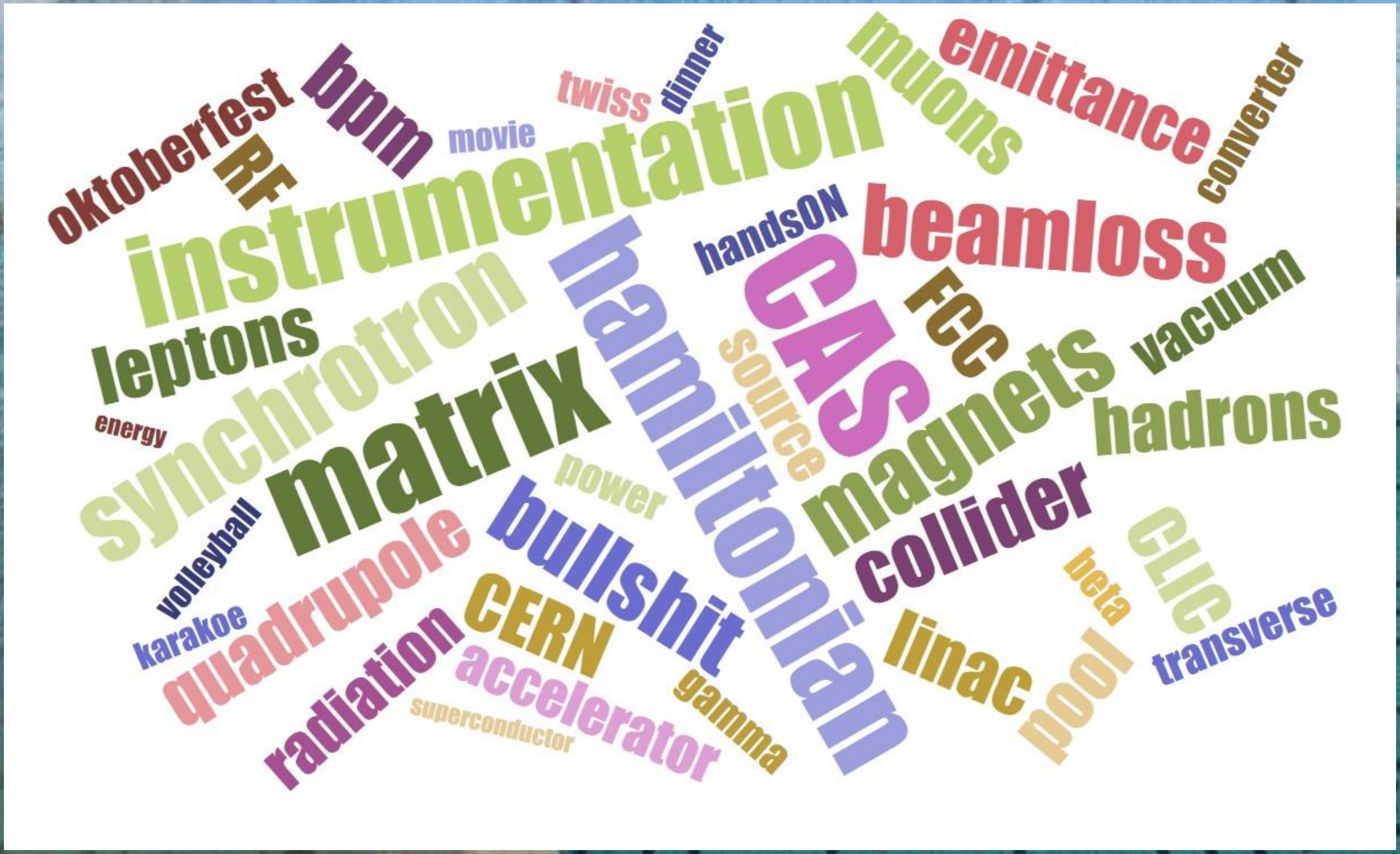
Recap of introductory Course (mainly beam dynamics)

H.Schmickler, Advanced CAS 2024



The CERN Accelerator School

Copyright statement and speaker's release for video publishing

The “minimum take-away”

- **Accelerators – past-today-future**

- **Beam dynamics**

  - what formalism to take?
  - phase-space, phase-space diagrams
  - focusing

- **Technologies**

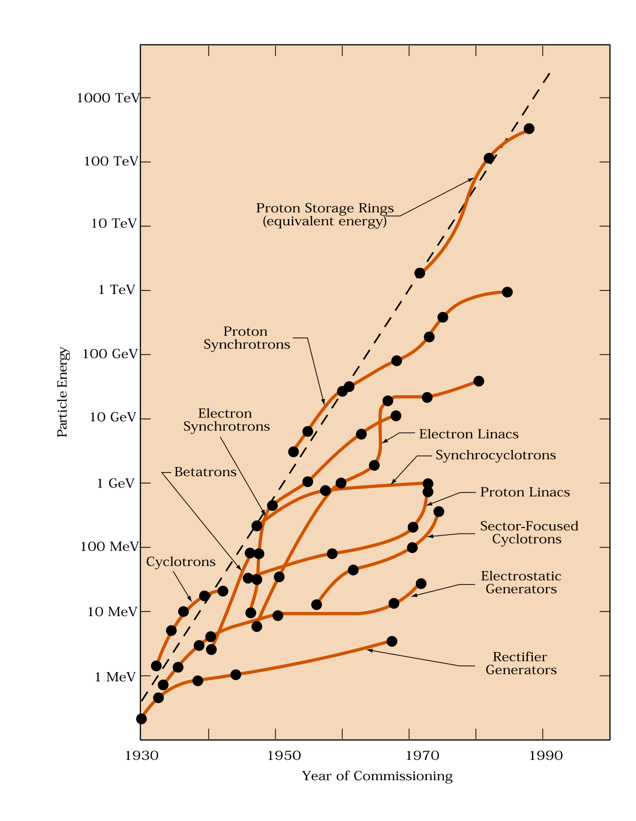
  - magnets
  - BI
  - RF

- **More advanced**

  - Non-linearities
  - Collective effects

Where do breakthrough technologies come from?

Many innovations emerge from interplay between curiosity driven research and societal need



https://www.symmetrymagazine.org/article/october-2009/deconstruction-livingston-plot

particle physics

vaccines, archaeology, etc…

proton therapy

radiotherapy, security

water, food, materials treatment, sterilisation

John Womersley, former CEO of STFC (UK) said:

"Particle physics is unreasonable. It makes unreasonable demands on technology. And when those technologies, those inventions, those innovations happen, they spread out into the economy, and they generate a huge impact."

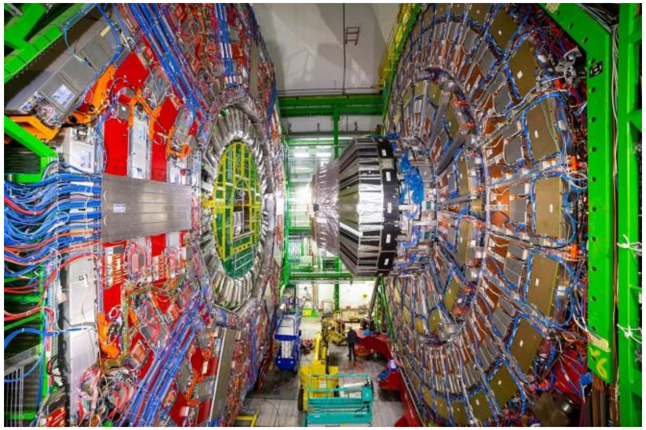
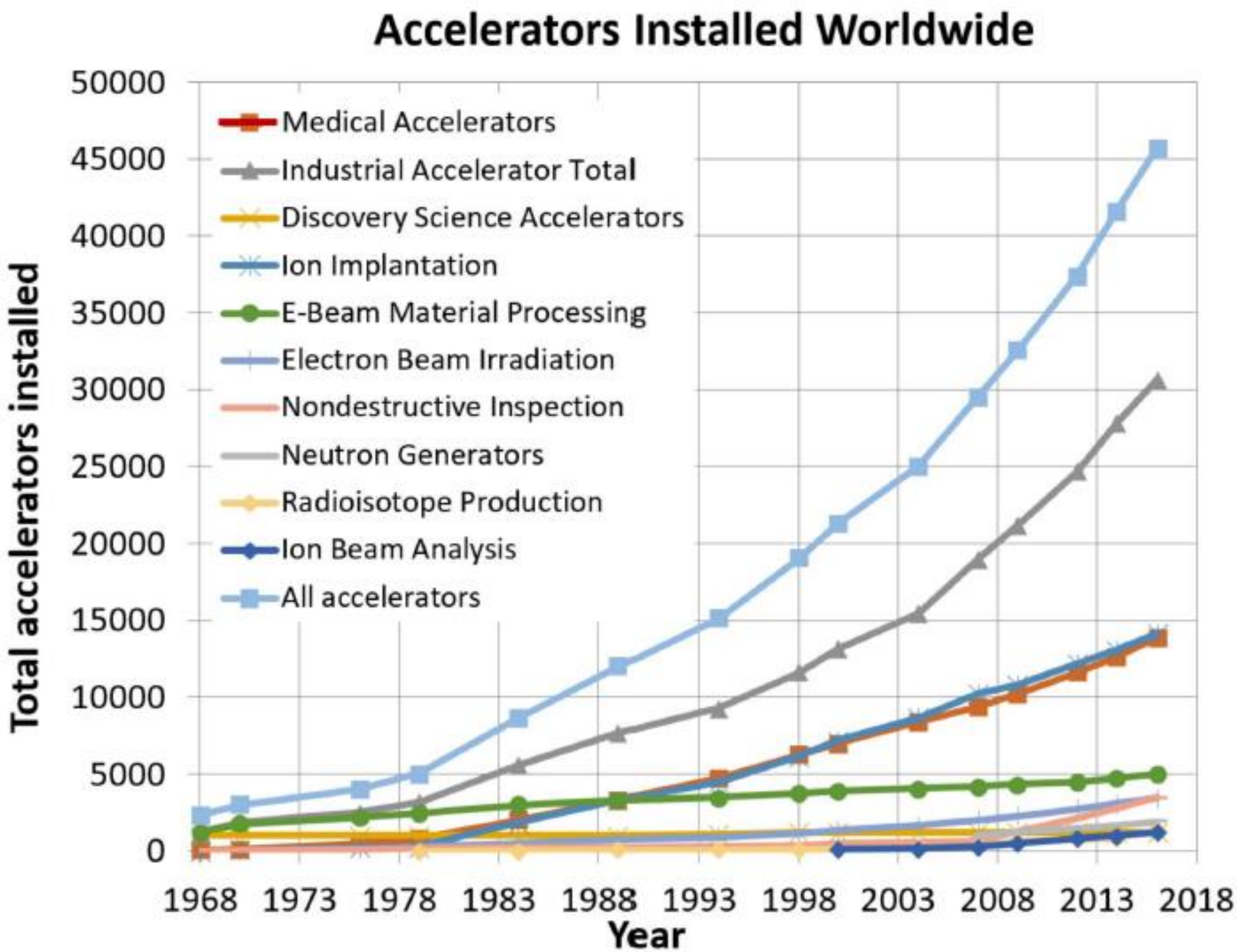


Image: CMS, CERN



Doyle, McDaniel, Hamm, *The Future of Industrial Accelerators and Applications, SAND2018-5903B*

Basics

- Relativity…we remember from school
- With what force do we act on the beams:

→ Lorentzforce

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Electric force   *Magnetic force*

1. Acceleration with electric field
2. Transverse forces with magnetic fields
   → dipoles: bending
   → quadrupoles: focusing
   → sextupoles: correction of momentum dependent focusing errors

Relativistic momentum $p = mv = \gamma m_0 v = \gamma m_0 \beta c$

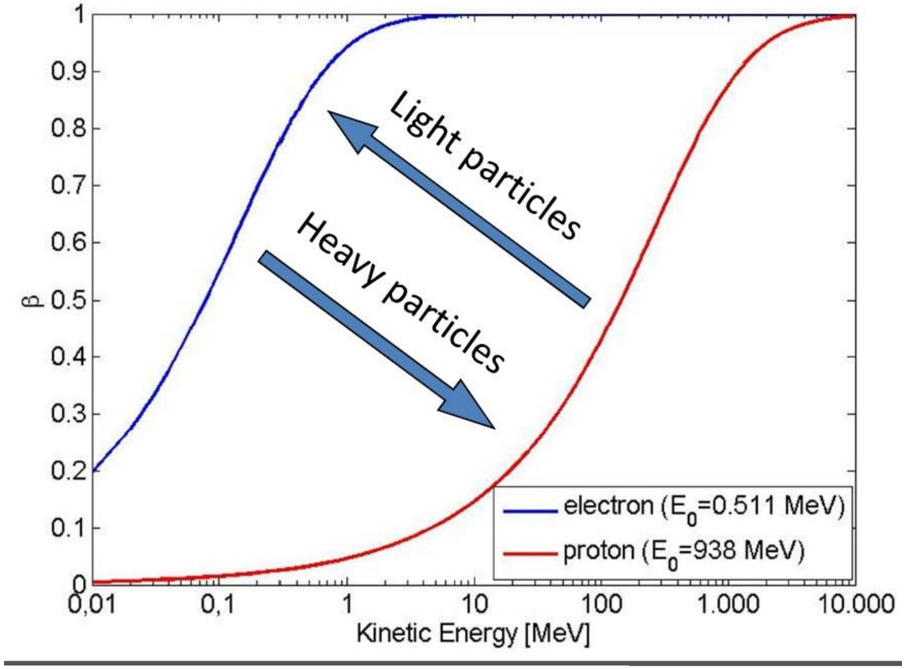
From page before (squared):

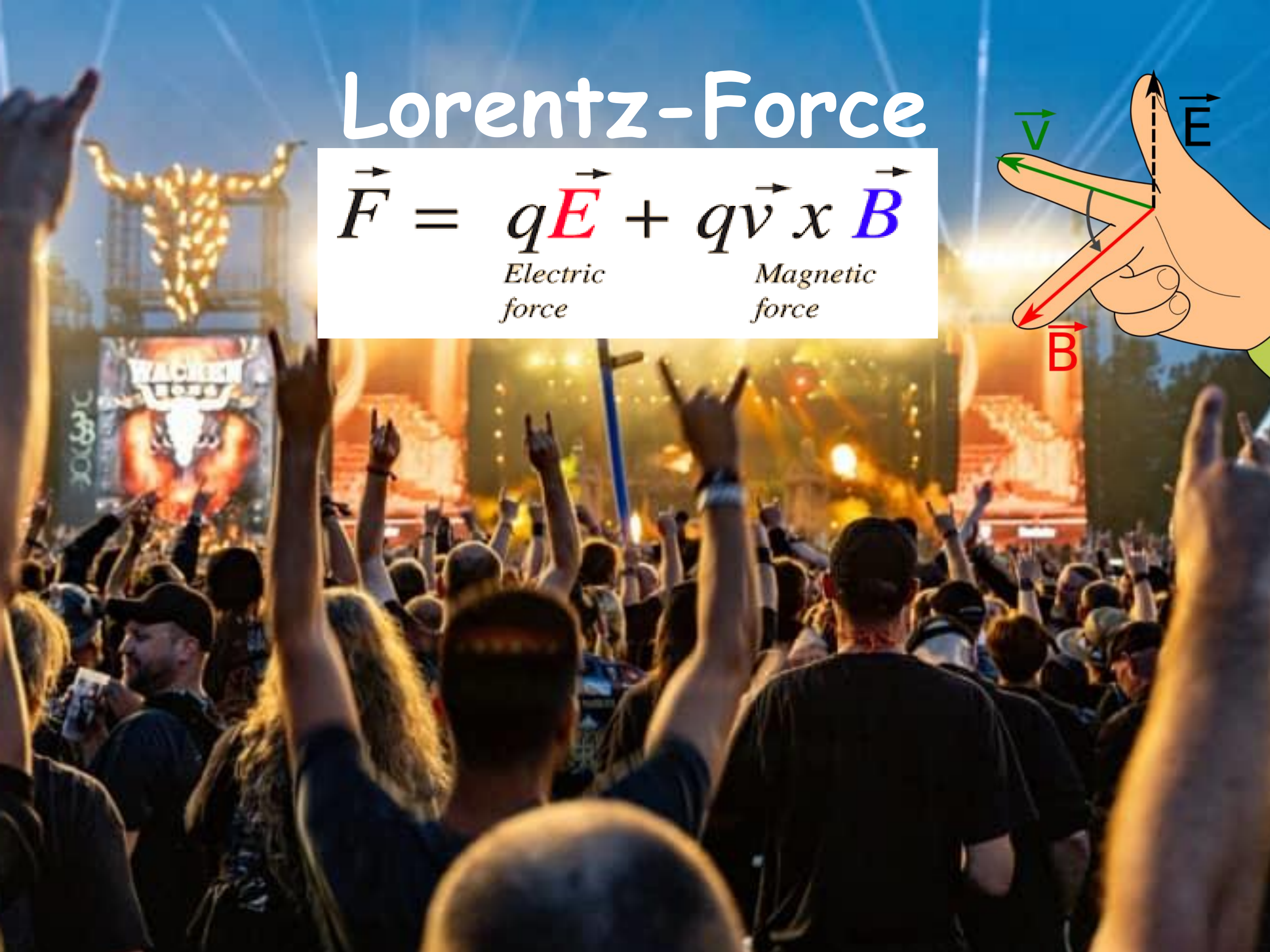
$$E^2 = m^2c^4 = \gamma^2 {m_0}^2 c^4 = \left(\frac{1}{1-\beta^2}\right){m_0}^2c^4 = \left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right){m_0}^2c^4 = (1+\gamma^2\beta^2){m_0}^2c^4$$

$$E^2 = (m_0c^2)^2 + (pc)^2 \Rightarrow \frac{E}{c} = \sqrt{(m_0c)^2 + p^2}$$

Or by introducing new units [E] = eV ; [p] =eV/c ; [m] = eV/c² $E^2 = {m_0}^2 + p^2$

Due to the small rest mass electrons reach already almost the speed of light with relatively low kinetic energy, but protons only in the GeV range





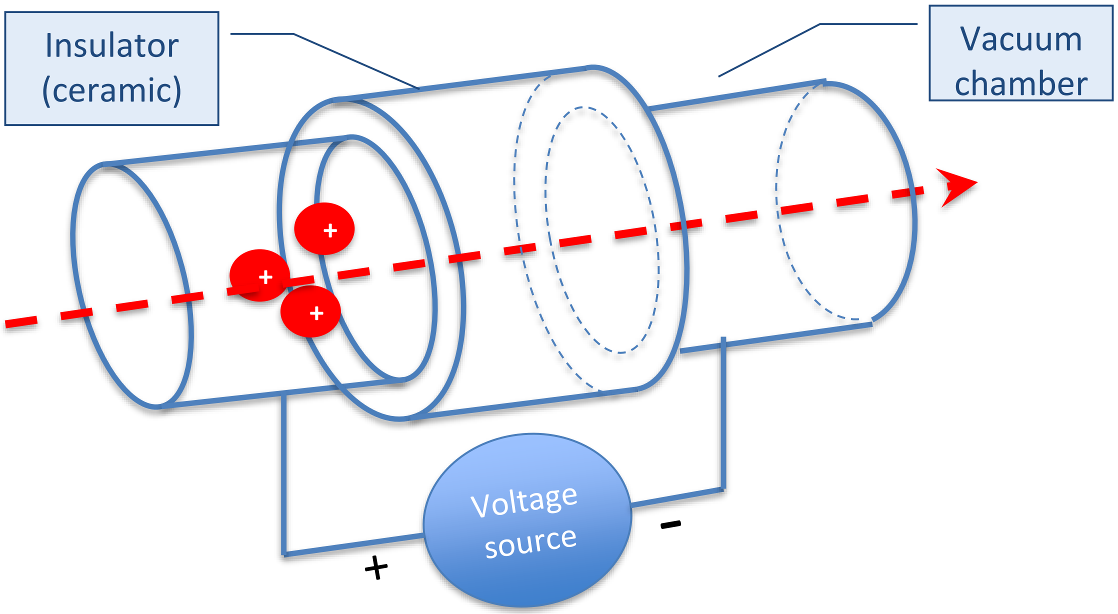
Lorentz-Force

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic force}}$$

Methods of Acceleration in circular accelerators

Electrostatic field limited by insulation, magnetic field doesn't accelerate at all.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$



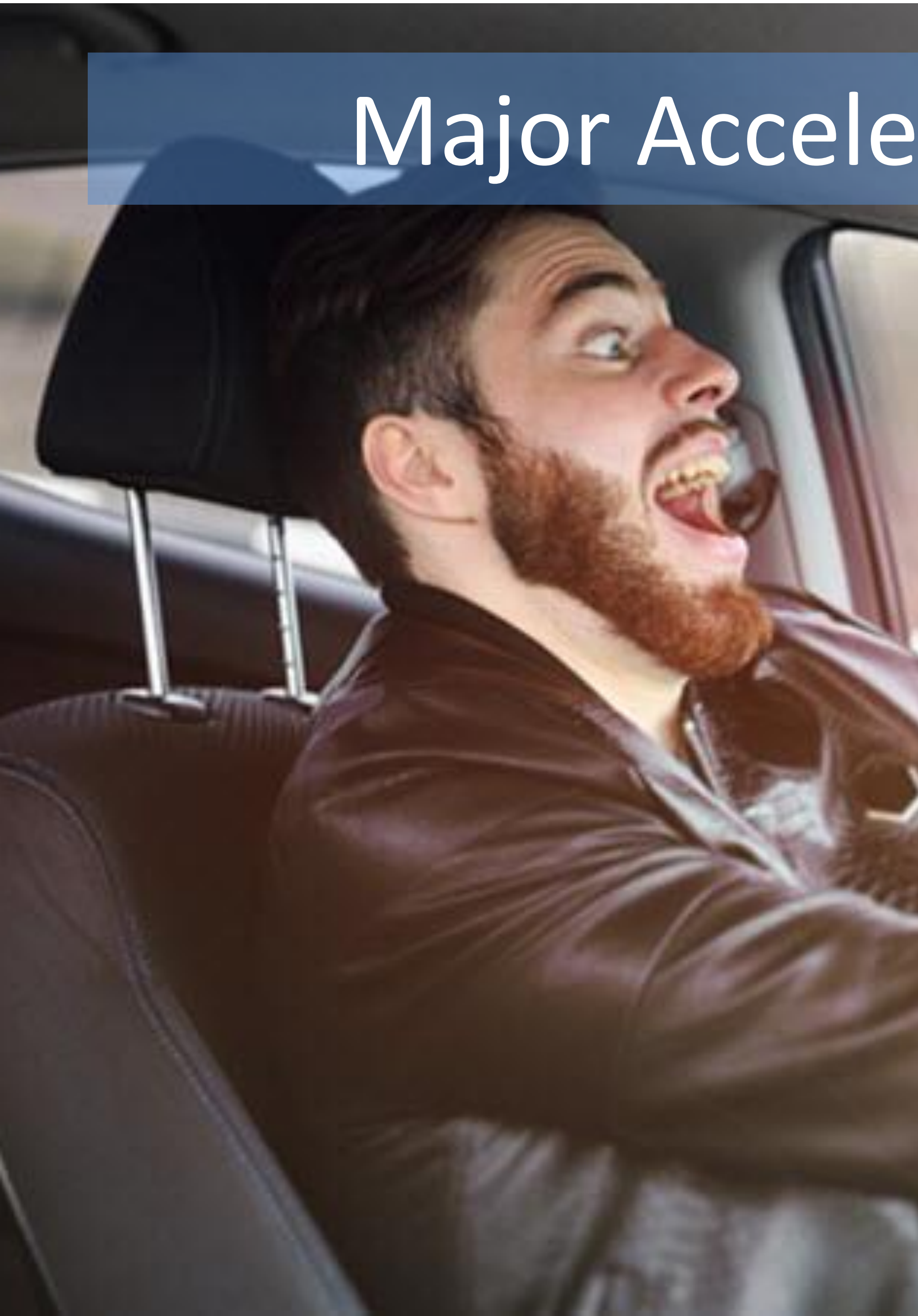
➡ **no Acceleration**

The electric field is derived from a scalar potential φ and a vector potential A
The time variation of the magnetic field H generates an electric field E

The solution: => time varying electric fields
- Induction
- RF frequency fields

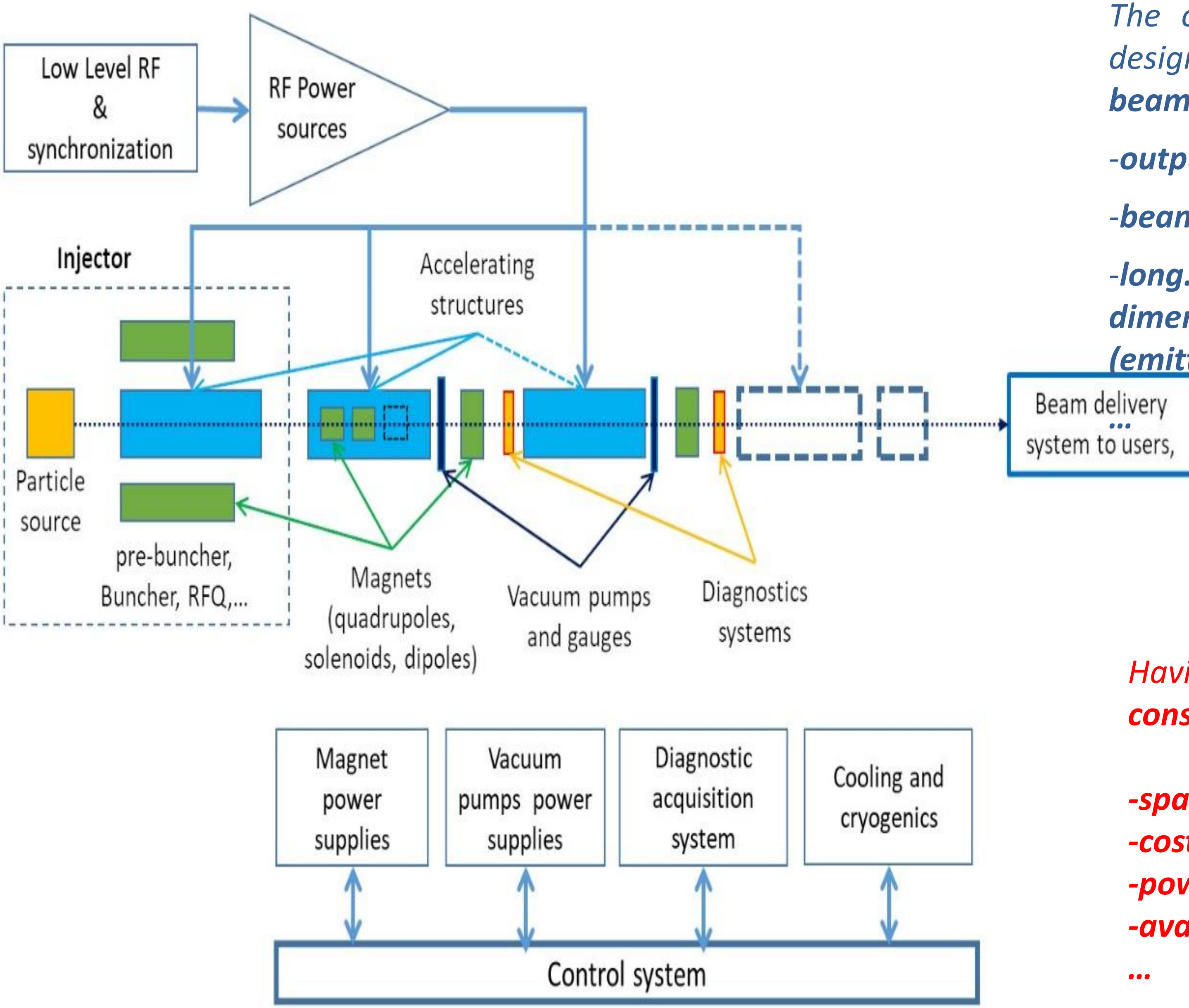
$$\oint \vec{E} \cdot \mathrm{d}\vec{s} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{A}$$

Major Accelerator Types



- DC beam electrostatic acce
- Linear Accelerators (linacs)
- Betatron
- Cyclotrons
- Synchrotrons
- Lightsources
  - synchrotron radiation
  - undulator radiation
- Colliders
  - linear
  - circular
- Test facilities for future con

LINAC OVERVIEW



*The overall LINAC has to be designed to obtain the **desired beam parameters** in term of:*

*-**output energy/energy spread***

*-**beam current (charge)***

*-**long. and transverse beam dimensions/divergence (emittance)***

*Having, in general, **constraints** in term of:*

*-**space***
*-**cost***
*-**power consumption***
*-**available power sources***
…

Acceleration by Induction: The Betatron

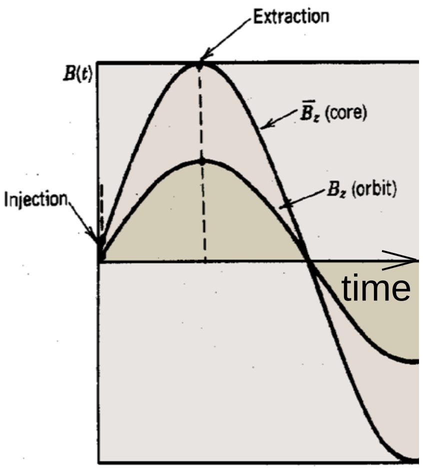
It is based on the principle of a transformer:
- primary side: large electromagnet - secondary side: electron beam.

