LONGITUDINAL DYNAMICS
RECAP

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The CERN Accelerator School

Advanced Accelerator Physics Course
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Summary of the 2 lectures:

- Acceleration methods
- Accelerating structures
- Linac: Phase Stability + Energy-Phase oscillations
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron, Transition
- Stability and Longitudinal Phase Space Motion
- Hamiltonian
- Stationary Bucket
- Injection Matching

Including selected topics from other CAS lectures :

- Linacs - *Alessandra Lombardi*
- RF Systems - *Erk Jensen, me*
- Timing, Synchronization & Longitudinal Aspects - *Heiko Damerau*
- Electron Beam Dynamics - *Lenny Rivkin*

Continued by

- RF Manipulations – *Helga Timko*
- RF Feedbacks – *Heiko Damerau*
- Beam Loading – *Heiko Damerau*
- Beam Instabilities – Longitudinal – *Giovanni Rumolo*

Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:

- electrons reach a constant velocity (~speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy

  -> need different types of resonators, optimized for different velocities

  -> the **revolution frequency will vary**, so the RF frequency will be changing

  -> magnetic field needs to follow the momentum increase

Particle rest mass m_0:

electron	0.511 MeV
proton	938 MeV
^{239}U	~220000 MeV

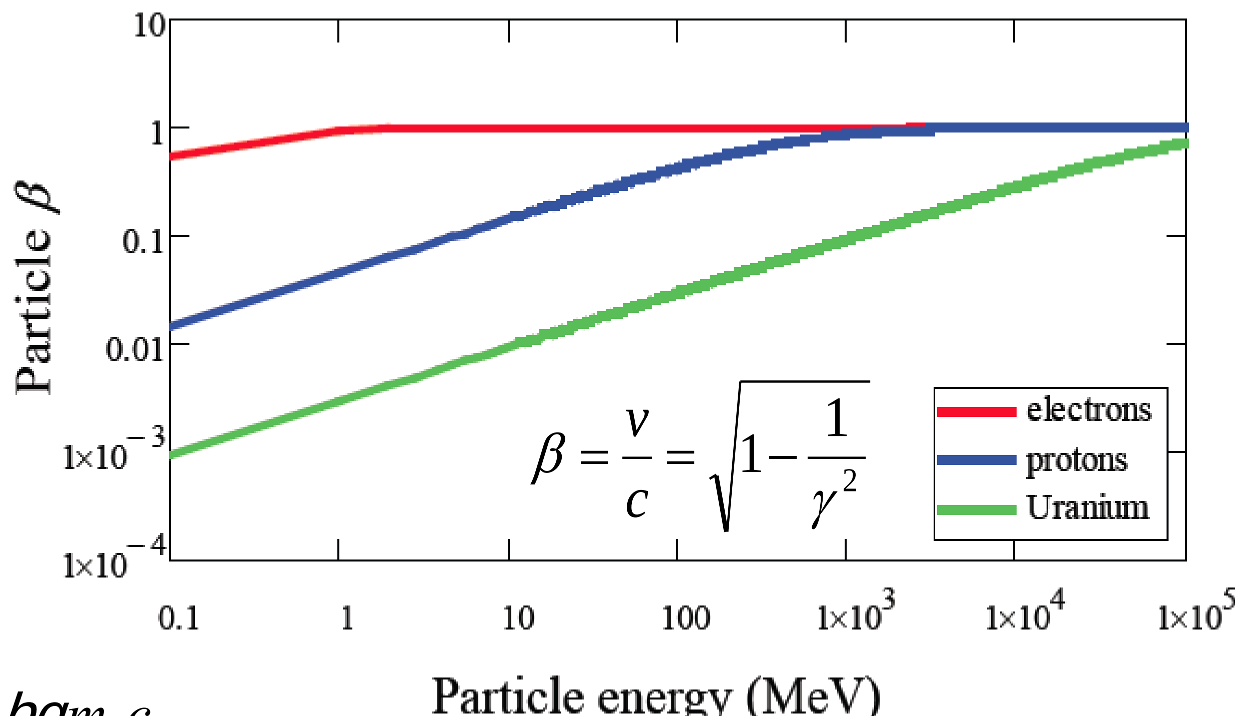
Total Energy: $E = g m_0 c^2$

Relativistic gamma factor:

$$g = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - b^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2} bc = b\frac{E}{c} = bg m_0 c$$



Revolution frequency variation

The revolution and RF frequency will be changing during acceleration
Much more important for lower energies (values are kinetic energy - protons).

