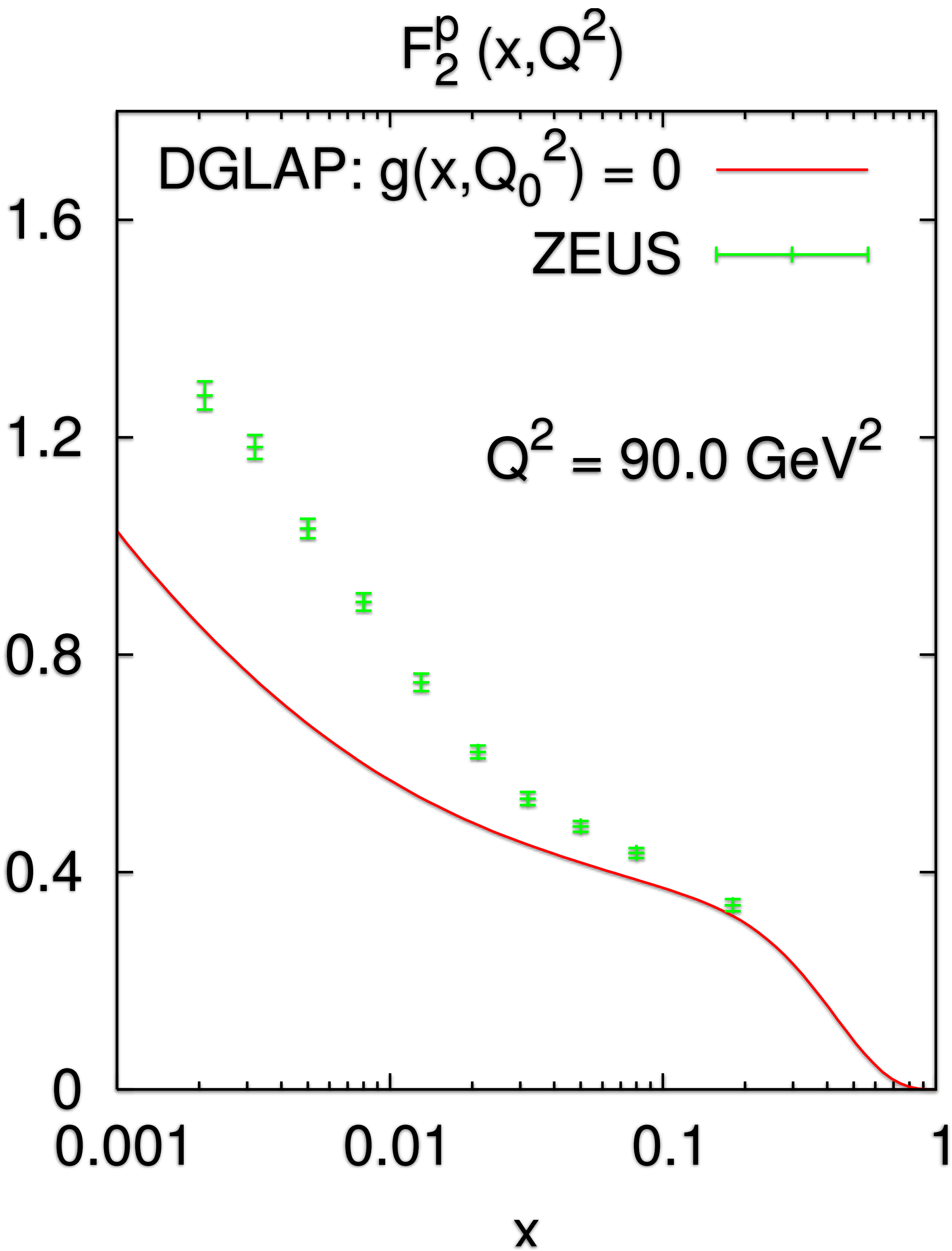

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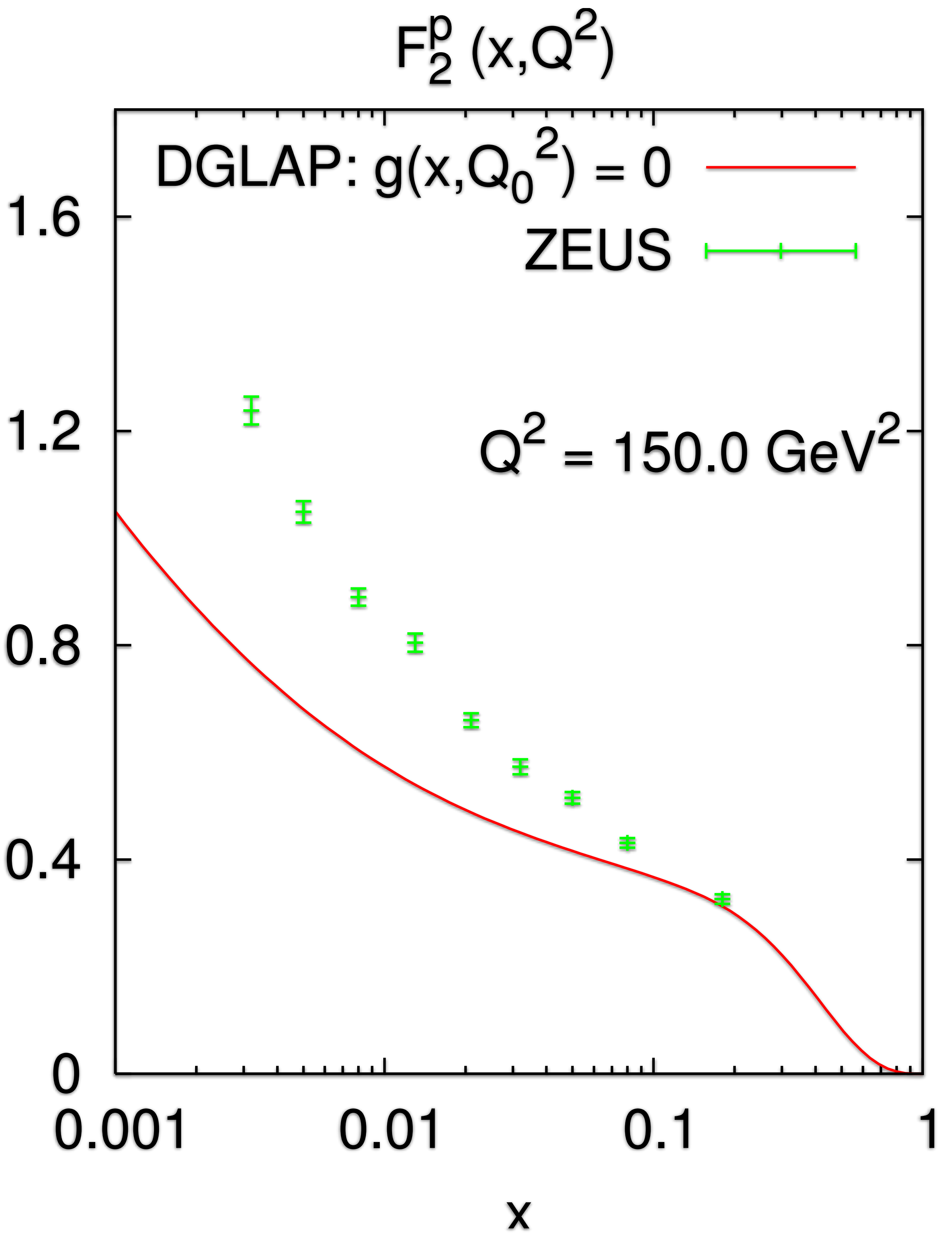

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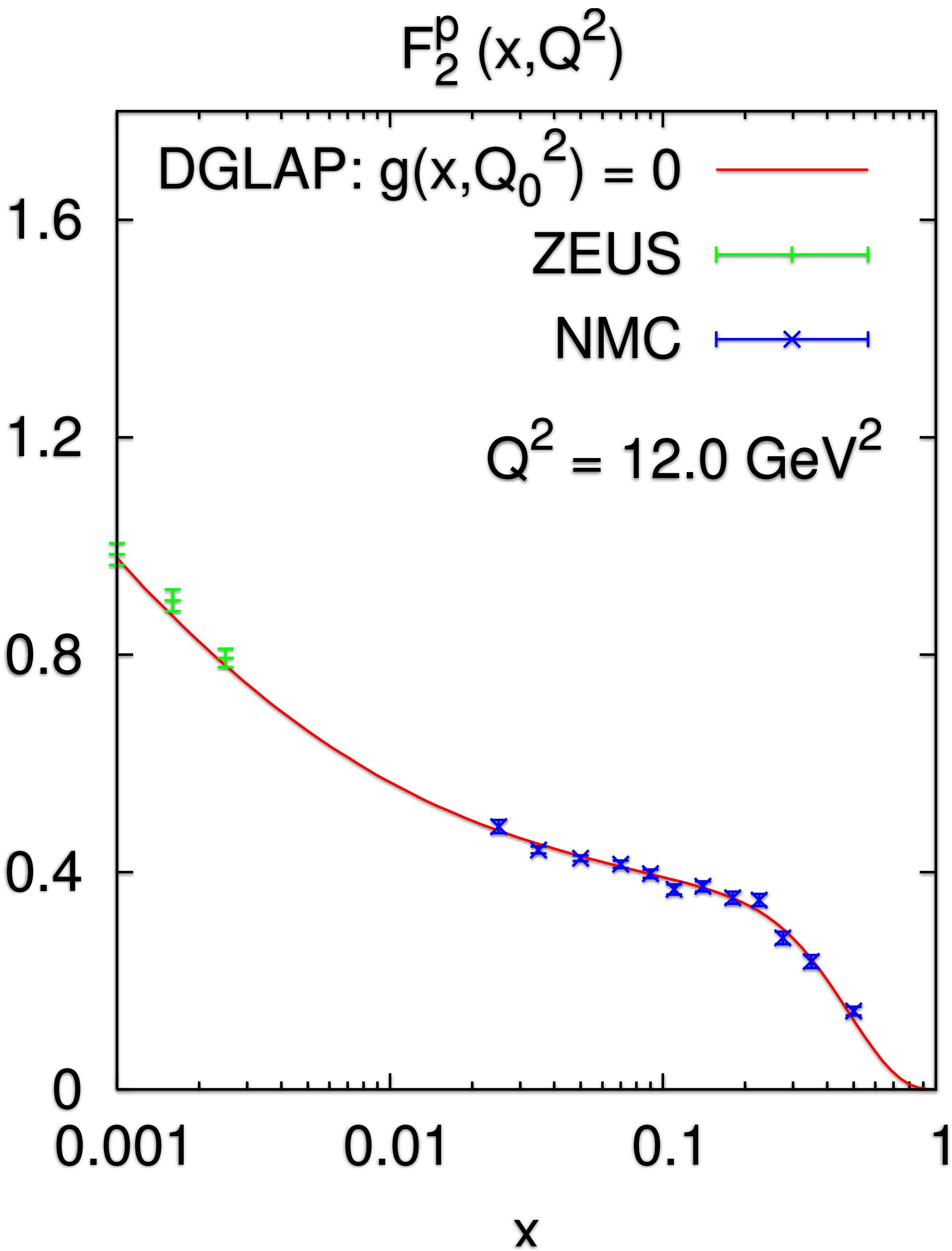
COMPLETE FAILURE
to reproduce data evolution

Consider DIS data — $F_2(x,Q^2)$ — with specially tuned gluon



If gluon $\neq 0$, splitting

$$g \to q\bar{q}$$

generates extra quarks at large Q2 ➠ faster rise of F2

Global PDF fits (CT, MMHT, NNPDF, etc.) choose gluon distribution that leads to the correct Q2 evolution.

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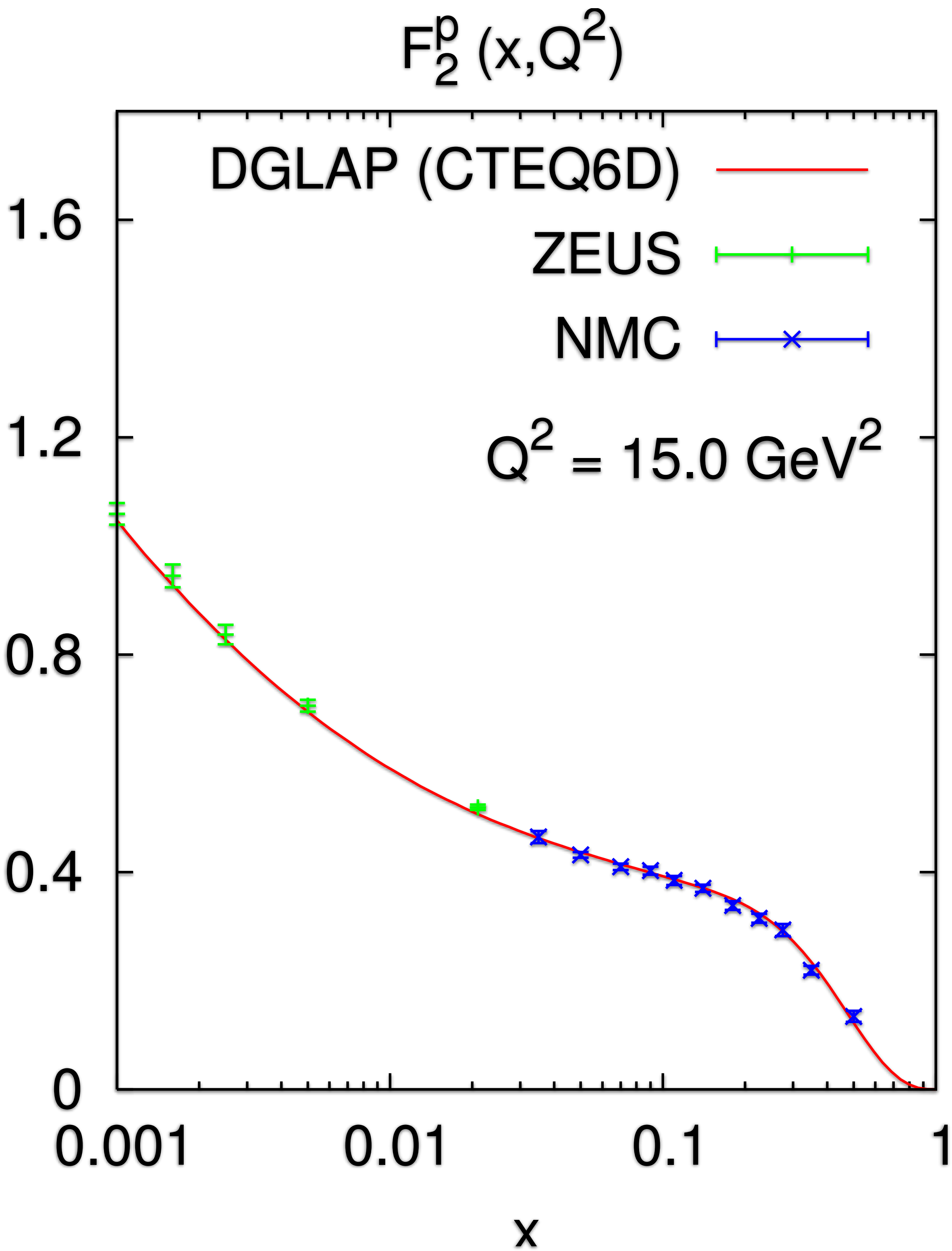

