Lattice Design

G. Sterbini, BE-ABP-INC, CERN

10-22 November 2024, CAS, Spa, Belgium

guido.sterbini@cern.ch

Lattice Design

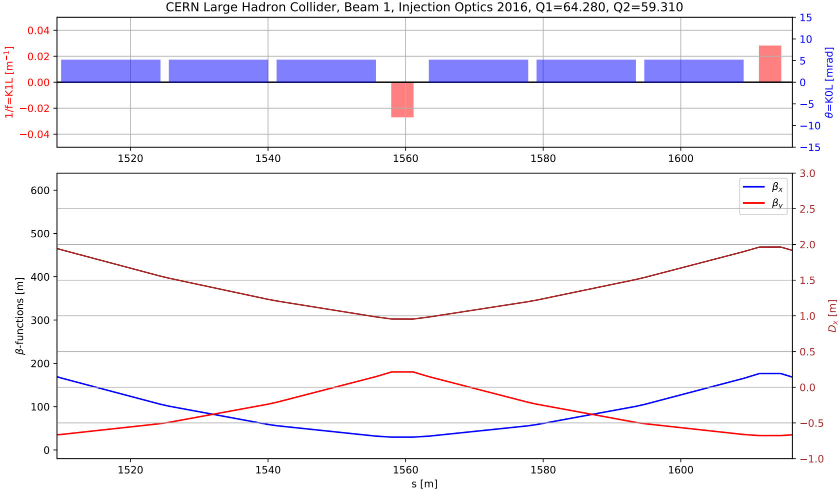
The analysis of the cell stability and betatron functions can be done via an algorithmic approach using the method presented yesterday [1]:

1. compute *symbolically* the M_{OTM},
2. diagonalize it $M_{OTM} = PDP^{-1}$, with $det(P) = -i$ and $P_{11} = P_{12}$,
3. impose that all the eigenvalues amplitude is 1 to get the stability condition,
4. study P to get the periodic solution for β and α at the start of the cell,
5. propagate the solution from the start of the cell along the different lattice element.

We will start considering a **FODO cell**.

[1] LatticeCellStudies.ipynb

The CERN Large Hadron Collider FODO cell



The FODO cell description

Let's consider a FODO cell of length L_{cell} in **thin lens approximation**, where

1. the space of the focusing (F) and defocusing (D) quadrupoles is equal to $L_{cell}/2$ and
2. the focal length of the F and D quadrupoles equal in module, that is $f_D = -f_F$ with $f_F > 0$.

For convenience, we will start and end the FODO cell with half of an F quadrupole (i.e., with focal lenght $2 \times f_F$) and we will consider, as first step, the horizontal plane.

The FODO M_{OTM} diagonalization

Using symbolic tools (e.g., *sympy*) one can compute

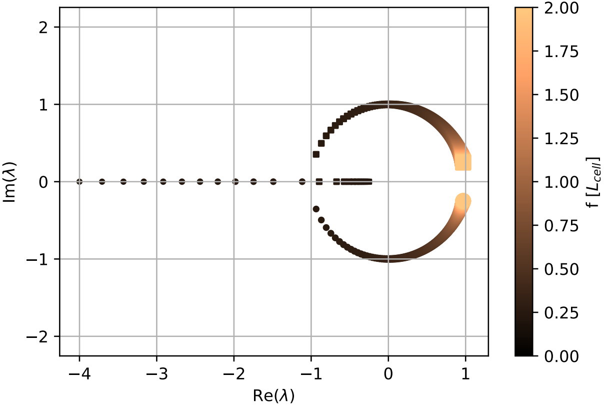
$$M_{OTM} = \begin{bmatrix} -\frac{L_{cell}^2}{8f^2} + 1 & \frac{L_{cell}^2}{4f} + L_{cell} \\ \frac{L_{cell}(L_{cell}-4f)}{16f^3} & -\frac{L_{cell}^2}{8f^2} + 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{-L_{cell}^2 + L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} & 0 \\ 0 & \frac{-L_{cell}^2 - L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}(-L_{cell}+4f)}} & \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}(-L_{cell}+4f)}} \\ -\frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}\sqrt{L_{cell}^2-16f^2}} & \frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}\sqrt{L_{cell}^2-16f^2}} \end{bmatrix}$$

The FODO stability I

The stability on the horizontal plane is achieved if λ_1 and λ_2 have unitary module.



The FODO stability II

This implies $-1 < \frac{\lambda_1+\lambda_2}{2} = \cos\mu < 1$, that is

$$\boxed{\frac{L_{cell}}{4} < f}.$$

The stability condition in the vertical plane is exactly equivalent, since D(f)=D(-f).

The stability condition of a FODO lattice (thin lens approximation and no dipoles) imposes an F quadrupole with f larger than $L_{cell}/4$.