PS Booster: 50 MeV (β= 0.314) -> 1.4 GeV (β=0.915)
(pre LS2) 602 kHz -> 1746 kHz => **190% frequency increase**
(post LS2): 160 MeV (β= 0.520) -> 2 GeV (β=0.948) => **95% increase**

PS: 1.4 GeV (β=0.915) -> 25.4 GeV (β =0.9994)
(pre LS2) 437 KHz -> 477 kHz => **9% increase**
(post LS2): 2 GeV (β=0.948) -> 25.4 GeV (β =0.9994) => **5% increase**

SPS: 25.4 GeV -> 450 GeV (β=0.999998)
=> **0.06% frequency increase**

LHC: 450 GeV -> 7 TeV (β= 0.999999991)
=> only **2 10^{-6} increase**

RF system needs more flexibility in lower energy accelerators.

Question: What about electrons and positrons?

Acceleration + Energy Gain

May the force be with you!

To accelerate, we need a **force in the direction of motion**!

Newton-Lorentz Forc
on a charged particle

n always perpendicular
on => no acceleration

Hence, it is necessary
(preferably) along the
which changes the mo

$$\boxed{\frac{dp}{dt} = qE_z}$$

In relativistic dynami

nked by

$$E^2 = E_0^2 + p^2c^2$$

$dE = c^2 mv / E \, dp = v dp)$

The rate of energy ga

z) is then:

$$\frac{dE}{dz} = v$$

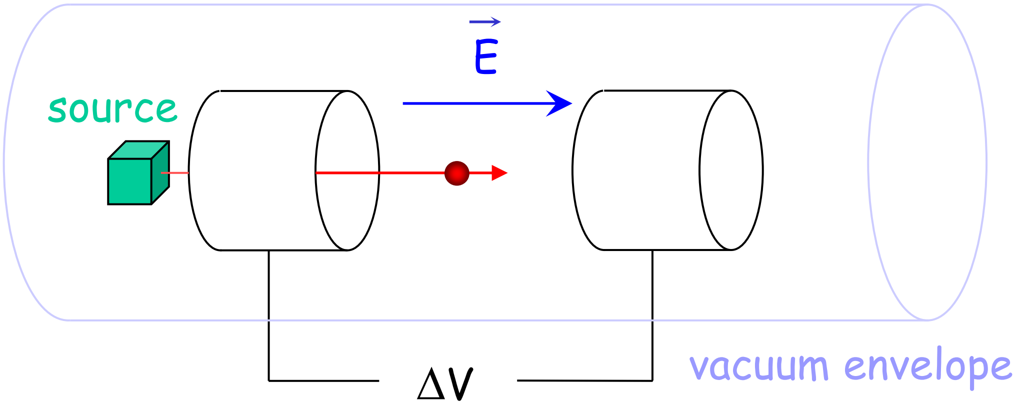
and the kinetic energ

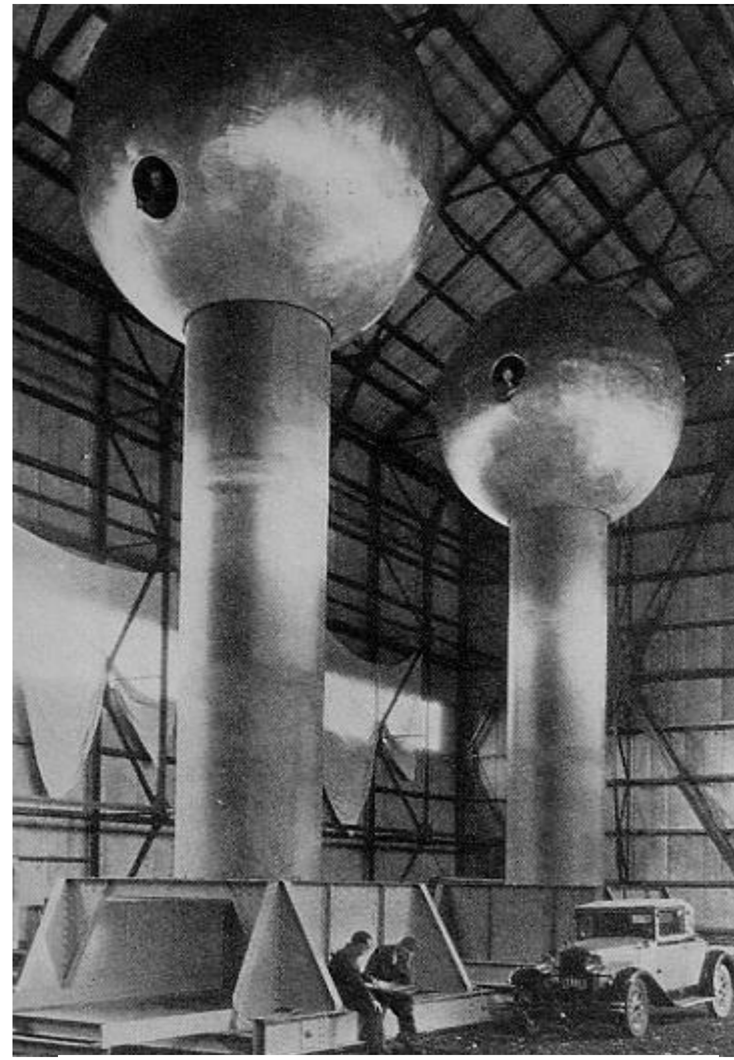
is:

$$dW = dE = qE_z$$

- V is a potential
- q the charge

Electrostatic Acceleration





Van-de-Graaf generator at MIT

Electrostatic Field:

Force: $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = q\,\vec{E}$

Energy gain: W = q ΔV

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Methods of Acceleration: Time varying fields

The electrostatic field is limited by insulation,
the magnetic field does not accelerate.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$

From Maxwell's Equations:

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \mu\vec{H} = \vec{\nabla} \times \vec{A} \qquad \text{or} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The electric field is derived from a scalar potential φ and a vector potential *A*
The time variation of the magnetic field H generates an electric field E

The **solution**: => ***time varying electric fields !***

1) Induction
2) RF frequency fields

Consequence: We can only accelerate bunched beam!

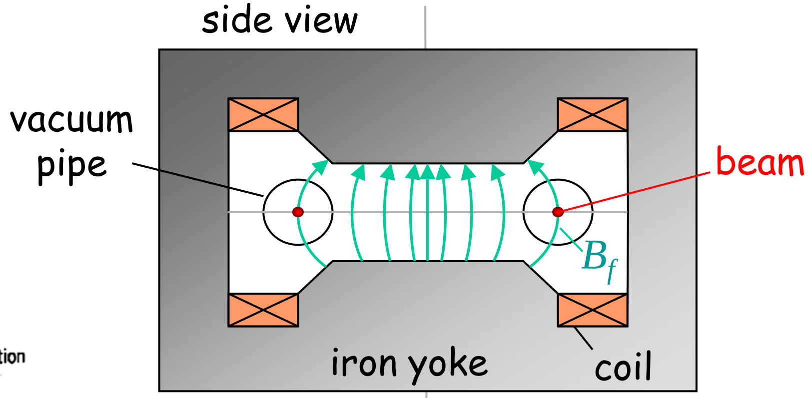
Acceleration by Induction: The Betatron

It is based on the principle of a transformer:
- primary side: large electromagnet
- secondary side: electron beam.