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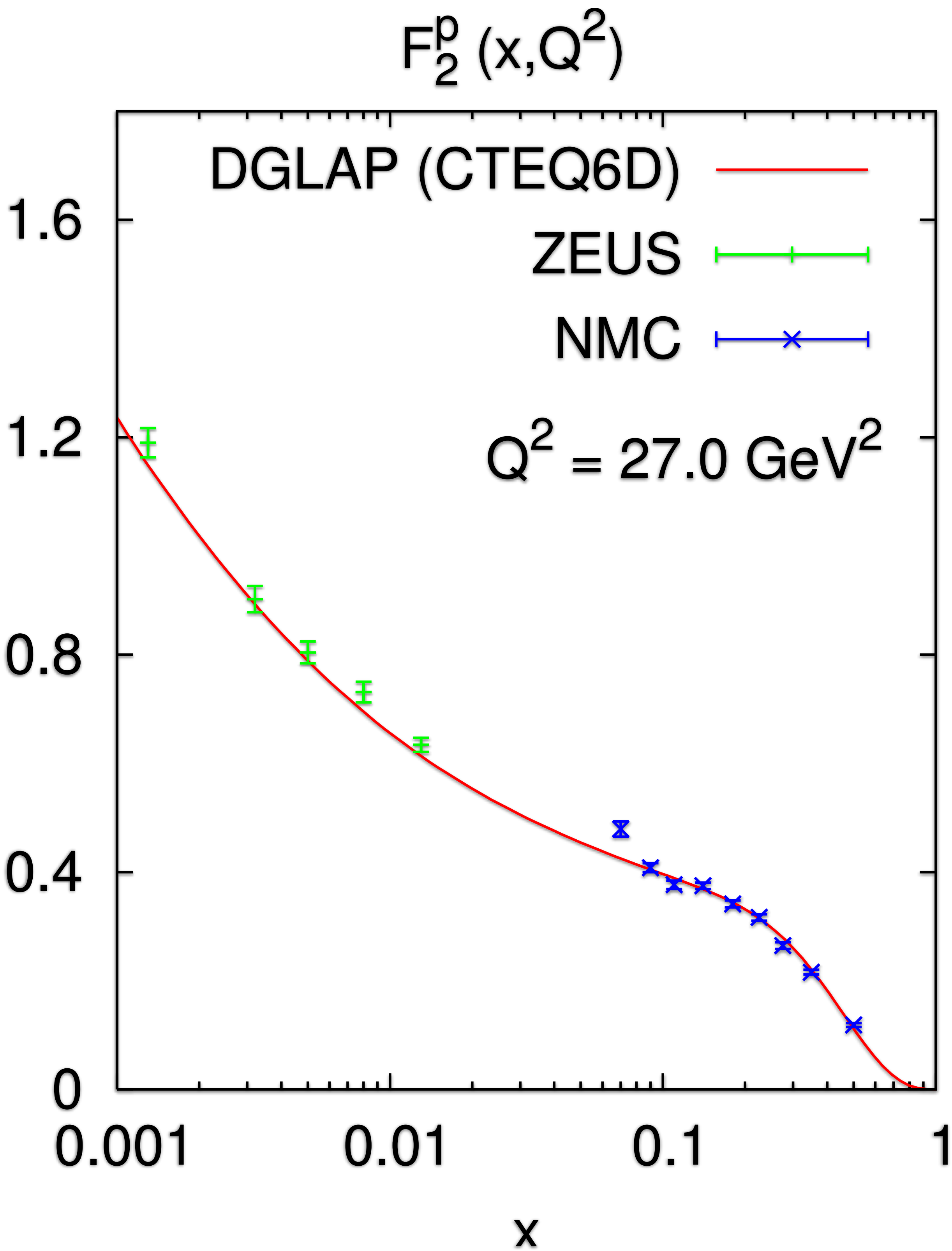

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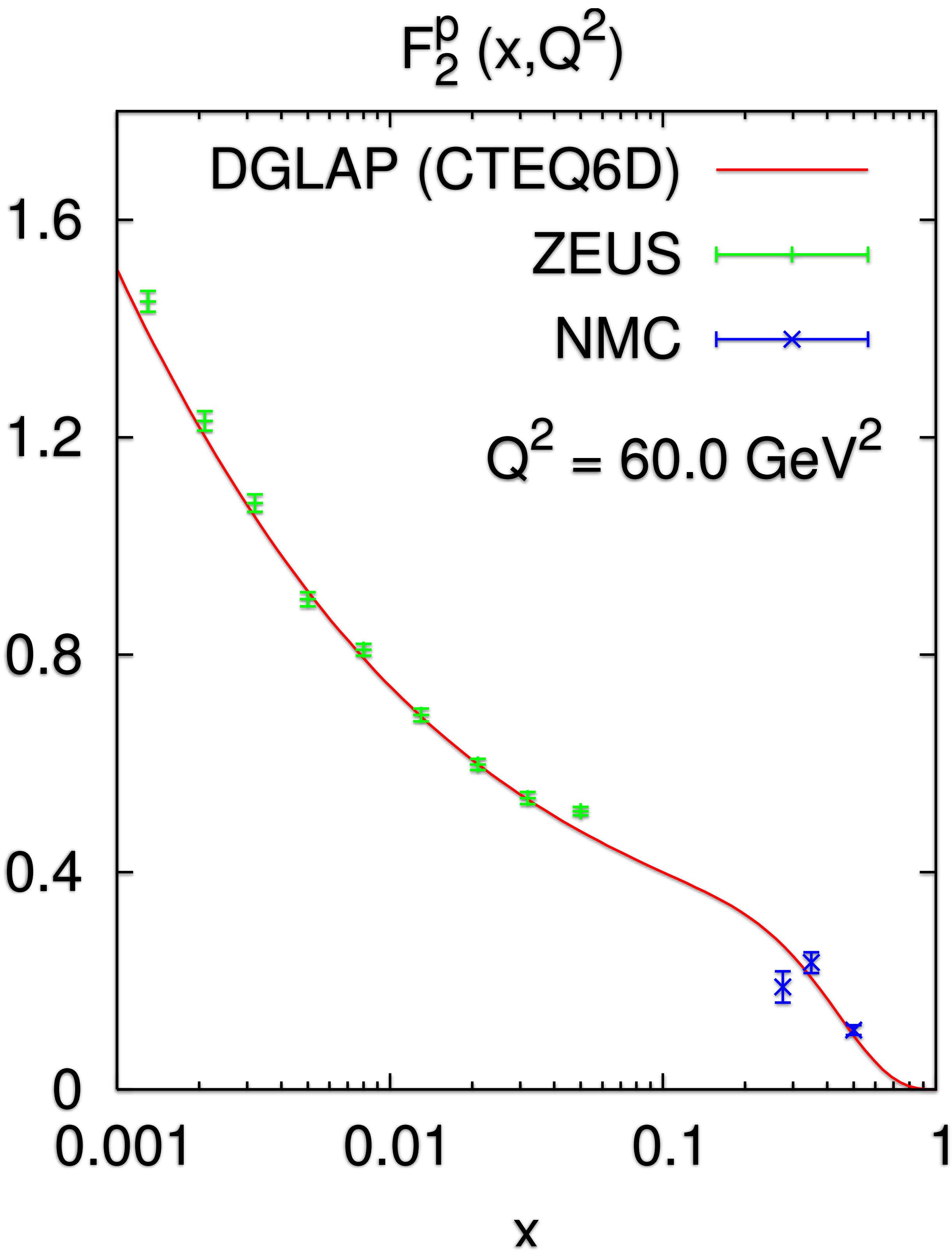

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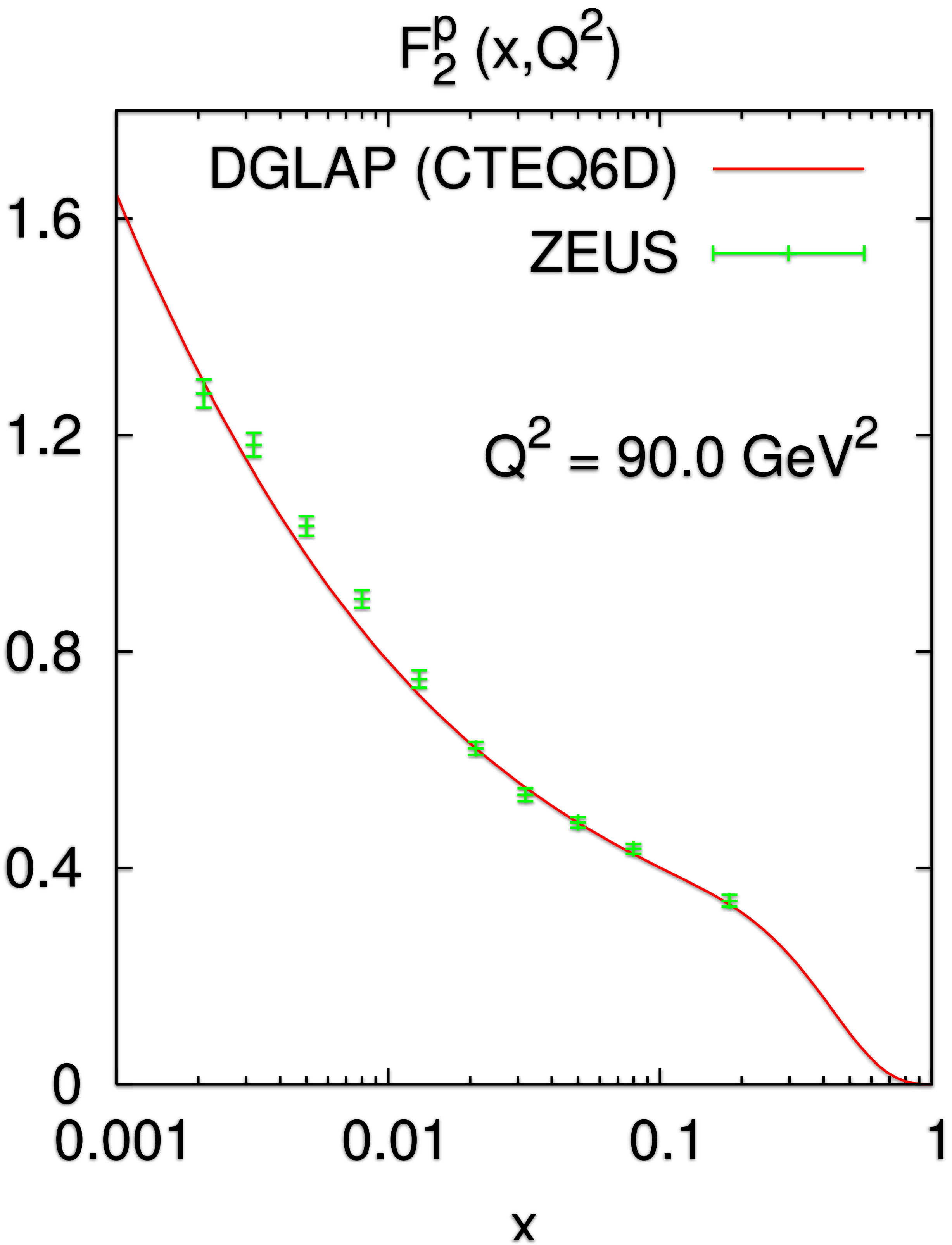

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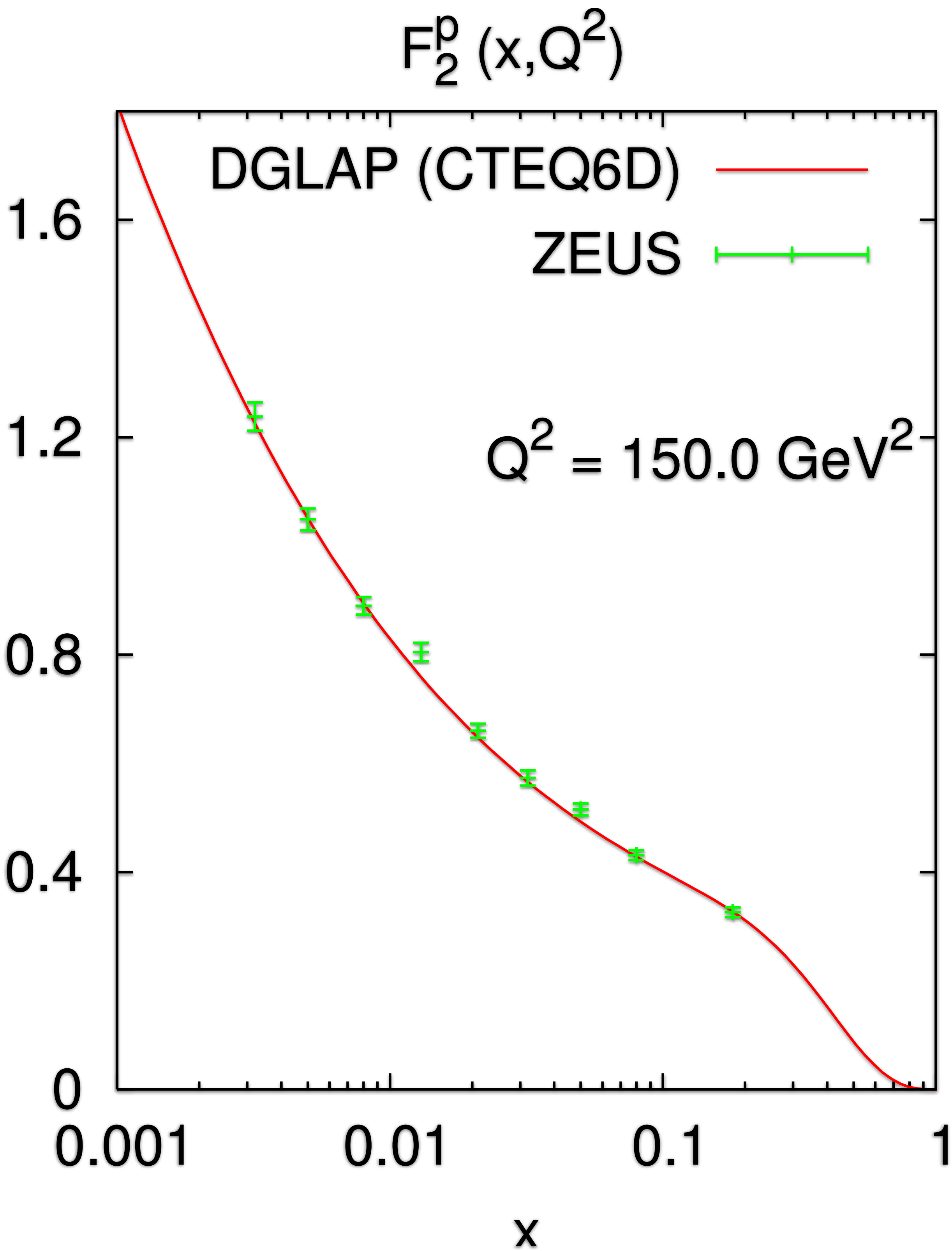

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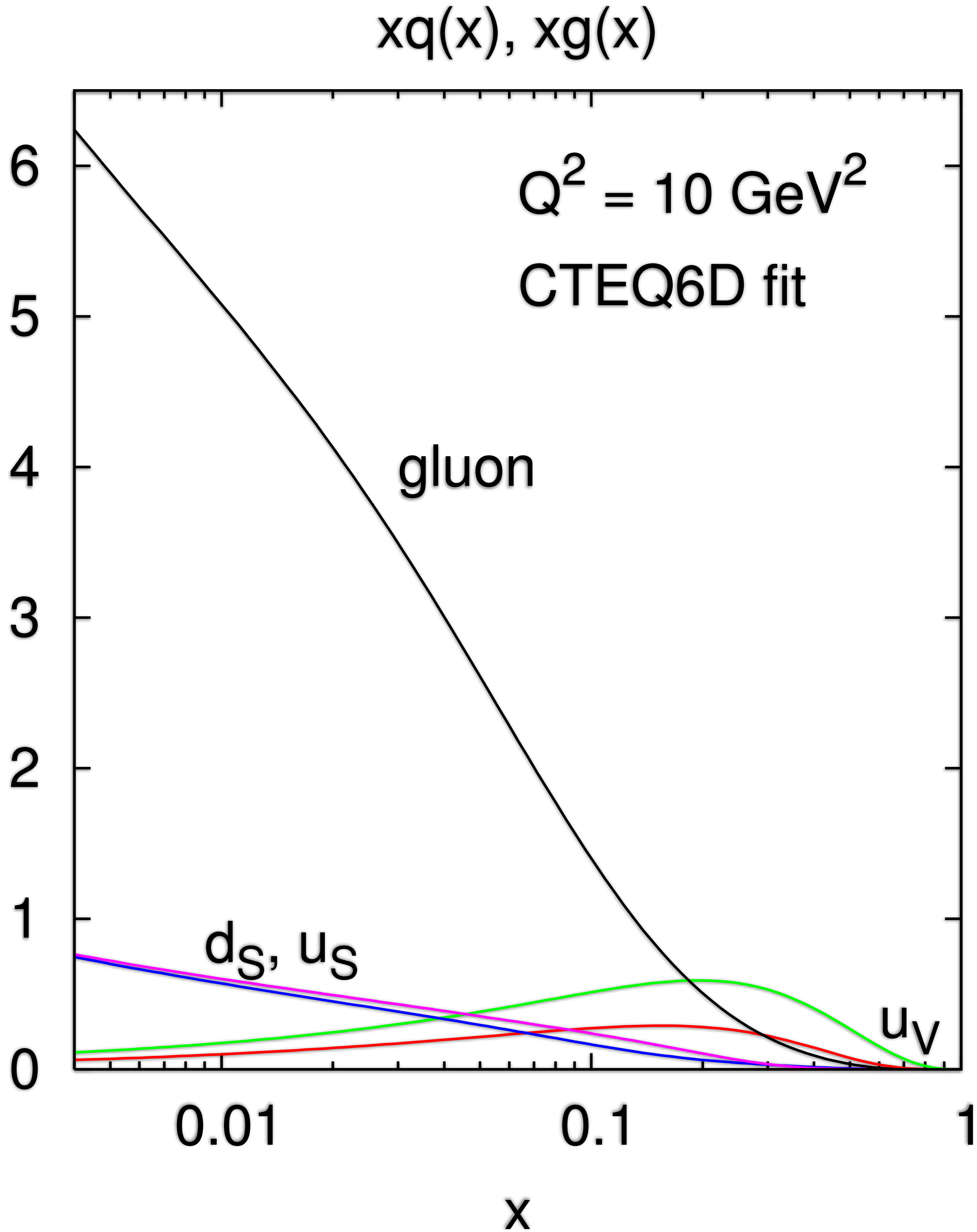
SUCCESS