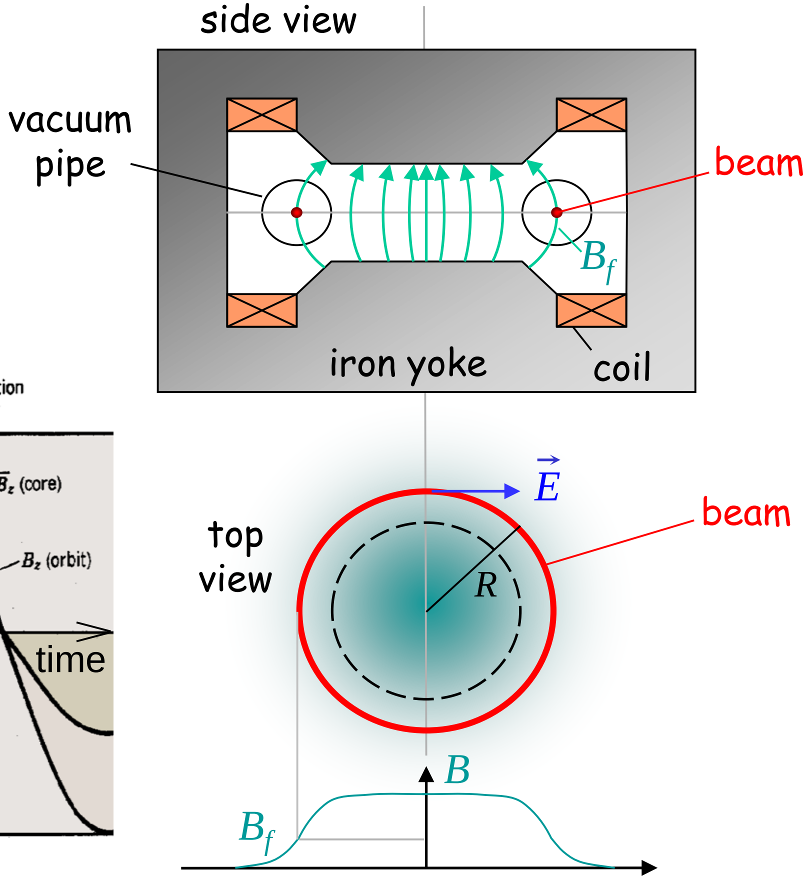
The ramping magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

Limited by saturation in iron (~300 MeV e-)

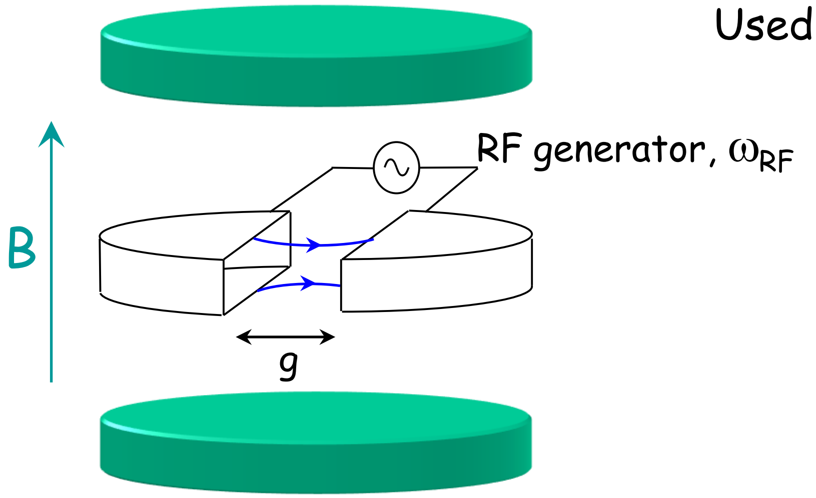
Used in industry and medicine, as they are compact accelerators for electrons

Donald Kerst with the first betatron, invented at the University of Illinois in 1940





Circular accelerators: Cyclotron



Used for protons, ions

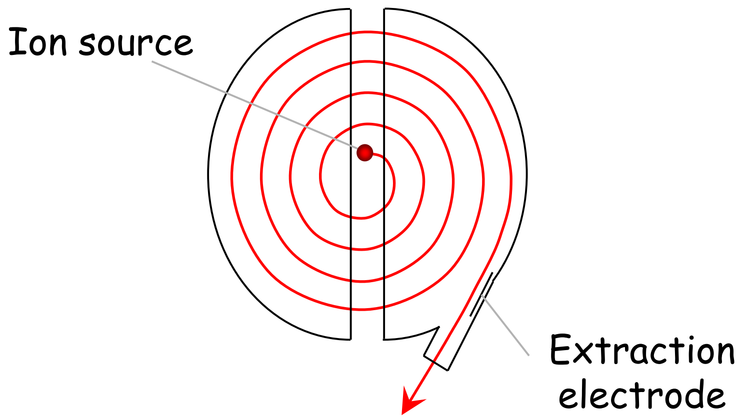
B = constant

ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$

$$2\pi\ \rho = v_s\ T_{RF}$$



Cyclotron frequency $\omega = \dfrac{q\,B}{m_0\,\gamma}$

1. γ increases with the energy
   ⇒ no exact synchronism
2. if **v << c** ⇒ **$\gamma \cong 1$**

Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php

Cyclotron / Synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma\ \omega_{RF}$ = constant

ω_{RF} decreases with time

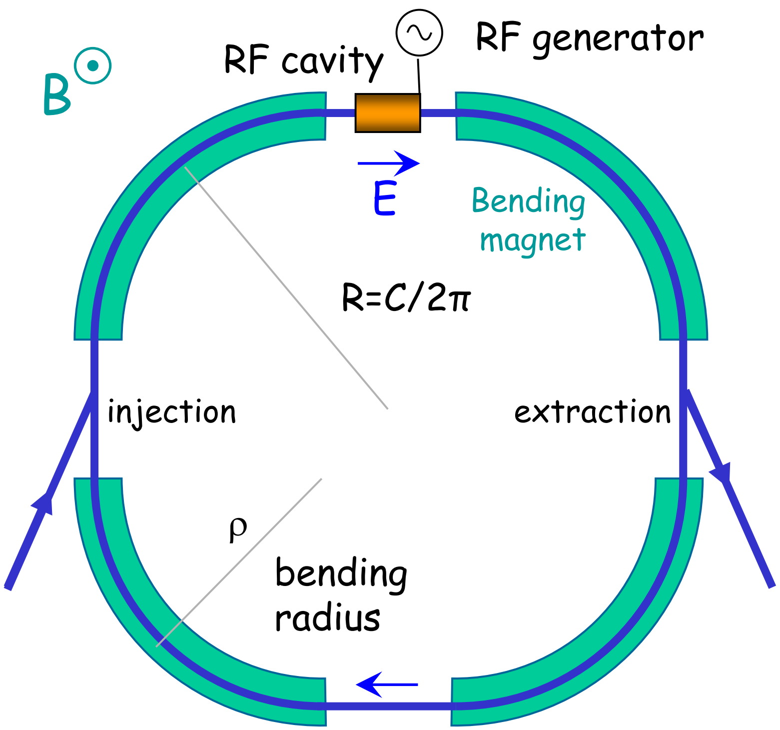
More in lectures by Mike Seidel

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\ B}{m_0\ \gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition

$$T_s = h\, T_{RF}$$

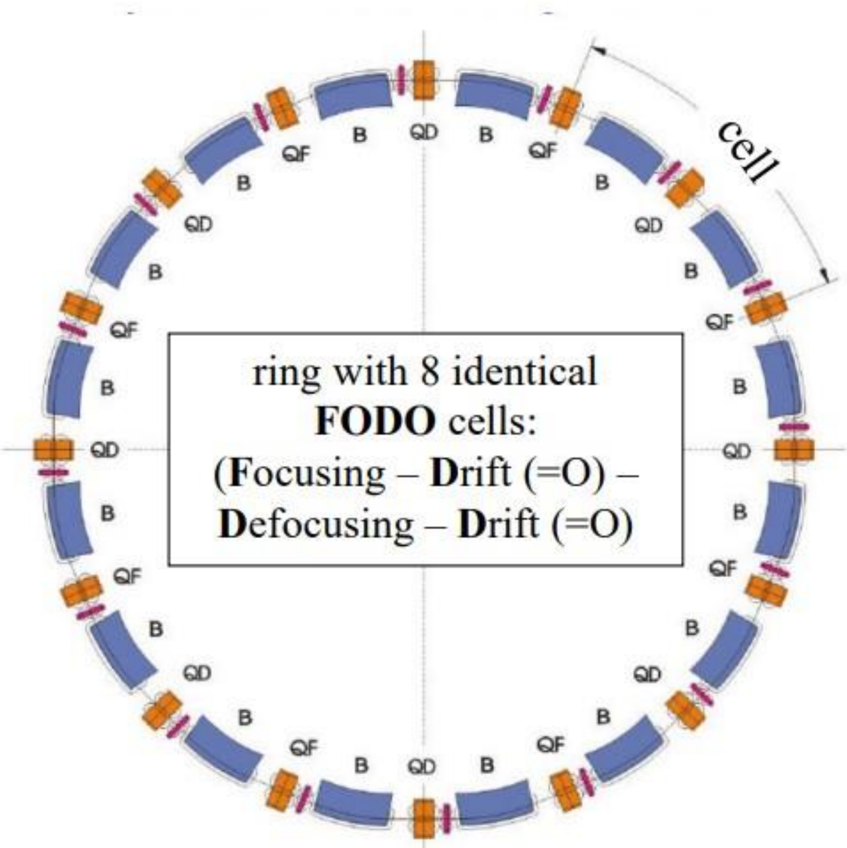
$$\frac{2\pi R}{v_s} = h\, T_{RF}$$

h integer,
harmonic number:
number of RF cycles per revolution

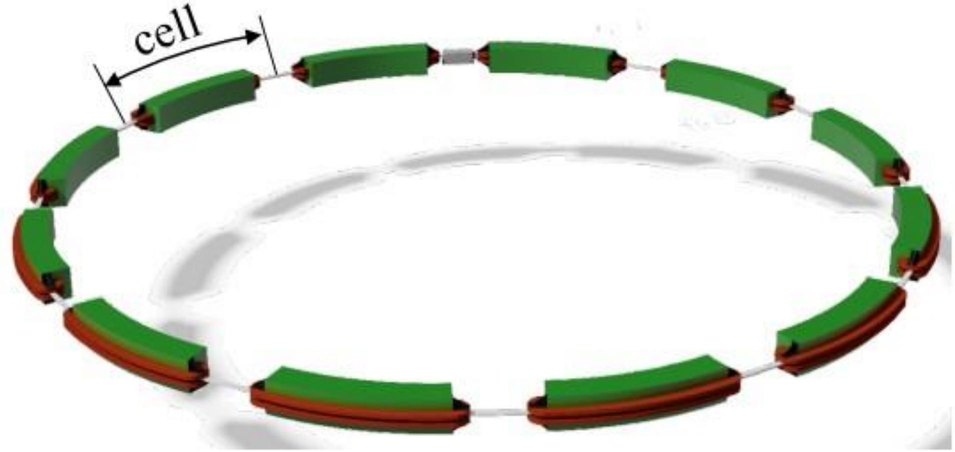
h is the maximum number of bunches in the synchrotron.
Normally less bunches due to gaps for kickers, collision constraints,…

AG Synchrotron

FODO lattice



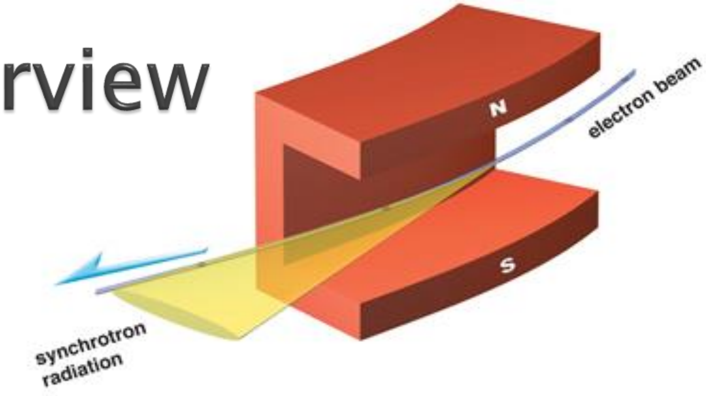
Identical combined function AG magnets



ring with 12 identical combined function FD magnets

Important: due to periodicity, we can choose any position s_0 to define a periodic cell ($s_0 \rightarrow s$) and its transfer matrix $\mathbf{M}(s,s_0) \equiv \mathbf{M}(s-s_0) = \mathbf{M}(L)$

Synchrotron radiation overview



- Accelerated charged particles emit electromagnetic radiation following Maxwell equations

- In the case of radially accelerated charges, the associated radiation is called **synchrotron radiation.**

- This phenomenon occurs in bending magnets and was first observed in synchrotron facilities, where the beam energy and magnet dipole strengths are ramped up *synchronously* → hence the name "synchrotron radiation"

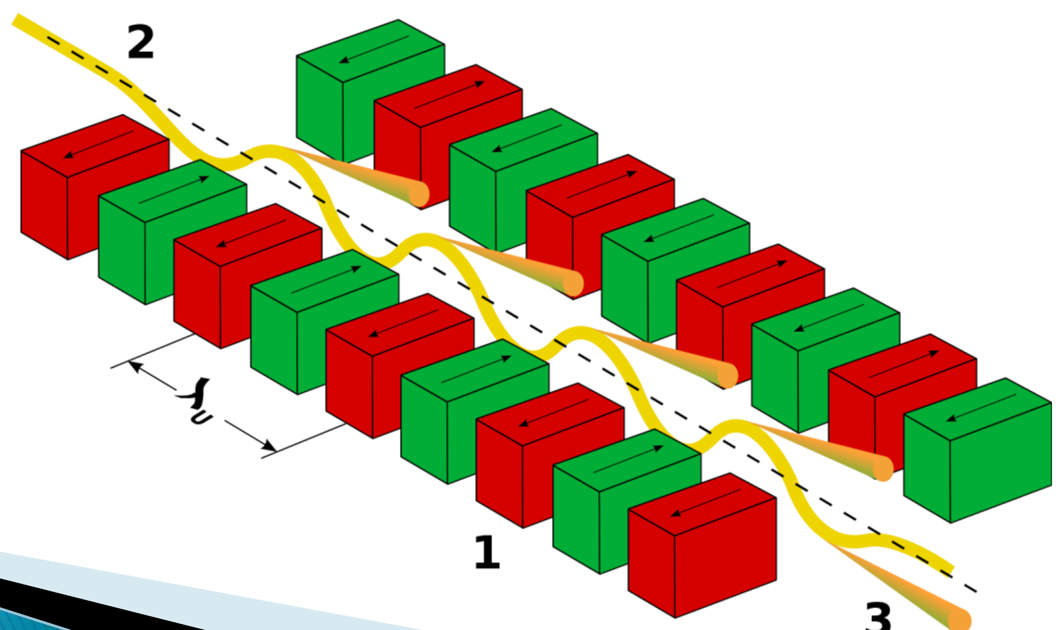
- The radiated power is proportional to m^{-4} (m: charged particle mass) → in practice only relevant for electron machines!

- For electron machines, synchrotron radiation (SR) is boon and bane:
  - SR is the main obstacle for circular machines to reach higher energies
  - But SR (today) is also the main application of circular electron machines and thus the primary motivation to build them!
    → most of recent design work has gone into optimizing SR for experimental and industrial use

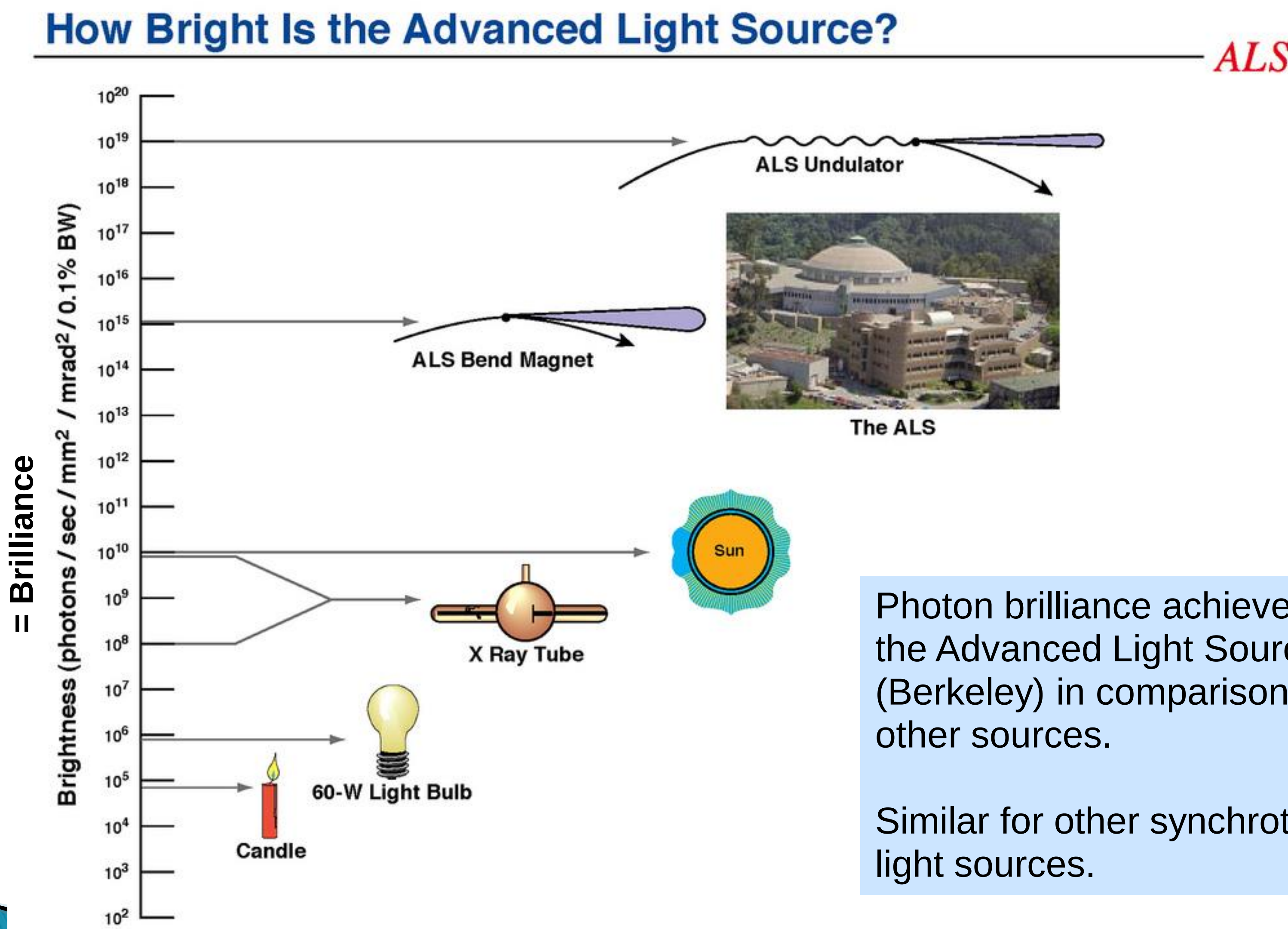
    → also the reason why many particle physics laboratories have become photon science laboratories (SLAC, DESY, PSI, Cornell...)

Undulator radiation

- Undulators are periodic structures of dipole magnets with alternating polarity. An undulator is defined by the number of bending magnets N and the period λ_u (with typical values of few cms).
- The radiation emitted in undulators has higher power and better quality than the radiation emitted in an individual bending magnet.
- A main advantage: the deflection alternates so that the global electron trajectory is straight (in contrast to the curved trajectory in bending magnets) → increase of the radiation flux at the experimental station



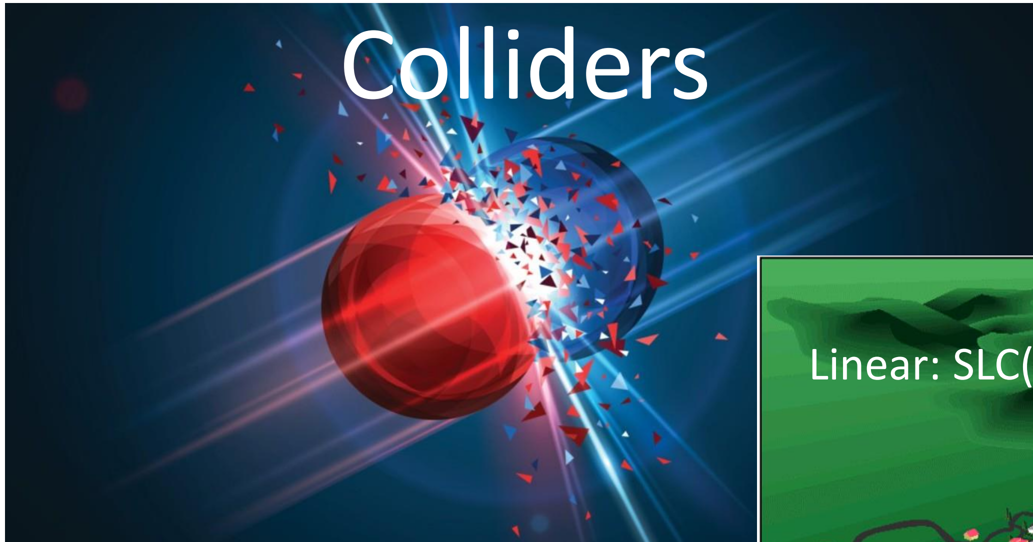
Brilliance comparison

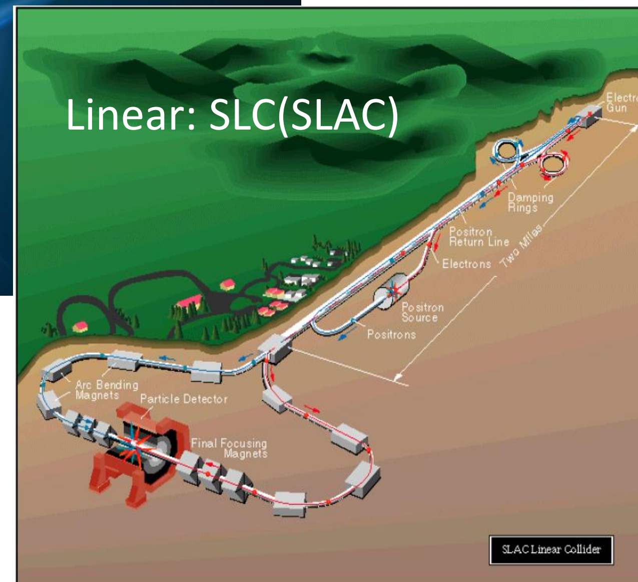


Photon brilliance achieved at the Advanced Light Source (Berkeley) in comparison with other sources.

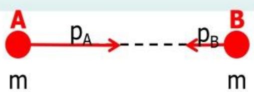
Similar for other synchrotron light sources.

Colliders





Fixed-target vs head-on beam collisions



- Relativistic invariant $(\Sigma m)^2c^4 = (\Sigma E)^2 - (\Sigma p)^2c^2$
- In the laboratory frame $4m^2c^4 = (E_A + E_B)^2 - (\overrightarrow{p_A} + \overrightarrow{p_B})^2c^2$
- Let E^* be the total energy available in the collision
- In the center-of-mass frame $\overrightarrow{p^*} = \overrightarrow{p_A}* + \overrightarrow{p_B}* \equiv 0$

$$4m^2c^4 = E^{*2}$$

$$\boxed{E^{*2} = (E_A + E_B)^2 - (\overrightarrow{p_A} + \overrightarrow{p_B})^2c^2}$$

- Fixed-target

$$p_B = 0\,;\ E_B = mc^2$$

$$E^{*2} = {E_A}^2 - {p_A}^2c^2 + m^2c^4 + 2E_Amc^2$$

$$E^{*2} = 2m^2c^4 + 2E_Amc^2 \approx 2E_Amc^2$$

$$\boxed{E^* \approx \sqrt{2E_Amc^2}}$$

- Head-on collision

$$\boxed{E^* = E_A + E_B}$$

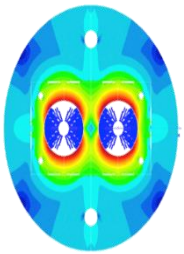
Options towards higher energies

Hadron (p) circular collider

$$p = e \cdot R \cdot B_y$$

Increase bending field
SC bend magnet work (FCC-hh)

Increase radius = size (FCC-hh)



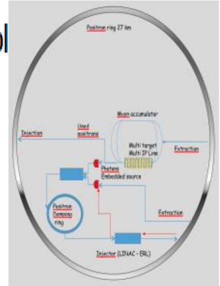
Lepton (e-,e+) circular collider

$$p \propto E_0 \cdot \sqrt[4]{\rho \cdot U_0}$$

Increase supplied RF vol
(FCC-ee)

Increase mass of acc. particle (muon)

Increase radius = size (FCC-ee)



Lepton (e-,e+) linear collider

$$p = L \cdot G_{acc}$$

Increase length (ILC, CLIC)

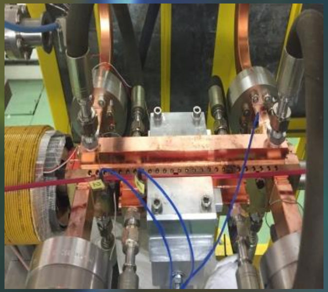
(b) New regime of ultra-high gradients (plasma, dielectric accelerators)

Compact , Cost Effective and Sustainability

High Gradient Options

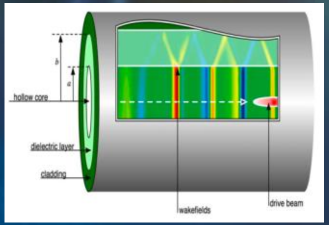
Metallic accelerating structures =>

100 MV/m < E_{acc} < 1 GV/m



Dielectrict structures, laser or particle driven =>

E_{acc} < 10 GV/m



Plasma accelerator, laser or particle driven =>

E_{acc} < 100 GV/m

Related Issues: Power Sources and Efficiency, Stability, Reliability, Staging, Synchronization, Rep. Rate and short (fs) bunches with small (μm) spot to match high gradients

Beam Quality Requirements

Future accelerators will require also high quality beams :

==> High Luminosity & High Brightness,

==> High Energy & Low Energy Spread

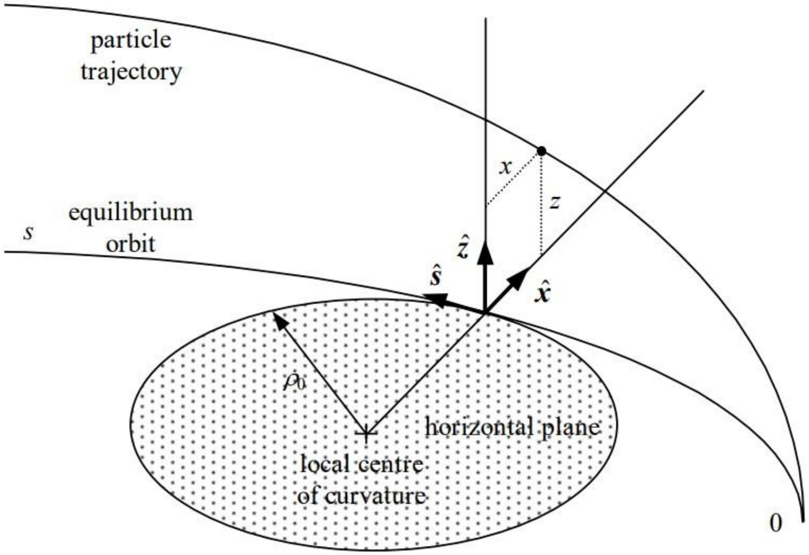
$$L = \frac{N_{e+} N_{e-} f_r}{4 p s_x s_y}$$

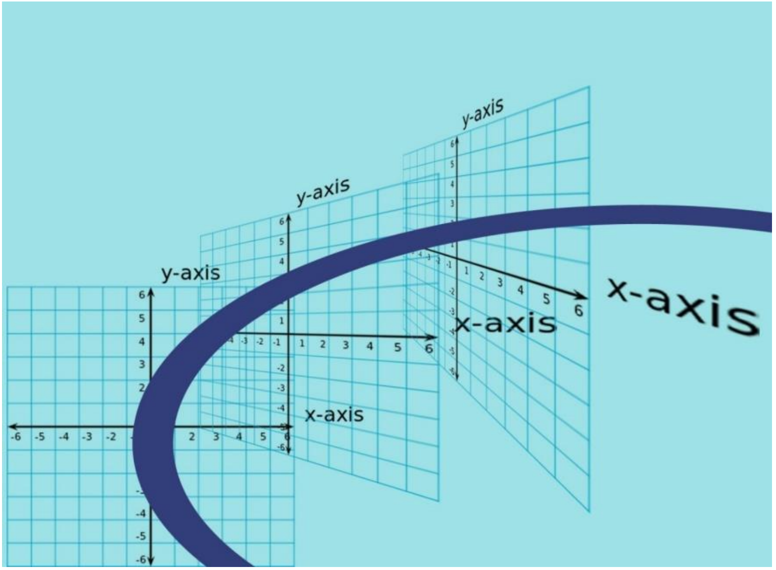
$$B_n » \frac{2I}{e_n^2}$$

- -N of particles per pulse => 10^9
- -High rep. rate f_r=> bunch trains
- -Small spot size => low emittance
- -Short pulse (ps => fs)
- -Little spread in transverse momentum and angle => low emittance

Describing particle motion in an accelerator

- Frenet-Serret co-ordinate system:



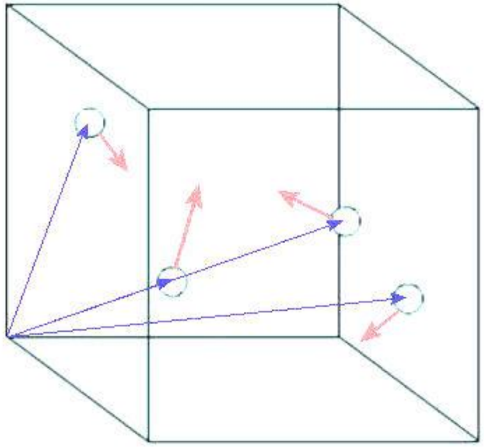


- Phase-Space

- Beam Emittance

Phase Space

- We are used to describe a particle by its 3D position (x,y,z in carth. Coordinates) (blue arrows below)
- In order to get the dynamics of the system, we need to know the momentum (px, py, pz); read arrows below
- In accelerators we describe a particle state as a 6D phase space point.
  Below the projection into a 2 D phase space plot.
  The points correspond to the x-position (q_x) and the x component of the p-vector (p_x).