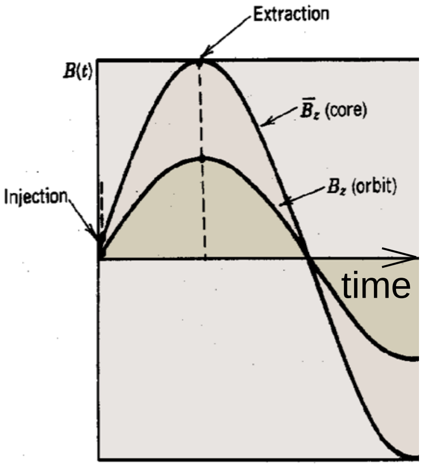
The ramping magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

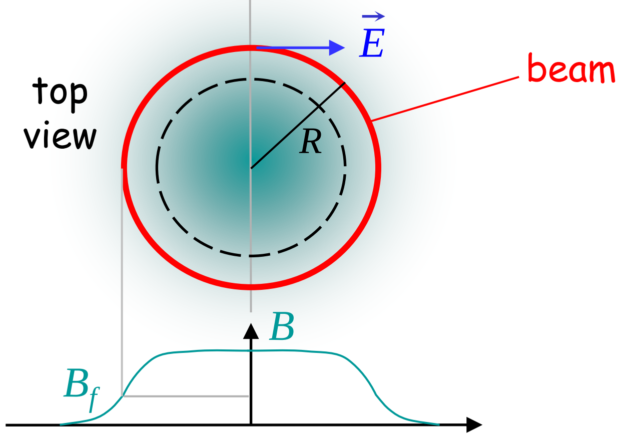
Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons

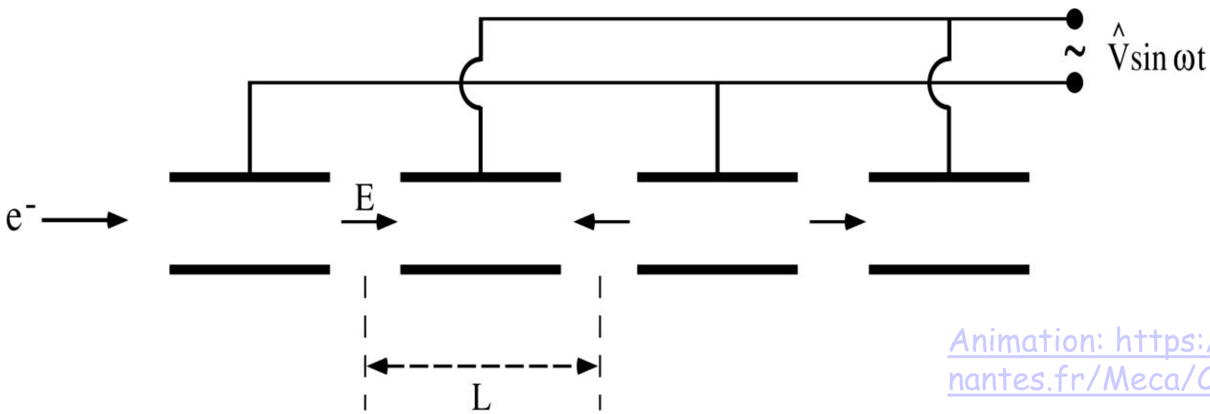


Donald Kerst with the first betatron, invented at the University of Illinois in 1940