Resulting gluon distribution, compared to quarks



Resulting gluon distribution is **HUGE!**

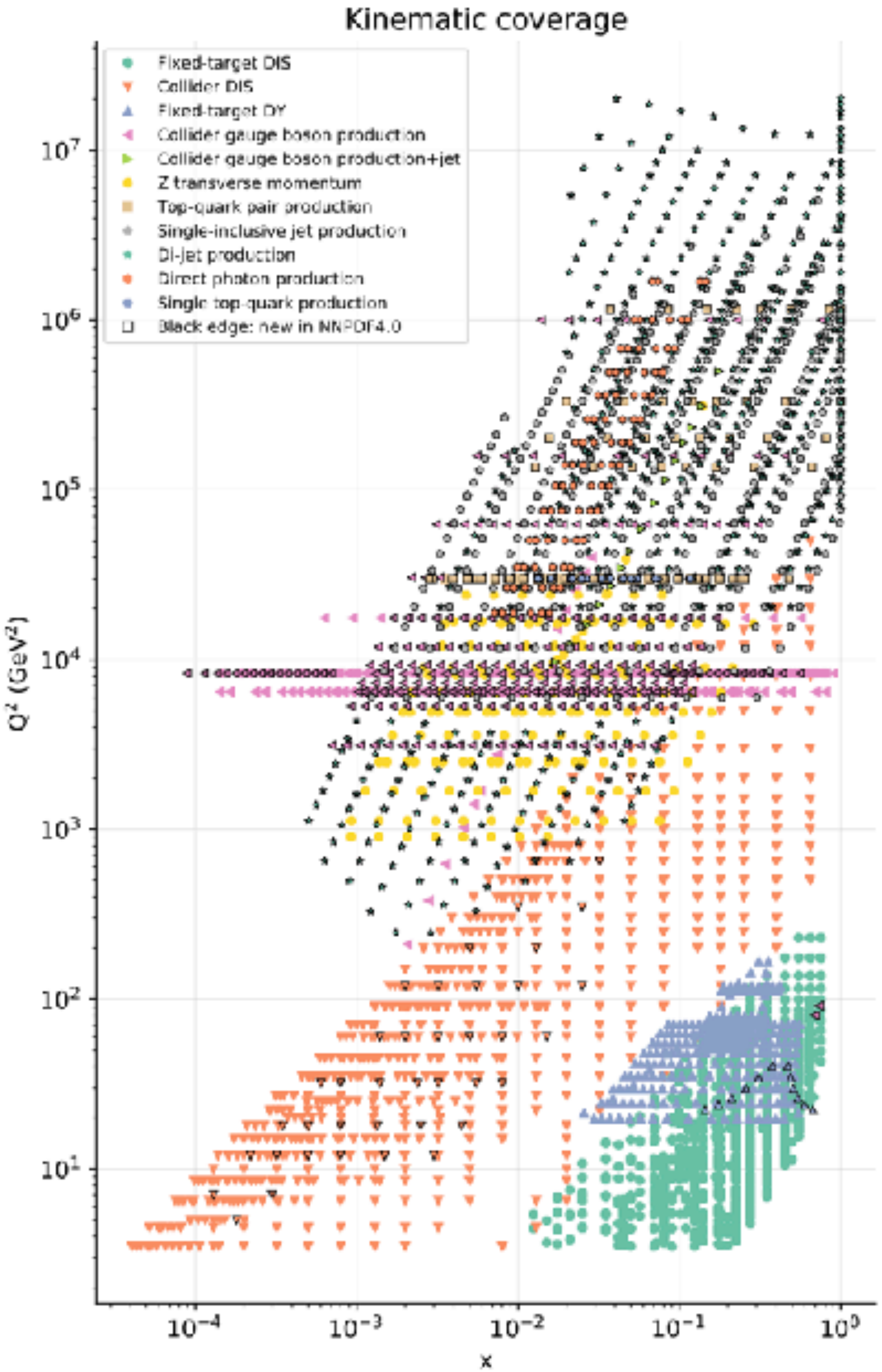
Carries 47% of proton's momentum
(at scale of 100 GeV)

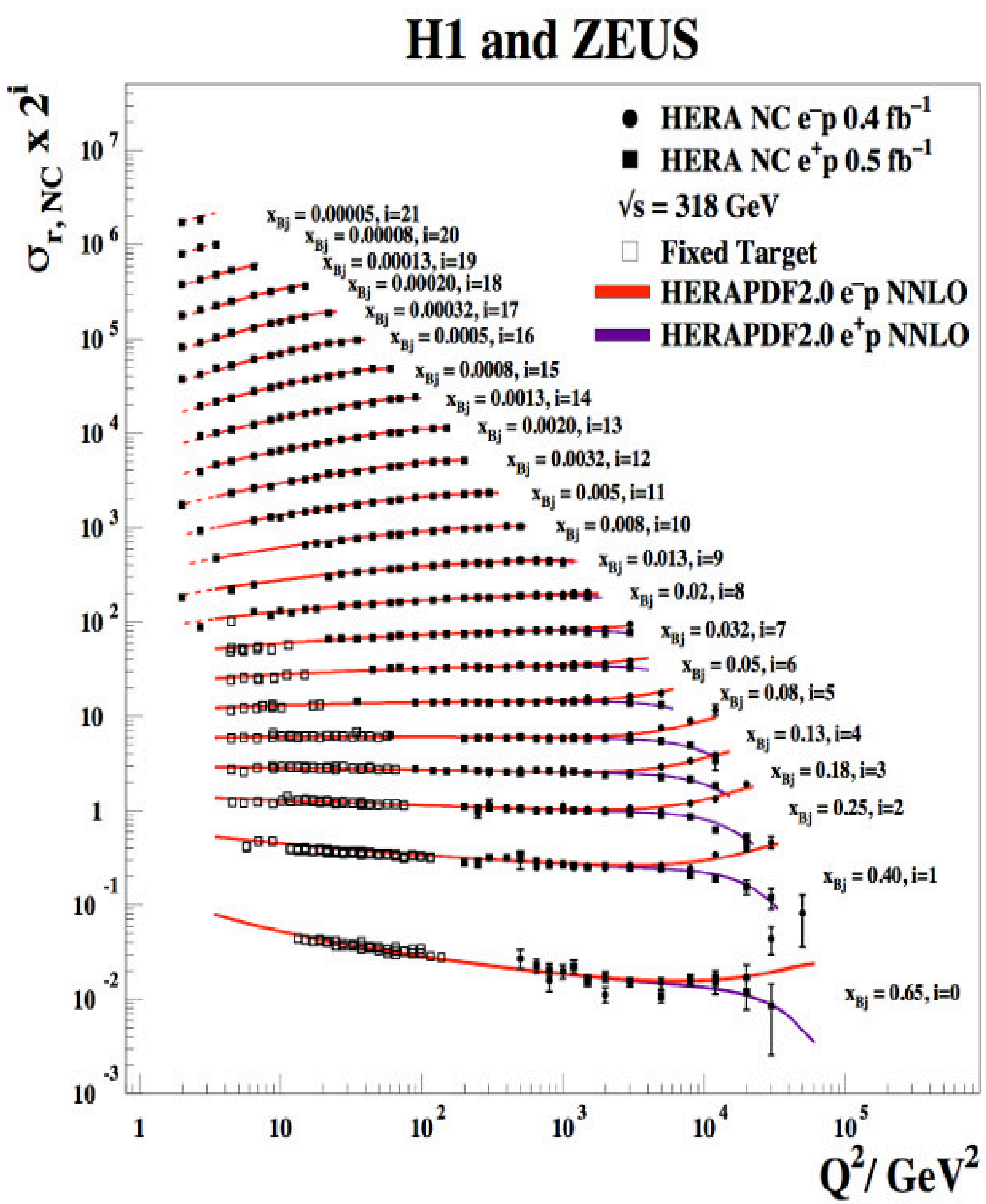
Crucial in order to satisfy momentum sum rule.

Large value of gluon has big impact on phenomenology

Today's PDF fits: huge array of data (and choices about which data to use)

NNPDF4.0 dataset





Today's PDF fits: huge array of data (and choices about which data to use)

Data set	NLO	NNLO
BCDMS μp F_2 [49]	169.4/163	180.2/163
BCDMS μd F_2 [49]	135.0/151	146.0/151
NMC μp F_2 [50]	142.9/123	124.1/123
NMC μd F_2 [50]	128.2/123	112.4/123
NMC $\mu n/\mu p$ [51]	127.8/148	130.8/148
E665 μp F_2 [52]	59.5/53	64.7/53
E665 μd F_2 [52]	50.3/53	59.7/53
SLAC ep F_2 [53, 54]	29.4/37	32.0/37
SLAC ed F_2 [53, 54]	37.4/38	23.0/38
NMC/BCDMS/SLAC/HERA F_L [49, 50, 54, 146–148]	79.4/57	68.4/57
E866/NuSea pp DY [149]	216.2/184	225.1/184
E866/NuSea pd/pp DY [150]	10.6/15	10.4/15
NuTeV νN F_2 [55]	43.7/53	38.3/53
CHORUS νN F_2 [56]	27.8/42	30.2/42
NuTeV νN xF_3 [55]	37.8/42	30.7/42
CHORUS νN xF_3 [56]	22.0/28	18.4/28
CCFR $\nu N \to \mu\mu X$ [57]	73.2/86	67.7/86
NuTeV $\nu N \to \mu\mu X$ [57]	41.0/84	58.4/84
HERA e^+p CC [84]	54.3/39	52.0/39
HERA e^-p CC [84]	80.4/42	70.2/42
HERA e^+p NC 820 GeV [84]	91.6/75	89.8/75
HERA e^+p NC 920 GeV [84]	553.9/402	512.7/402
HERA e^-p NC 460 GeV [84]	253.3/209	248.3/209
HERA e^-p NC 575 GeV [84]	268.1/259	263.0/259
HERA e^-p NC 920 GeV [84]	252.3/159	244.4/159
HERA ep F_2^{charm} [26]	125.6/79	132.3/79
DØ II $p\bar{p}$ incl. jets [125]	117.2/110	120.2/110
CDF II $p\bar{p}$ incl. jets [124]	70.4/76	60.4/76
CDF II W asym. [90]	19.1/13	19.0/13
DØ II $W \to \nu e$ asym. [151]	44.4/12	33.9/12
DØ II $W \to \nu\mu$ asym. [152]	13.9/10	17.3/10
DØ II Z rap. [153]	15.9/28	16.4/28
CDF II Z rap. [154]	36.9/28	37.1/28
DØ W asym. [21]	13.1/14	12.0/14

Table 6: The values of χ^2/Npts. for the non-LHC data sets included in the global fit at NLO and NNLO.

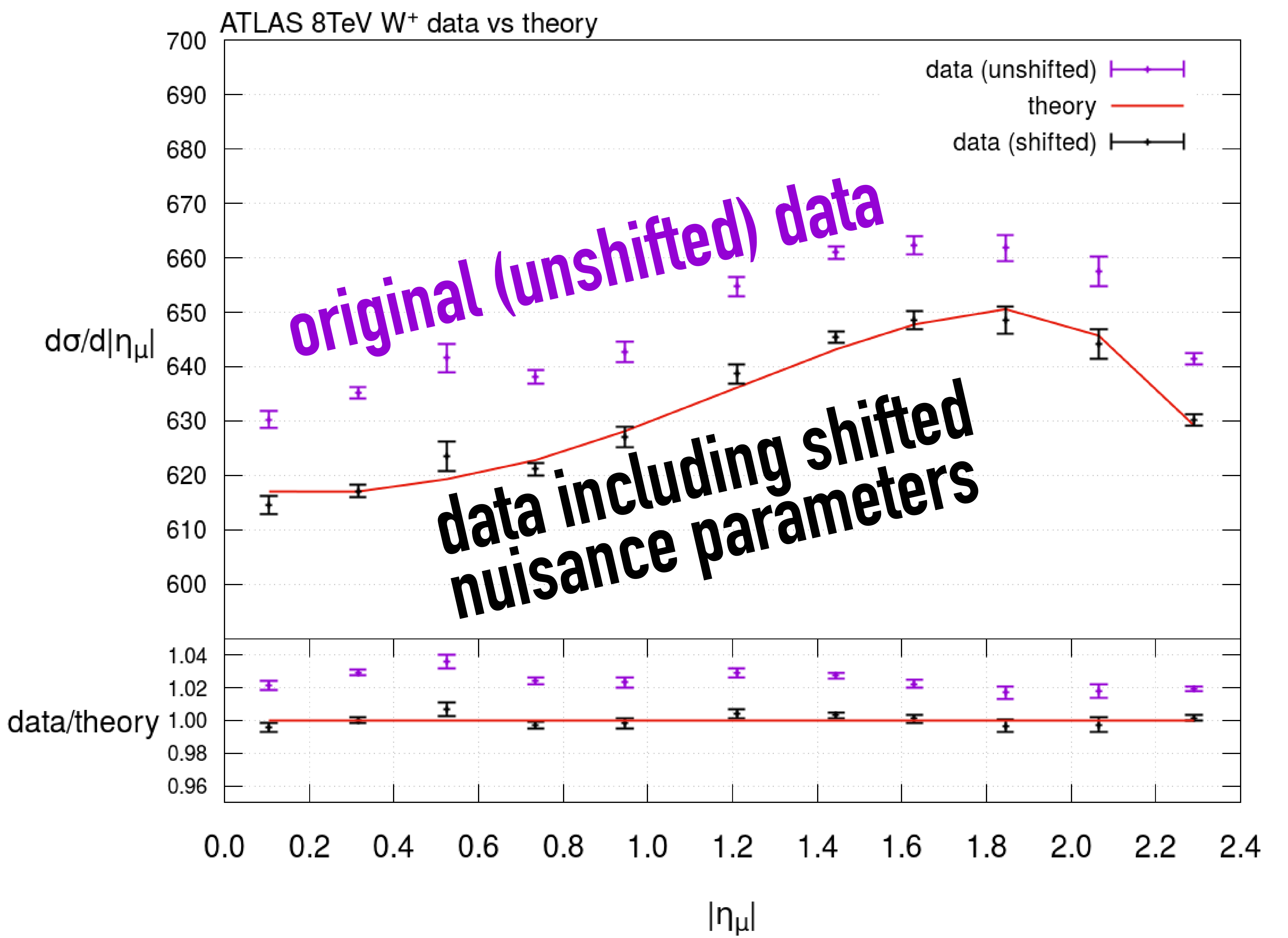
MSHT20 data sets & χ^2

Data set	NLO	NNLO
ATLAS W^+, W^-, Z [119]	34.7/30	29.9/30
CMS W asym. $p_T > 35$ GeV [155]	11.8/11	7.8/11
CMS asym. $p_T > 25, 30$ GeV [156]	11.8/24	7.4/24
LHCb $Z \to e^+e^-$ [157]	14.1/9	22.7/9
LHCb W asym. $p_T > 20$ GeV [158]	10.5/10	12.5/10
CMS $Z \to e^+e^-$ [159]	18.9/35	17.9/35
ATLAS High-mass Drell-Yan [160]	20.7/13	18.9/13
CMS double diff. Drell-Yan [72]	222.2/132	144.5/132
Tevatron, ATLAS, CMS $\sigma_{t\bar{t}}$ [93]–[94]	22.8/17	14.5/17
LHCb 2015 W, Z [95, 96]	114.4/67	99.4/67
LHCb 8 TeV $Z \to ee$ [97]	39.0/17	26.2/17
CMS 8 TeV W [98]	23.2/22	12.7/22
ATLAS 7 TeV jets [18]	226.2/140	221.6/140
CMS 7 TeV $W + c$ [99]	8.2/10	8.6/10
ATLAS 7 TeV high precision W, Z [20]	304.7/61	116.6/61
CMS 7 TeV jets [100]	200.6/158	175.8/158
CMS 8 TeV jets [101]	285.7/174	261.3/174
CMS 2.76 TeV jet [107]	124.2/81	102.9/81
ATLAS 8 TeV Z p_T [75]	235.0/104	188.5/104
ATLAS 8 TeV single diff $t\bar{t}$ [102]	39.1/25	25.6/25
ATLAS 8 TeV single diff $t\bar{t}$ dilepton [103]	4.7/5	3.4/5
CMS 8 TeV double differential $t\bar{t}$ [105]	32.8/15	22.5/15
CMS 8 TeV single differential $t\bar{t}$ [108]	12.9/9	13.2/9
ATLAS 8 TeV High-mass Drell-Yan [73]	85.8/48	56.7/48
ATLAS 8 TeV W [106]	84.6/22	57.4/22
ATLAS 8 TeV W + jets [104]	33.9/30	18.1/30
ATLAS 8 TeV double differential Z [74]	157.4/59	85.6/59
Total	5822.0/4363	5121.9/4363

Table 7: The values of χ^2/Npts. for the LHC data sets included in the global fit and the overall global fit χ^2/N at NLO and NNLO. The corresponding values for the non-LHC data sets are shown in Table 6, and the total value corresponds to the sum over both tables.

data is precise, correlations between systematics are crucial

e.g. from MSHT20 (2012.04684)



A generic function $f(x)$ involves an infinite number of degrees of freedom. How can you fit this with a finite number of data points?