The FODO phase advance

Remembering that

$$\mu = \arccos \frac{\lambda_1 + \lambda_2}{2},$$

one gets

$$\boxed{\mu = \arccos\left(1 - \frac{L_{cell}^2}{8f^2}\right)},$$

or, equivalently, from[2]

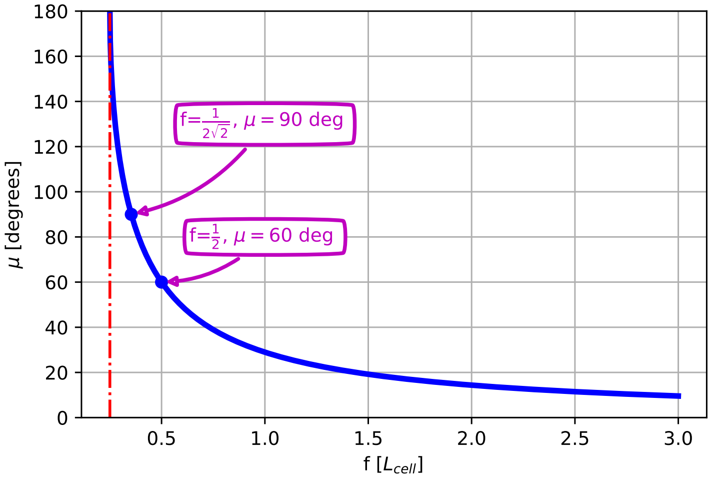
$$\sin\left(\frac{\arccos(1-x)}{2}\right) = \sqrt{\frac{x}{2}}$$

we get

$$\boxed{\sin\left(\frac{\mu}{2}\right) = \frac{L_{cell}}{4f}}.$$

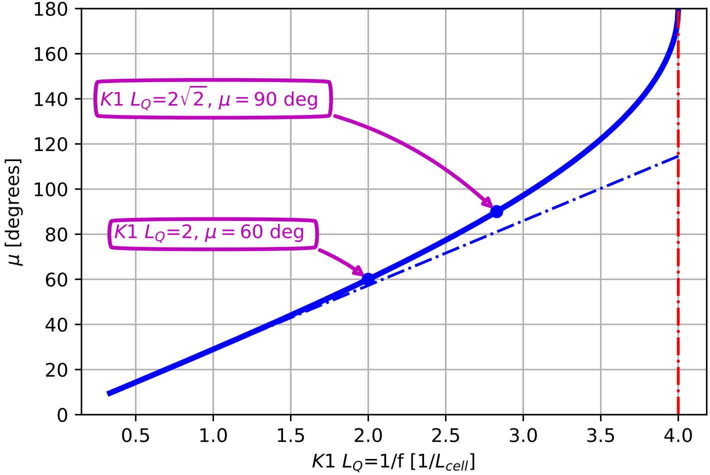
[2] LatticeCellStudies.ipynb

μ vs f and 1/f



180
160
140
120
100
80
60
40
20
0
0.5
1.0
1.5
2.0
2.5
3.0
μ [degrees]
f [L_{cell}]
f=$\frac{1}{2\sqrt{2}}$, $\mu = 90$ deg
f=$\frac{1}{2}$, $\mu = 60$ deg

μ vs f and 1/f



$K1\ L_Q$=2$\sqrt{2}$, $\mu = 90$ deg
$K1\ L_Q$=2, $\mu = 60$ deg
μ [degrees]
$K1\ L_Q$=1/f [1/L_{cell}]
180
160
140
120
100
80
60
40
20
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

FODO Optics Functions I

Remembering that

$$P = \begin{pmatrix} \sqrt{\frac{\beta}{2}} & \sqrt{\frac{\beta}{2}} \\ \frac{-\alpha + i}{\sqrt{2\beta}} & \frac{-\alpha - i}{\sqrt{2\beta}} \end{pmatrix}$$

we have

$$\boxed{\beta(0) = 2\ P_{11}^2} \text{ and } \boxed{\alpha(0) = -P_{11}(P_{21} + P_{22})}.$$

FODO Optics Functions II

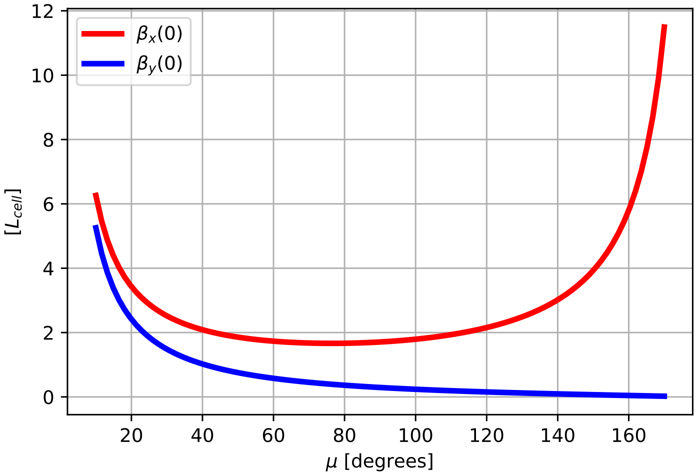
This yields

$$\beta_x(0) = \frac{2f\sqrt{4f + L_{cell}}}{\sqrt{4f - L_{cell}}} = L_{cell}\frac{1+\sin(\mu/2)}{\sin(\mu)}$$
$$\alpha_x(0) = 0.$$