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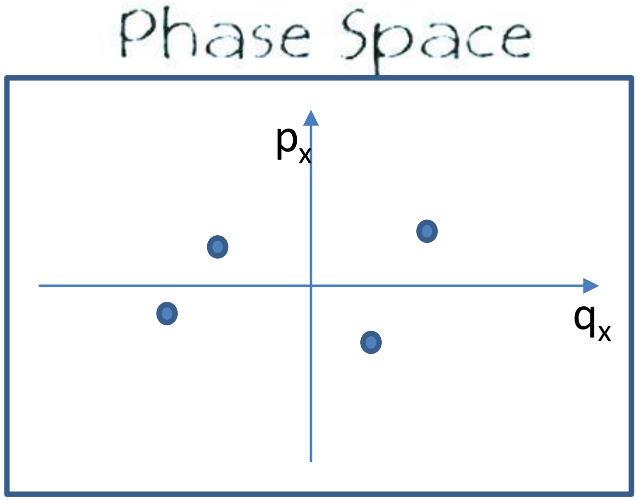
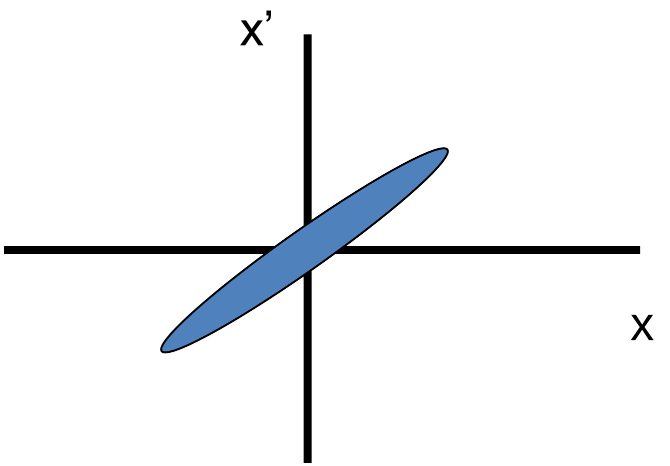


This shows one of the three possible phase space projections

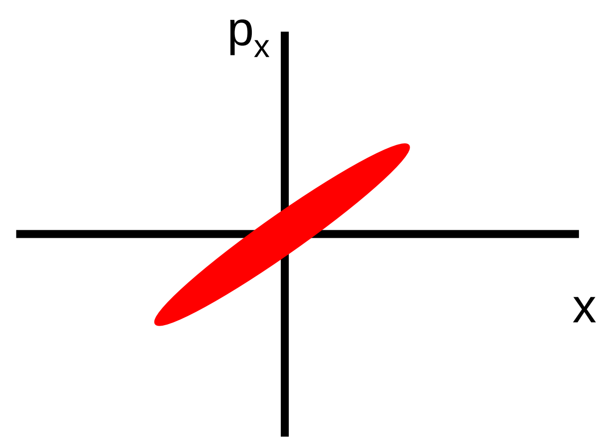
Warning: We often use the term phase space for the 6N dimensional space defined by x, x' (space, angle), but this the "trace space" of the particles.
At constant energy phase space and trace space have similar physical interpretation

Trace space



$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta_s}$$

Phase space



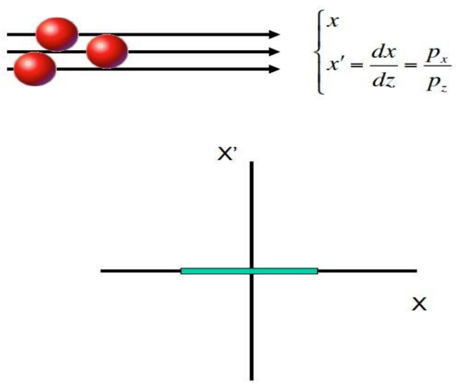
$$p_x = m_o \, c \, \gamma_{rel} \, \beta_x$$

An important argument to use the trace space is that in praxis we can measure angles of particle trajectories, but it is very difficult to measure the momentum of a particle.

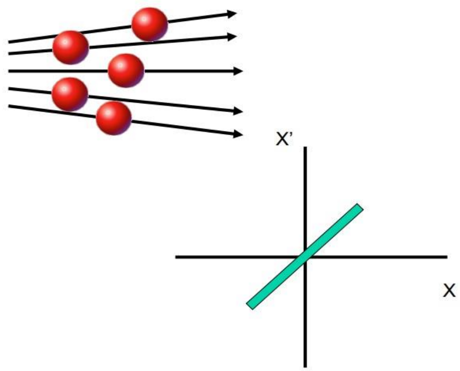
Most important beam parameters

4) beam size …the most complex part!

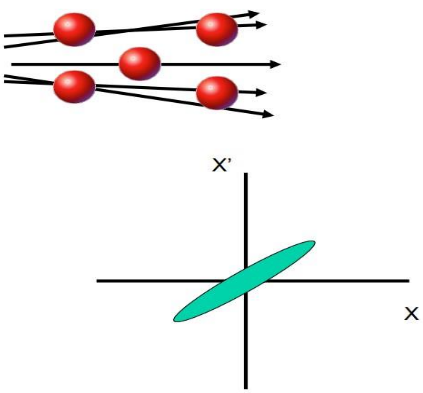
Description of beams in **trace space**:= space – angle coordinate system



Ideal beam

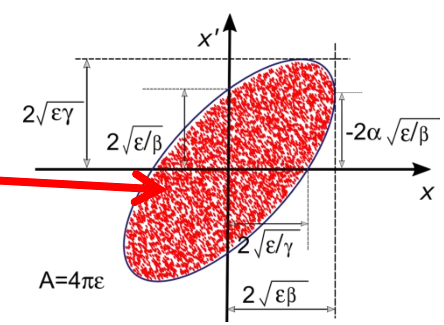


laminar beam



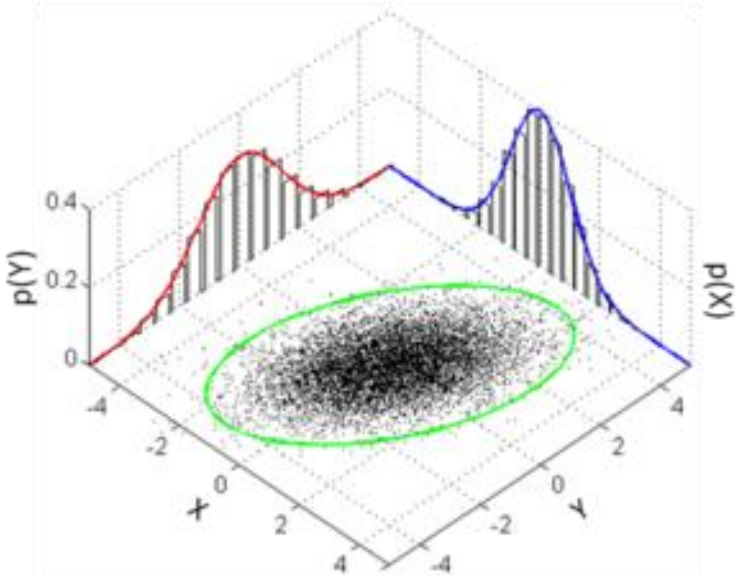
non-laminar beam

Describe real beam by its surface in Phase/trace space:= geometrical emittance



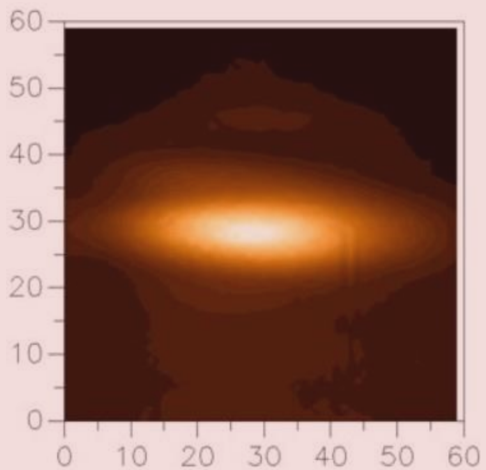
!! In a conservative system (energy conservation) the beam emittance is preserved !!
← Based on Liouville's theorem

What do we normally measure from the phase-space ellipse?



- At a given location in the accelerator we can measure the position of the particles, normally it is difficult to measure the angle…so we measure the projection of the phase space ellipse onto the space dimension:
  →called a profile monitor

Attention! The standard 2 D image of a synchrotron light based beam image is NOT a phase space measurement



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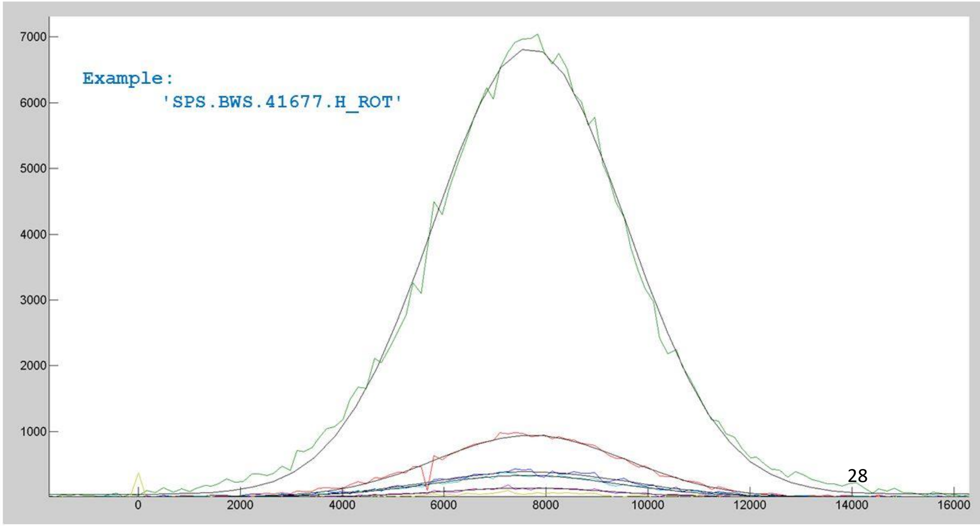


Constants of motion always give new means of describing particle motion: “action”-functional S

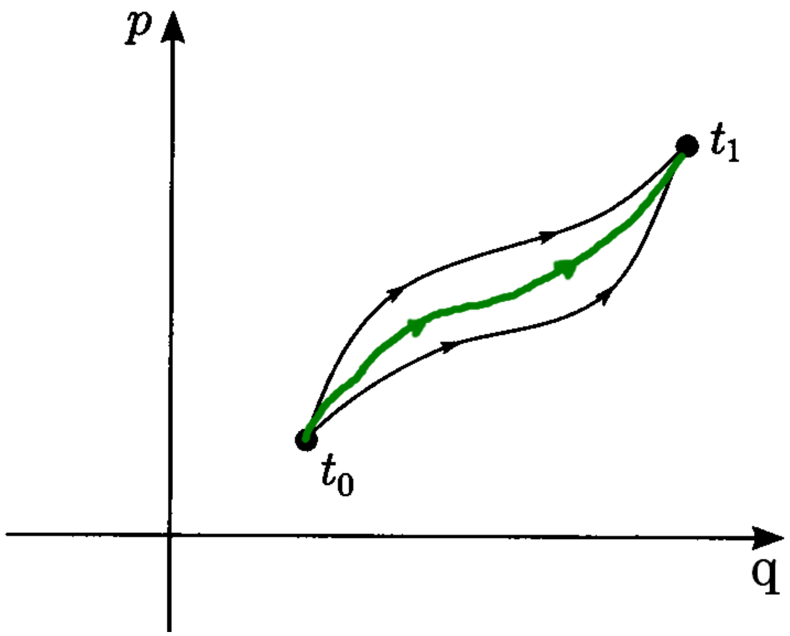
Define action S:= $\int_{t_1}^{t_2} p\, dq$

No immediate physical interpretation of S

Much more important:

“Stationary” action principle:=
Nature chooses path from t_1 to t_2 such that the action integral is a minimum and stationary

→ we have a new invariant, which we can use to study the dynamics of the system



Different Mathematical descriptions…a real pain?

We use differential equations, matrices,maps, tensors, Hamiltonians

- Is there a right or wrong?
- Is it personal likings?

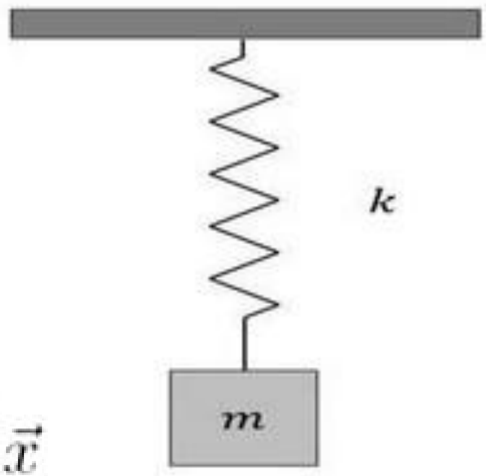
→ Depending on the problem to solve (or the phenomenon to describe) one mathematical tool is more adequate than the other.

→ One should be aware of many of them in order to be able to choose the most adequate one.

In the following slides we will look at the very simple example of the classical spring-oscillator and describe it with a differential equation, with a matrix formalism and by using the Hamiltonian equations of motion.

Harmonic oscillator (1/3)

Solved by using a **Differential equation**



Starting from:
Newton's Kraftansatz (F = m * a)   and
Hook's law (F = - k * x)

$$\vec{F} = m \cdot \vec{a} = -k \cdot \vec{x} \quad \text{or} \quad \ddot{\vec{x}} = \frac{k}{m}\vec{x}$$

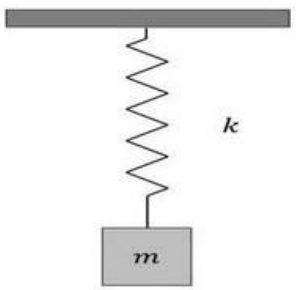
As at school we "guess" the solution:

$$x(t) = A_0 \cdot \cos \omega t$$

And we find that with the angular frequency $\omega = \sqrt{\frac{k}{m}}$
We have found a description of the motion of
our system.

Harmonic oscillator (2/3)

Solved by using a **matrix formalism**



The general solution to the previous differential equation is a linear combination of a cosinus- and a sinus-term.
So after an additional differentiation we get:

$$x(t) = A_c \cdot \cos\omega t + A_s \cdot \sin\omega t$$
$$\dot{x}(t) = -\omega A_c \cdot \sin\omega t + \omega A_s \cdot \cos\omega t$$

Furthermore we have to introduce initial conditions x(0) = x_0 and $\dot{x}(0) = \dot{x}_0$ and the classical momentum $p = m \cdot \dot{x}$; ($p_0 = m \cdot \dot{x}_0$) which then yields:

$$x(t) = A_c \cdot \cos\omega t + A_s \cdot \sin\omega t$$
$$p(t) = -m\omega A_c \cdot \sin\omega t + p_0 \cdot \cos\omega t$$

By comparing coefficients we get $A_c = x_0$ and $A_s = p_0/m\omega$, which finally produces:

$$x(t) = x_0 \cdot \cos\omega t + \frac{p_0}{m\omega} \cdot \sin\omega t$$
$$p(t) = -m\omega x_0 \cdot \sin\omega t + p_0 \cdot \cos\omega t$$

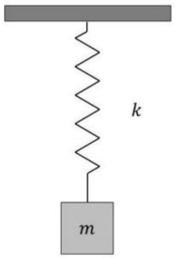
or in matrix annotation:

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \frac{1}{m\omega}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

So we can stepwise develop our solution from a starting point x_0, p_0

Harmonic oscillator (3/3)

$$H = T + V = \frac{1}{2}\text{k}\,x^2 + \frac{p^2}{2m} = \text{E}$$



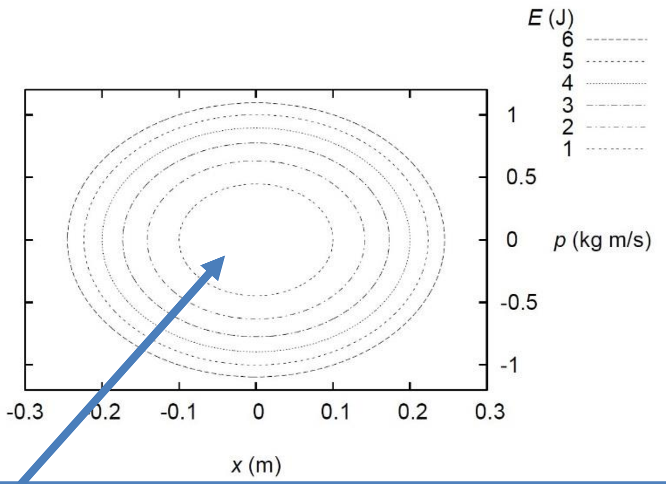
Hamiltonian formalism

Hamiltonian formalism to obtain the equations of motion:

$$\frac{\delta x}{\delta t} = \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \text{ or } p = m\dot{x} = mv$$

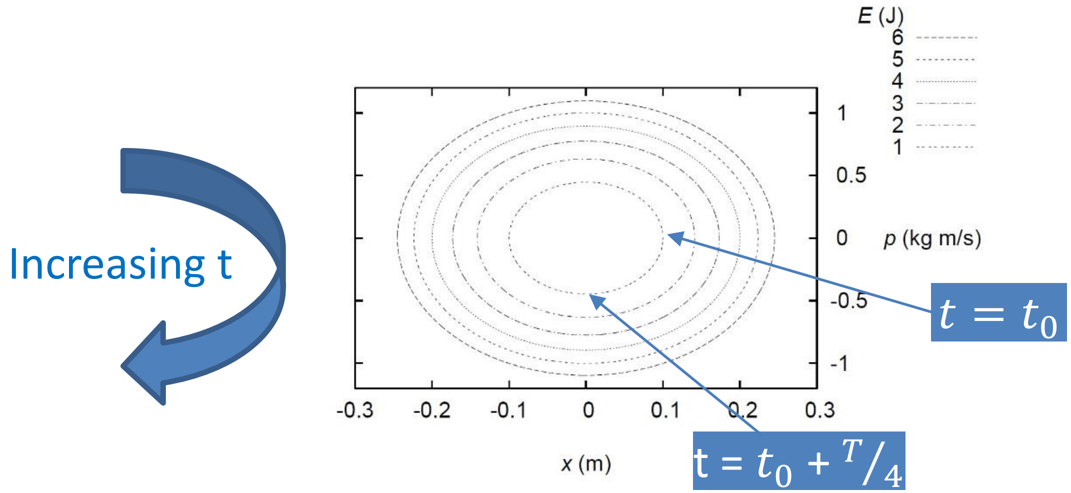
$$\frac{\delta p}{\delta t} = \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

This brings us back to the differential equation of solution 1:
$F = ma = m\ddot{x} = -kx$
With the well known "guessed" sinusoidal solution for x(t).



Instead of guessing a solution for x(t) we look at the trajectory of the system in phase space.
In this simple case the Hamiltonian itself is the equation of an ellipse.

Outlook on Hamiltonian treatments



- In the example, the free parameter along the trajectory is time (we are used to express the space-coordinate and momentum as a function of time)
- This is fine for a linear one-dimensional pendulum, but it is not an adequate description for transverse particle motion in an accelerator.
  → we will choose "s", the path length along the particle trajectory as free parameter
- Any linear motion of the particle between two points in phase space can be written as a matrix transformation: $\begin{pmatrix} x \\ x' \end{pmatrix}(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}(s_0)$
- In matrix annotation we define an action "J" as product $J := \frac{1}{2} \begin{pmatrix} x \\ x' \end{pmatrix}(s) \begin{pmatrix} x \\ x' \end{pmatrix}(s_0)$.
- J is a motion invariant and describes also an ellipse in phase space. The area of the ellipse is $2\pi J$

Why all this? This somewhat mathematically more complex approach allows us **more complex systems**. The focus on **motion invariants** will give us access to important beam observables (ex: emittance)

Why CAS focuses on "Hamiltonian" treatment?

- Why not just Newton's law and Lorentz force?
  Newton requires rectangular coordinates and time ; for curved trajectories one needs to introduce "reaction forces".
- Several people use Hill's equation as starting point, but
  - always needs an "Ansatz" for a (periodic) solution:

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right) x = 0 \qquad \frac{d^2y}{ds^2} + k_1(s)\, y = 0$$

  No real accelerator is built fully periodically
  - Hill's equation follows directly out of a simplified Hamiltonian description

  - no direct way to extend the treatment to non-linearities

- Hamiltonian equations of motion are two systems of first order <-> Lagrangian treatment yields one equation of second order.
- Hamiltonian equations use the canonical variables p and q,
  Lagrangian description uses q and $\partial q / \partial t$ and t
  p, q are independent, the others not.

Step by step through the accelerator

- From each point in an accelerator we can come to the next point by applying a map (or in the linear case a matrix).

$\binom{x}{x'}(s)= \mathrm{M} \binom{x}{x'}(s_0)$ Linear case: $\binom{x}{x'}(s)= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \binom{x}{x'}(s_0)$

- The map M must be symplectic ← energy conservation
- The maps can be calculated from the Hamiltonian of the corresponding accelerator component.
- We “know” the Hamiltonian for some specific accelerator components (drift, dipole, quadrupole…)
- This way we generate a piecewise description of the accelerator instead of trying to find a general continuous mathematical solution.

This is ideal for implementation in a computer code.

- It needs some complex mathematical framework to be able to derive the formalism on how to get symplectic maps from the Hamiltonian.

Map for quadrupole

- Consider the 1D quadrupole Hamiltonian

$$H = \frac{1}{2}(k_1 x^2 + p^2)$$

- For a quadrupole of length L, the map is written as

$$e^{\frac{L}{2}:(k_1 x^2 + p^2):}$$

- Its application to the transverse variables is

This course !

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}x = \sum_{n=0}^{\infty}\left(\frac{(-k_1L^2)^n}{(2n)!}x - L\frac{(-k_1L^2)^n}{(2n+1)!}p\right)$$

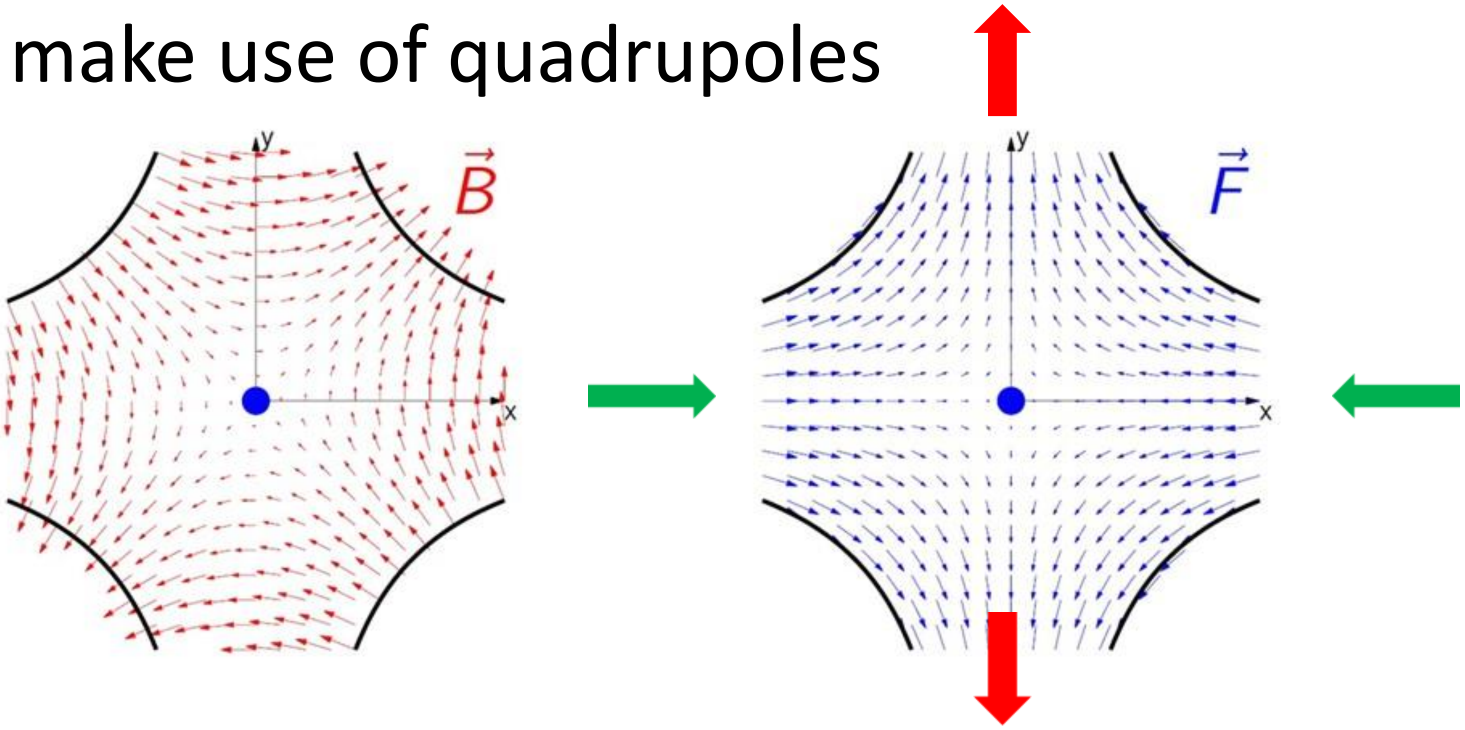
$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}p = \sum_{n=0}^{\infty}\left(\frac{(-k_1L^2)^n}{(2n)!}p - \sqrt{k_1}\frac{(-k_1L^2)^n}{(2n+1)!}p\right)$$

- This finally provides the usual quadrupole matrix

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}x = \cos(\sqrt{k_1}L)x + \frac{1}{\sqrt{k_1}}\sin(\sqrt{k_1}L)p$$
$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}p = -\sqrt{k_1}\sin(\sqrt{k_1}L)x + \cos(\sqrt{k_1}L)p$$

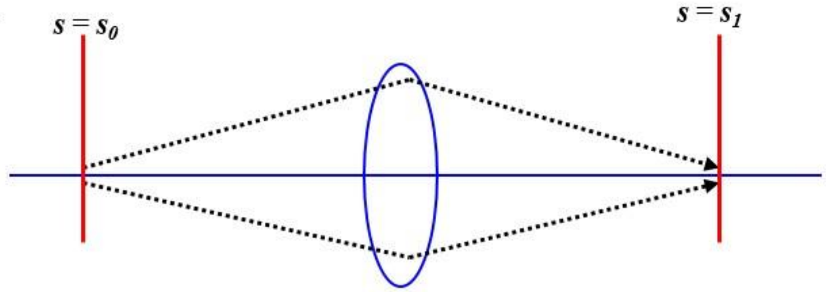
Let's focus!