CT / MSHT use parameterisations with hand-picked number of terms, e.g. up to $n = 6$ in Chebyshev series:

$$xf(x, Q_0^2) \;=\; A(1-x)^\eta x^\delta \left(1 + \sum_{i=1}^{n} a_i T_i^{\mathrm{Ch}}(y(x))\right)$$

NNPDF use a ***neural network*** as a generic fit function, and separate data into training / validation. Fit is done using just the training subset, and stops when χ^2 on training + validation starts to increase. (Supplemented with closure tests)

today's PDF fits: uncertainty estimation

With fits to O(60) data sets, chances are they won't all be consistent (plainly inconsistent data sets may simply be excluded, but that can be biased)

CT / MSHT do a Hessian fit, with error eigenfunctions, scaled by a tolerance T that is like replacing $\Delta\chi^2 = 1$ with $\Delta\chi^2 = T$.

Squared error on a cross section is obtained by summing squared variations from each of the eigenfuntions.

NNPDF fits ***Monte Carlo replica data sets***
i.e. fluctuate the data according to errors, and fit the fluctuated data; repeat over and over, to get O(100) replica fits; prediction for any cross section is then average and std.dev. across the replicas

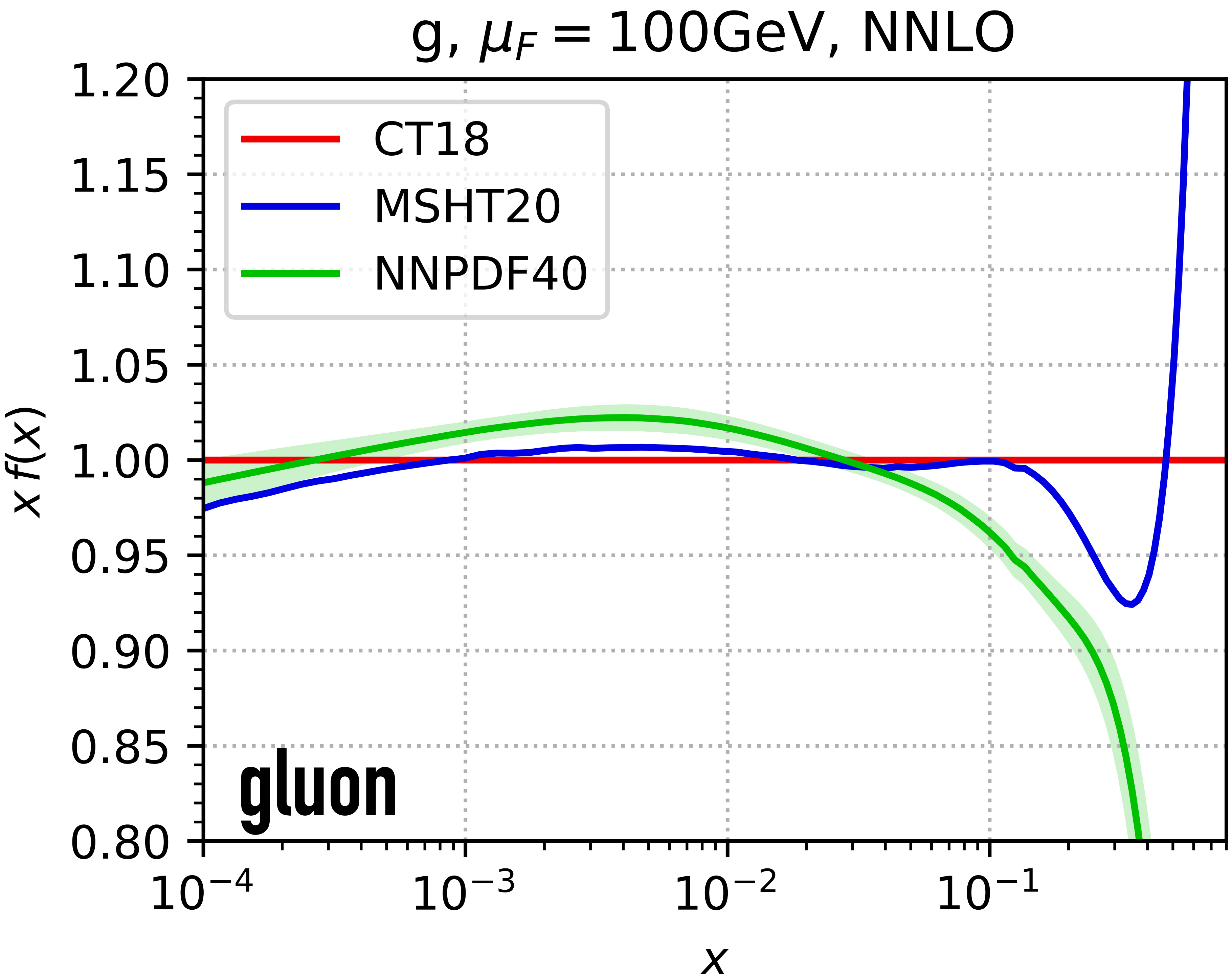
Charm-quark mass is around 1.5 GeV. Is this perturbative enough to treat it as purely perturbative generated? Or should one fit the charm as a light flavour?

CT / MSHT treat charm perturbatively, turning on its evolution from (almost zero) at the charm mass.

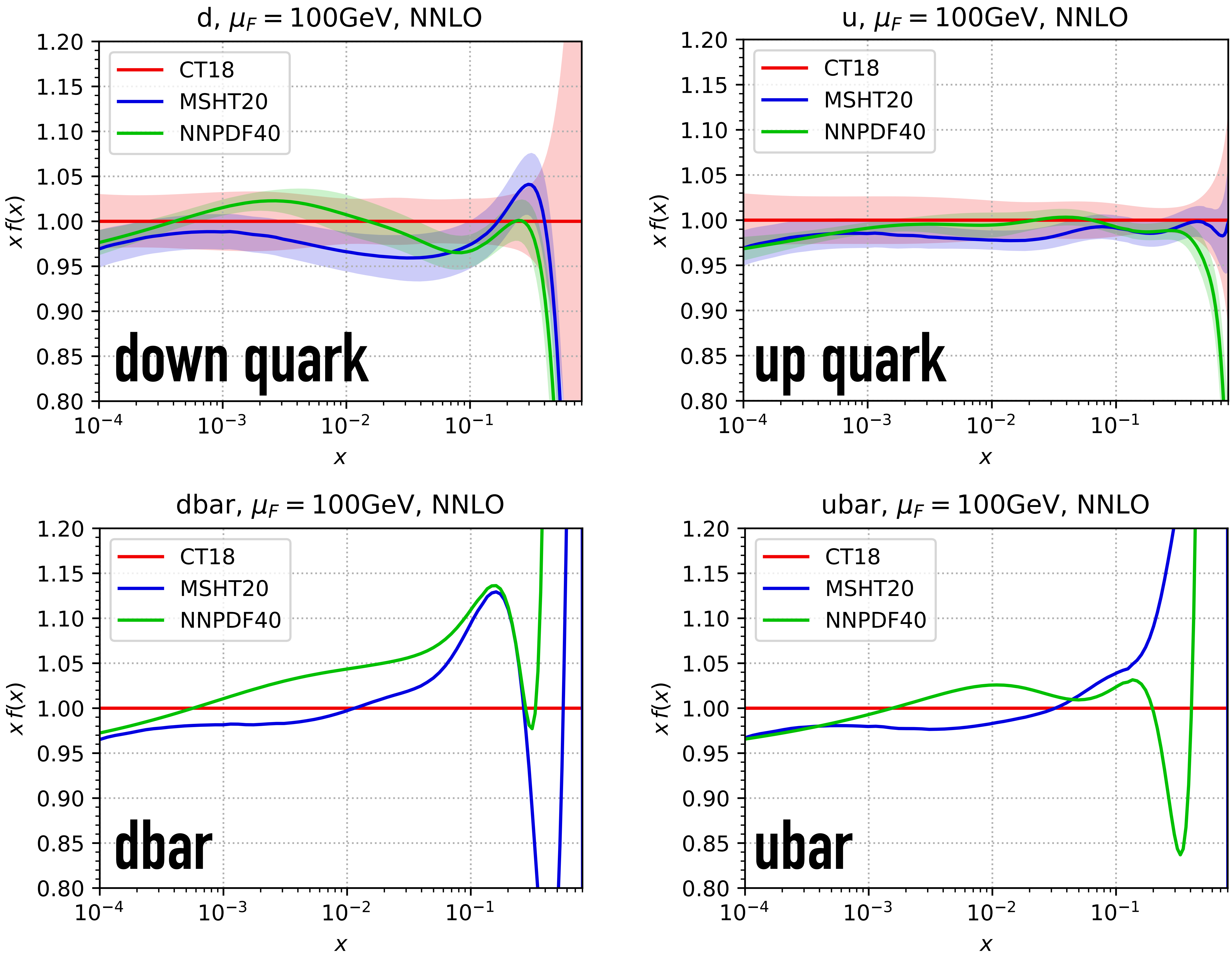
NB: CT also explores "fitted" charm

NNPDF fits by default treat the charm as light, but also provide PDF sets with perturbative charm

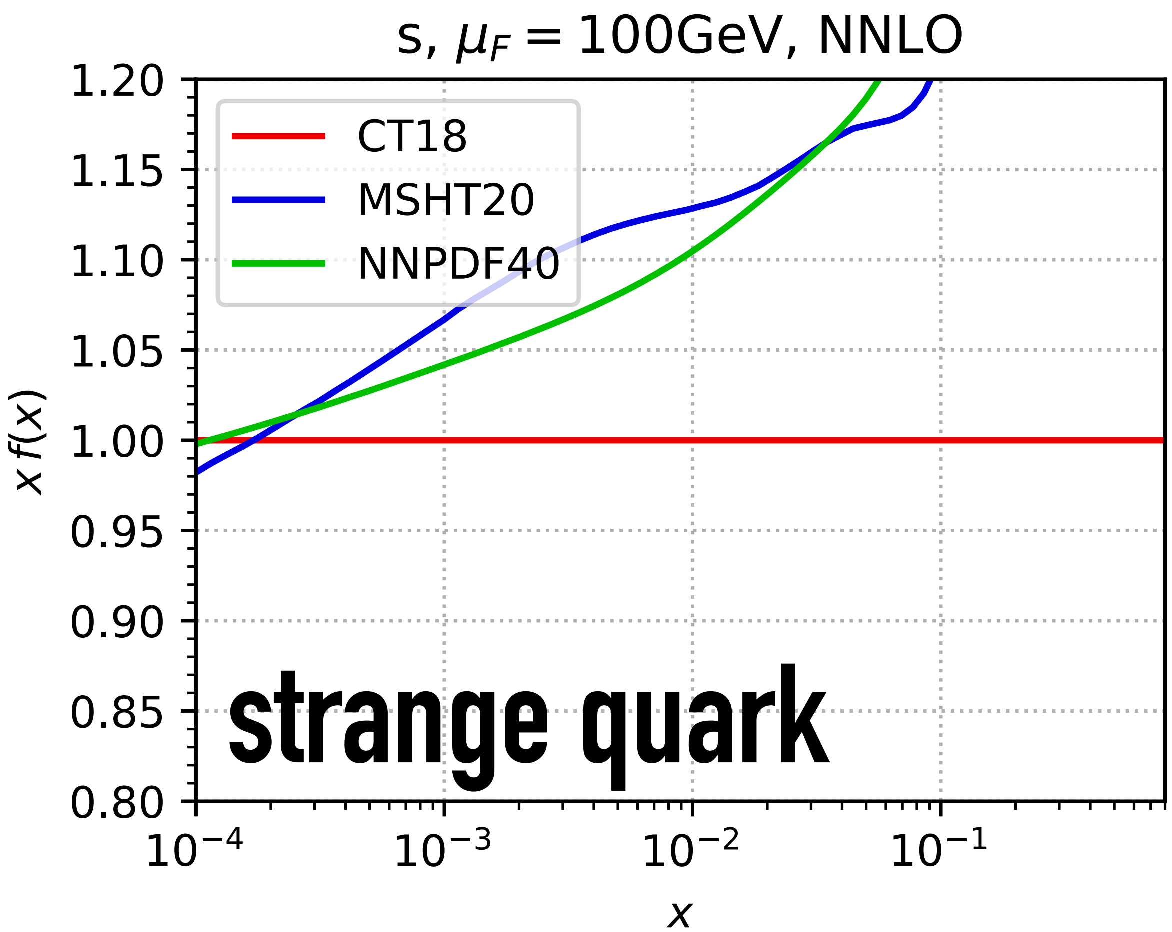
Comparing different sets: the gluon

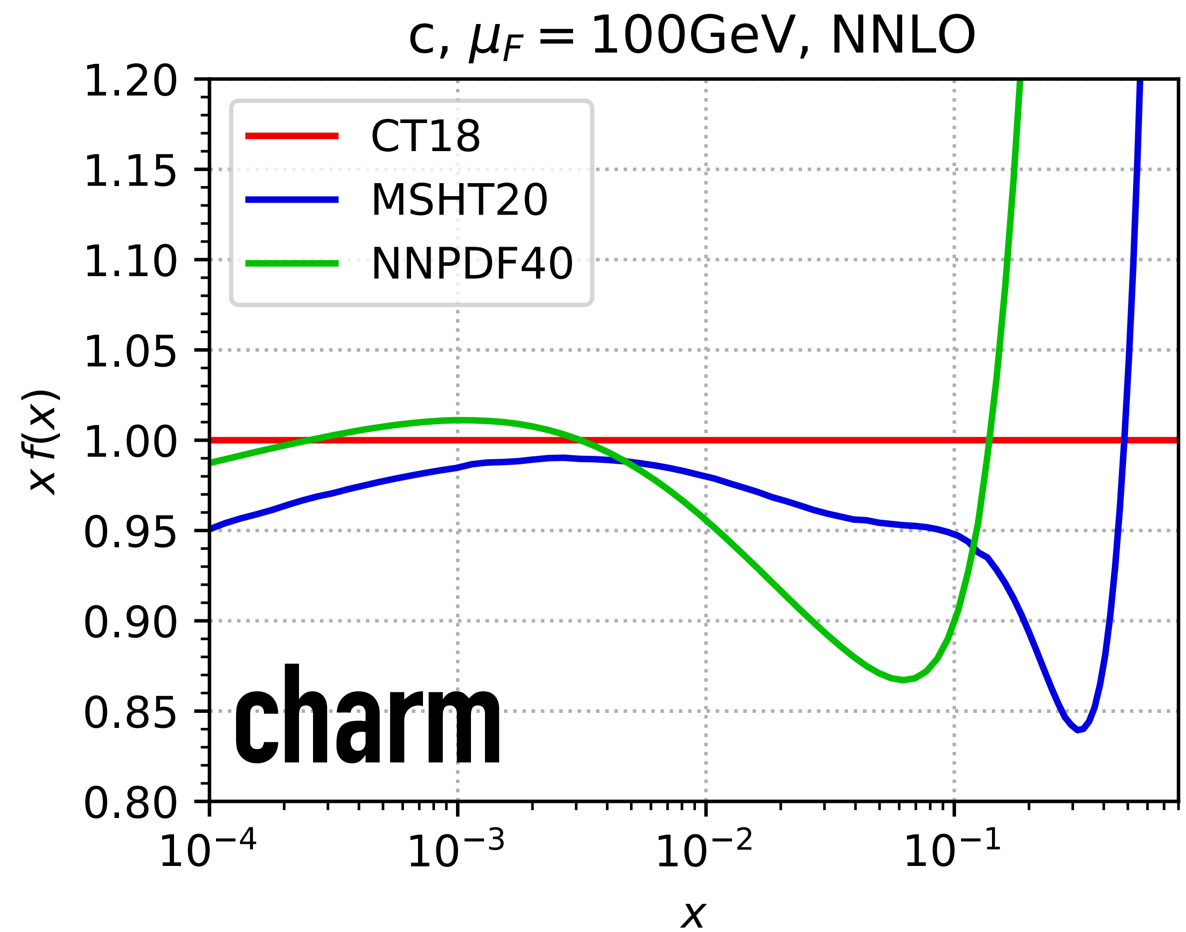


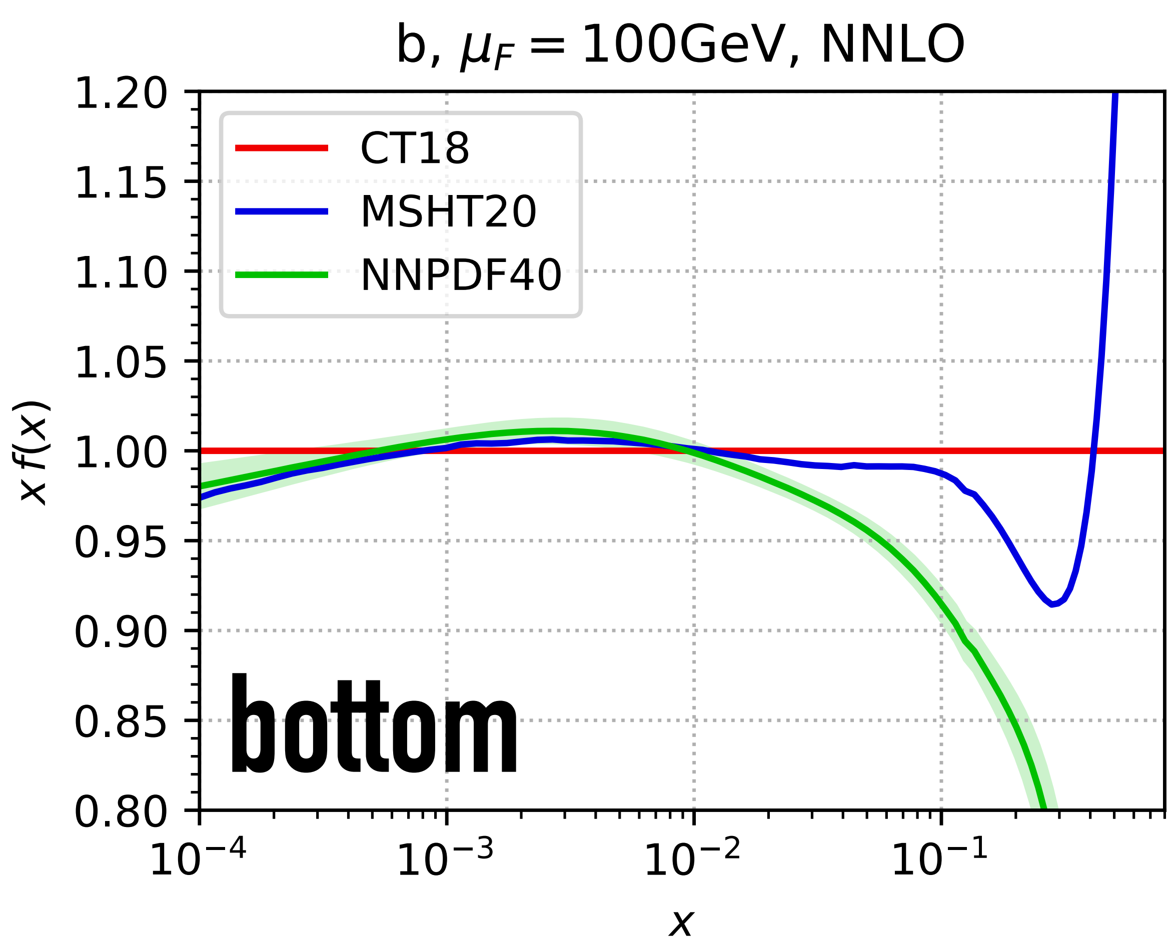
In much of region relevant to LHC, uncertainty is in the 1-2% range