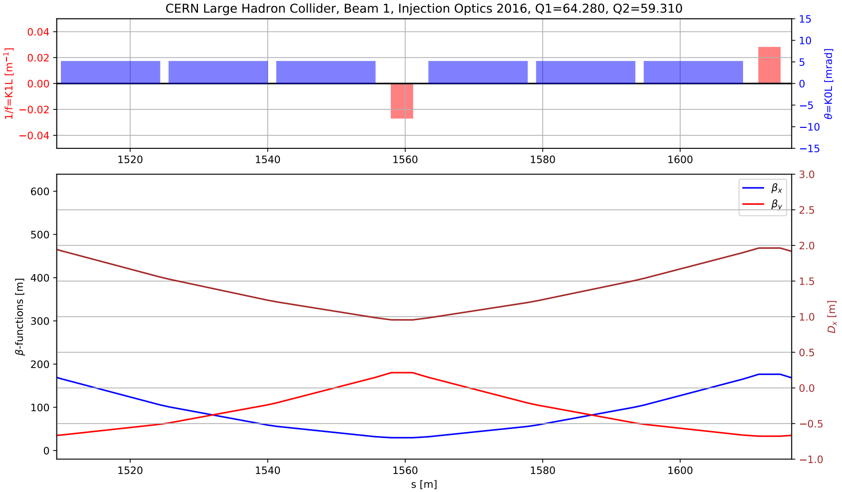
With a similar approach, we can compute the y-plane optical functions by considering P(-f), getting

$$\beta_y(0) = \frac{2f\sqrt{4f - L_{cell}}}{\sqrt{4f + L_{cell}}} = L_{cell}\frac{1-\sin(\mu/2)}{\sin(\mu)}$$
$$\alpha_y(0) = 0.$$

β-function vs μ



β-function vs μ



Chromaticity of a FODO I

The definition of the linear chromaticity is

$$\xi = \frac{\Delta Q}{\frac{\Delta p}{p_0}} = \frac{1}{2\pi}\frac{\Delta\mu}{\frac{\Delta p}{p_0}}. \tag{1}$$

From the relation

$$f\left(\frac{\Delta p}{p_0}\right) = f \times \left(1 + \frac{\Delta p}{p_0}\right) \tag{2}$$

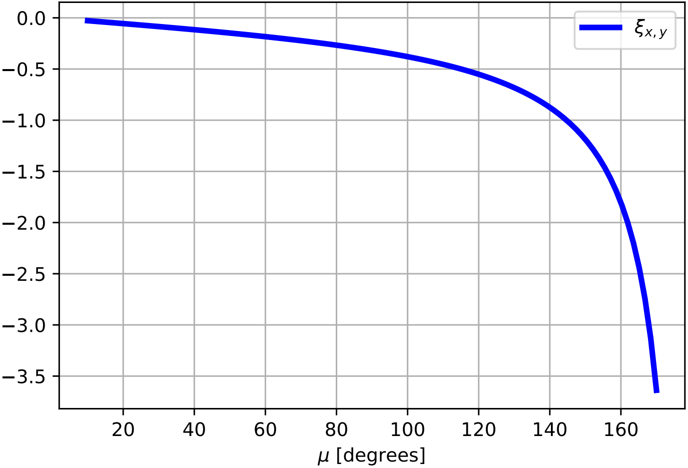
and from

$$\sin\left(\frac{\mu}{2}\right) = \frac{L_{cell}}{4f}, \tag{3}$$

one can compute the FODO lattice chromaticity

$$\xi = -\frac{1}{4\pi}\frac{L_{cell}}{f}\frac{1}{\cos(\mu/2)} = \boxed{-\frac{1}{\pi}\tan\left(\frac{\mu}{2}\right)} \tag{4}$$

Chromaticity of a FODO II



FODO flavours I

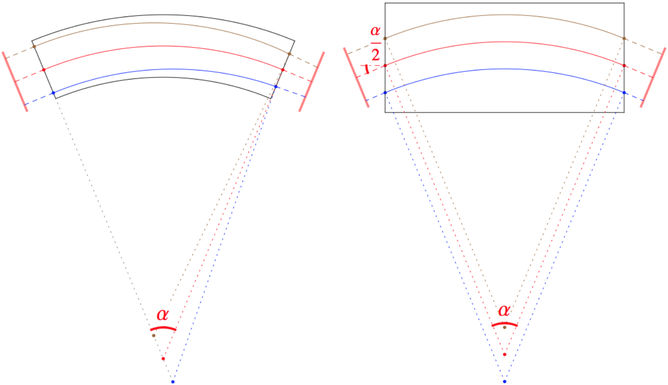
From the FODO lattice we can define at least two additional "flavours":

1. different focal length in the F and D quadrupoles,
2. uneven distance between quadrupoles.

The stability of the two cases is discussed in LatticeCellStudies.ipynb In addition, effects of dipole edge focusing

FODO flavours II

(e.g. sector and rectangular bends) and thick quadrupoles can be computed using `Xsuite`.