Longitudinal focusing → lecture of Frank
Transverse focusing
→ make use of quadrupoles



But: **focusing in one plane, defocusing in other plane**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

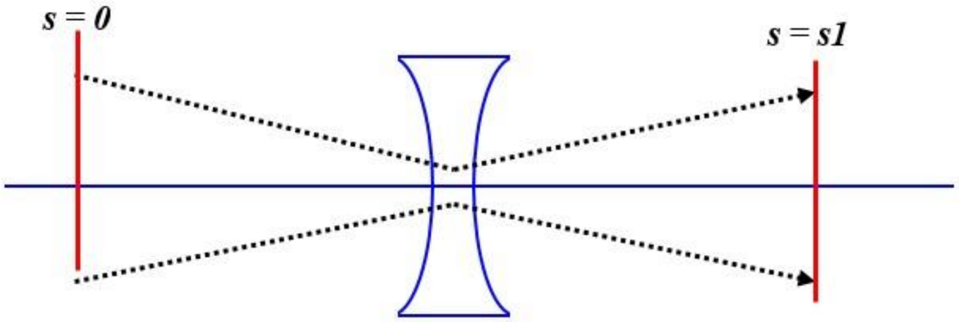
limes: $l_q \rightarrow 0$ *while keeping* $k\, l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Negative = focusing

The negative sign in the Hamiltonian makes the same quadrupole defocusing in the other plane.

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$



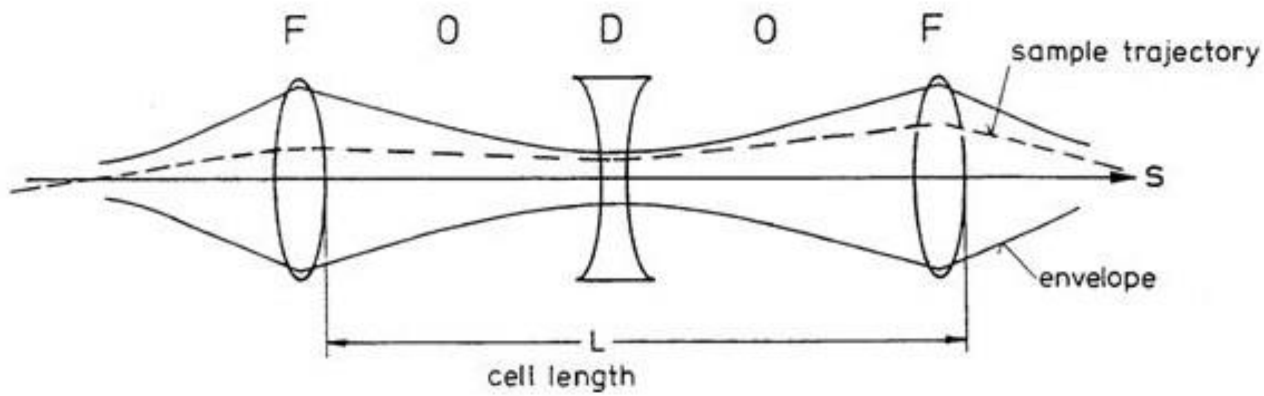
$$f = \frac{1}{kl_q} >> l_q$$ *... focal length of the lens is much bigger than the length of the magnet*

limes: $l_q \rightarrow 0$ *while keeping* $k\,l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Positive = defocusing

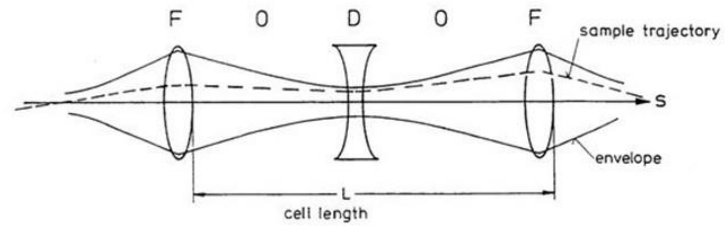
Consider an alternating sequence of focussing (F) and defocussing (D) quadrupoles separated by a drift (O)



The transfer matrix of the basic FODO cell reads

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{L}{2f} & L\left(1 + \frac{L}{4f}\right) \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

Strong transverse focusing (FODO)



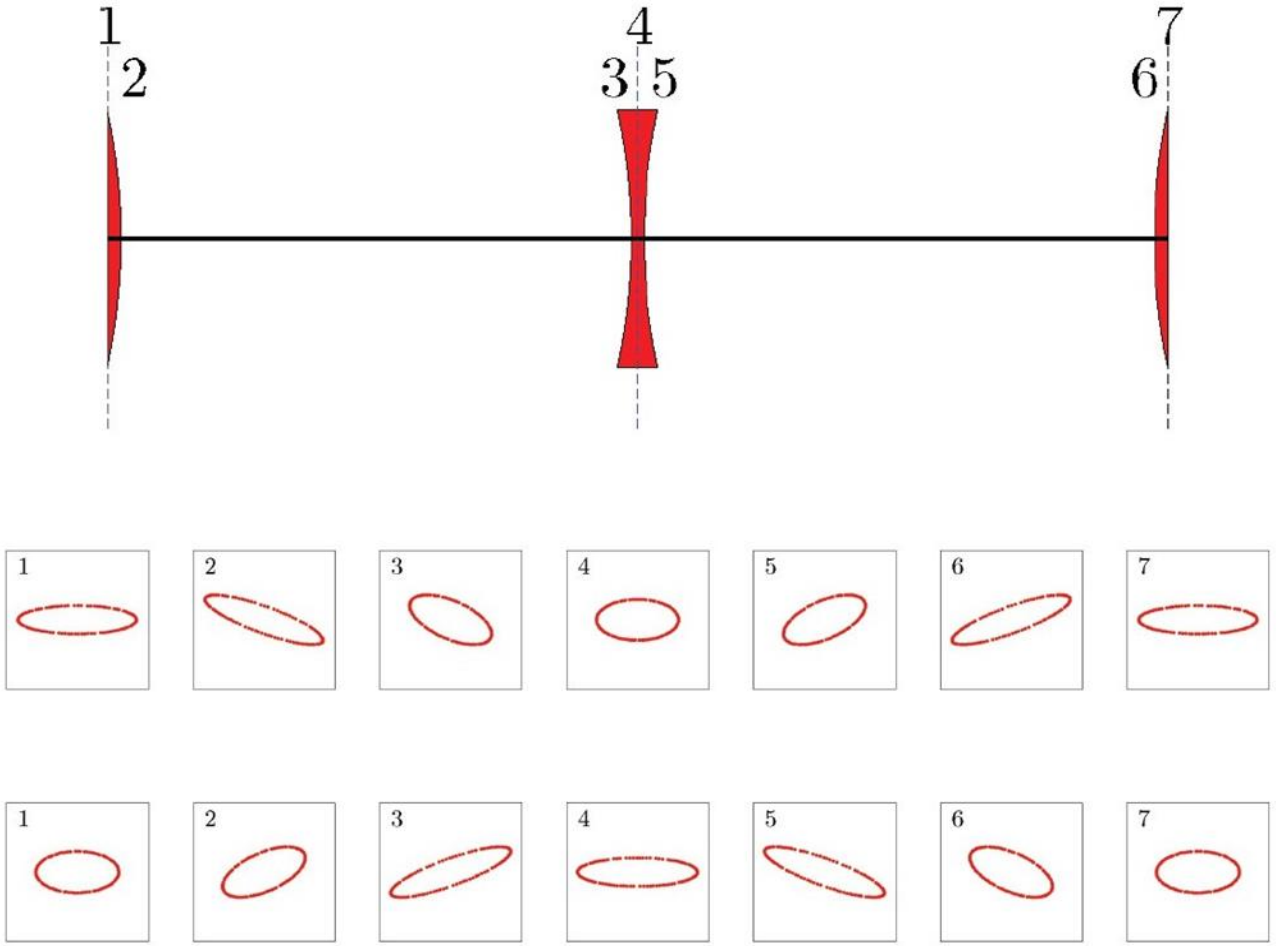
In order to calculate numbers one usually defines a FODO cell from the middle of the first F-quadrupole up to the middle of the last F-quadrupole. Hence the resulting transfer matrix looks:

M= $M_Q(2f_0)\cdot M_D(L)\cdot\ M_Q(-f_0)\cdot M_D(L)\cdot\ M_Q(2f_0)$

$$\begin{pmatrix} 1-\frac{L^2}{2f_0^2} & \frac{L}{f_0}(L+2f_0) & 0 & 0 & 0 & 0 \\ \frac{L}{4f_0^3}(L-2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f_0^2} & -\frac{L}{f_0}(L-2f_0) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f_0^3}(L+2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Negative = overall focusing

Evolution of the Phase Space Ellipse in a FODO Cell



“Bending” a transfer line to make a synchrotron

The previous example can easily be extended to several consecutive FODO cells. This describes very well a regular transport line or a linac (in which we have switched off the cavities).

If we add dipoles into the drift-spaces, the situation for the transverse particle motion does not change

So actually with the previous description we also describe a very simple regular synchrotron.
The phase space ellipse (action J) we can compute provided we know the total transfer map (matrix) M_{tot}: (C:= circumference of accelerator)

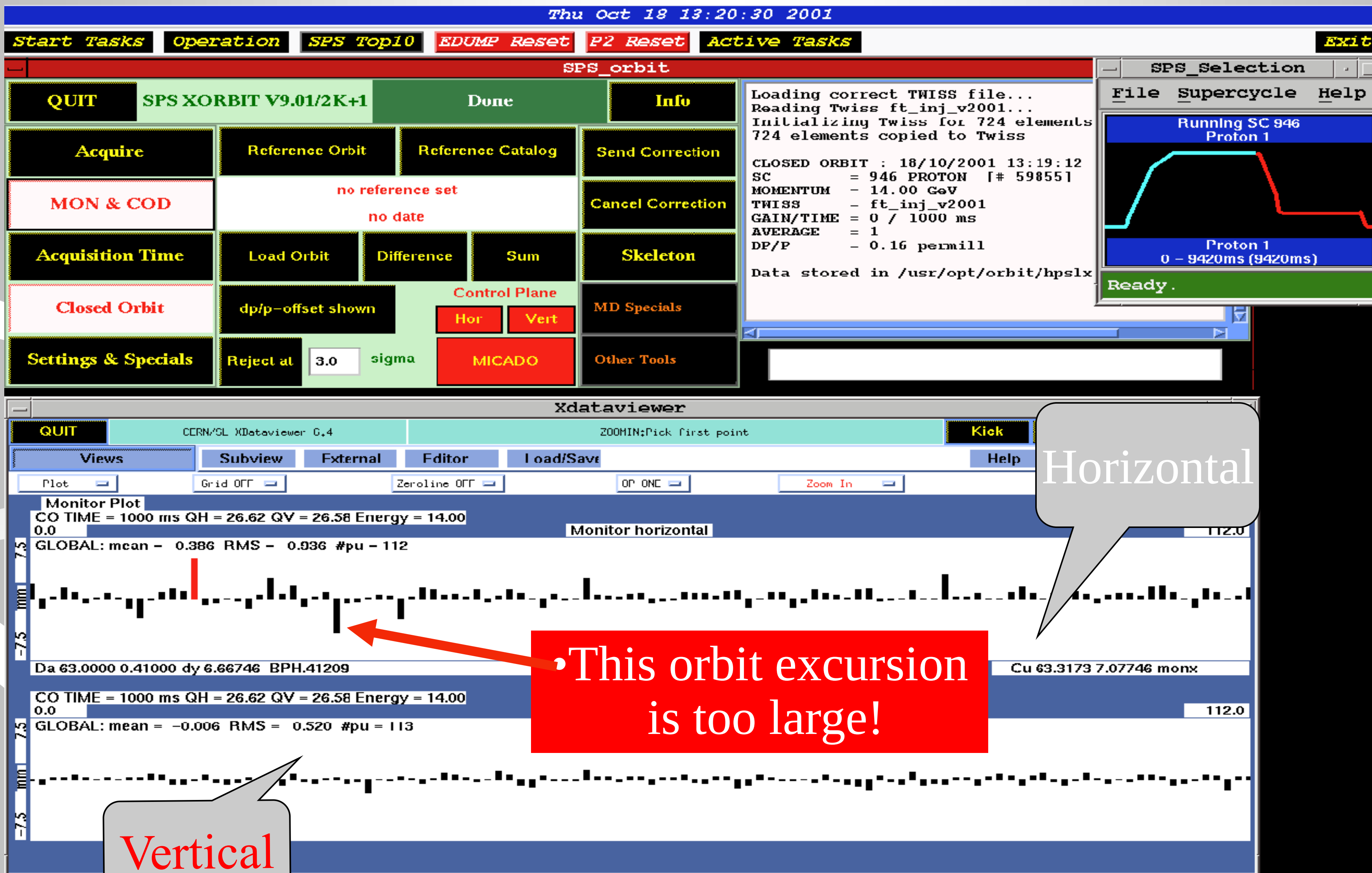
$$J = \frac{1}{2}\binom{x}{x'}(s_0)\ \binom{x}{x'}(s_0 + C) = \frac{1}{2}\binom{x}{x'}(s_0)\ \mathrm{Mtot}\ \binom{x}{x'}(s_0\ )$$

The phase space plots will look qualitatively the same as in the previous case.

Definition: trajectory (single passage) or closed orbit (multiple passages): (1)

Fix point of the transfer matrix…in our cases so far the “0” centre of all ellipses.

Orbit Acquisition

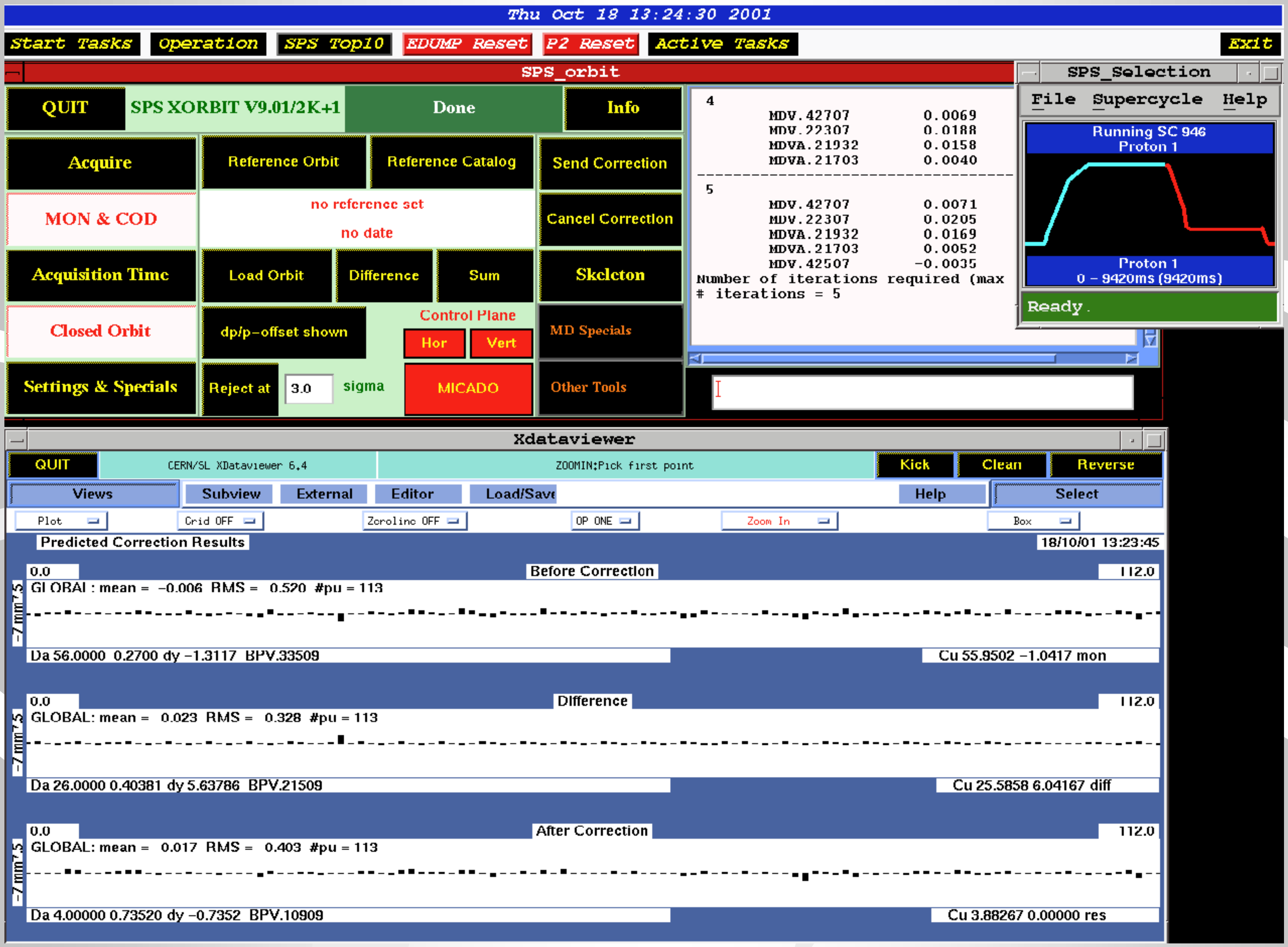


Horizontal

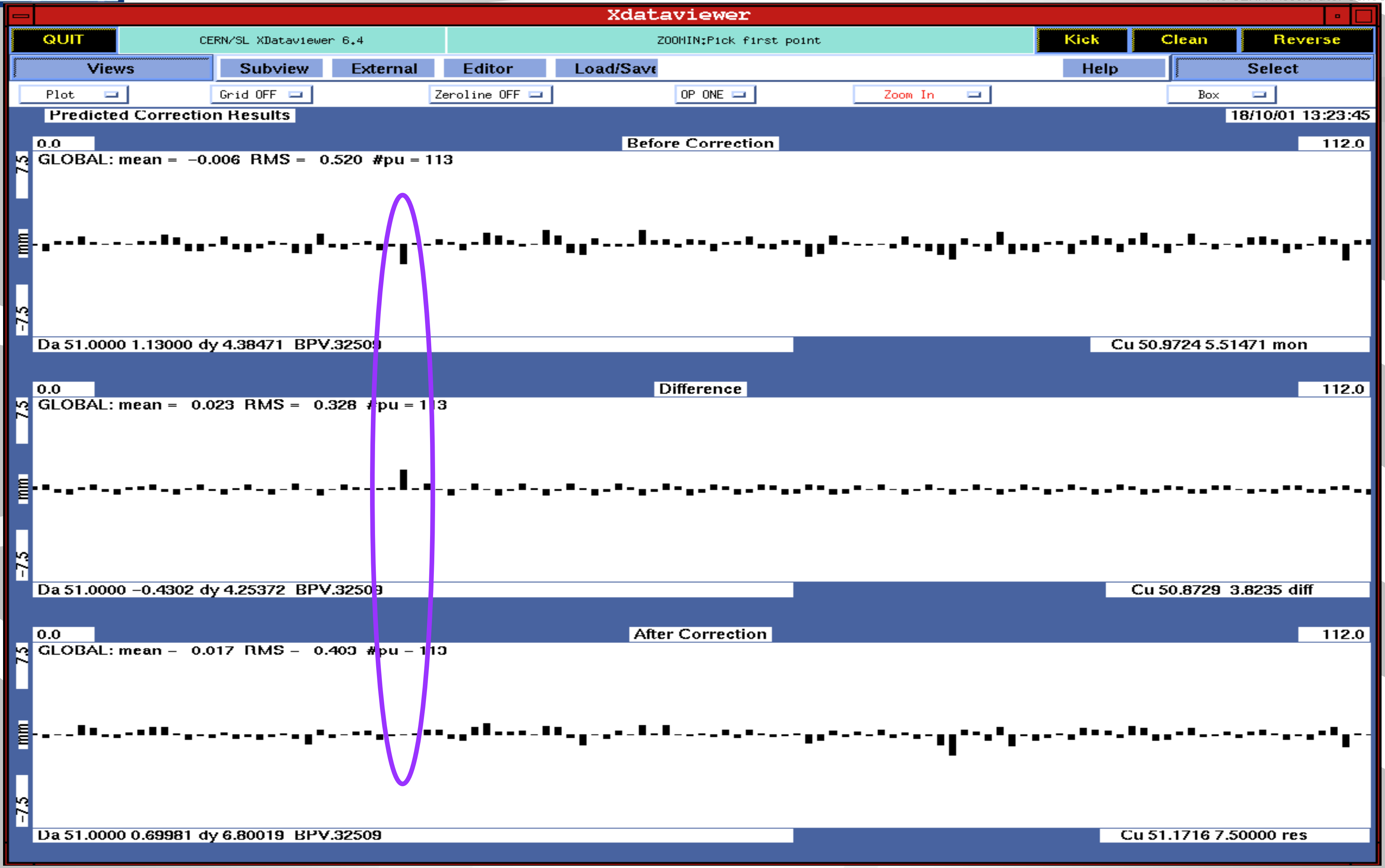
• This orbit excursion is too large!

Vertical

Orbit Correction (Operator Panel)



Orbit Correction (Detail)



Courant – Snyder formalism / Twiss parameters

- Same beam dynamics
- Introduced in the late 50's
- The classical way to parametrize the evolution of the phase space ellipse along the accelerator

Basic concept of this formalism:

1) Write the transfer matrix in this form (2 dimensional case):

$$M = I\,cos\mu + S\cdot A\,sin\mu$$

I $=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$; $S = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$; A$=\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}$

2) M must be symplectic → $\beta\gamma - \alpha^2 = 1$

3) Four parameters: $\alpha(s);\ \beta(s);\ \gamma(s) and\ \mu(s)$, with one interrelation (2)
→ Three independent variables

4) Again, the preserved action variable J describes an ellipse in phase-space:

$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x\mathrm{p} + \beta p^2)$$

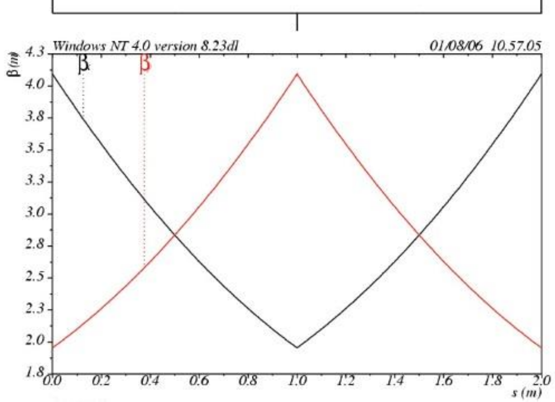
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} \qquad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

Once we know the transport matrix between individual places, we also know how the twiss parameters $\alpha(s)$; $\beta(s)$; $\mu(s)$ transform

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

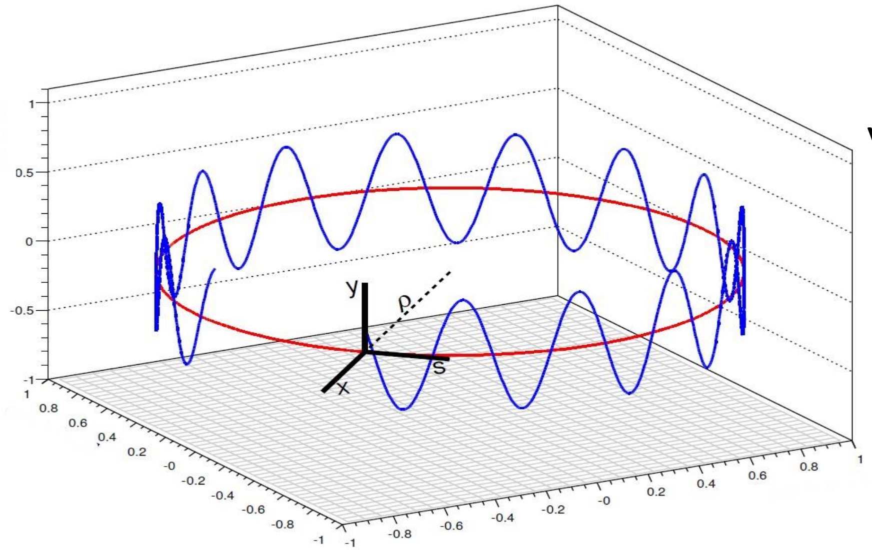
Interpretation of the Twiss parameters (1/2)

1) Horizontal and vertical beta function $\beta_{H,V}\ (s)$:

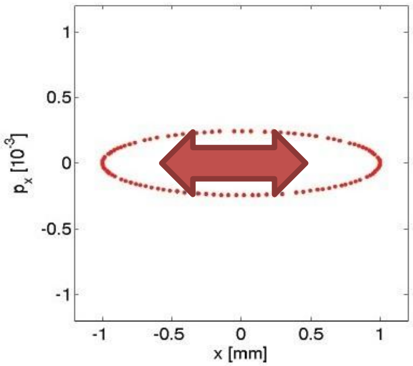


- Proportional to the square of the projection of the phase space ellipse onto the space coordinate
- Focusing quadrupole → low beta values

Although the shape of phase space changes along s, the rotation of the particle on the phase space ellipse projected onto the space co-ordinate looks like an harmonic oscillation with variable amplitude: called **BETATRON-Oscillation**



$$y(s) = const \cdot \sqrt{\beta(s)} \cdot cos\{\mu(s) + \varphi\}$$



Interpretation of the Twiss parameters (2/2)

2.) $\alpha = -\frac{1}{2}\frac{d\beta}{ds}$

α indicates the rate of change of β along s
α zero at the extremes of beta (waist)

3.) $\mu = \int_{s1}^{s2} \frac{1}{\beta} \mathrm{ds}$

Phase Advance: Indication how much a particle rotates in phase space when advancing in s

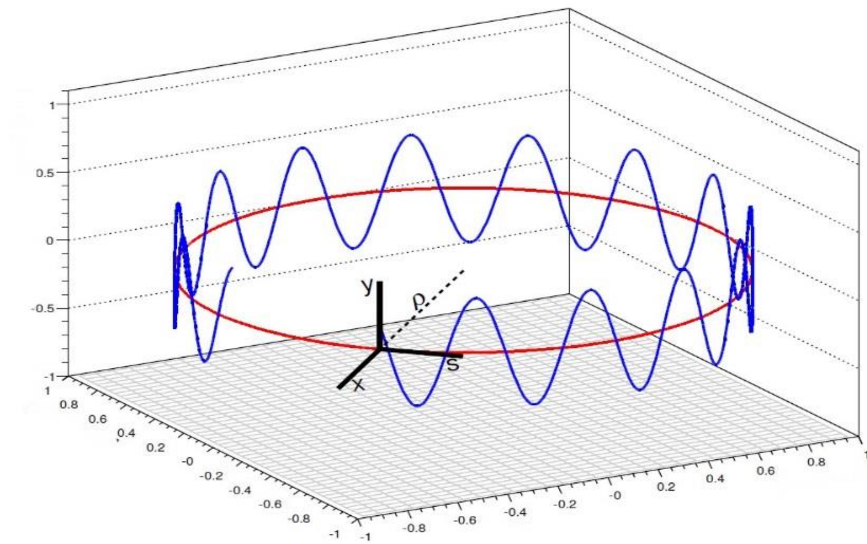
Of particular importance: Phase advance around a complete turn of a circular accelerator, called the betatron tune Q (H,V) of this accelerator

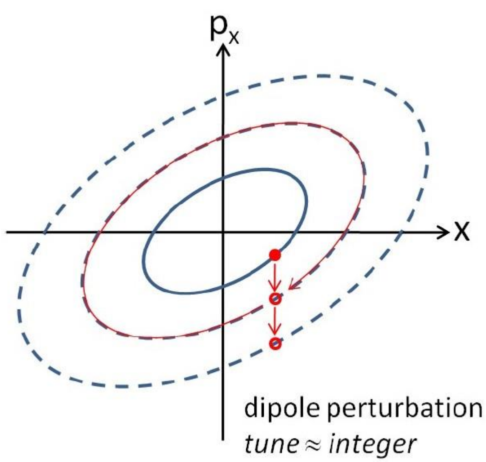
$$Q_{H,V} = \frac{1}{2\pi}\int_0^C \frac{1}{\beta_{H,V}}\, ds$$

The betatron tunes $Q_{H,V}$

- **One of the most important parameters of a circular accelerator**
- **It is the phase advance over one turn in each respective plane.**

- In large accelerators the betatron tunes are large numbers (LHC ˜ 65), i.e. the phase space ellipse turns about 65 times in one machine turn.
- We measure the tune by exciting transverse oscillations and by spectral analysis of the motion observed with one pickup.
  This way we measure the **fractional part of the tune; often called $q_{H,V}$**





- Integer tunes (fractional part= 0) lead to resonant infinite growth of particle motion even in case of only small disturbances.