d, $\mu_F = 100$GeV, NNLO
u, $\mu_F = 100$GeV, NNLO
dbar, $\mu_F = 100$GeV, NNLO
ubar, $\mu_F = 100$GeV, NNLO
CT18
MSHT20
NNPDF40
down quark
up quark
dbar
ubar
$x\,f(x)$
x







- strange (anti-)quark is least well known PDF (small charge, few good experimental handles)
- charm: current debate about intrinsic charm
- bottom: mostly driven by gluon

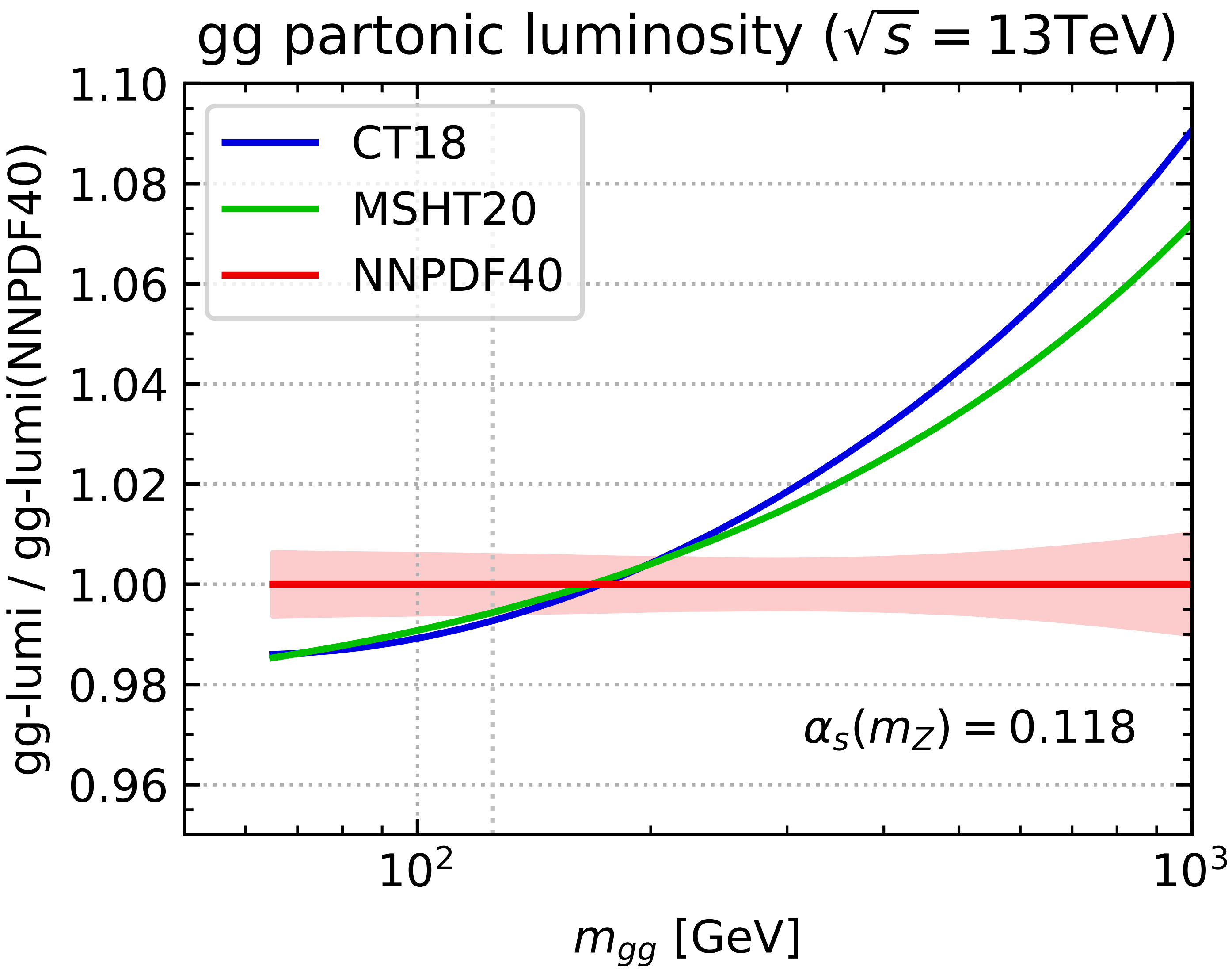
"Think" at Leading Order (LO) in QCD:

- collide protons at CoM energy $\sqrt{s}$,
- take momentum fractions x_1 and x_2 from the two protons
- producing a system of mass m requires $x_1 x_2 s = m^2$

Number of parton-parton collisions with flavours i and j is proportional to **partonic luminosity** $\mathscr{L}_{ij}(m^2)$

$$\mathscr{L}_{ij}(m^2) = \int dx_1 dx_2\, f_{i,p}(x_1, \mu_F^2)\, f_{j,p}(x_2, \mu_F^2)\, \delta(x_1 x_2 s - m^2)$$

comparing PDF "luminosities"



gg-lumi, ratio to PDF4LHC15

PDF4LHC15	1.0000	± 0.0184
PDF4LHC21	0.9930	± 0.0155
CT18	0.9914	± 0.0180
MSHT20	0.9930	± 0.0108
NNPDF40	0.9986	± 0.0058

× 3

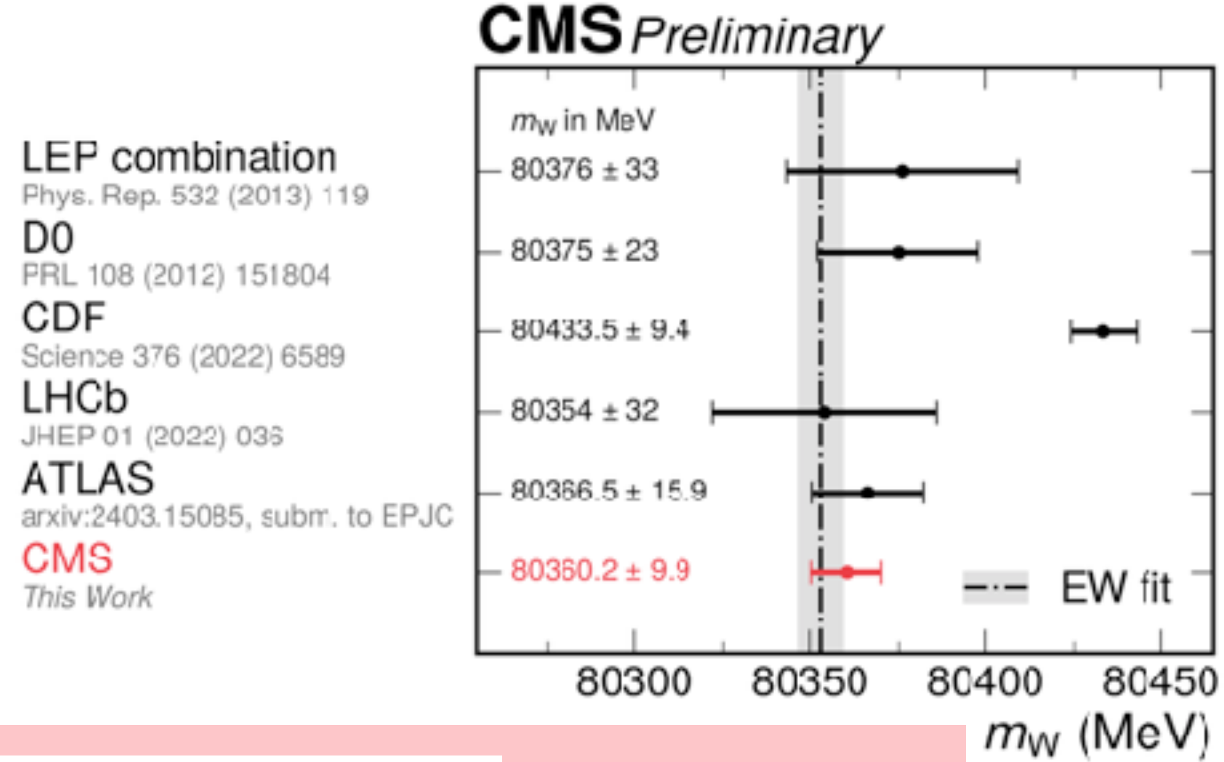
NB: PDF4LHC21 uses CT18/MSHT20/NNPDF31

Amazing that MSHT20 & NNPDF40 are reaching %-level precision

At this level, QED effects probably no longer optional (MSHT20QED: 0.9870)

Example: W mass

W mass is one area where LHC is unexpectedly competitive. Depending on the extraction method, PDFs can be critical



ATLAS

arXiv:2403.15085

PDF set	Combined m_W [MeV]
CT14	80363.6 ± 15.9
CT18	80366.5 ± 15.9
CT18A	80357.2 ± 15.6
MMHT2014	80366.2 ± 15.8
MSHT20	80359.3 ± 14.6
ATLASpdf21	80367.6 ± 16.6
NNPDF3.1	80349.6 ± 15.3
NNPDF4.0	80345.6 ± 14.9

CMS PAS SMP-23-002

	npact (MeV)			
Source of uncertainty	Nominal		Global	
	in m_Z	in m_W	in m_Z	in m_W
Muon momentum scale	5.6	4.8	5.3	4.4
Muon reco. efficiency	3.8	3.0	3.0	2.3
W and Z angular coeffs.	4.9	3.3	4.5	3.0
Higher-order EW	2.2	2.0	2.2	1.9
p_T^V modeling	1.7	2.0	1.0	0.8
PDF	2.4	4.4	1.9	2.8
Nonprompt background	–	3.2	–	1.7
Integrated luminosity	0.3	0.1	0.2	0.1
MC sample size	2.5	1.5	3.6	3.8
Data sample size	6.9	2.4	10.1	6.0
Total uncertainty	13.5	9.9	13.5	9.9

LHCb

arXiv:2109.01113

NNPDF31 $m_W = 80362 \pm 23_{\text{stat}} \pm 10_{\text{exp}} \pm 17_{\text{theory}} \pm 9_{\text{PDF}}$ MeV,

CT18 $m_W = 80350 \pm 23_{\text{stat}} \pm 10_{\text{exp}} \pm 17_{\text{theory}} \pm 12_{\text{PDF}}$ MeV,

MSHT20 $m_W = 80351 \pm 23_{\text{stat}} \pm 10_{\text{exp}} \pm 17_{\text{theory}} \pm 7_{\text{PDF}}$ MeV,

FINAL REMARKS ON PDFS

- In range $10^{-3} < x < 0.1$, core PDFs (up, down, gluon) known to ~ 1–3% accuracy
- For many LHC applications, you can use PDF4LHC21 set, which merges CT18, MSHT20, NNPDF31, all available at NNLO
- first N3LO sets also indicate
- Situation is not full consensus:
  - differences in errors (e.g. NNPDF40 v. CT18),
  - differences in central values (ABMP; approx N3LO v. NNLO)

SO FAR

- We discussed the "Master" formula

$$\sigma\left(h_1 h_2 \to W + X\right) = \sum_{n=0}^{\infty} \alpha_s^n \left(\mu_R^2\right) \sum_{i,j} \int dx_1 dx_2\, f_{i/h_1}\left(x_1, \mu_F^2\right) f_{j/h_2}\left(x_2, \mu_F^2\right) \times \hat{\sigma}^{(n)}_{ij\to W+X}\left(x_1 x_2 s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),$$

- and its main inputs
  - the strong coupling α_s
  - Parton Distribution Functions (PDFs)
- **Next:** we discuss the actual scattering cross section

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$$\times\, \hat{\sigma}^{(n)}_{ij \to W+X}\left(x_1 x_2 s, \mu_R^2, \mu_F^2\right) + \mathcal{O}\left(\frac{\Lambda^2}{M_W^4}\right),$$

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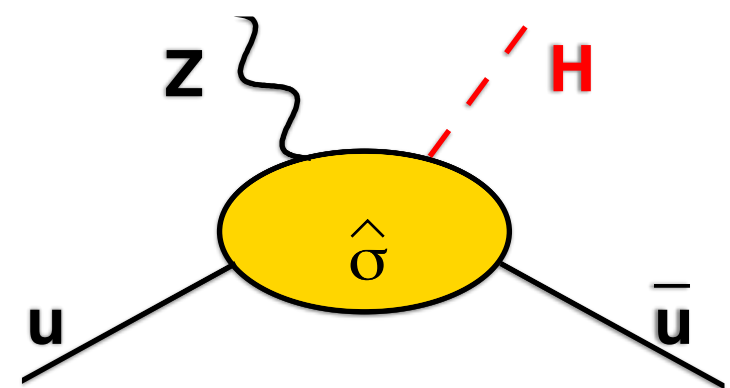


- **Next:** we discuss the actual scattering cross section

the hard cross section

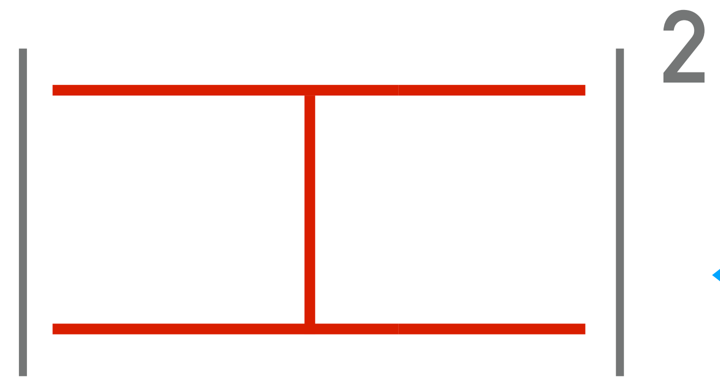
$$\sigma \sim \sigma_2 \alpha_s^2 + \sigma_3 \alpha_s^3 + \sigma_4 \alpha_s^4 + \sigma_5 \alpha_s^5 + \cdots$$

LO NLO NNLO N3LO

INGREDIENTS FOR A CALCULATION (generic 2→2 process)

LO

Tree
2→2



to illustrate the concepts, we don't care what the particles are — just draw lines

INGREDIENTS FOR A CALCULATION (generic 2→2 process)