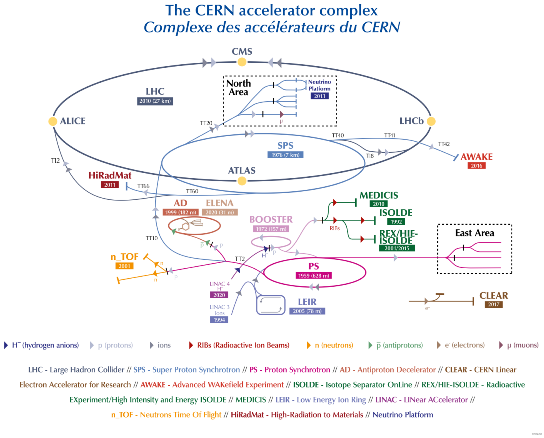


Triplet cell

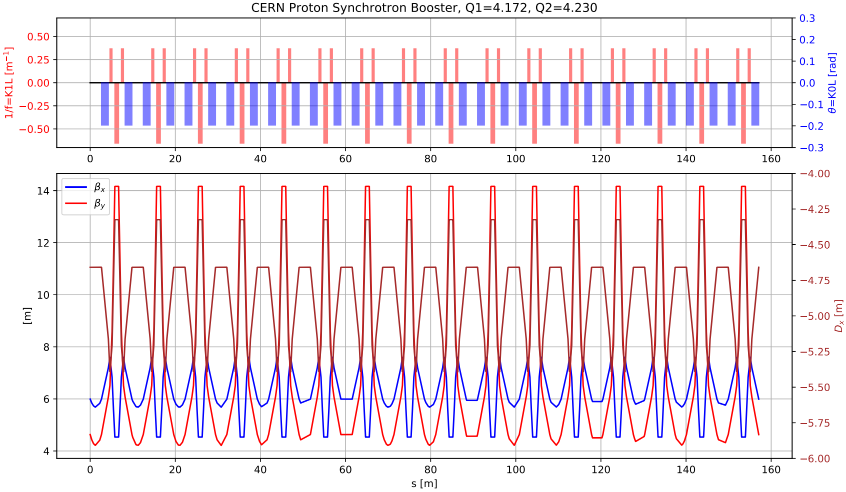
Starting from the FODO we can consider other lattice cells. As an example, by putting back-to-back two OFOD's, we have a triplet cell (OFODDOFO).

An example of triplet lattice analysis is presented in LatticeCellStudies.ipynb, where the stability condition is discussed.

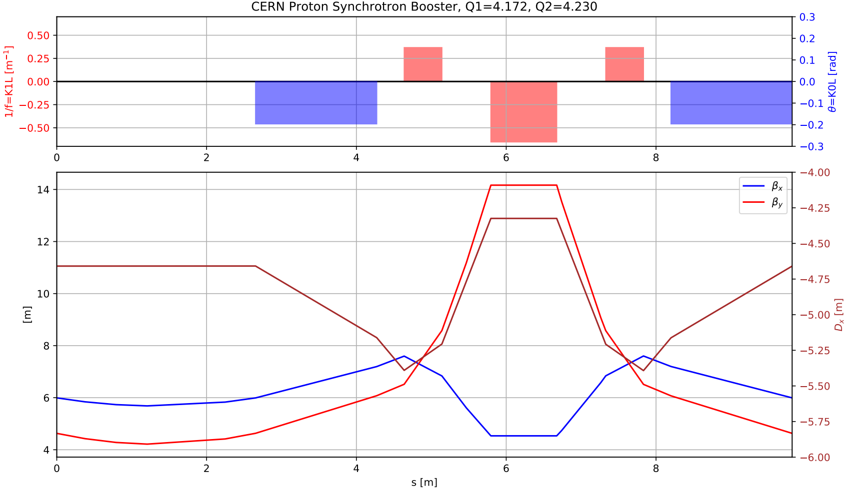
In the following we present few of the CERN Accelerator Complex optics (acc-models.web.cern.ch).