Importance of betatron tunes

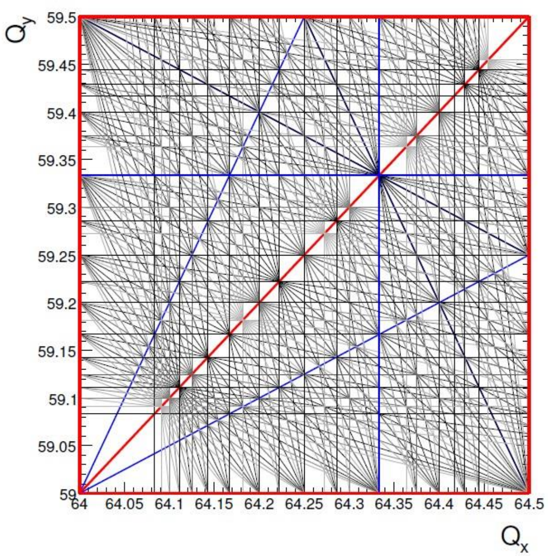
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy:

$$m_x \nu_x + m_y \nu_y = \ell,$$

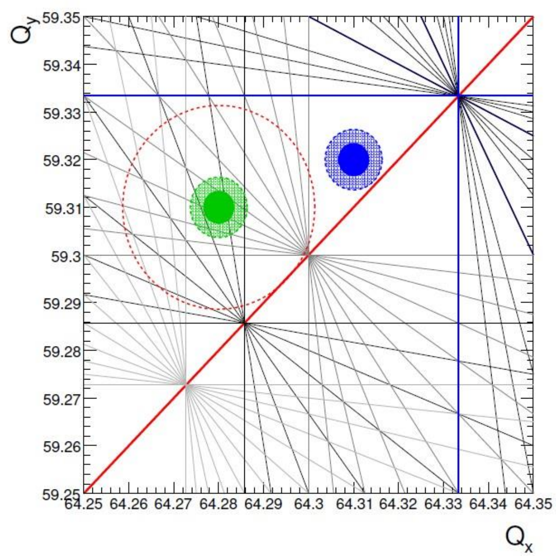
where m_x, m_y and ℓ are integers.

The order of the resonance is $|m_x| + |m_y|$.

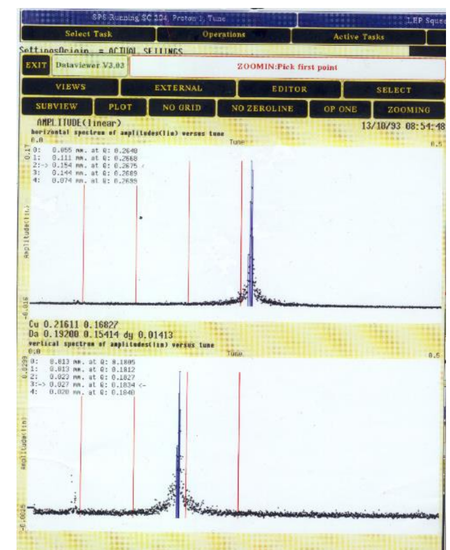
The couple (Q_H, Q_V) is called the working point of the accelerator. Below: tune measurement example from LEP



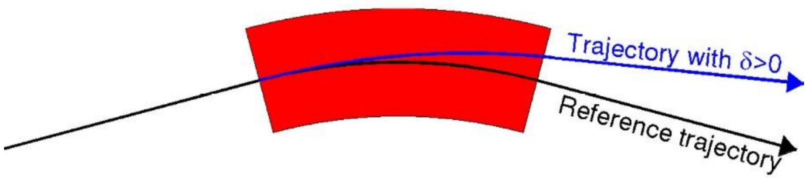
(a) Full tune diagram



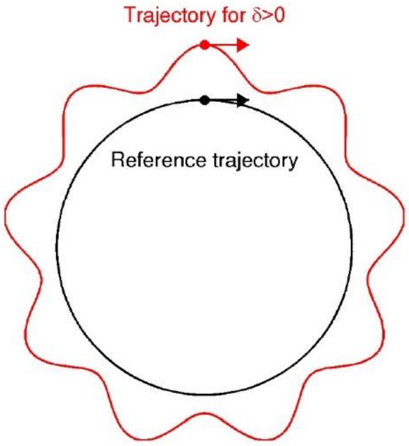
(b) Zoom around LHC Q working points



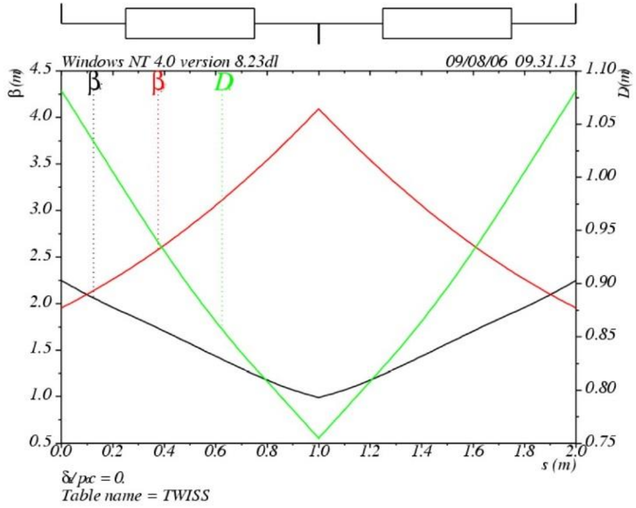
“off-momentum” particles in a synchrotron



What happens: A particle with a momentum deviation $\delta = \frac{\delta p}{p} > 0$ gets bent less in a dipole.



- In a weakly focusing synchrotron it would just settle to another circular orbit with a bigger diameter
- In an alternate gradient synchrotron it is more complicated: The focusing/defocusing is also dependent on the momentum, so the resulting orbit follows the optics of the accelerator.

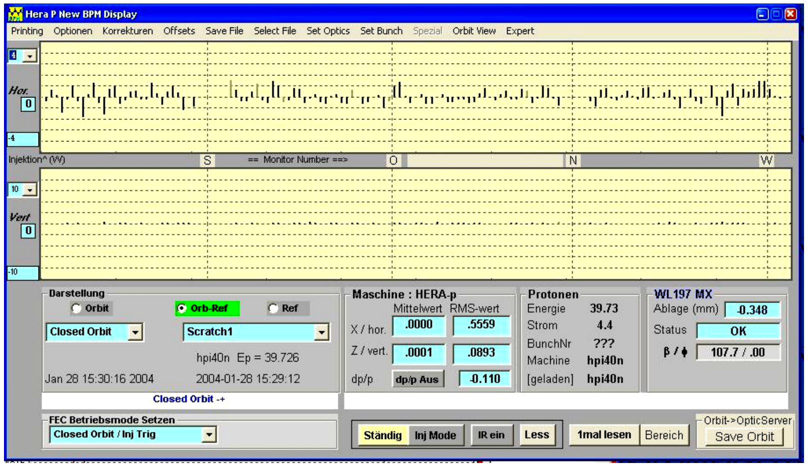


We describe the dispersion as a function of s as $D(s)$; the resulting position of a particle is thus simply:

$$x_{\delta p} = x_0 + D(s)\,\frac{\delta p}{p}$$

Typical values of D(s) are some meters, with $\frac{\delta p}{p} = 10^{-3}$ the orbit deviation becomes millimeters

Measurement example



HERA Standard Orbit

This gives also an example of an orbit measurement.

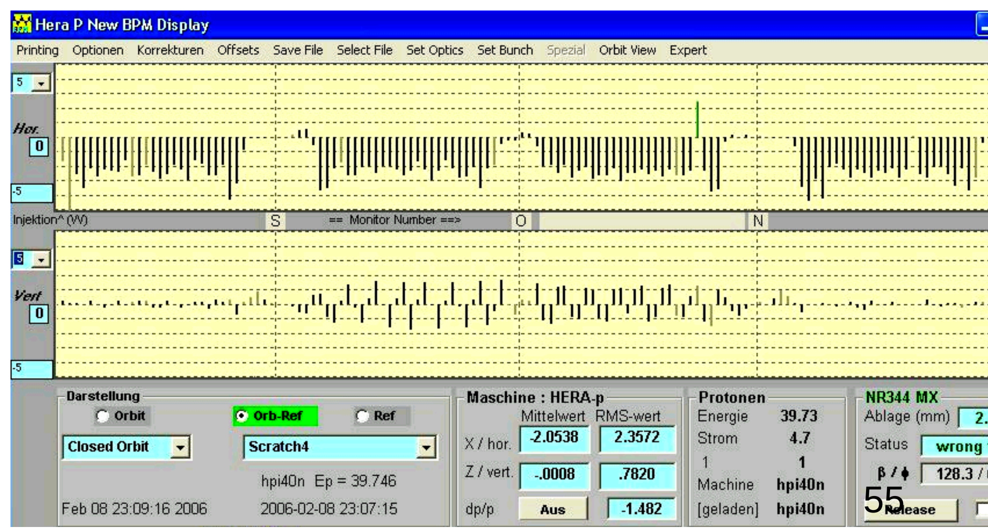
More on this: again R.Jones (BI)

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

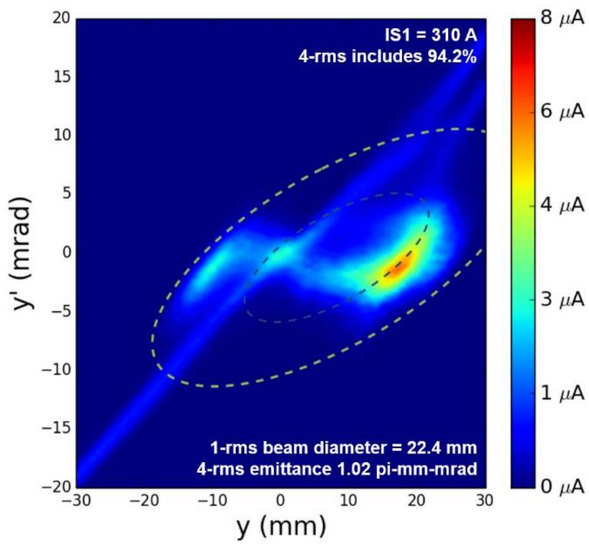
$$x_D = D(s) * \frac{\partial p}{p}$$



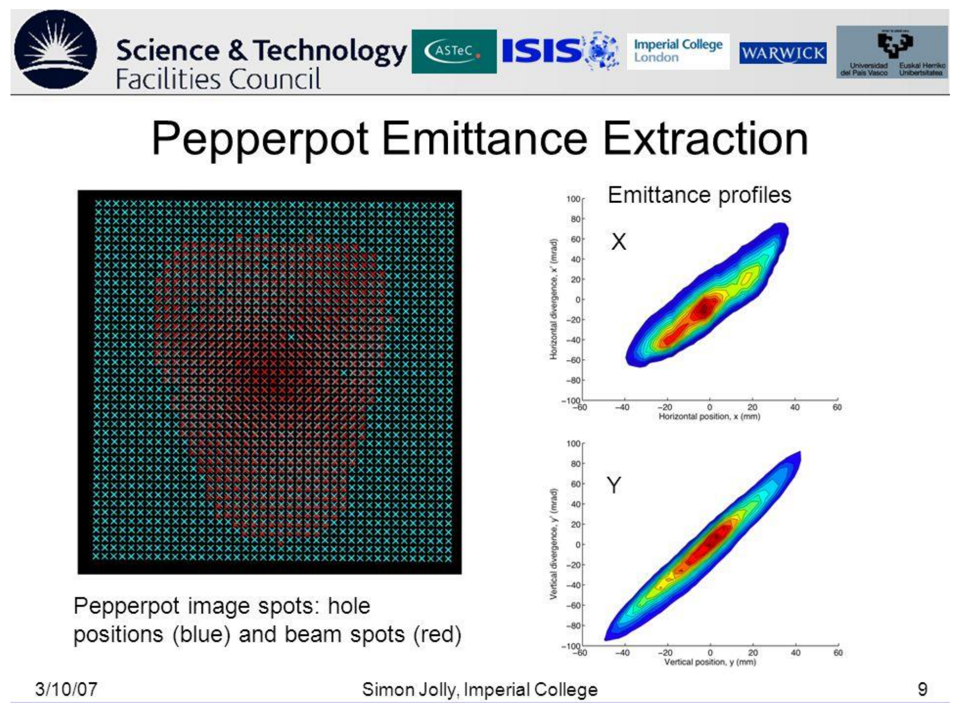
Putting in a beam

We focus on “bunched” beams, i.e. many (10^{11}) particles bunched together longitudinally (much more on this in the RF classes).

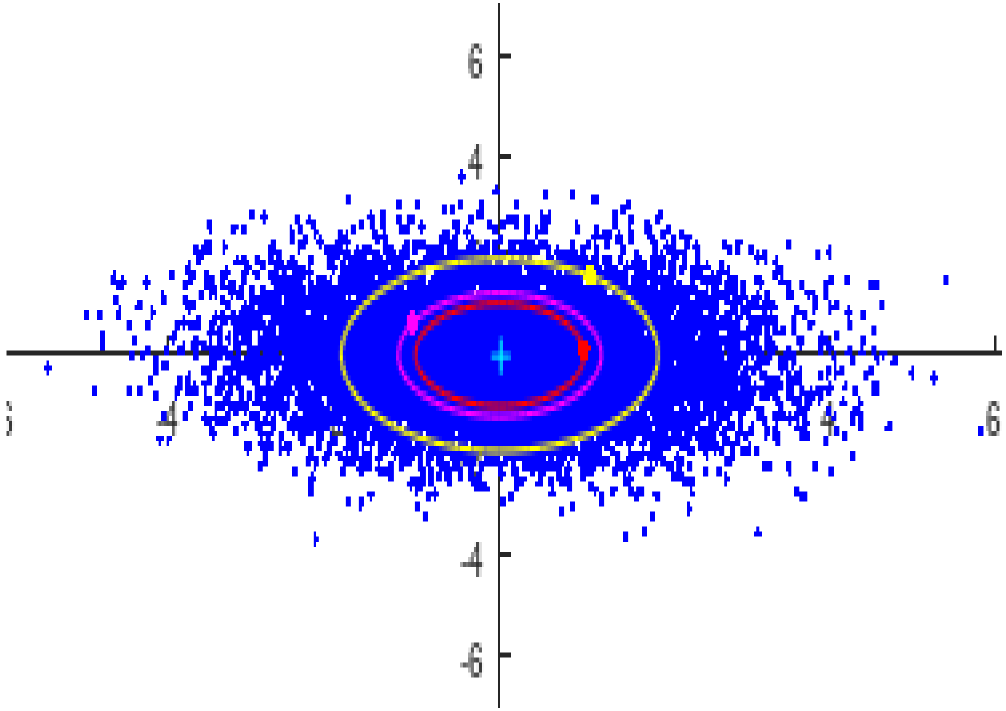
From the generation of the beams the particles have transversally a spread in their original position and momentum.



Source: ISODAR (Isotope at rest experiment)

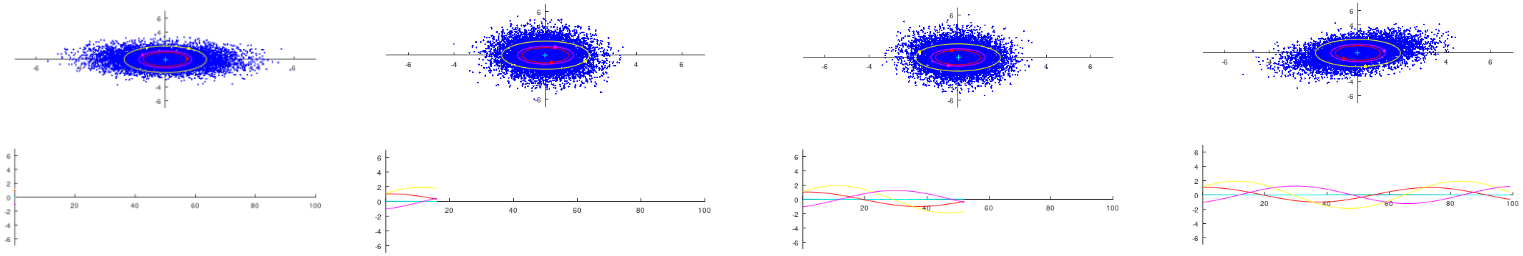


A beam (bunch): Motion of individual particles (1/4)

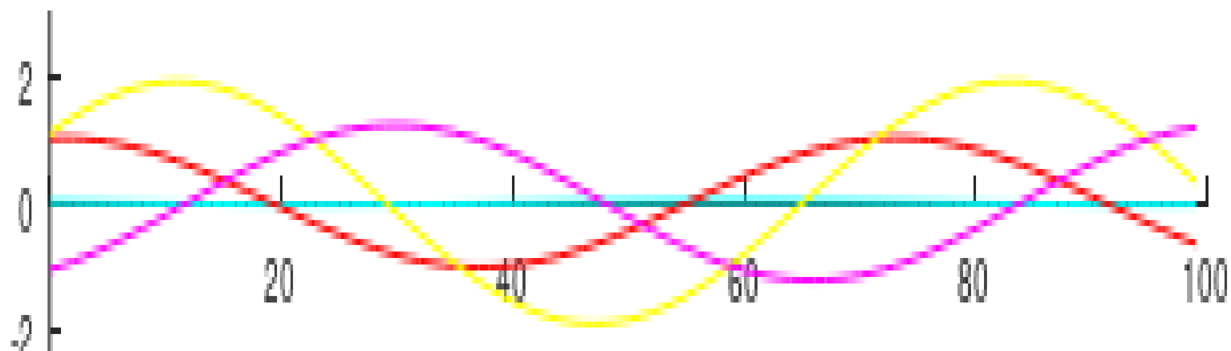


- Generate 10000 particle as a Gaussian distribution in x and p_x
- For illustration mark 3 particle in colours red, magenta and yellow
- The average (centre of charge) is indicated as cyan cross
- Make some turns (100 turns with 3 degrees phase advance par turn)

A beam (bunch): Motion of individual particles (2/4)



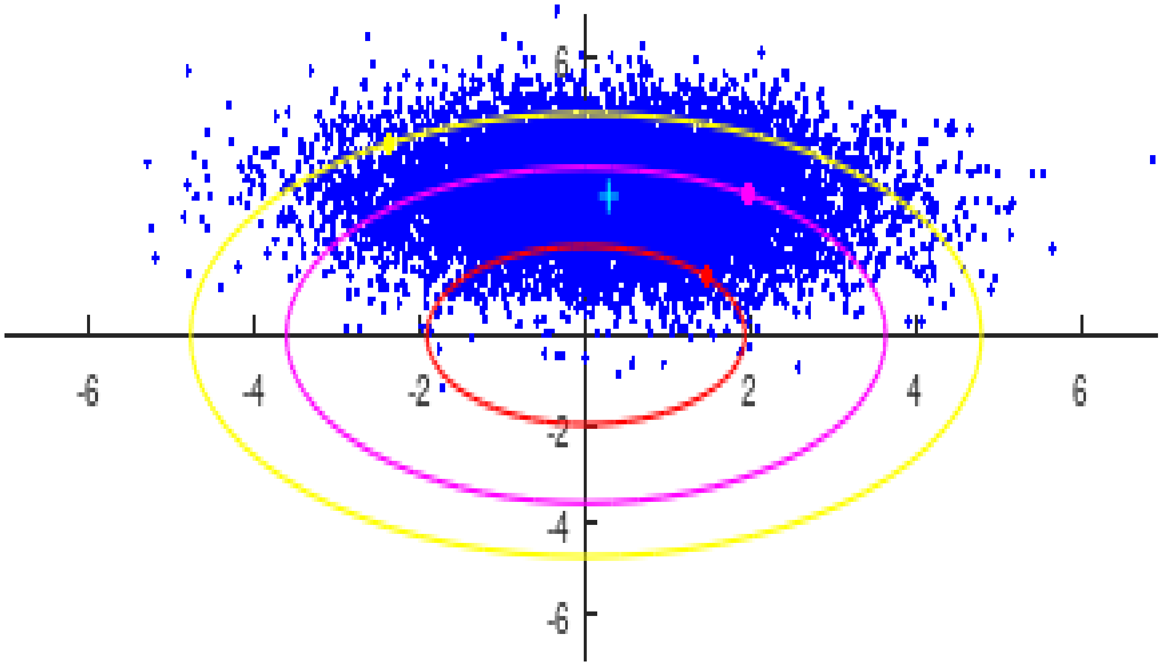
turn 0 | turn 10 | turn 53 | turn 100



Trajectory in x over 100 turns

Individual particles perform betatron oscillations (incoherently!), the whole beam is “quiet”. No coherent betatron motion.

A beam (bunch): Motion of individual particles (3/4)



- The whole bunch receives (at injection) a transverse kick (additional momentum q) of 2 units
- Tracing over 100 turns as before

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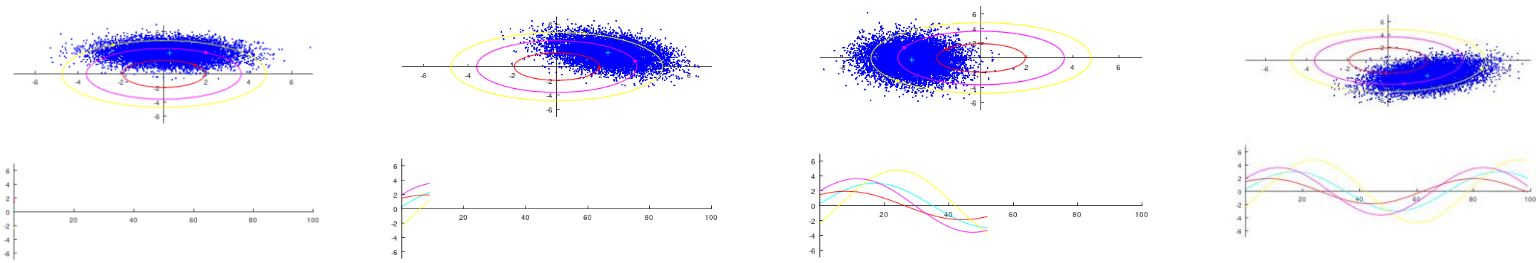
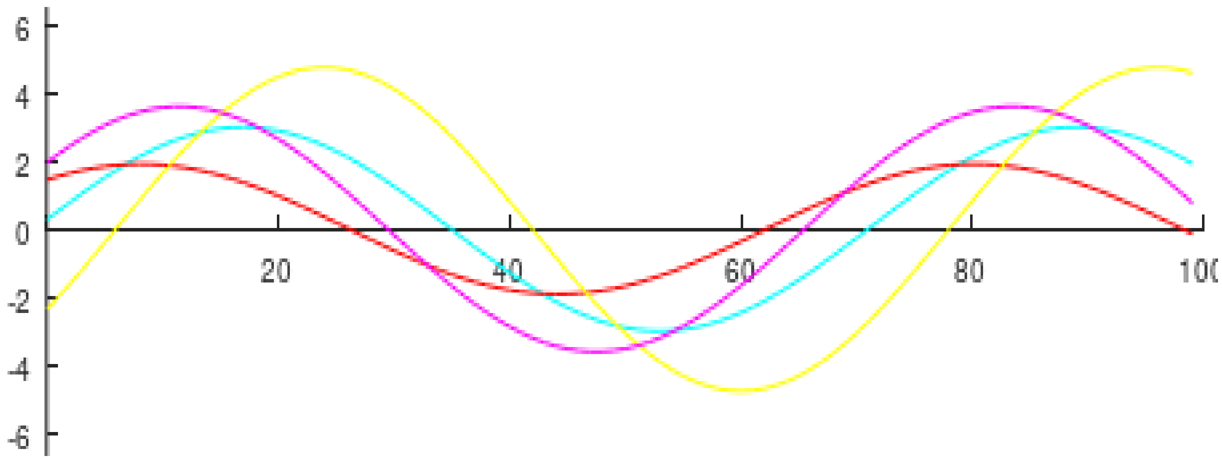
A beam (bunch): Motion of individual particles (4/4)



Turn 0 Turn 10 Turn 53 Turn 100



The incoherent motion of the particles remains the same, but this time the center of charge also moves (cyan curve). **The beam beforms a betatron oscillation.**

Technologies

- Magnets
- RF
- BI
- Kickers-Septa-Dumps
- Vacuum
- Power converters
- Control system
- Offline analysis/AI/modeling

In most cases we find isolated multipole magnets in an accelerator…not any arbitrary shapes of magnetic fields, but classified field types by making reference to a multipole expansion of magnetic fields:

In the usual notation:

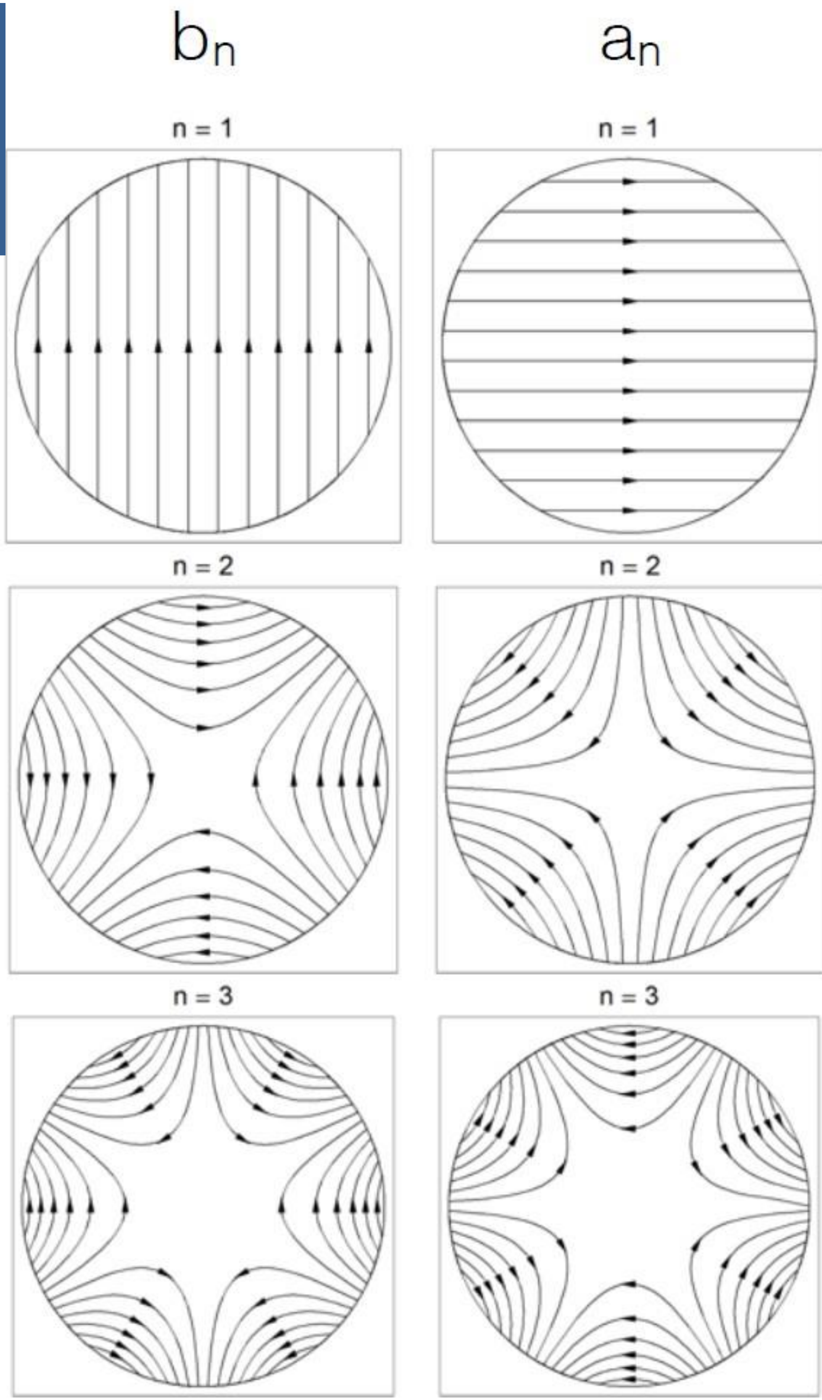
$$B_y + iB_x = B_{ref} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

b_n are "normal multipole coefficients" (LEFT)
and a_n are "skew multipole coefficients" (RIGHT)
'ref' means some reference value

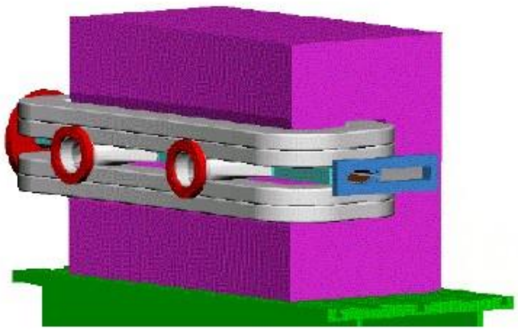
n=1, dipole field
n=2, quadrupole field
n=3, sextupole field

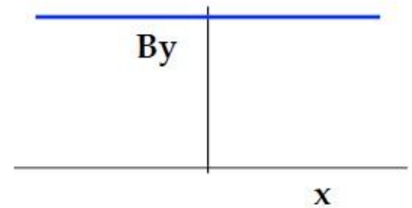
True in the rest of the world,
in the US n=0 dipole….!!!