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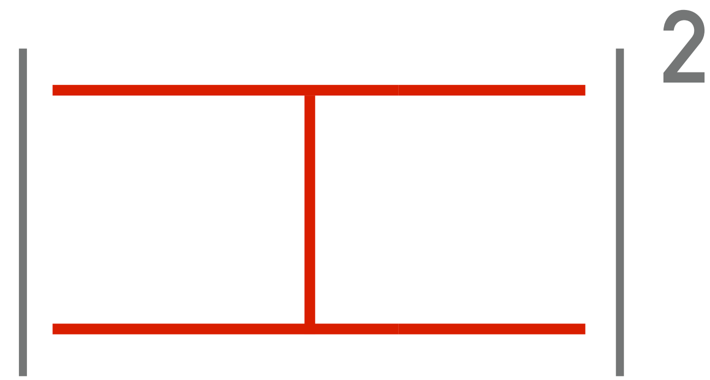
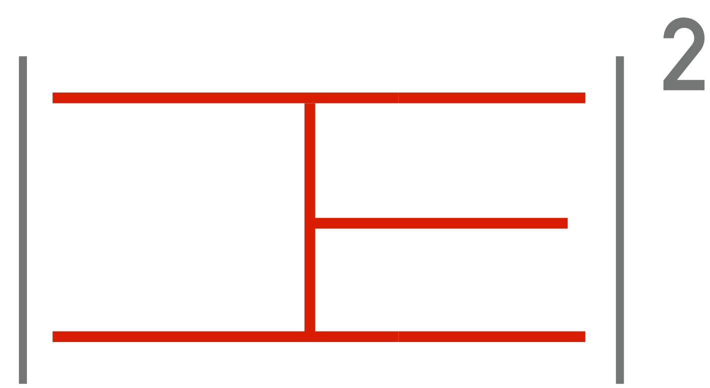


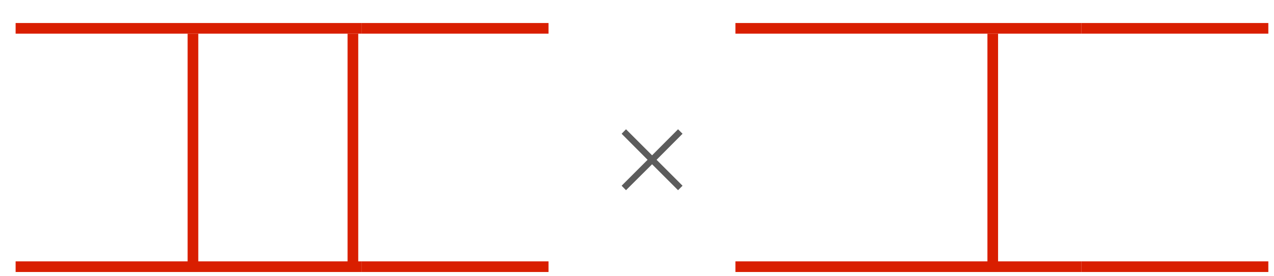
to illustrate the concepts, we don't care what the particles are — just draw lines

NLO

Tree
2→3



1-loop
2→2

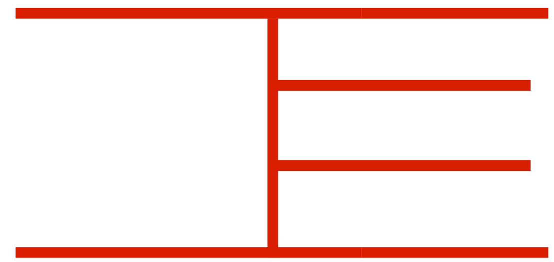


$+$ *complex conj.*

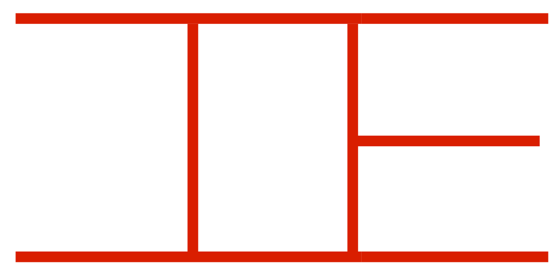
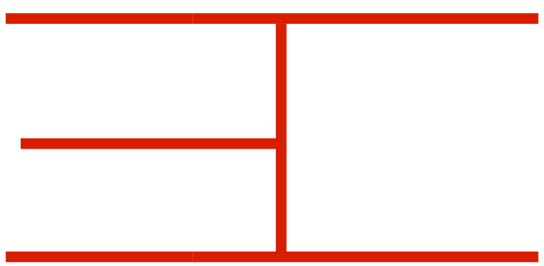
INGREDIENTS FOR A CALCULATION (generic 2→2 process)

NNLO

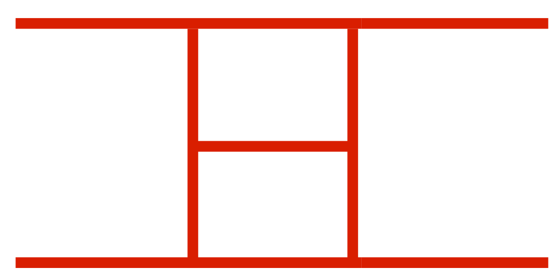
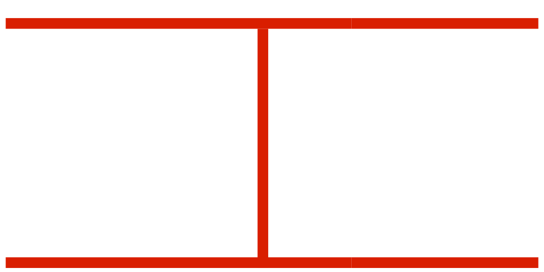
Tree
2→4

$\Bigg|$  $\Bigg|^2$

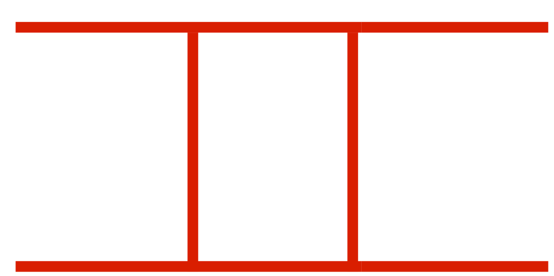
1-loop
2→3

 $\times$  $+$ *complex conj.*

2-loop
2→2

 $\times$  $+$ *complex conj.*

1-loop
2→2

$\Bigg|$  $\Bigg|^2$

EXAMPLE SERIES #1

$$\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \qquad [\alpha_s \equiv \alpha_s(\sqrt{s_{e^+e^-}})]$$

$$= R_0\left(1 + 0.32\alpha_s + 0.14\alpha_s^2 - 0.47\alpha_s^3 - 0.59316\alpha_s^4 + \cdots\right)$$

Baikov et al., 1206.1288
(numbers for γ-exchange only)

This is one of the few quantities calculated to N4LO

Good convergence of the series at every order
(at least for $\alpha_s(M_Z) = 0.118$)

EXAMPLE SERIES #2

$$\sigma(pp \to H) = (961\,\mathrm{pb}) \times (\alpha_s^2 + 10.4\alpha_s^3 + 38\alpha_s^4 + 48\alpha_s^5 + \cdots)$$

$$\alpha_s \equiv \alpha_s(M_H/2)$$

$$\sqrt{s_{pp}} = 13\,\mathrm{TeV}$$

Anastasiou et al., 1602.00695 (ggF, hEFT)

pp→H (via gluon fusion) is one of a few hadron-collider processes known at N3LO

(others are pp→H via weak-boson fusion, Drell-Yan production)

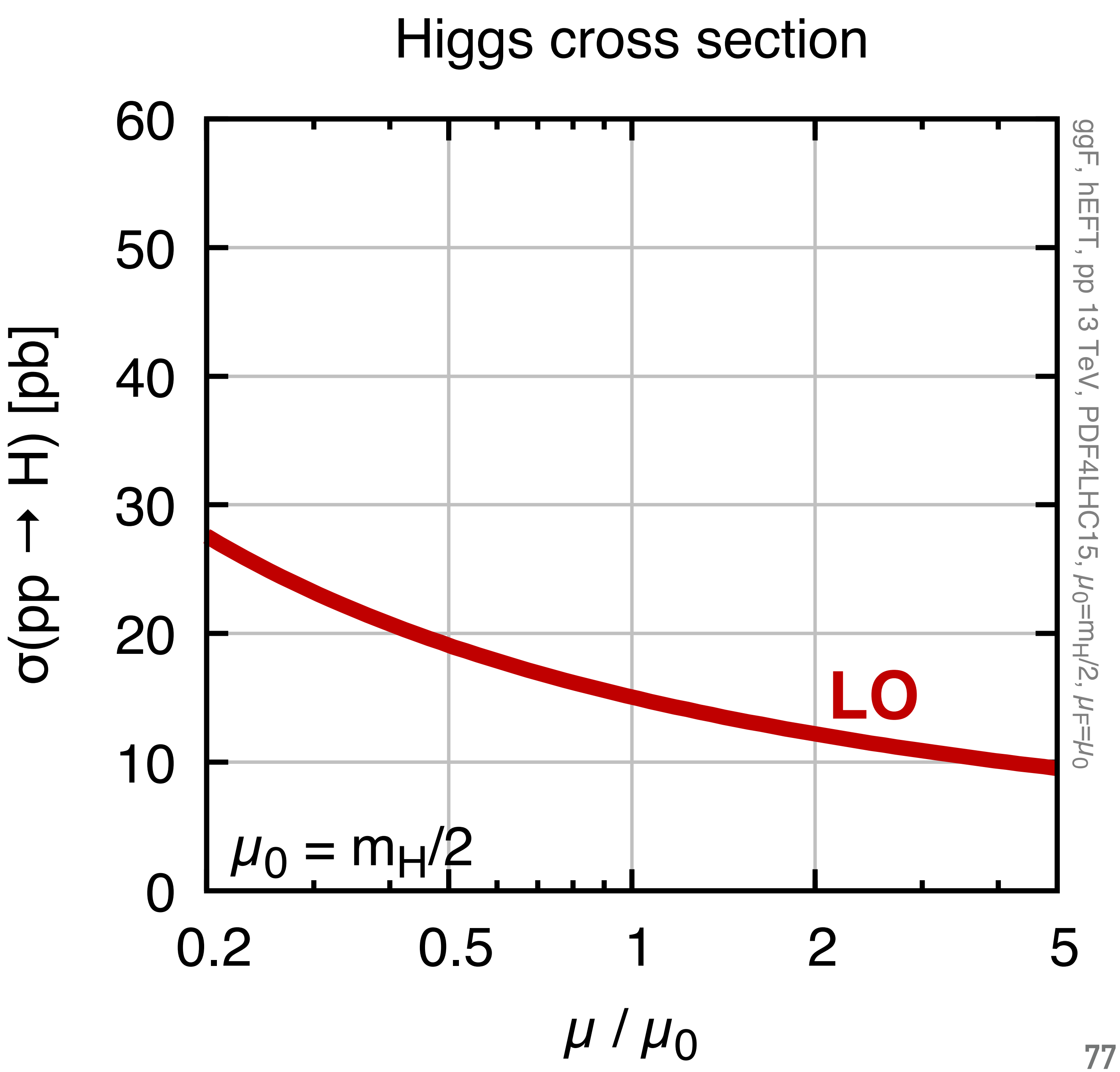
Series convergence is poor until last term

(explanations for why are only moderately convincing)

- On previous page, we wrote the series in terms of powers of $\alpha_s(M_H/2)$
- But we are free to rewrite it in terms of $\alpha_s(\mu)$ for any choice of "renormalisation scale" μ.

LO

$$\sigma(pp \to H) = \sigma_0 \times \alpha_s^2(\mu)$$



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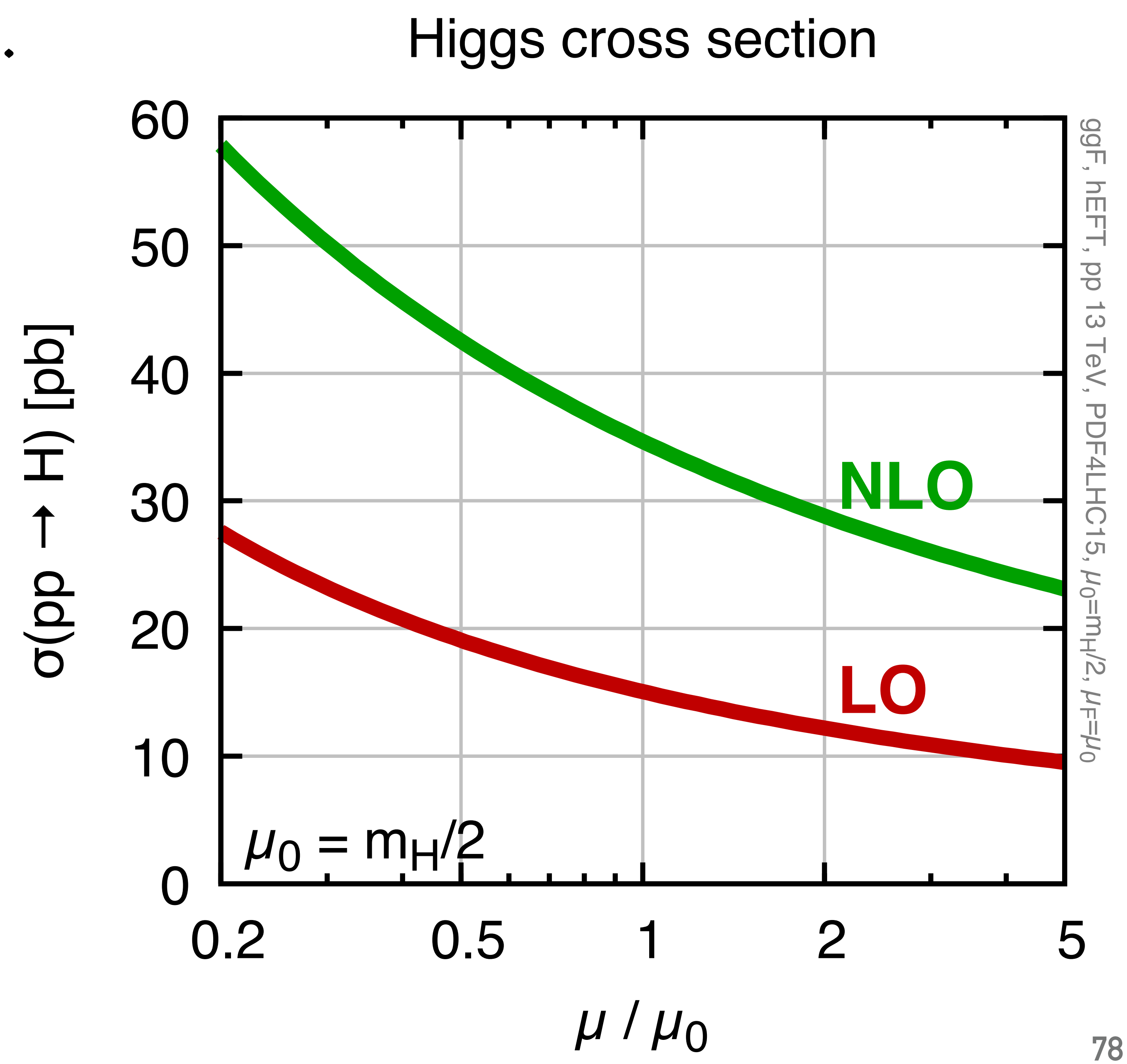
NLO

$$\sigma(pp \to H) = \sigma_0 \times \left(\alpha_s^2(\mu) + (10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2})\alpha_s^3(\mu)\right)$$