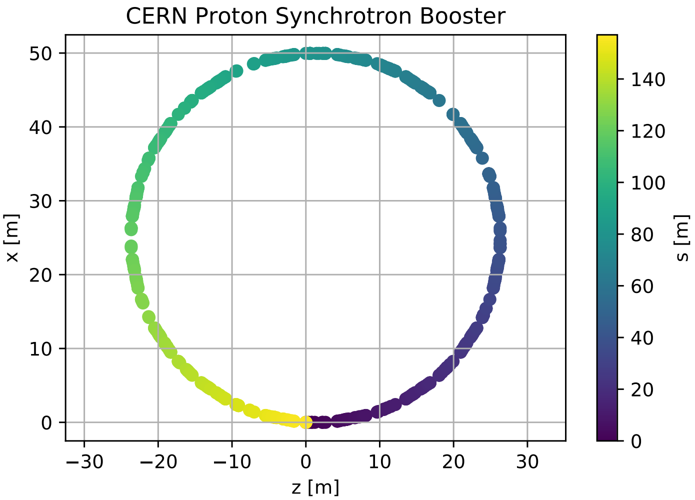
CERN Proton Synchrotron Booster



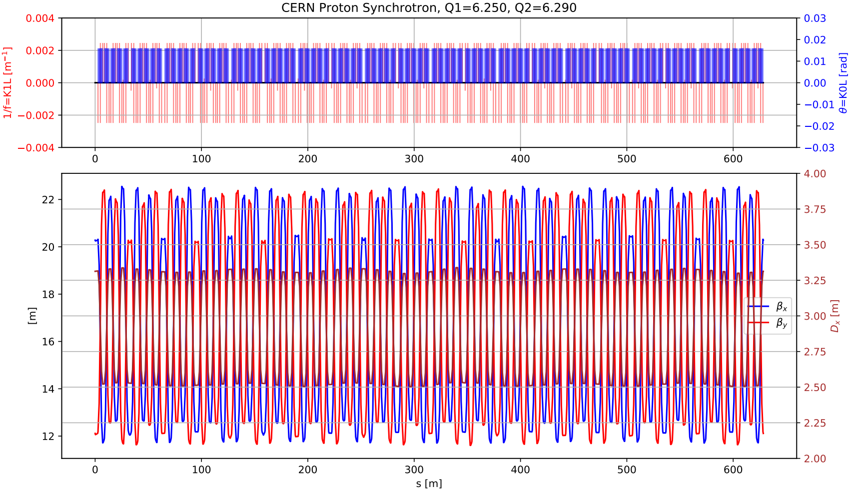
CERN Proton Synchrotron Booster



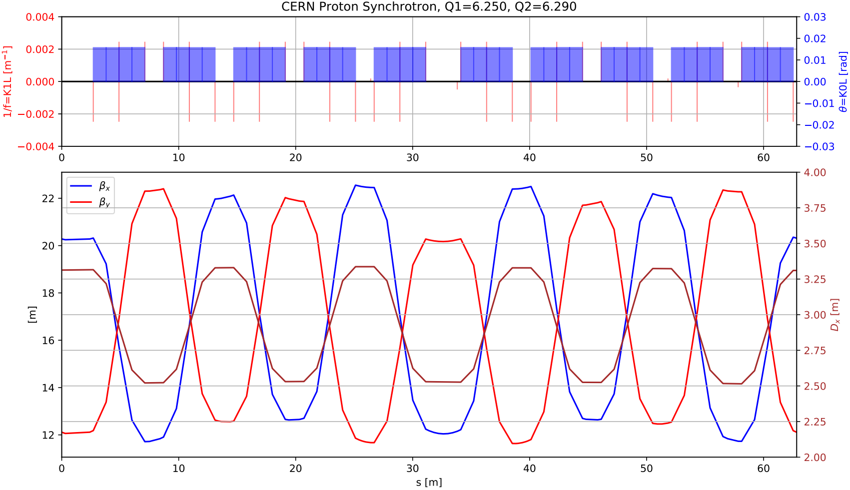
CERN Proton Synchrotron Booster



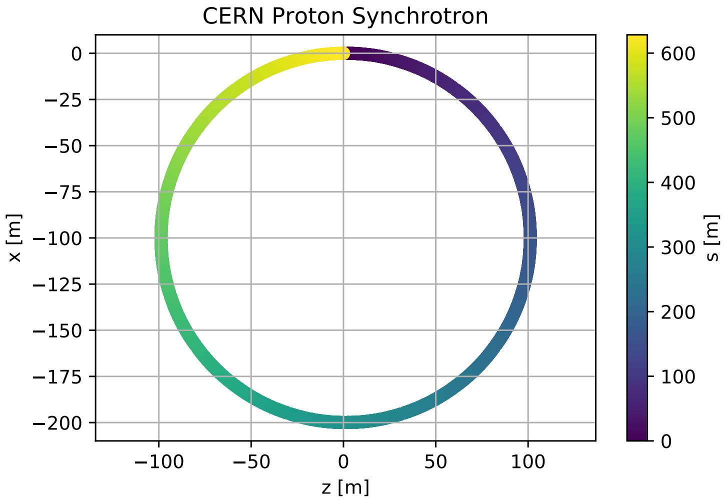
CERN Proton Synchrotron