Images: A. Wolski, https://cds.cern.ch/record/1333874



Multipole Magnets





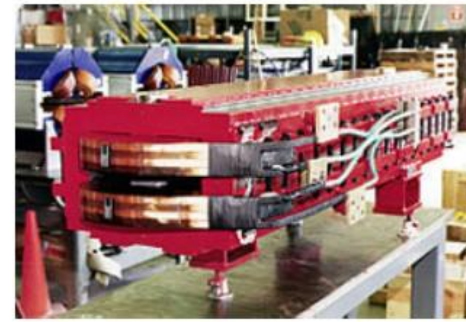
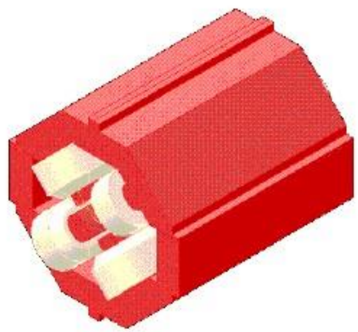


Image: Wikimedia commons



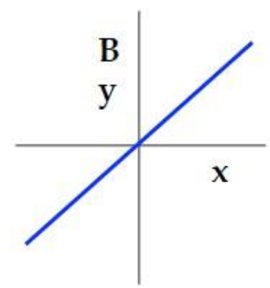
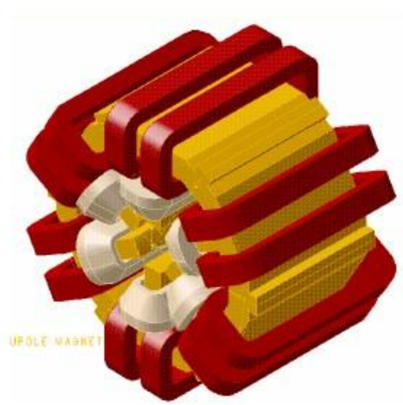
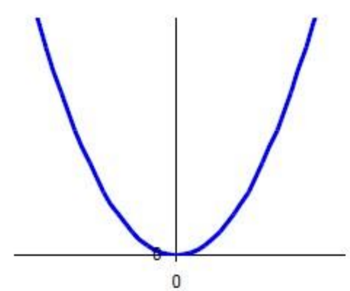


Image: STFC

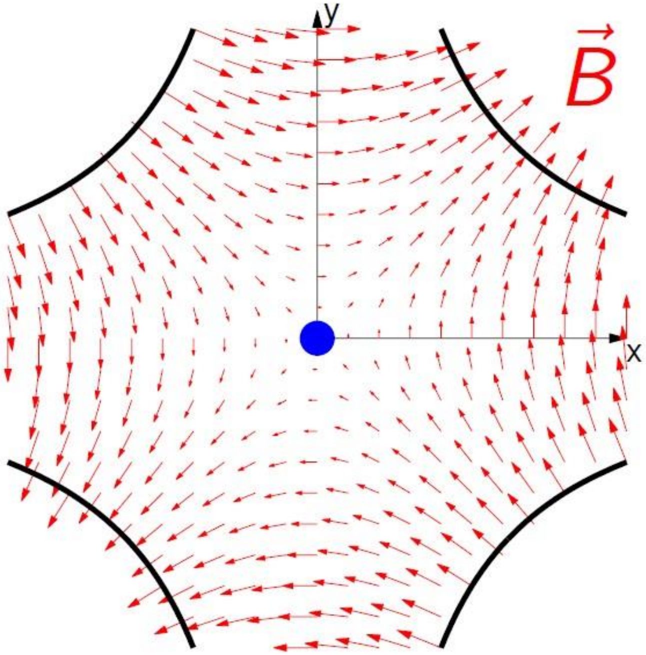


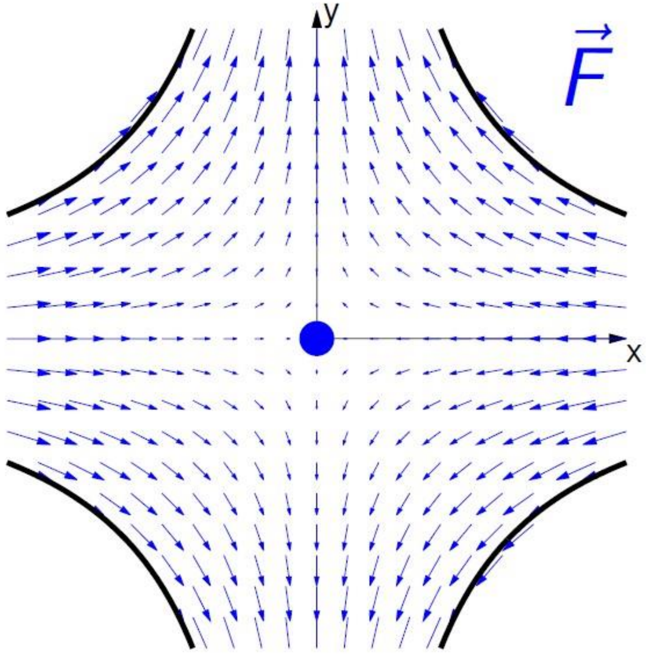


Images: Ted Wilson, JAI Course 2012

Image: Danfysik

Quadrupole Errors (1/2)

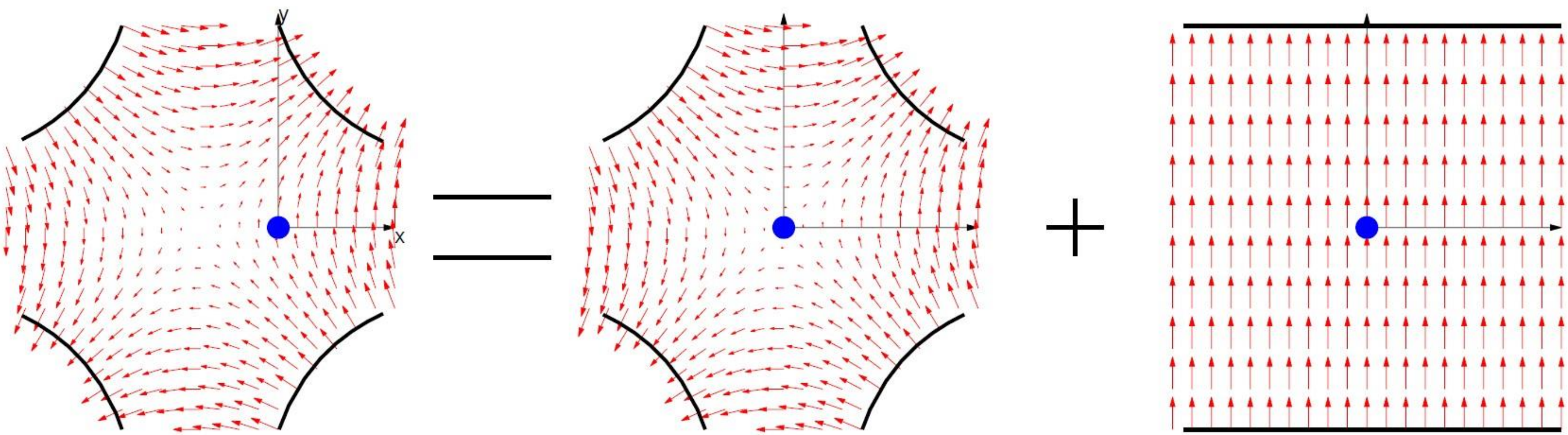




Note that $F_x = -kx$ and $F_y = ky$ making horizontal dynamics totally decoupled from vertical.

Quadrupole Errors 2/2

Error type	effect on beam	correction(s)
strength	Change in focusing, "beta-beating"	Change excitation current, Repair/Replace magnet
Lateral shift	Extra dipole kick	Excitation of a corrector dipole magnet
tilt	Coupling of the beam motion in the two planes	Excitation of a additional "skewed quadrupoles (45^0)



An offset quadrupole is seen as a centered quadrupole plus a dipole.

Magnet types, technological view

We can also classify magnets based on their technology

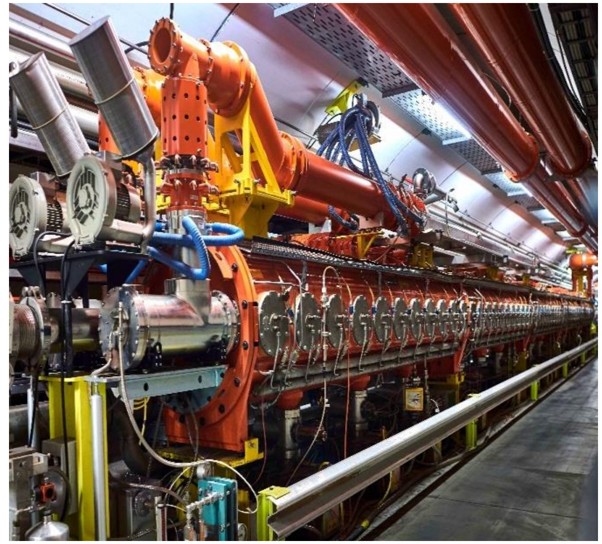
electromagnet	permanent magnet
iron dominated	coil dominated
normal conducting (resistive)	superconducting

static	cycled / ramped slow pulsed	fast pulsed

What is Radio Frequency (for accelerators)?

Source: en.wikipedia.org/wiki/Radio_spectrum

Band name	Abbreviation	ITU band number	Frequency and Wavelength
High frequency	HF	7	3–30 MHz 100–10 m
Very high frequency	VHF	8	30–300 MHz 10–1 m
Ultra high frequency	UHF	9	300–3,000 MHz 1–0.1 m
Super high frequency	SHF	10	3–30 GHz 100–10 mm
Extremely high frequency	EHF	11	30–300 GHz 10–1 mm



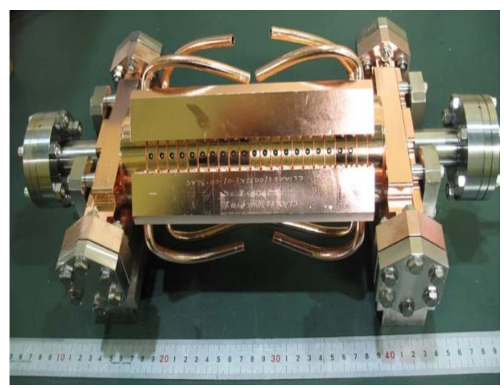
Travelling wave cavity, freq = 200 MHz
Total length: 12 & 16 m. (CERN SPS)

approx. 2 m

Accelerating Cavity, freq = 80 MHz
(CERN PS)

All pictures © CERN

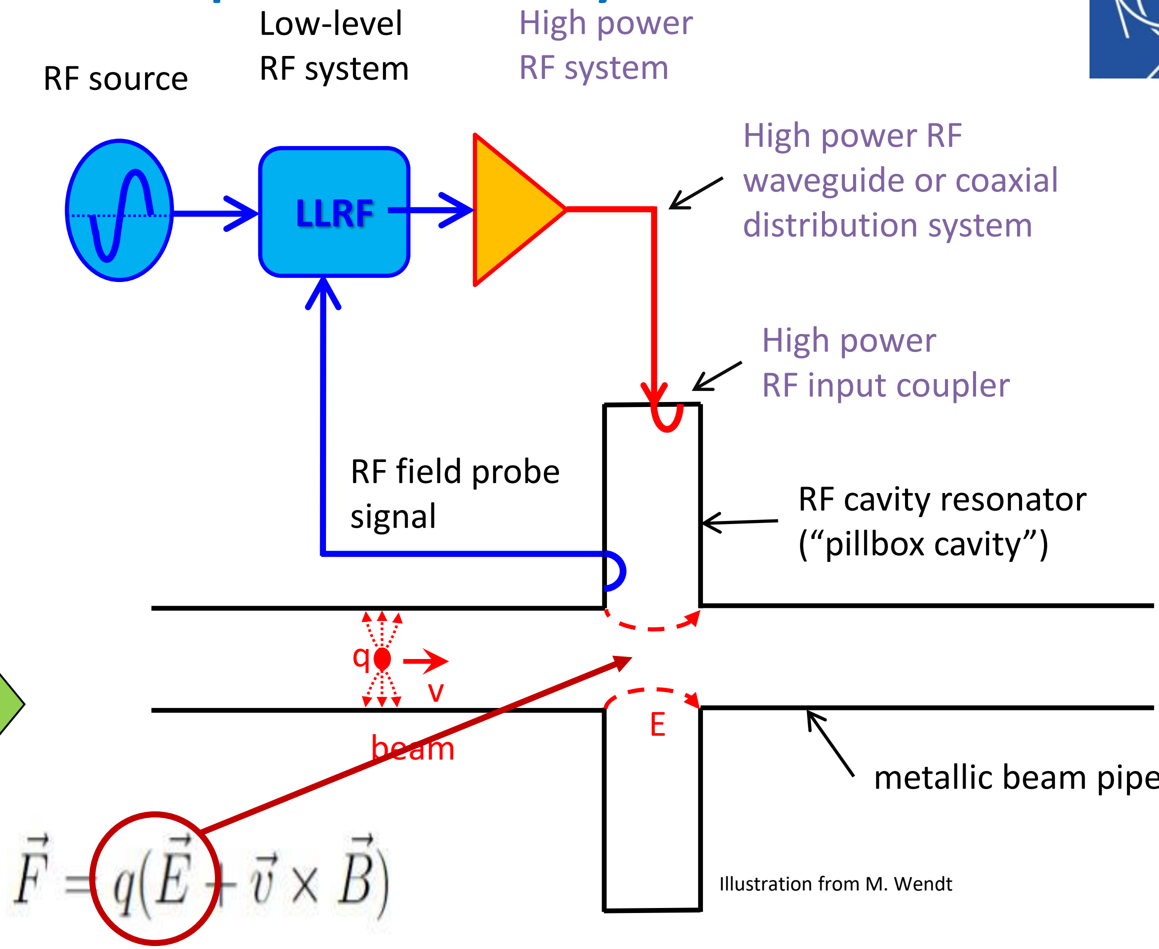
Ferrite Loaded Cavity,
freq = 3 – 8 MHz
(CERN PS Booster)



CLIC structure, freq = 12 GHz

A simplified RF System

PS single cell cavity (“pillbox”)



Recall: Lorenz force will only accelerate if the E-field is synchronized with the beam (synchronicity condition).

Main Instrument types

- intercepting the EM field of particles:

  beam position monitor: beam position and eam oscillations

  beam current transformer: bunch intensities, bunch length

- Using EM radiation (mostly light) emitted by the beam

  Synchrotron light telescope: 2D beam profile

  Streaking: bunch length

- Using the interaction of beam particle with the environment

  wire scanner: 1 D profile

  wire chambers: 2 D profile

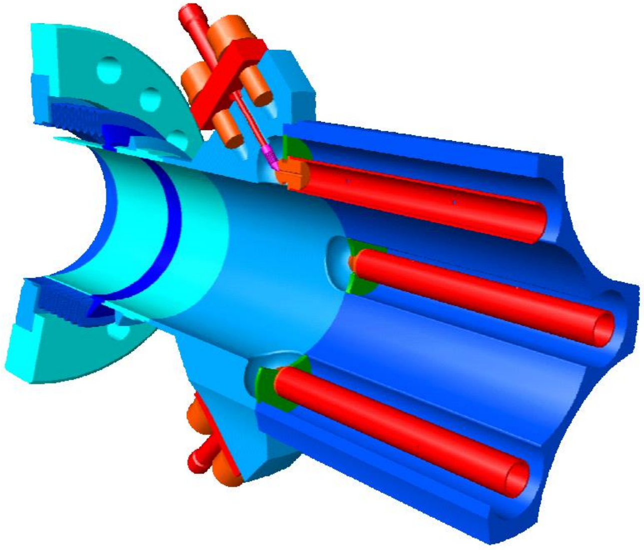
  beam loss monitors: beam loss

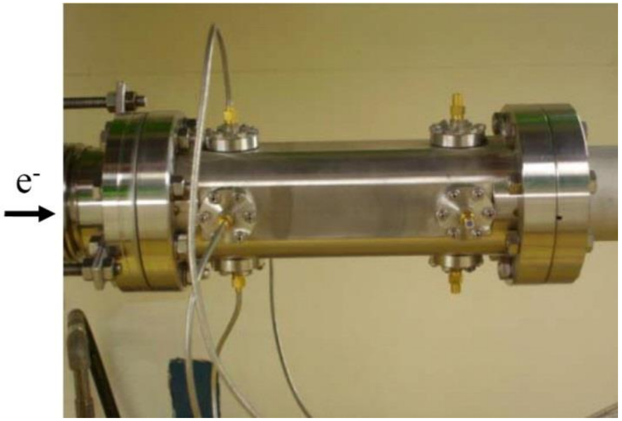
- Derived accelerator quantities: Tune, beta-function, emittance…

Comparison: Stripline and Button BPM (simplified)

	Stripline	Button
Idea	traveling wave	electro-static
Requirement	Careful $Z_{strip} = 50\ \Omega$ matching	
Signal quality	Less deformation of bunch signal	Deformation by finite size and capacitance
Bandwidth	Broadband, but minima	Highpass, but $f_{cut} < 1$ GHz
Signal strength	Large Large longitudinal and transverse coverage possible	Small Size <∅3cm, to prevent signal deformation
Mechanics	Complex	Simple
Installation	Inside quadrupole possible ⇒improving accuracy	Compact insertion
Directivity	**YES**	No

FLASH BPM inside quadrupole

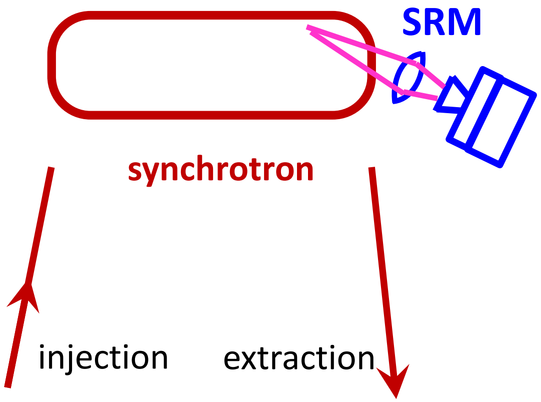


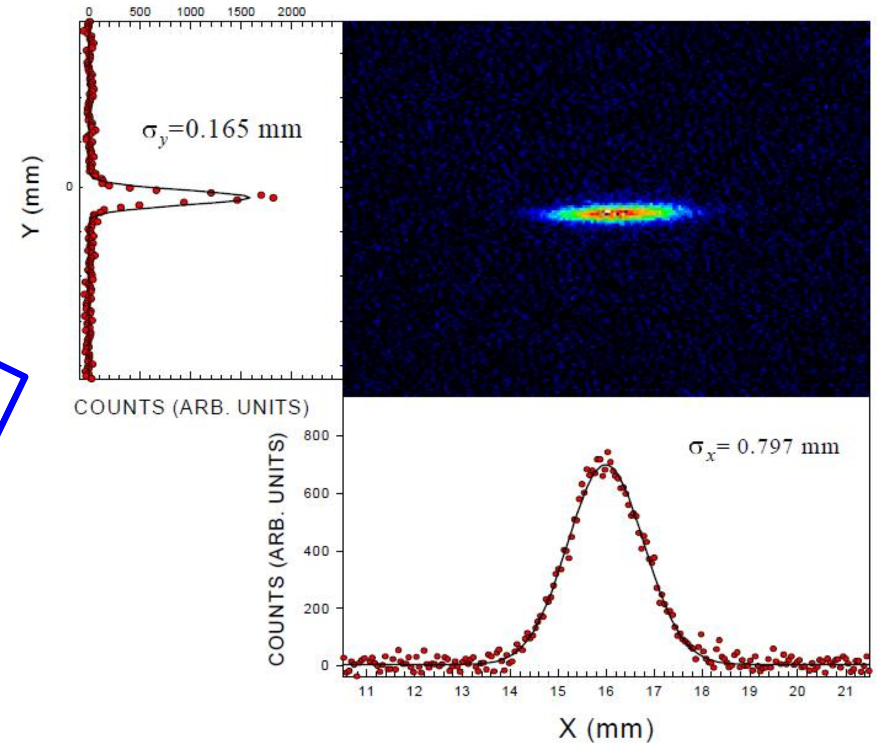


From . S. Vilkins, D. Nölle (DESY)

Result from a Synchrotron Light Monitor

Example: Synchrotron radiation facility APS accumulator ring and blue wavelength:





B.X. Yang (ANL) et al. PAC'97

Advantage: Direct measurement of 2-dim distribution, good optics for visible light

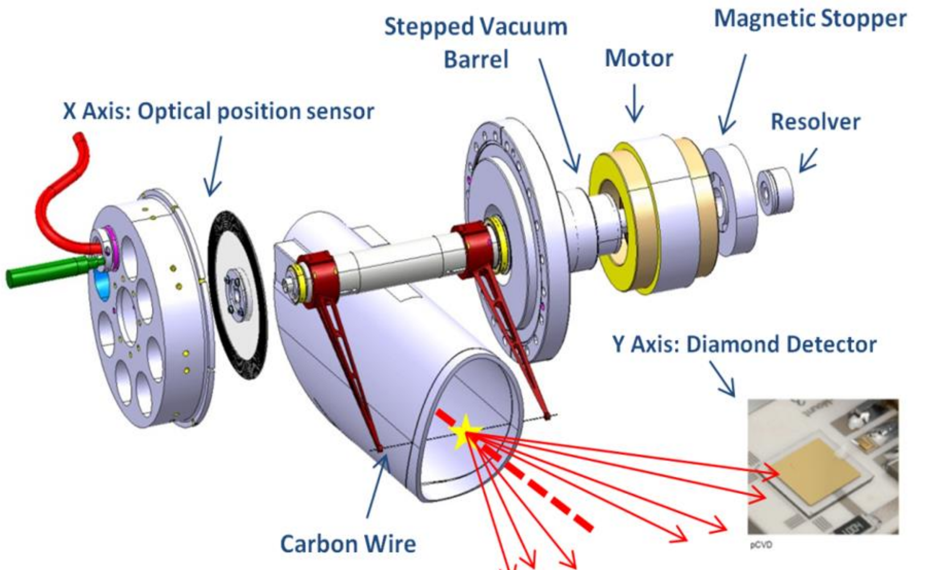
Realization: Optics outside of vacuum pipe

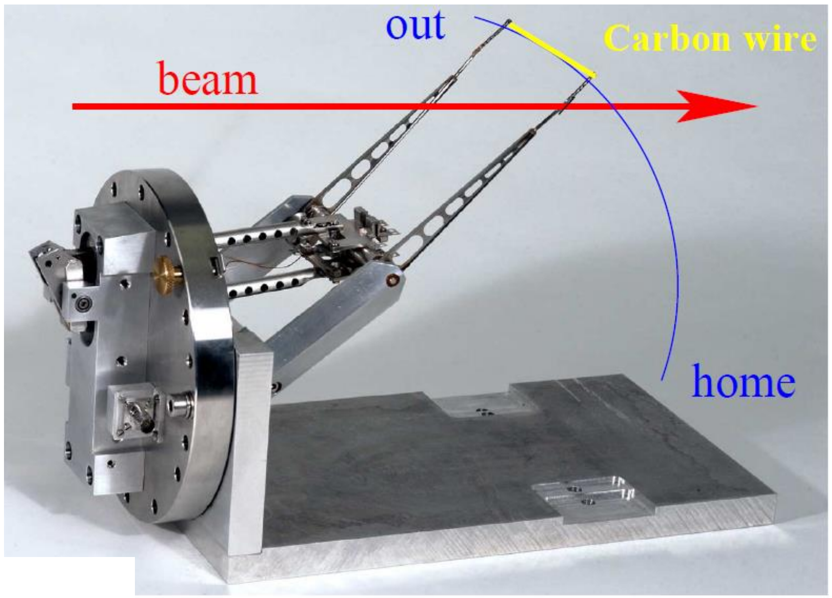
Disadvantage: Resolution limited by the diffraction due to finite apertures in the optics.

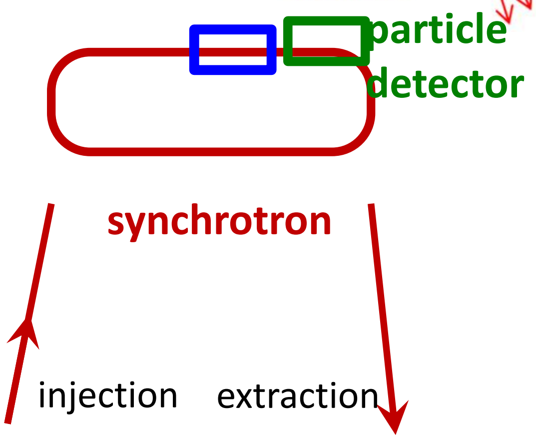
Fast, Flying Wire Scanner

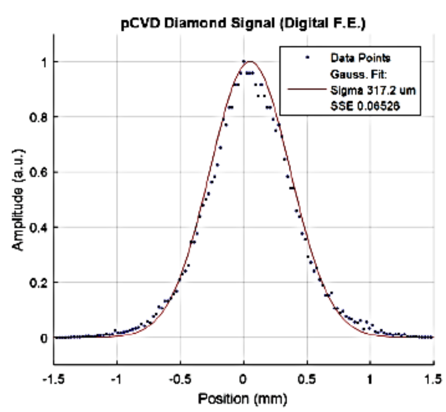
In a synchrotron ***one*** wire is scanned though the beam as fast as possible.