SCALE DEPENDENCE

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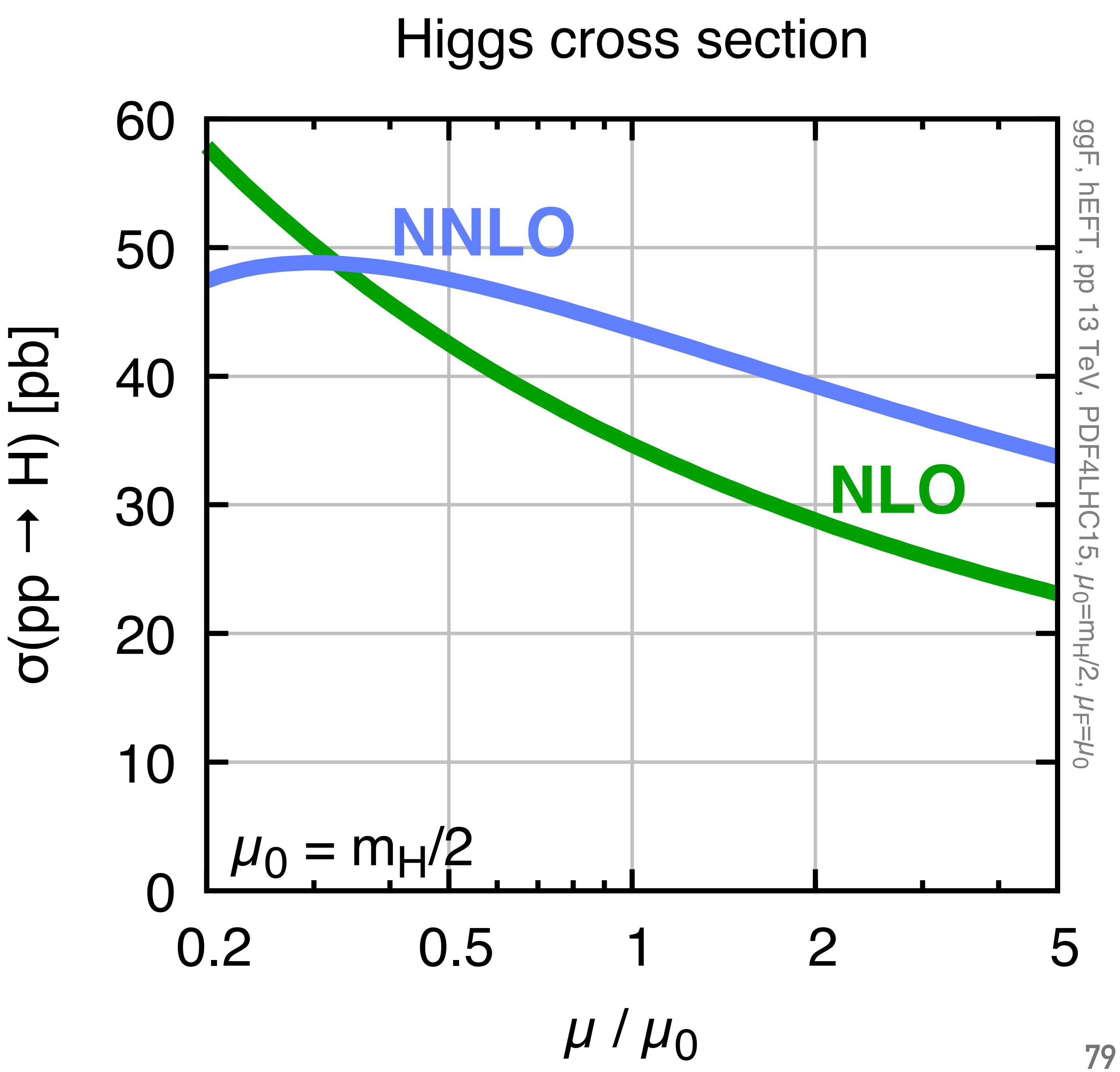
NNLO

$$\sigma(pp \to H) = \sigma_0 \times \left(\alpha_s^2(\mu)\right.$$

$$+(10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2})\alpha_s^3(\mu)$$

$$\left.+c_4(\mu)\alpha_s^4(\mu)\right)$$



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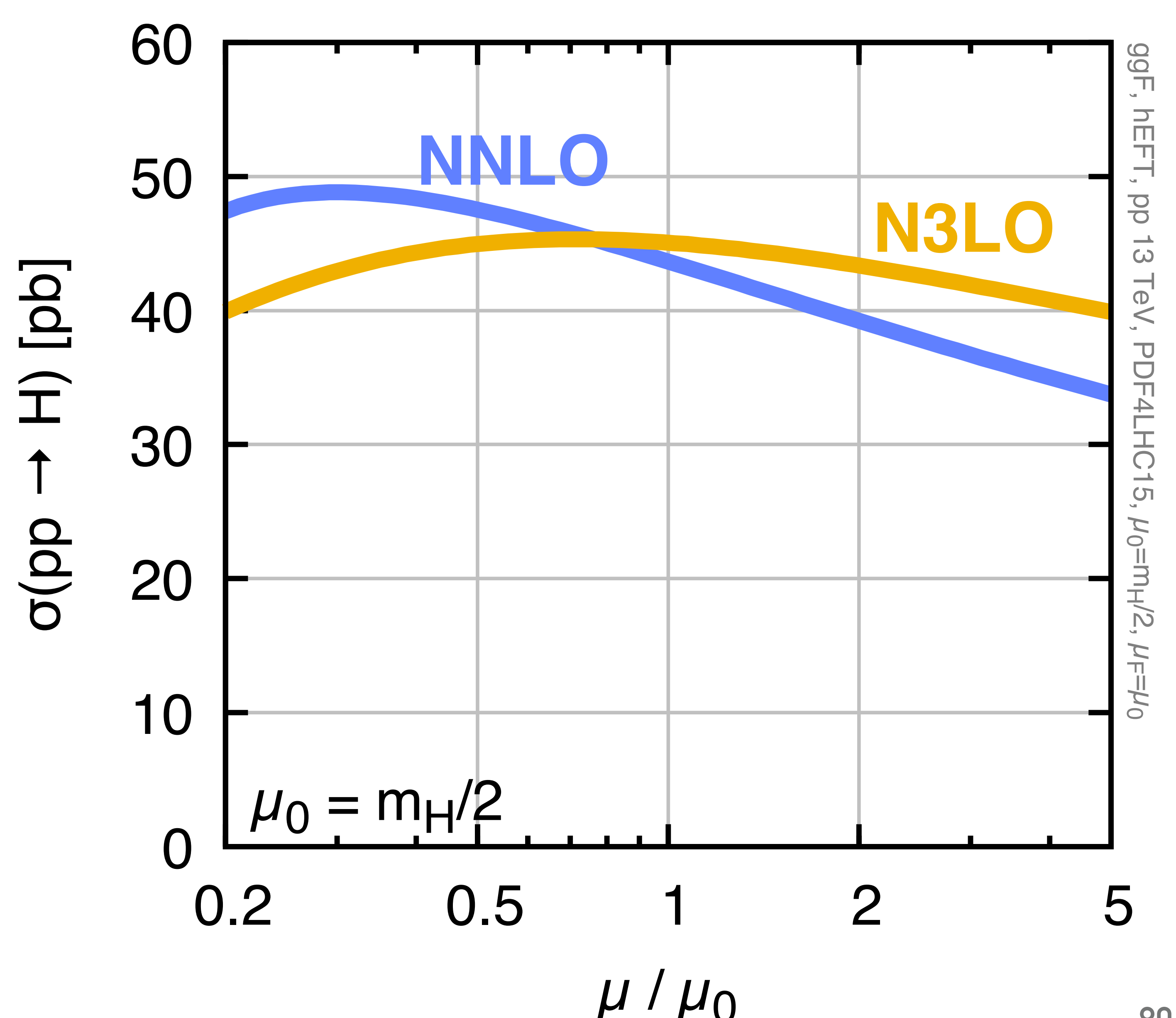
N3LO

$$\sigma(pp \to H) = \sigma_0 \times \Big(\alpha_s^2(\mu) + (10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2})\alpha_s^3(\mu) + c_4(\mu)\alpha_s^4(\mu) + c_5(\mu)\alpha_s^5(\mu)\Big)$$

Higgs cross section



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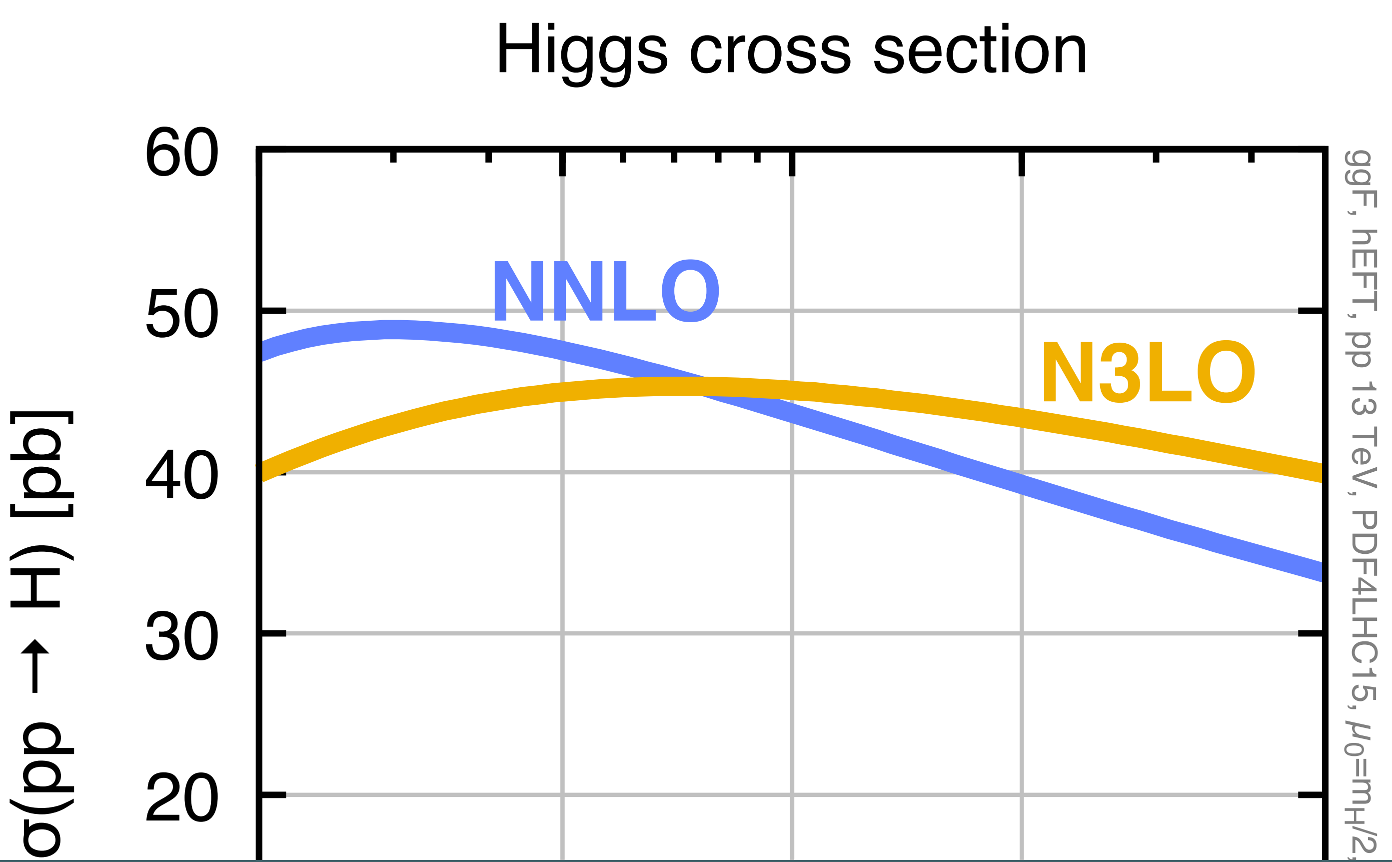
N3LO

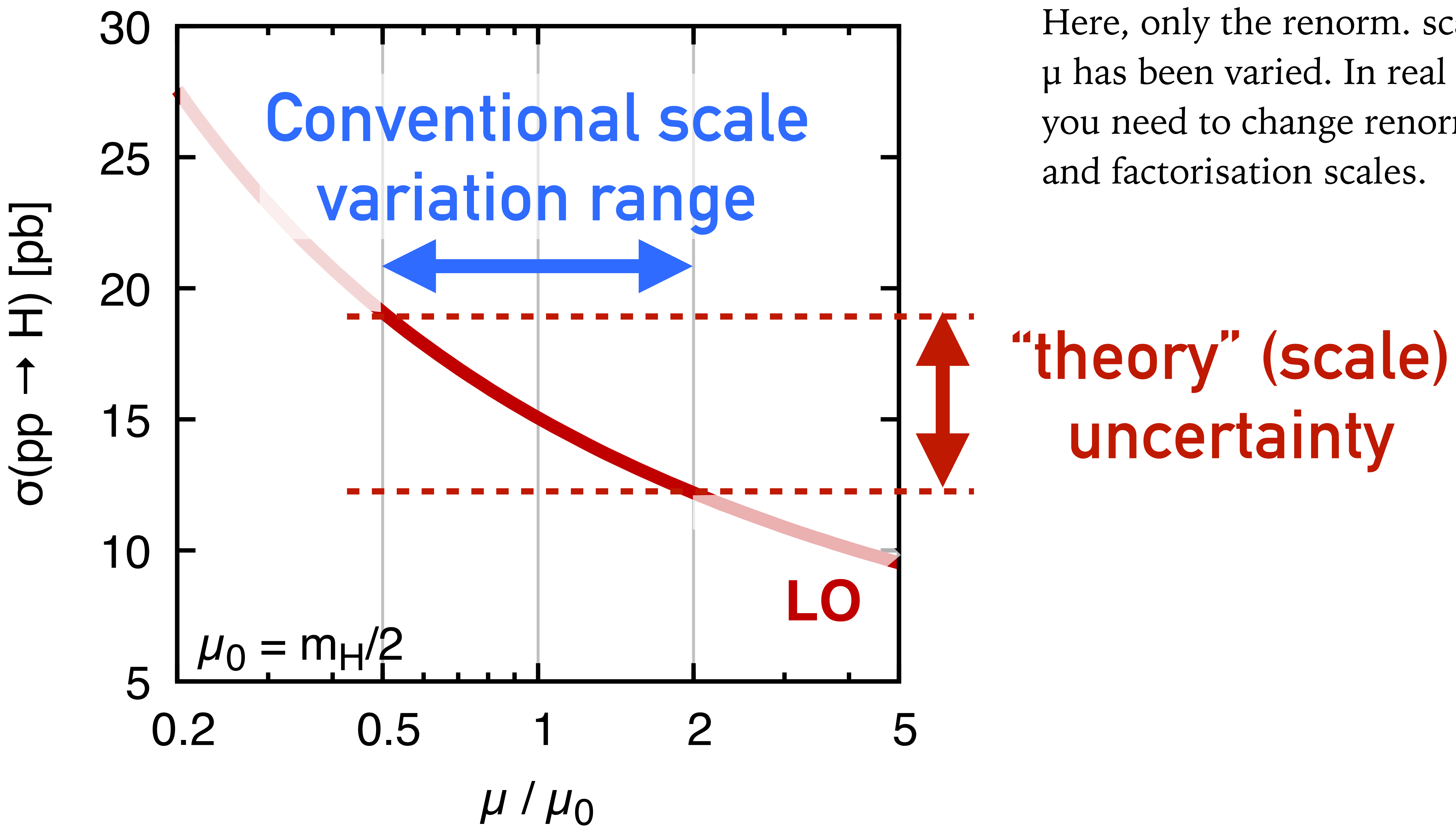
$$\sigma(pp \to H) = \sigma_0 \times \Big(\alpha_s^2(\mu)$$

$$+(10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2})\alpha_s^3(\mu)$$



scale dependence (an intrinsic uncertainty) gets reduced as you go to higher order

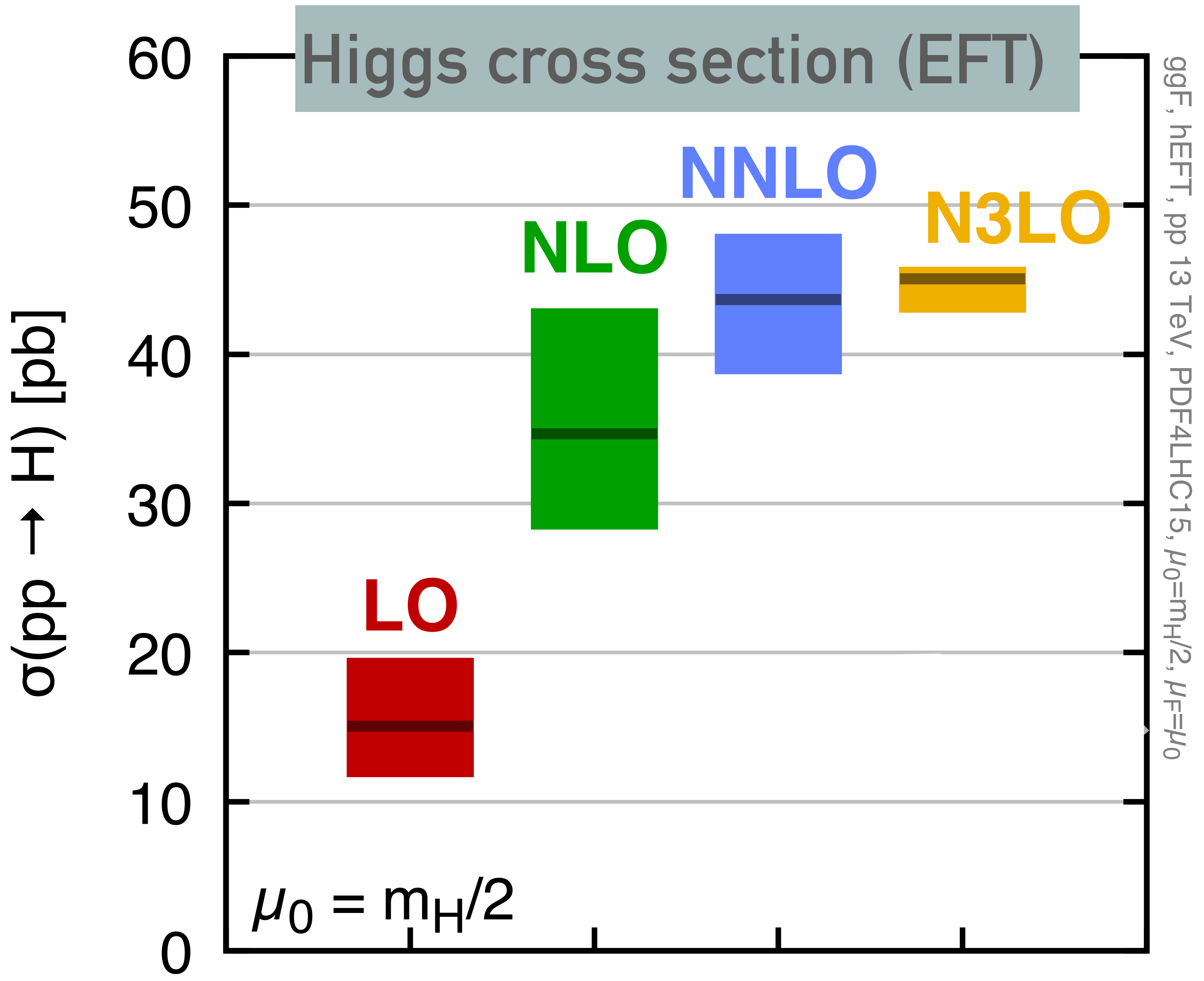
Scale dependence as the "THEORY UNCERTAINTY"



Here, only the renorm. scale μ has been varied. In real life you need to change renorm. and factorisation scales.