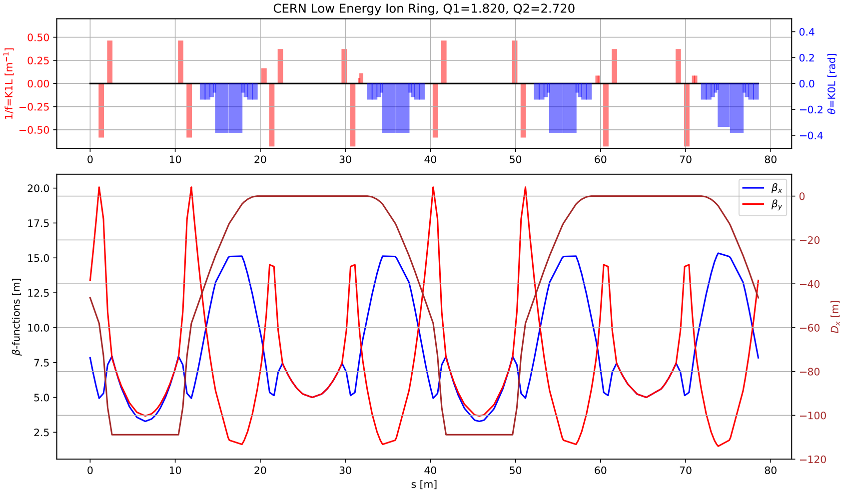
CERN Proton Synchrotron



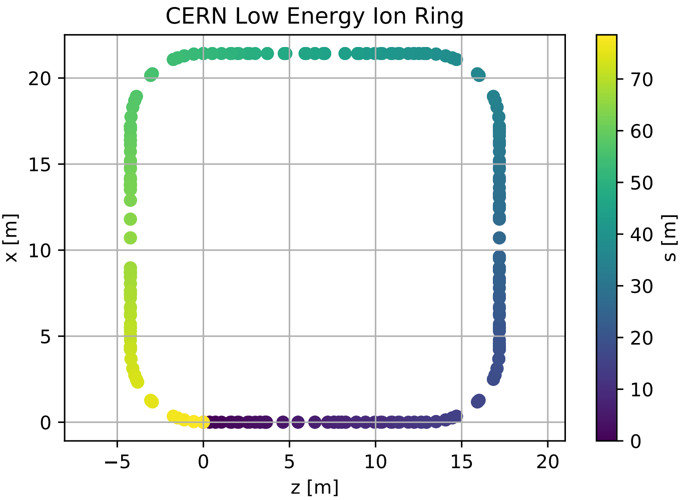
CERN Proton Synchrotron



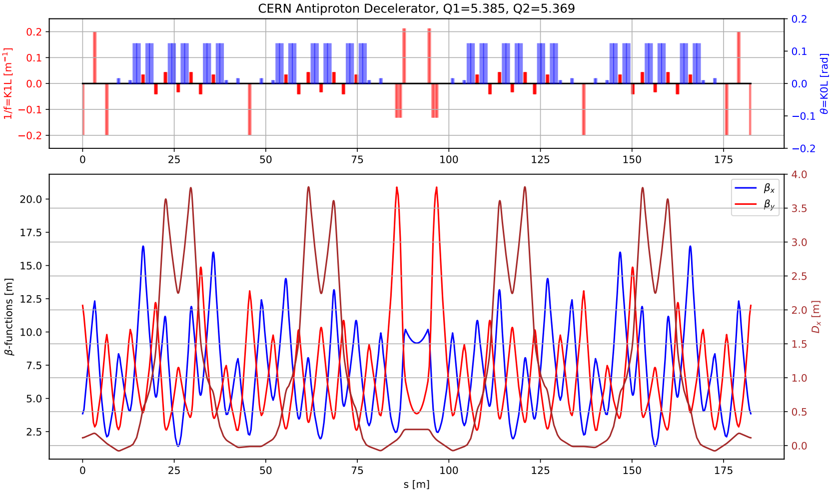
CERN Low Energy Ion Ring



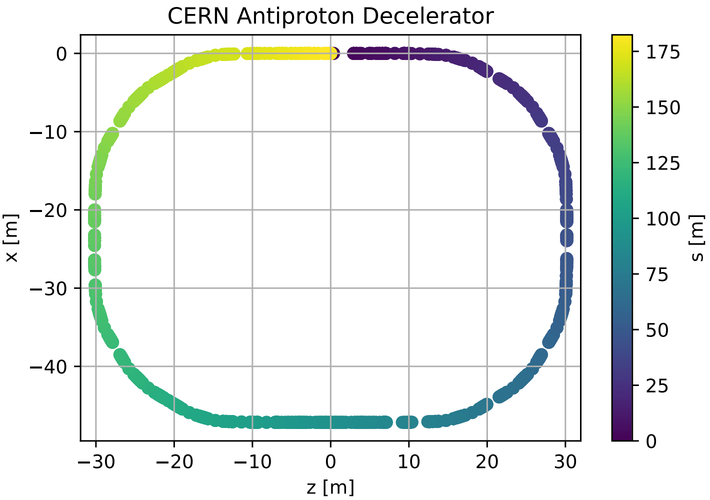
CERN Low Energy Ion Ring



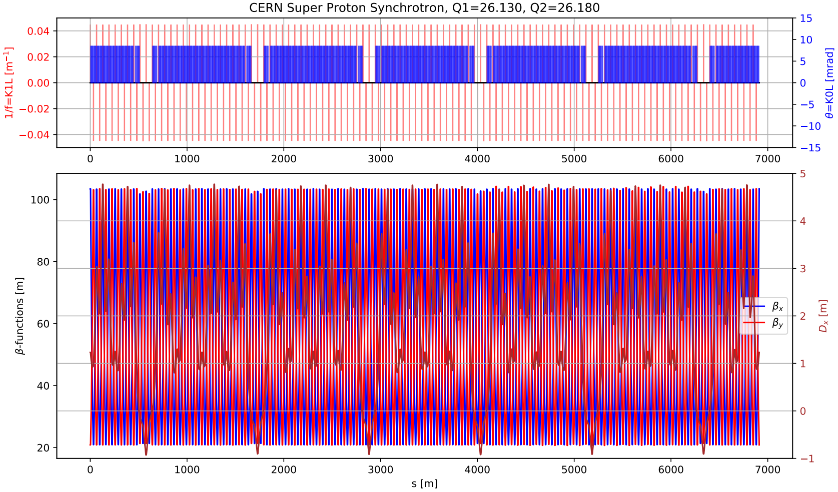