Fast pendulum scanner for synchrotrons; sometimes it is called ***'flying wire'***:





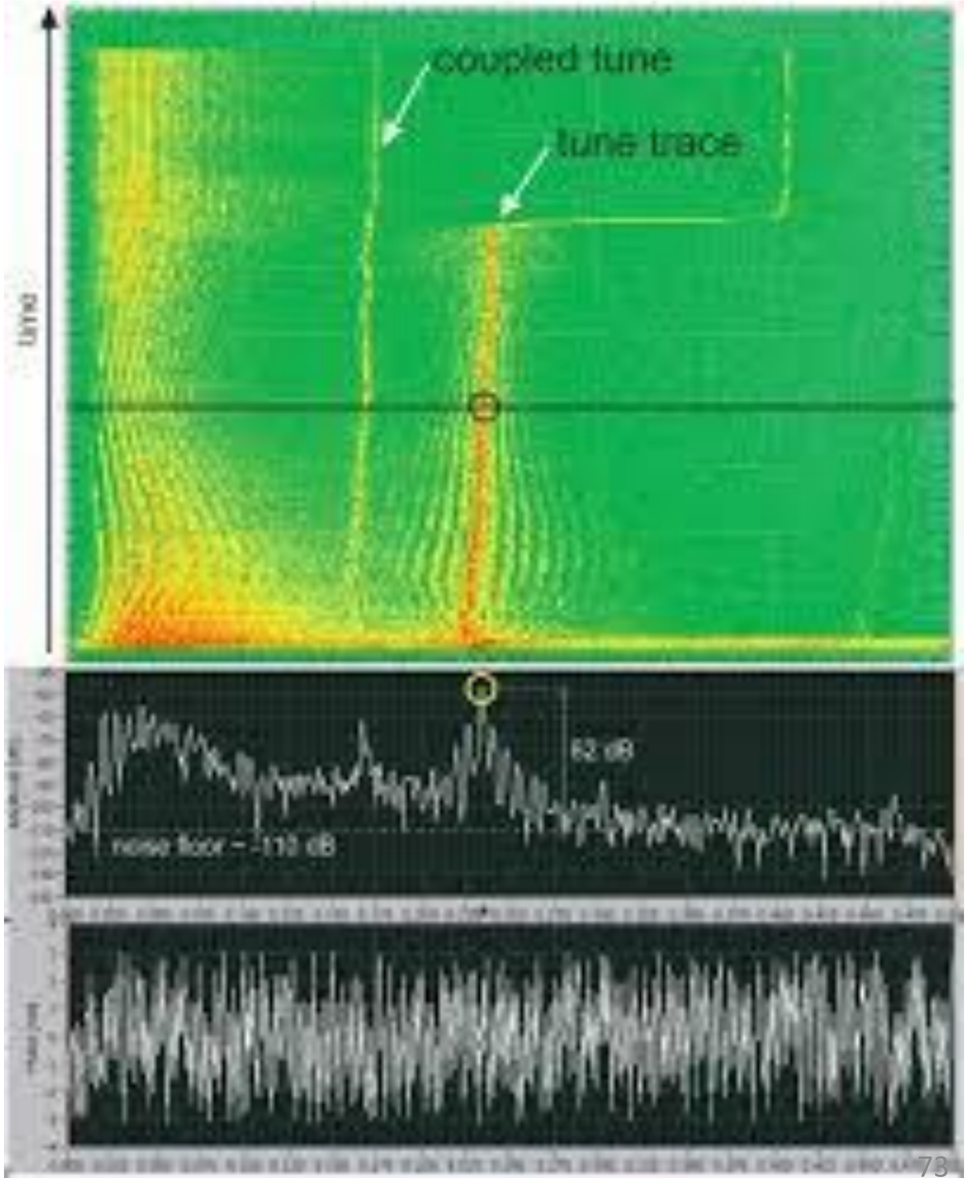




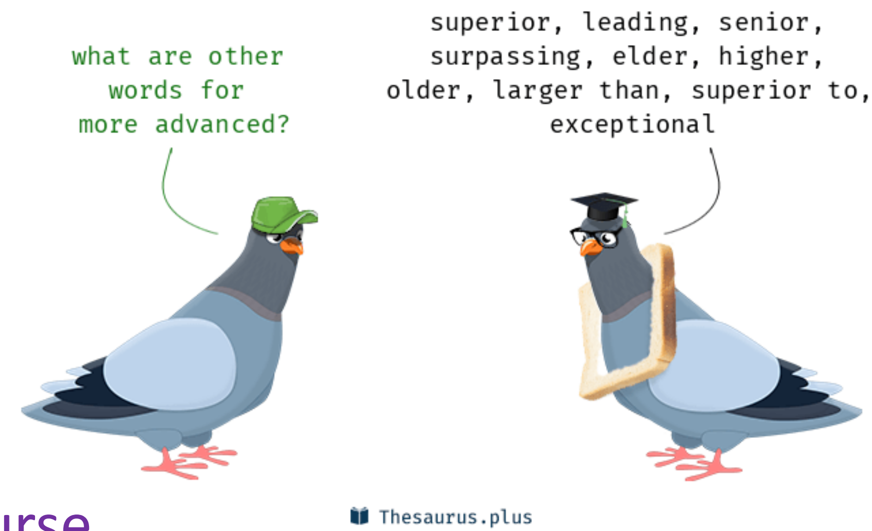
From https://twiki.cern.ch/twiki/bin/viewauth/BWSUpgrade/

STFT Measurement examples I

- A trace of a transverse tune signal over several seconds during the energy ramp of the CERN SPS proton accelerator.



Already somewhat advanced



Non-linearities…

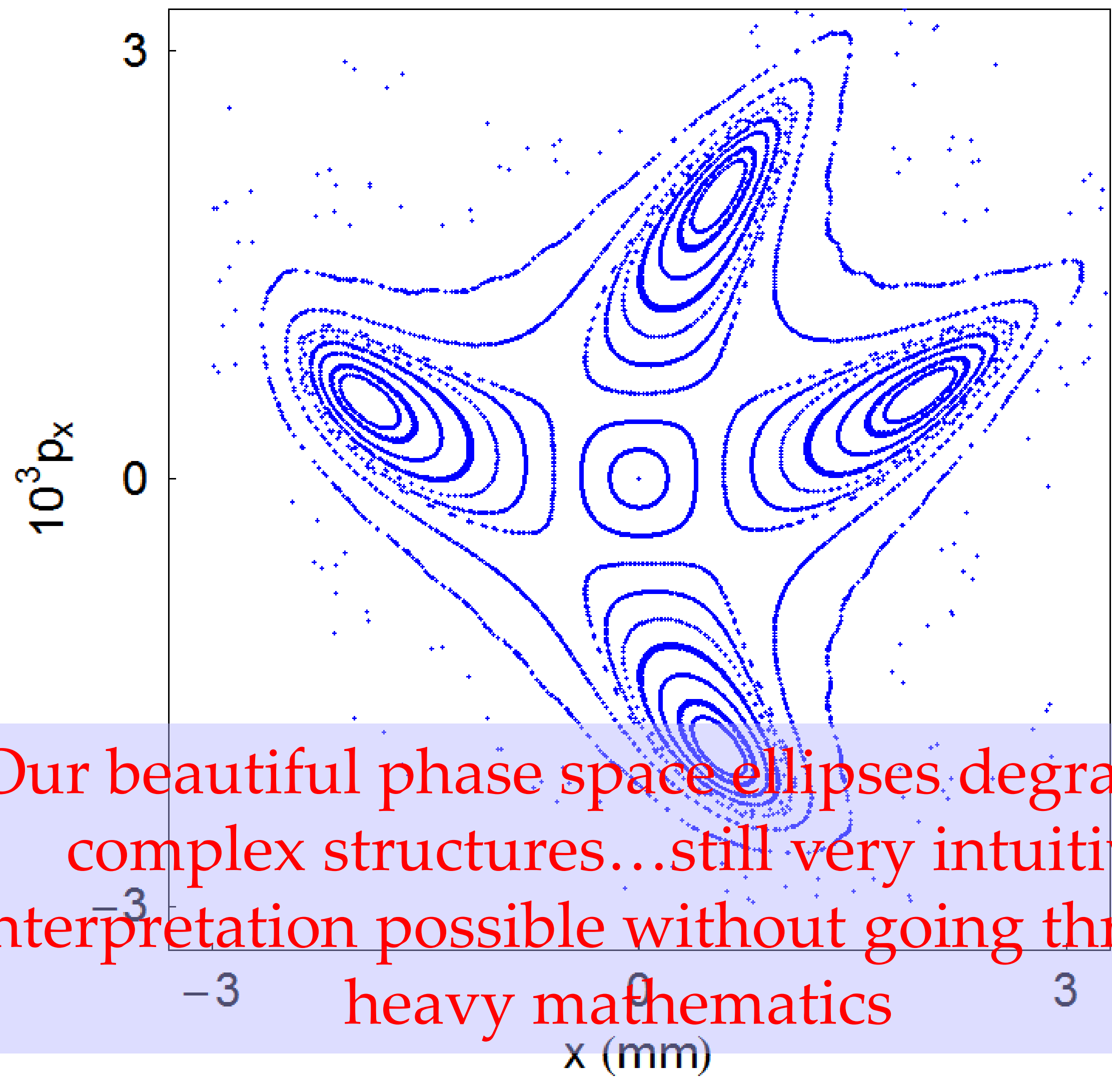
Just touched in the introductory course

Collective effects… also there is more to come!!

- Direct space charge tune shift
- Interaction of beam charges with the environment (impedances)

Example of a simple storage ring

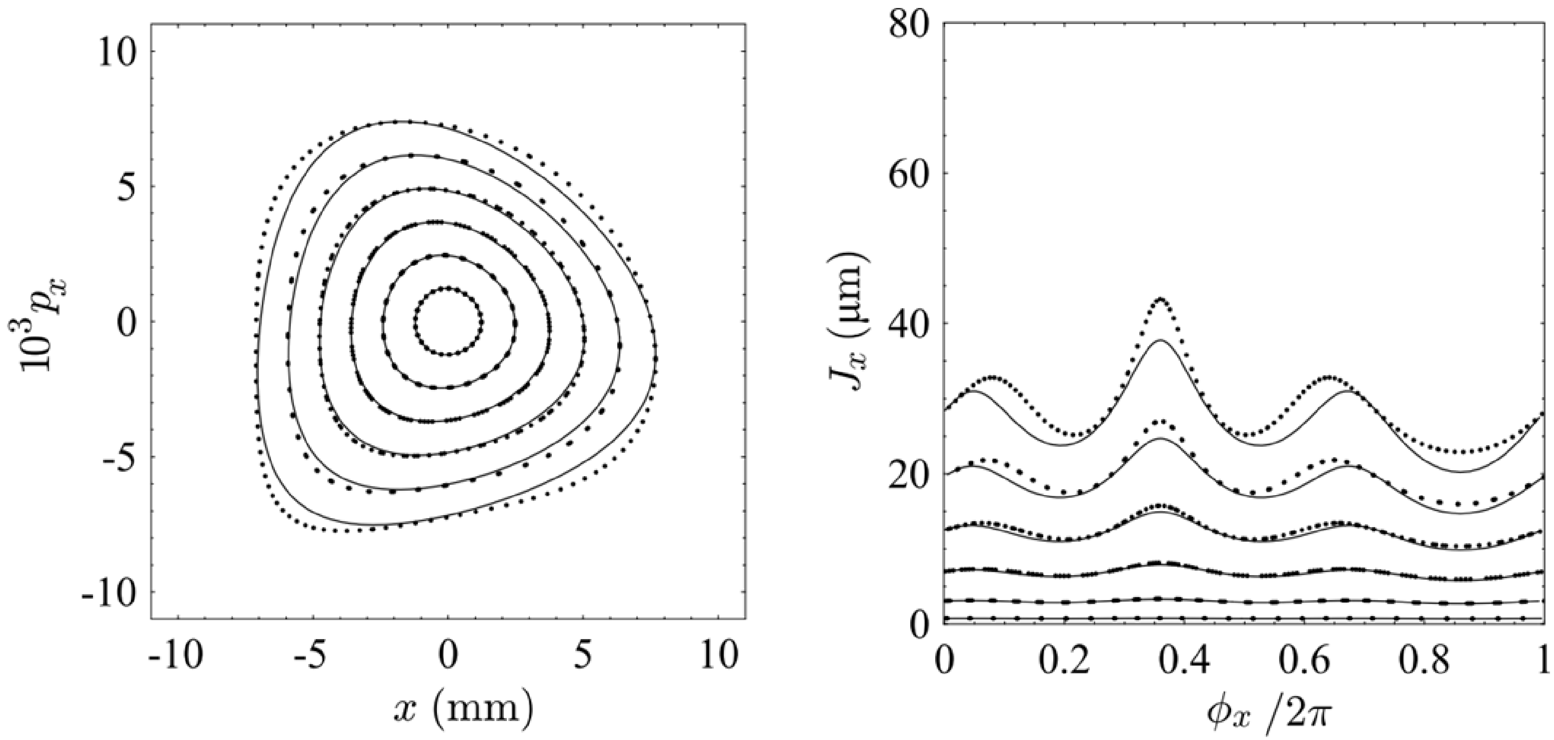
$$\mu_x = 0.252 \times 2\pi$$



Our beautiful phase space ellipses degrade to complex structures…still very intuitive interpretation possible without going through heavy mathematics

Normal form for sextupole

$$\mu_x = 0.28 \times 2\pi$$

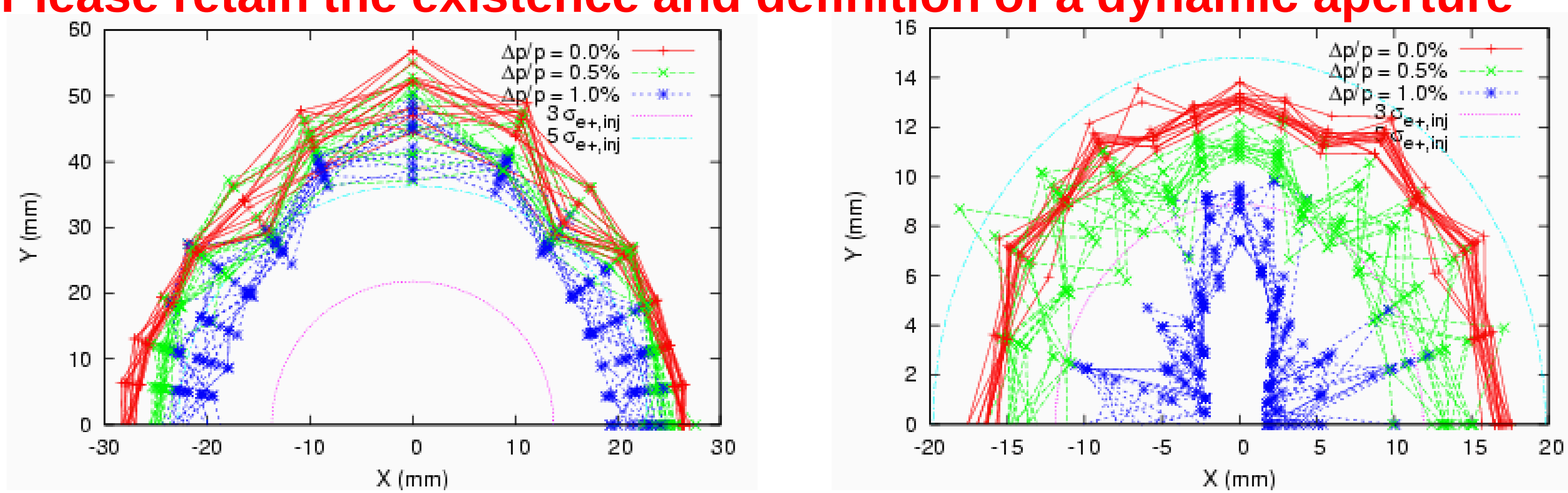


Normalforms: one step further in understand phase space plots.
Describing action and phase dependence of the non-linearity

Dynamic aperture (DA)

- The most direct way to evaluate the nonlinear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA), which is the **boundary of the stable region in co-ordinate space**
- Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns until particles start getting lost → this boundary defines the **Dynamic aperture**
- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space

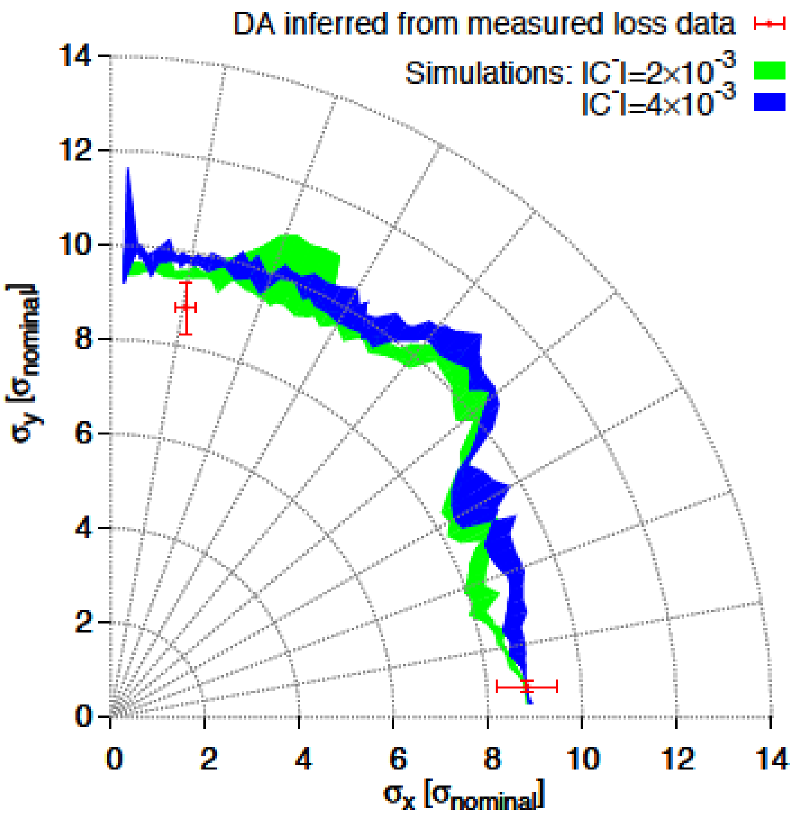
Please retain the existence and definition of a dynamic aperture



DA simulations for CLIC damping rings

Dynamic aperture - LHC

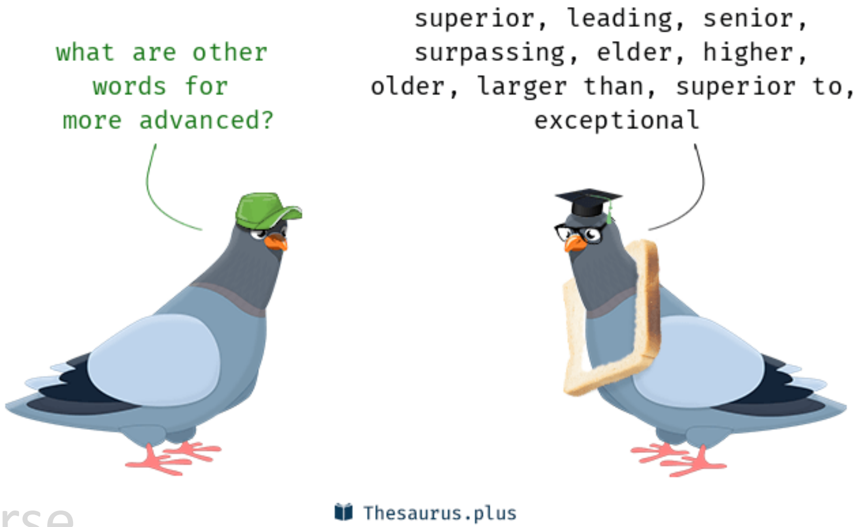
- LHC design was based on a large campaign of systematic DA simulations (including margin for stability)
  - The goal is to allow significant margin in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture



- A few years after LHC started operating, a measurement of the DA was performed (kicking the beam to large amplitudes)
- Very good agreement between tracking simulations and measurements in the machine

E.Mclean, PhD thesis, 2014

Already somewhat advanced



Non-linearities…
Just touched in the introductory course

Collective effects… also there is more to come!!

- Direct space charge tune shift
- Interaction of beam charges with the environment (impedances)

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Last not least: Collective effects

Collective effects:
= Summary term for all effects when the coulomb force of the particles in a bunch can no longer be neglected; in other words when there are too many particles…

We distinguish:
i) self interaction of the particles within a bunch:

1) space charge effects
2) Intra beam scattering
3) Touschek scattering

leads to emittance growth and particle loss

ii) Interaction of the particles with the vacuum wall
→concept of impedance of vacuum system
leads to instabilities of single bunches and multiple bunches

iii) Interaction of with particles from other counter-rotating beam
→ beam-beam effects (→ more later this school)

Space-charge Forces

In the rest frame of a bunch of charged particles, the bunch will expand rapidly (in the absence of external forces) because of the Coulomb repulsion between the particles.

The electric field around a single particle of charge q at rest is a radial field:

$$E_r = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

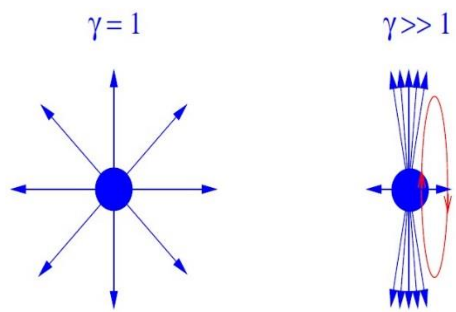
Applying a Lorentz boost along the z axis, with relativistic factor γ, the field becomes:

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left(x^2+y^2+\gamma^2 z^2\right)^{3/2}} \qquad E_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left(x^2+y^2+\gamma^2 z^2\right)^{3/2}} \qquad E_z = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{\left(x^2+y^2+\gamma^2 z^2\right)^{3/2}}$$

For large γ, the field is strongly suppressed, and falls rapidly away from $z = 0$. In other words, the electric field exists only in a plane perpendicular to the direction of the particle.

Space Charge: Scaling with energy

Example Coulomb field: (a charge moving with constant speed)



Recall from relativity

- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Electrical field : repulsive force between two charges of equal polarity
Magnetic field: attractive force between two parallel currents
after some work:

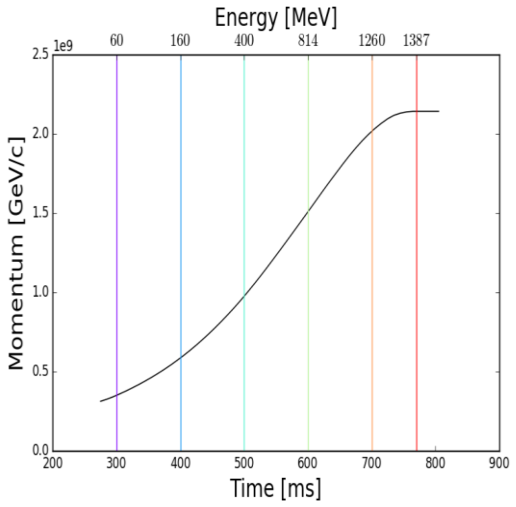
$$F_r = \frac{eI}{2\pi\varepsilon_0 \beta c}(1-\beta^2)\frac{r}{a^2} = \frac{eI}{2\pi\varepsilon_0\beta c}\frac{1}{\gamma^2}\frac{r}{a^2}$$

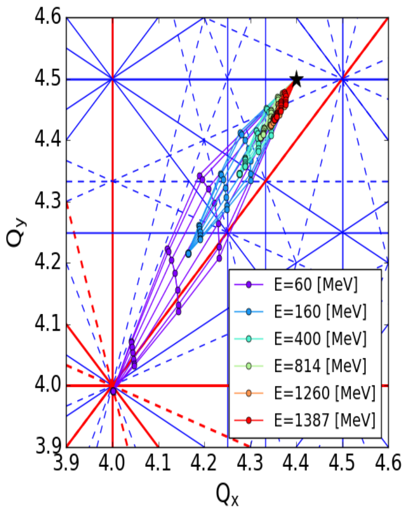
→ space charge diminishes with $^1/_{\gamma^2}$ scaling
→ each particle source immediately followed by a linac or RFQ for acceleration

Mitigation of direct space charge tune shift

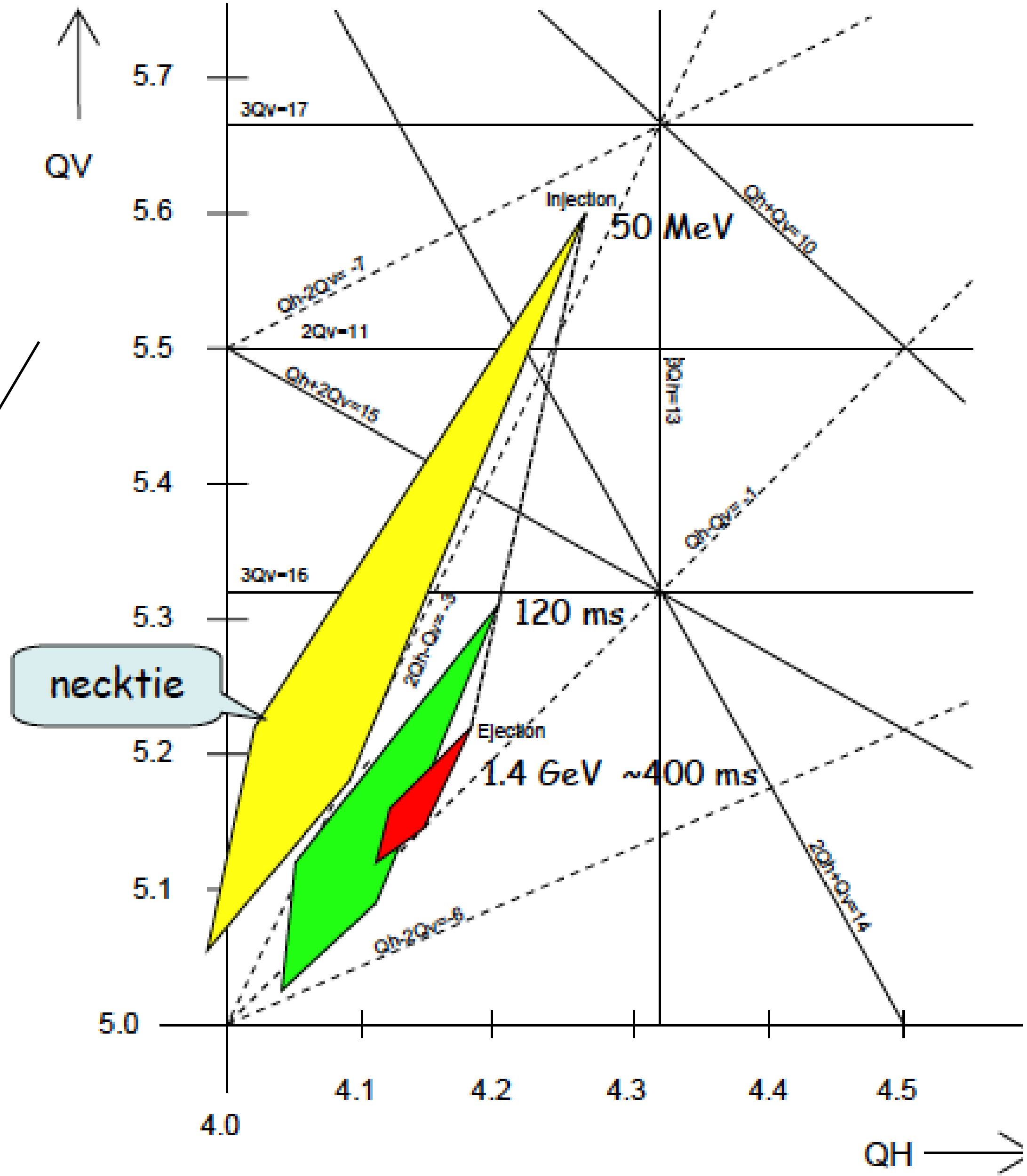
$$\Delta \hat{Q}_{x,y} = -\frac{r_0 C \hat{\lambda}}{2\pi e \beta \gamma^2} \frac{1}{2\,\varepsilon^n_{x,y}}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a specially configured (double harmonic) RF system
  - using **bunch distributions with small peak density** (e.g. parabolic instead of Gaussian)
  - reducing the central density of the particle distribution (e.g. “hollow bunches”)
- Increase the beam energy by
  - accelerating the beam as quickly as possible
  - increasing the injection energy (usually requires an upgrade of the pre-injector)





Space charge always defocusing



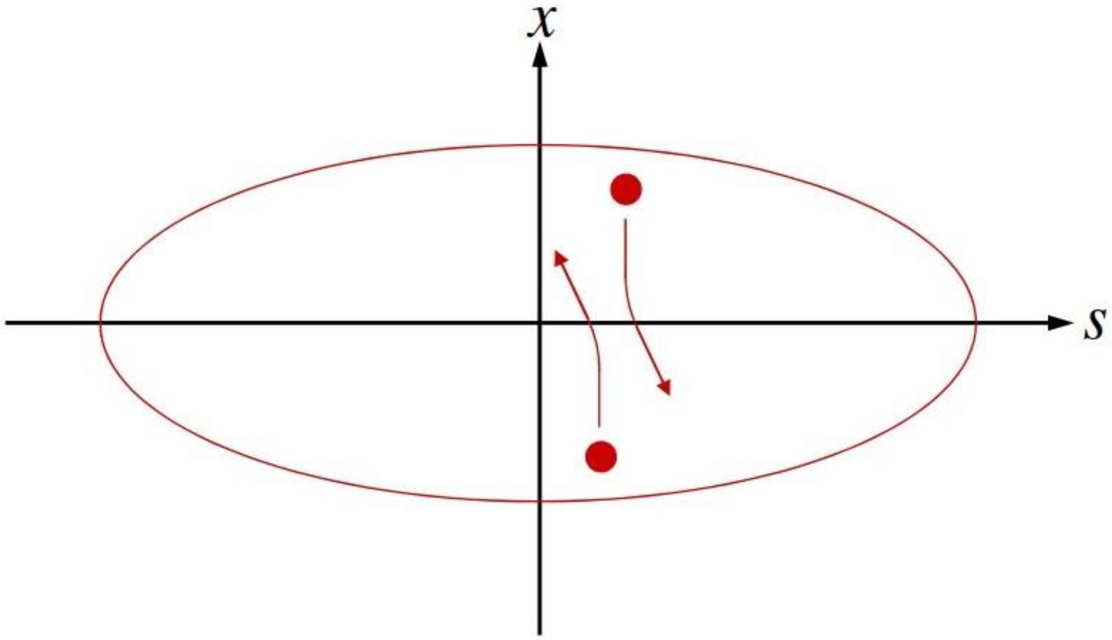
“footprint” of particles with space charge tune shift.