Convention: "theory uncertainty" (i.e. from missing higher orders) is estimated by change of cross section when varying μ in range 1/2 → 2 around central value

Scale dependence as the "THEORY UNCERTAINTY"



Here, only the renorm. scale $\mu (\equiv \mu_R)$ has been varied. In real life you need to change renorm. and factorisation (μ_F) scales.

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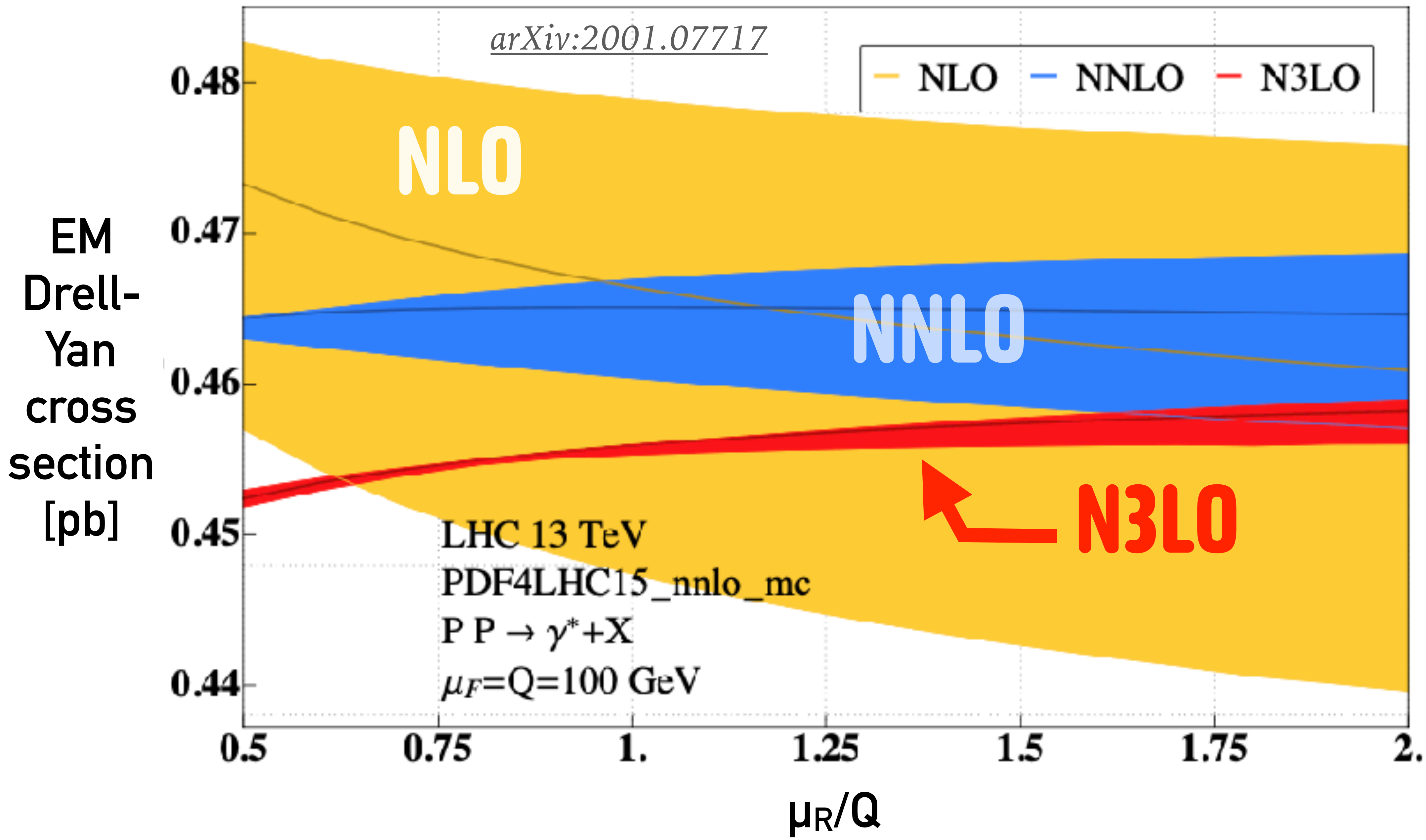
NLO/NNLO/N3LO for Drell-Yan process



Convergence from NNLO to N3LO is not so good.

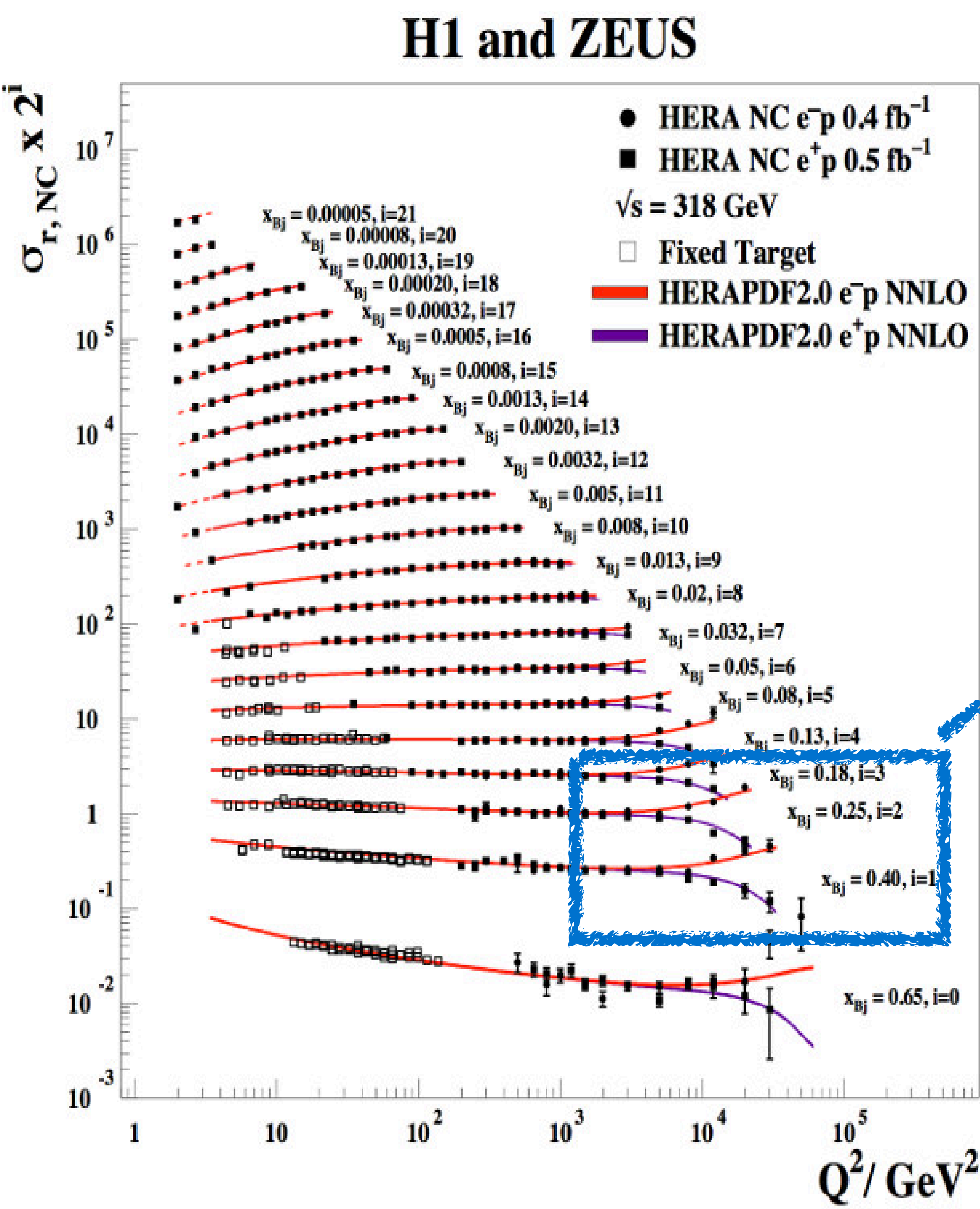
(Apparent good convergence from NLO to NNLO & small NNLO uncertainty were perhaps accident of cancellation between flavour channels)

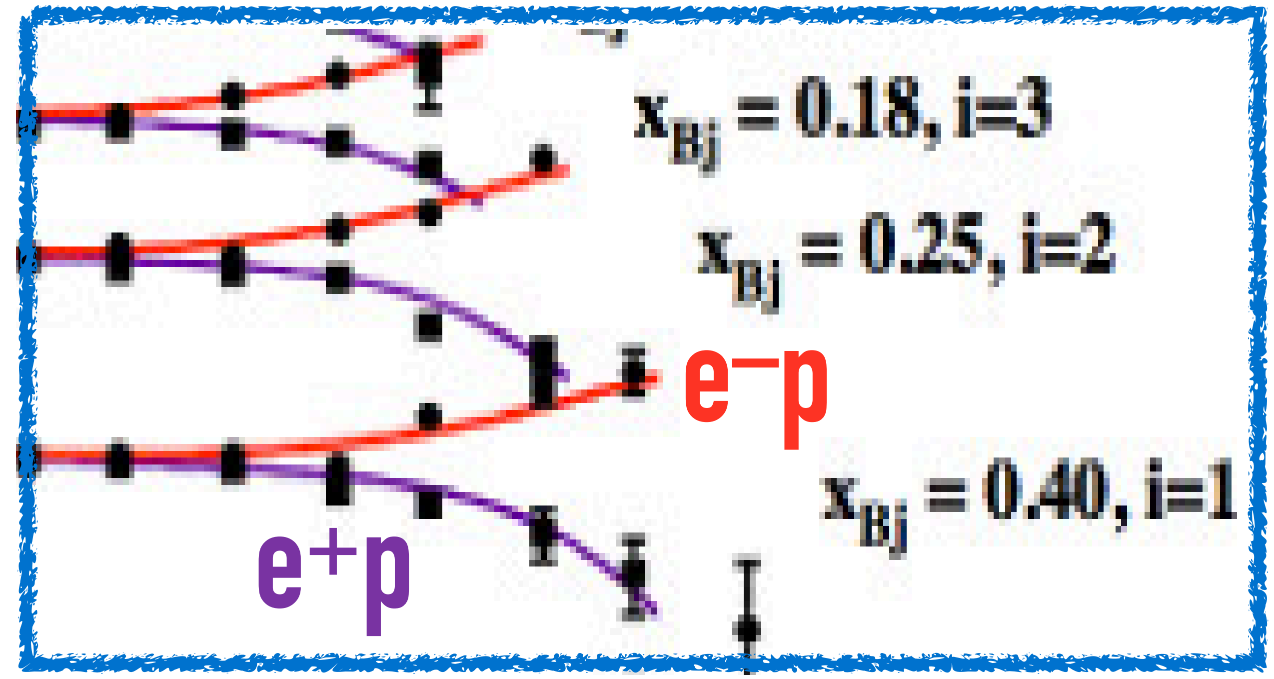
WHAT DO WE KNOW?

- LO: almost any process *(with MadGraph, Comix, ALPGEN, etc.)*
- NLO: most processes *(with MCFM, NLOJet++, MG5_aMC@NLO, POWHEG, OpenLoops/Blackhat/NJet/Gosam/Recola/etc.+Sherpa)*
- NNLO: all 2→1, most 2→2, and a few 2→3 (some approx) *(top++, DY/HNNLO, FEWZ, MATRIX, MCFM, NNLOJet, MINNLO, Geneva etc.)*
- N3LO: pp → Higgs and Drell Yan *some with approximations (EFT, $QCD_1 \times QCD_2$)*
- NLO EW corrections, i.e. relative α_{EW} rather than α_s: most 2→1, 2→2 and 2→3
- mixed NNLO (EW×QCD) for 2→1

BACKUP

Why the difference in e+p and e−p cross sections at high Q²? [1506.06042]





Answer: the F_3 (parity-violating) terms in the DIS cross section,

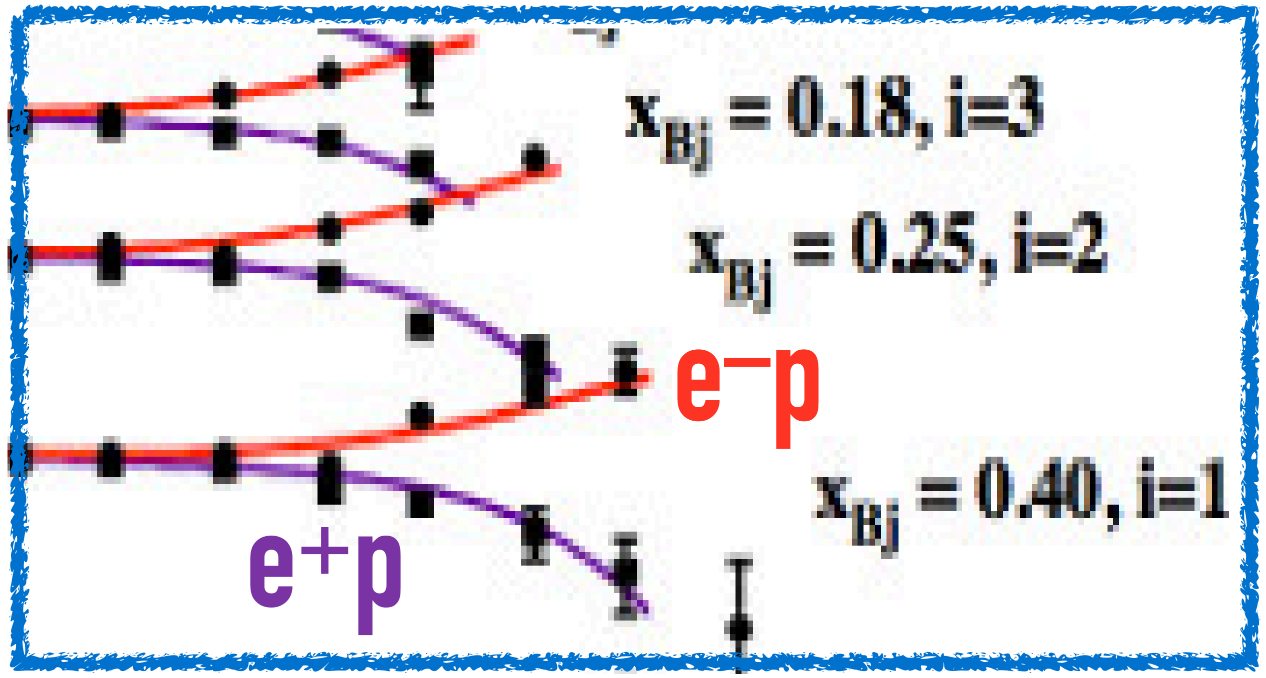
$$\sigma^{\pm}_{r,\mathrm{NC}} = \frac{\mathrm{d}^2\sigma^{e^\pm p}_{\mathrm{NC}}}{\mathrm{d}x_{\mathrm{Bj}}\mathrm{d}Q^2} \cdot \frac{Q^4 x_{\mathrm{Bj}}}{2\pi\alpha^2 Y_+} = \tilde{F}_2 \mp \frac{Y_-}{Y_+} x\tilde{F}_3 - \frac{y^2}{Y_+}\tilde{F}_{\mathrm{L}}$$

Why the difference in e^+p and e^-p cross sections at high Q^2? [1506.06042]

$$Y_\pm = 1 \pm (1-y)^2$$

$$x\tilde{F}_3 = -\kappa_Z a_e \cdot xF_3^{\gamma Z} + \kappa_Z^2 \cdot 2 v_e a_e \cdot xF_3^Z$$

$$\kappa_Z(Q^2) = Q^2 / [(Q^2 + M_Z^2)(4 \sin^2\theta_W \cos^2\theta_W)]$$



Answer: the F_3 (parity-violating) terms in the DIS cross section,

$$(xF_3^{\gamma Z}, xF_3^Z) \approx 2[(e_u a_u, v_u a_u)(xU - x\bar{U}) + (e_d a_d, v_d a_d)(xD - x\bar{D})],$$

$$xU = xu + xc, \qquad x\bar{U} = x\bar{u} + x\bar{c}, \qquad xD = xd + xs, \qquad x\bar{D} = x\bar{d} + x\bar{s}$$

$$\sigma^{\pm}_{r,NC} = \frac{d^2\sigma^{e^\pm p}_{NC}}{dx_{Bj}\,dQ^2} \cdot \frac{Q^4 x_{Bj}}{2\pi\alpha^2 Y_+} = \tilde{F}_2 \mp \frac{Y_-}{Y_+} x\tilde{F}_3 - \frac{y^2}{Y_+}\tilde{F}_L$$