CERN Antiproton Deceleration



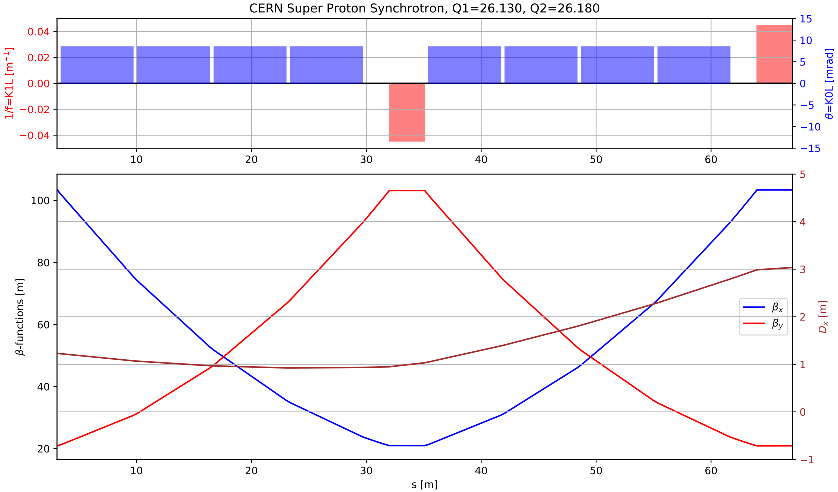
CERN Antiproton Deceleration



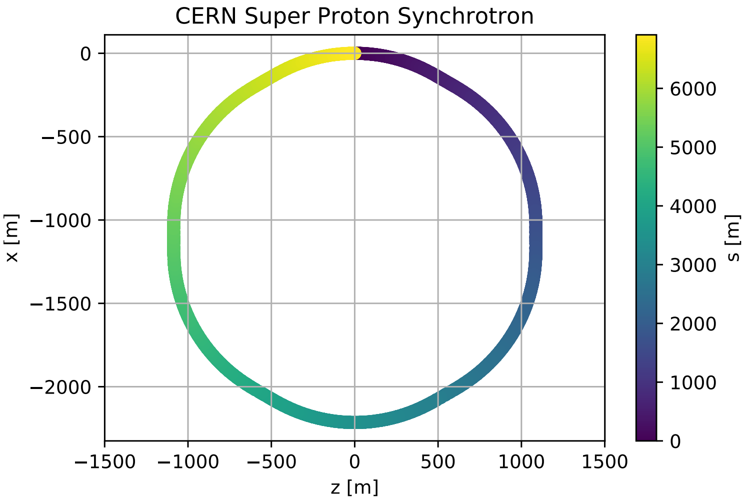
CERN Super Proton Synchrotron



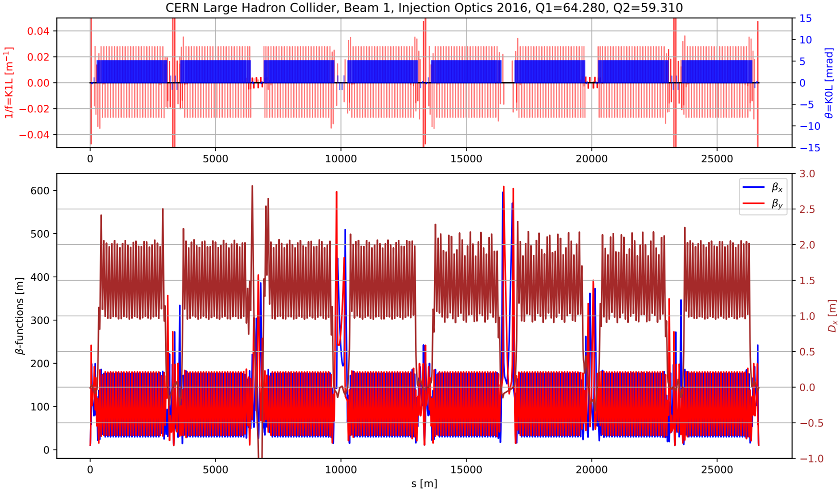
CERN Super Proton Synchrotron



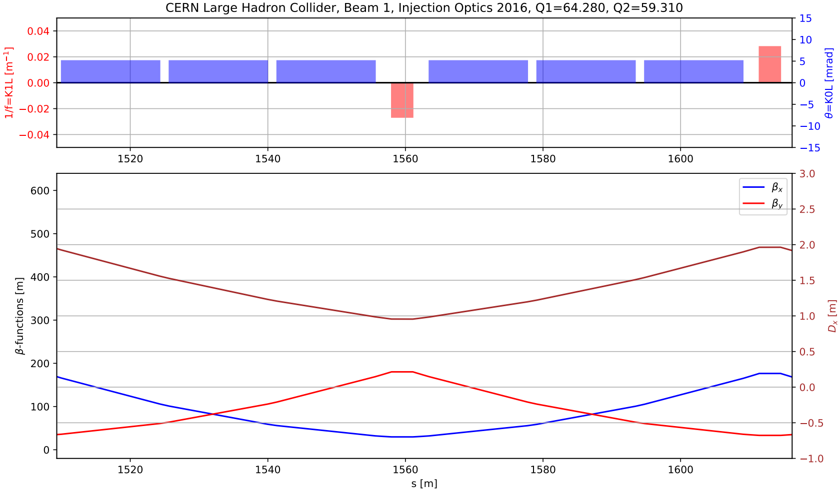