The effect dramatically reduces at higher energies

Intrabeam Scattering

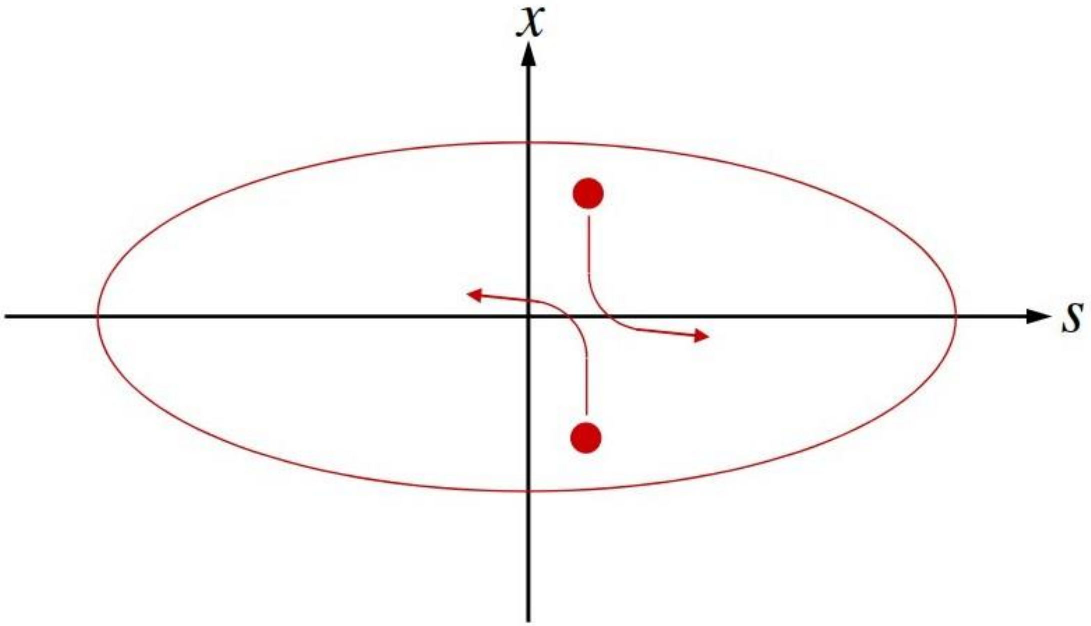
Particles within a bunch can collide with each other as they perform betatron and synchrotron oscillations. The collisions lead to a redistribution of the momenta within the bunch, and hence to a change in the emittances.

If a collision results in the transfer of transverse to longitudinal momentum at a location where the dispersion is non-zero, the result (after many scattering events) can be an increase in both transverse and longitudinal emittance.



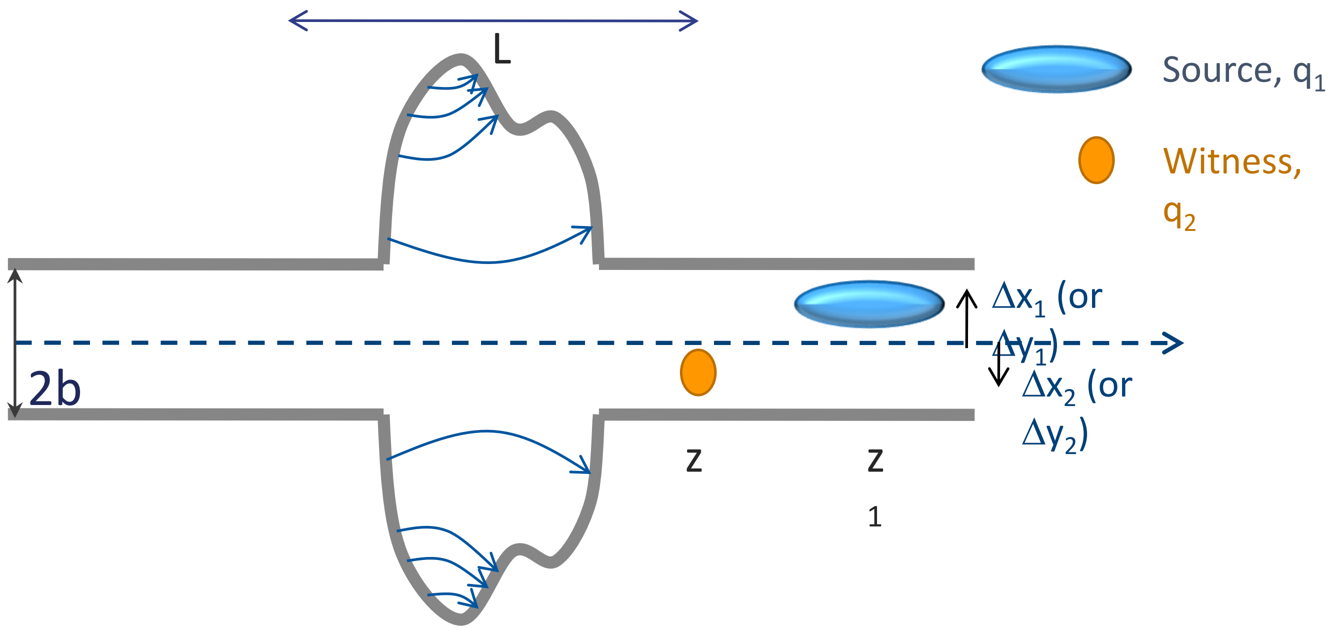
Touscheck effect

The Touschek effect is related to intrabeam scattering, but refers to scattering events in which there is a large transfer of momentum from the transverse to the longitudinal planes. IBS refers to multiple small-angle scattering; the Touschek effect refers to single large-angle scattering events.



If the change in longitudinal momentum is large enough, the energy deviation of one or both particles can be outside the energy acceptance of the ring, and the particles will be lost from the beam.

Wake potential for a distribution of particles



We define the **wake function as the integrated force** on the witness particle (associated to a change in energy):

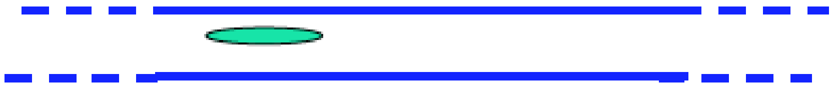
- For an extended particle distribution this becomes (superposition of all source terms)

$$\Delta E_2(z) = -\sum_i q_i\, q_2\, \boldsymbol{w(x_i, x_2, z - z_i)} \longrightarrow \int \lambda_1(x_1, z_1)\, \boldsymbol{w(x_1, x_2, z - z_1)}\, dx_1 dz_1$$

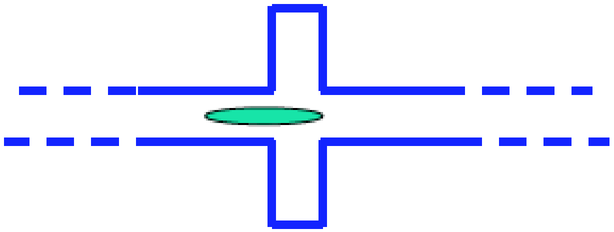
Forces become dependent on the **particle distribution function**

Interaction of beam with vacuum chamber

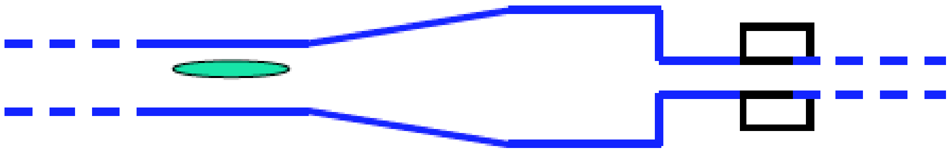
Resistive wall effect:
Finite conductivity



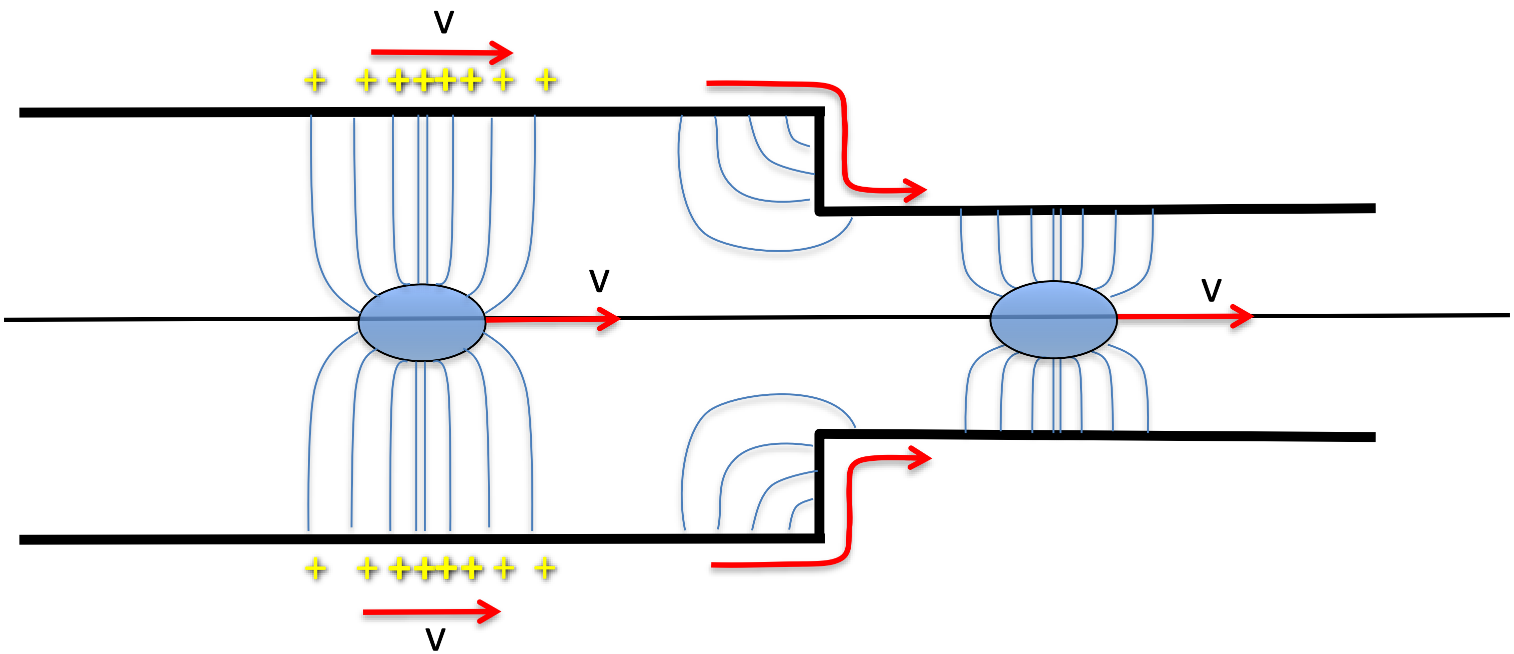
Narrow-band resonators:
Cavity-like objects



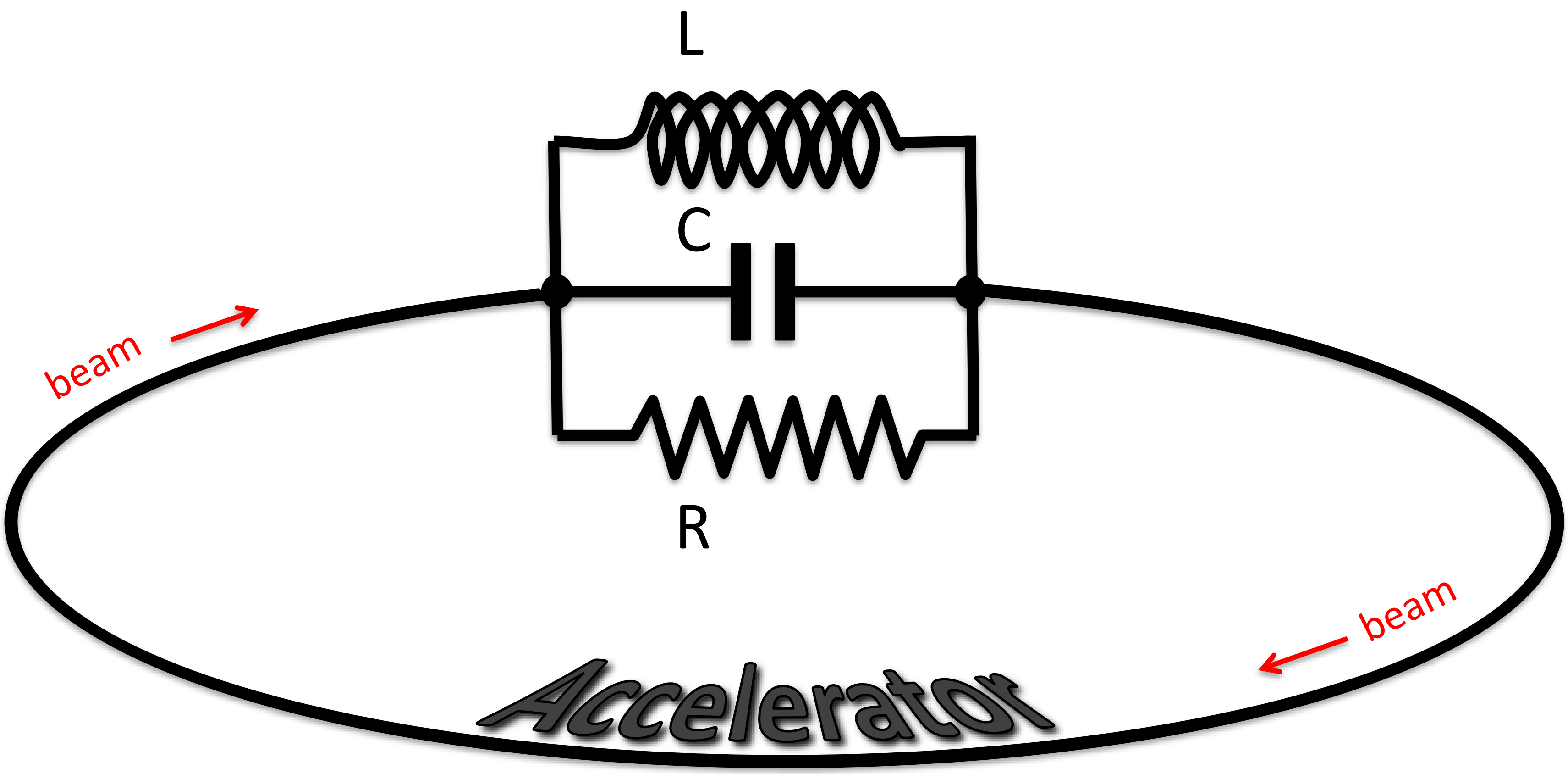
Broad-band resonators:
Tapers, other non-resonant structures

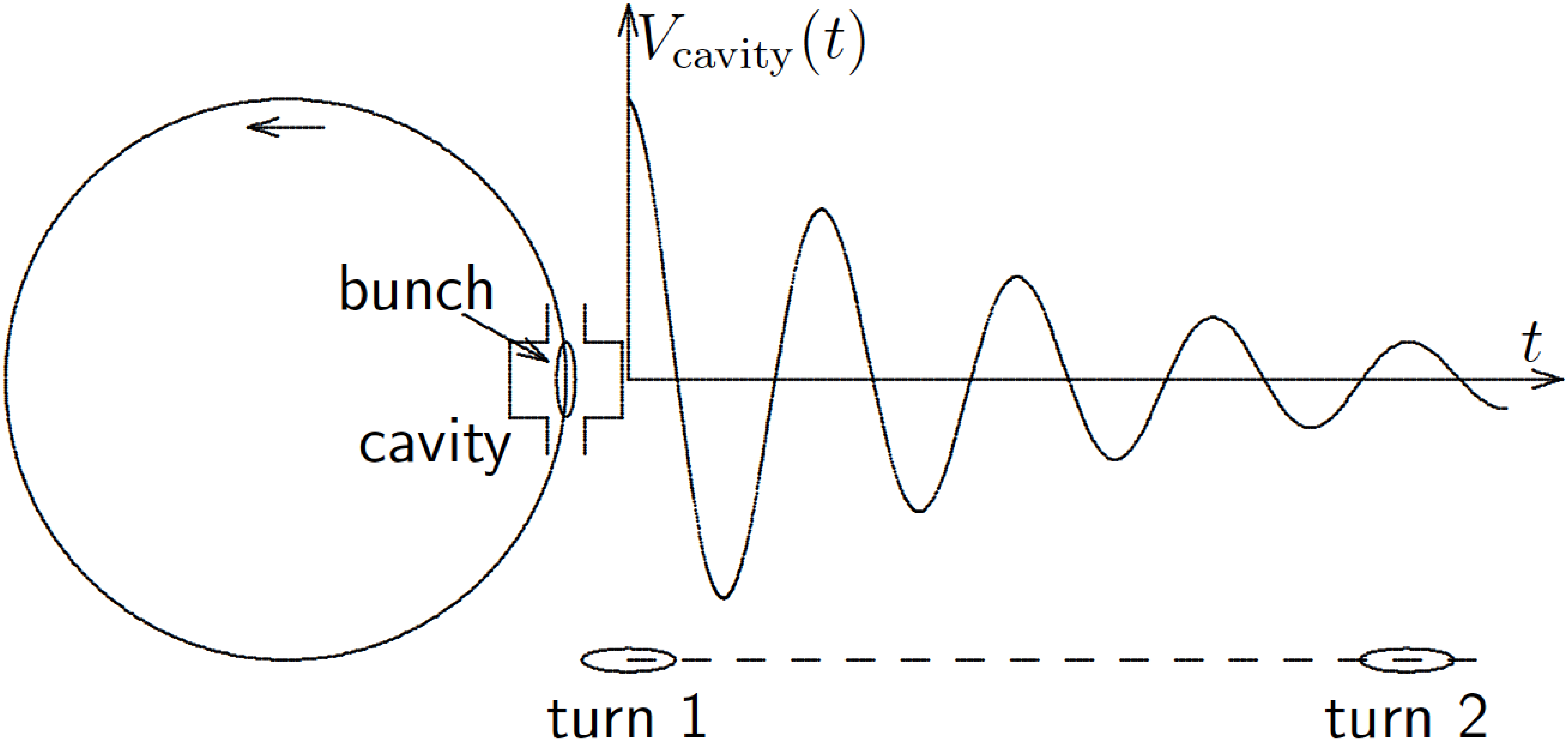


Bunch in a conducting pipe with sudden change

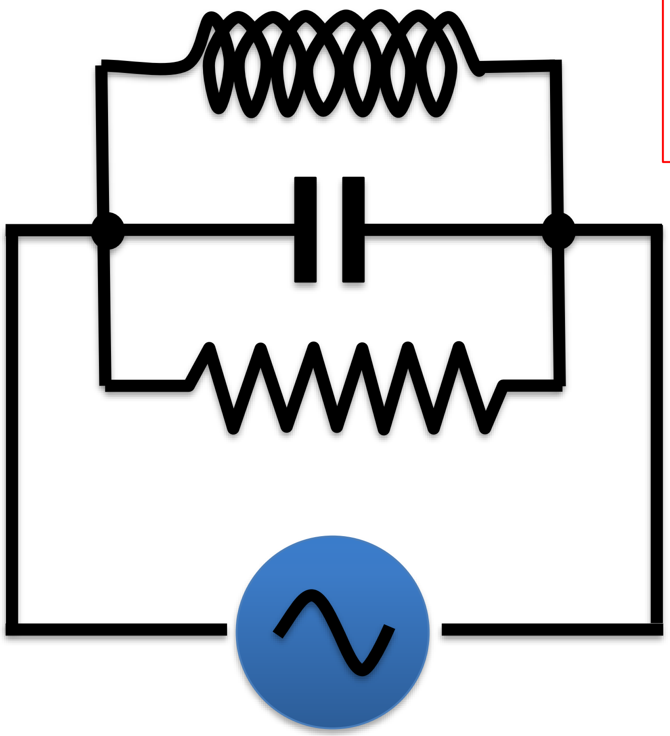


All together




$V_{\text{cavity}}(t)$
bunch
cavity
t
turn 1
turn 2

Impedance



$$I = \hat{I}\cos(\omega t)$$

Impedance

$$V(t) = Z_r(\omega)\hat{I}\cos(\omega t) - Z_i(\omega)\hat{I}\sin(\omega t)$$

$$Z_r(\omega) = R\frac{1}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

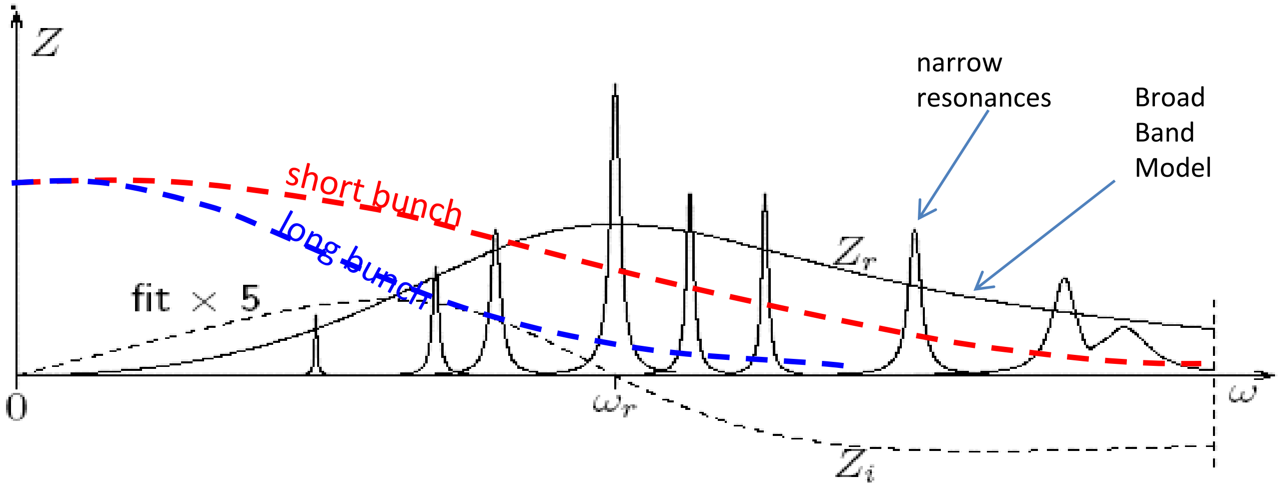
$$Z_i(\omega) = -R\frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

The real (resistive) part dissipates energy, the imaginary part creates instabilities

Consequences of impedances

Energy loss on pipes → heating (important in a superconducting accelerator)

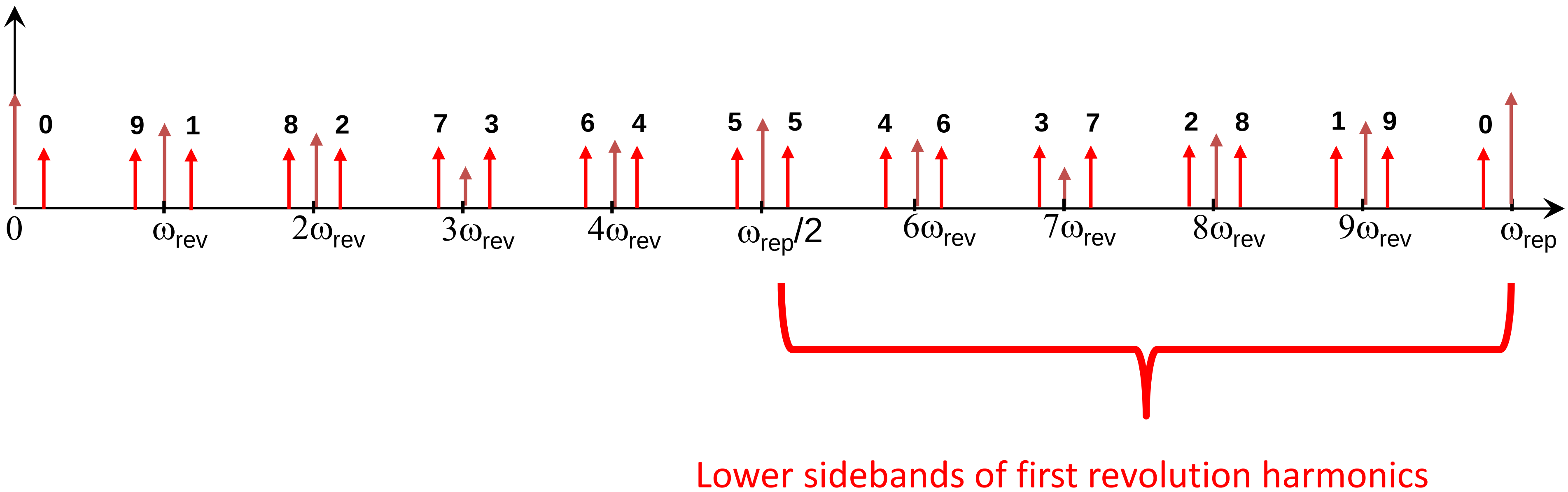
Tune shift



Single bunch instabilities (head-tail)

Multibunch instabilities

Multi-bunch modes sidebands summary for 10 equidistant bunches



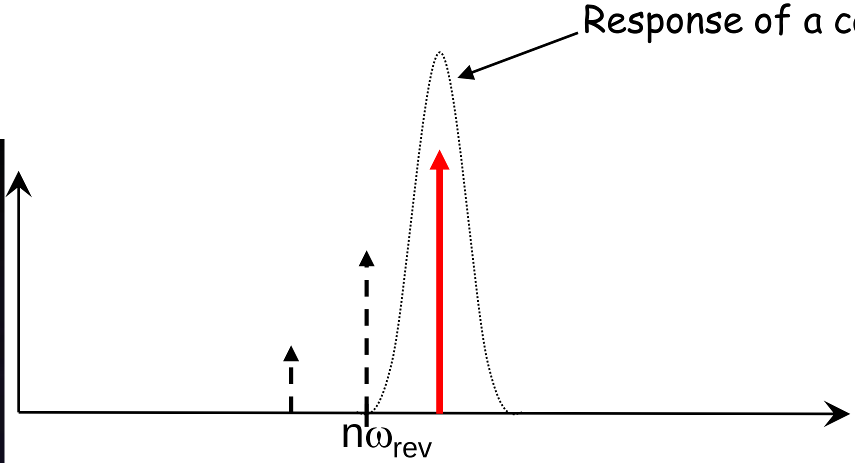
$$\omega_{SB} = p\,M\omega_{rev} \pm (m+q)\omega_{rev}$$

If the bunches have not the same charge, i.e. the buckets are not equally filled (uneven filling), the spectrum has frequency components also at the revolution harmonics (multiples of ω_{rev}). The amplitude of each revolution harmonic depends on the filling pattern over one machine turn

Multi-bunch modes: coupled-bunch instability

One multi-bunch mode can become unstable if one of its sidebands overlaps, for example, with the frequency response of a cavity high order mode (HOM). The HOM couples with the sideband giving rise to a coupled-bunch instability, with consequent increase of the sideband amplitude

Synchrotron Radiation Monitor showing the transverse beam shape



Effects of coupled-bunch instabilities:

- ☹ increase of the transverse beam dimensions
- ☹ increase of the effective emittance
- ☹ beam loss and max current limitation
- ☺ increase of lifetime due to decreased Touschek scattering (dilution of particles)