CERN Super Proton Synchrotron



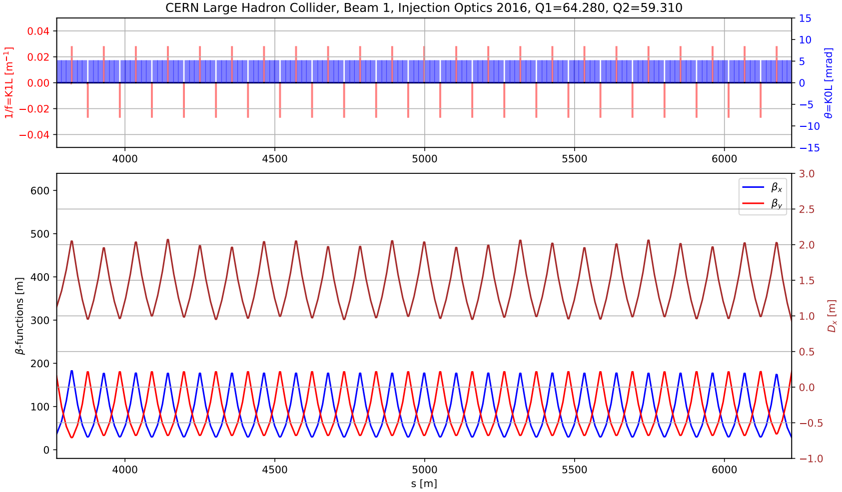
CERN Large Hadron Collider Beam 1



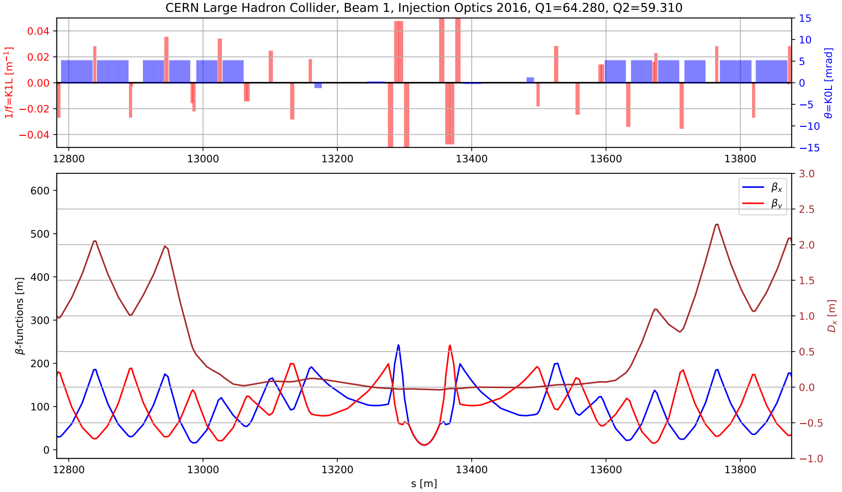
CERN Large Hadron Collider Beam 1



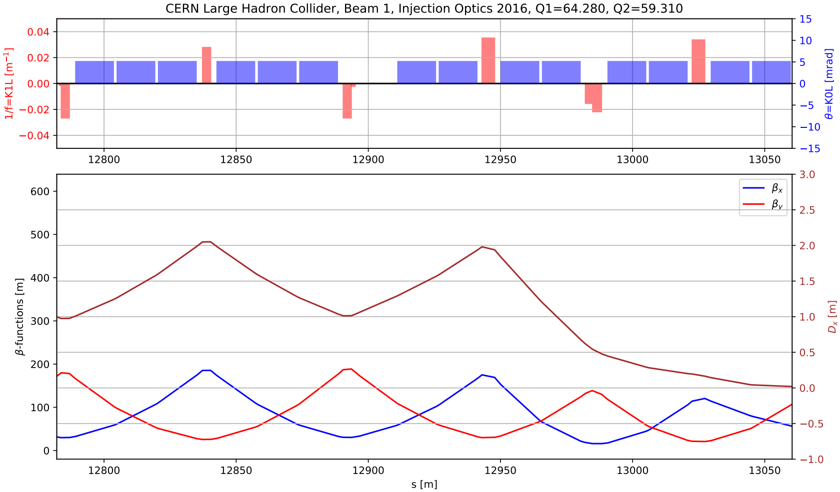
CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1