Radio-Frequency (RF) Acceleration



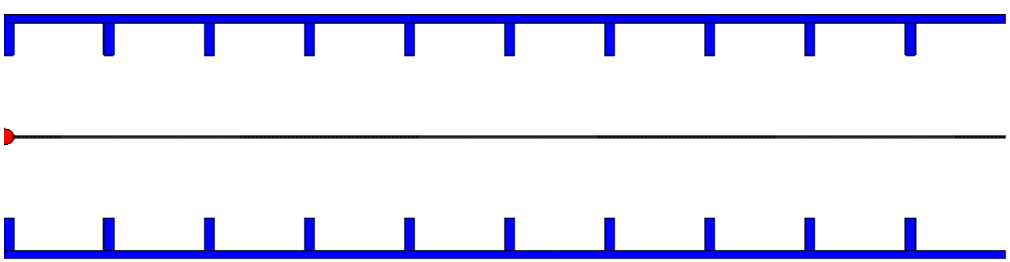
Widerøe-type structure

Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/linac.php

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition ⟶ $L = v\,T/2$

v = particle velocity
T = RF period



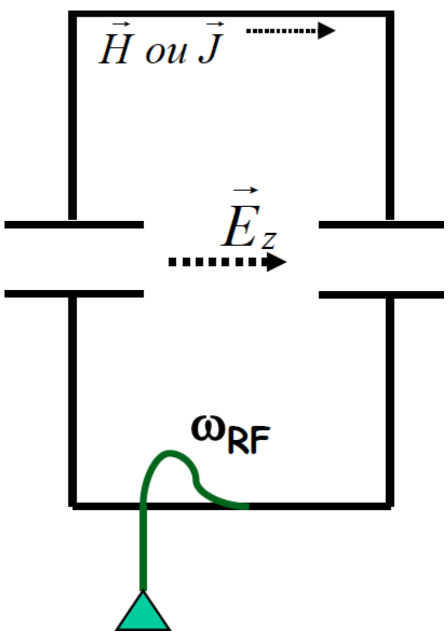
D.Schulte

Similar for standing wave cavity as shown (with $v \approx c$)

Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer, and one loses on the efficiency.
  => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
  => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

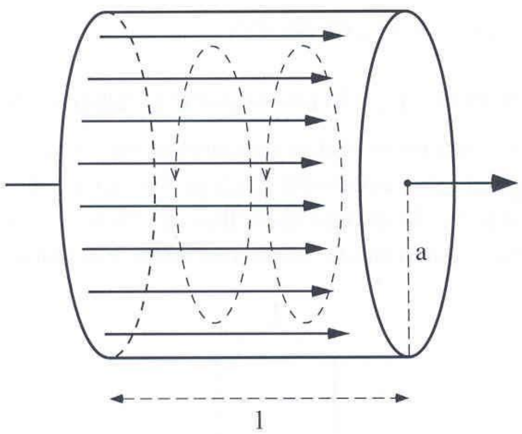


- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

The Pill Box Cavity



$\longrightarrow E_z \quad$ - - - -> H_θ

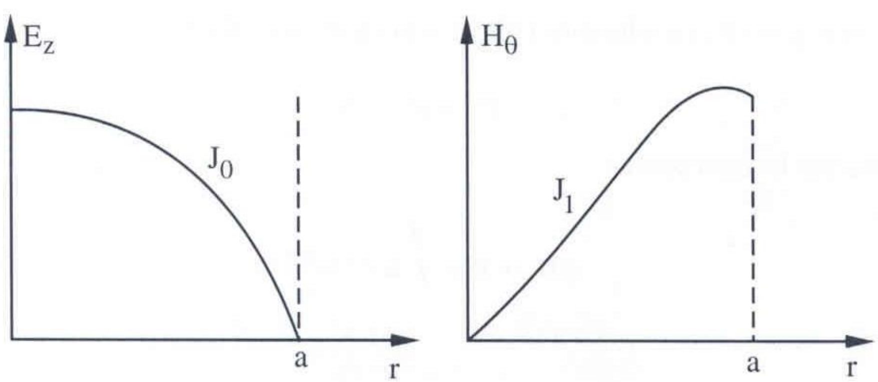
From Maxwell's equations one can derive the wave equations:

$$\nabla^2 A - \epsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ or } H)$$

Solutions for E and H are oscillating modes, at **discrete frequencies**, of types TM_{xyz} (transverse magnetic) or TE_{xyz} (transverse electric).

Indices linked to the number of field knots in polar co-ordinates φ, r and z.

For l<2a the simplest mode, TM_{010}, has the lowest frequency, and has only two field components:



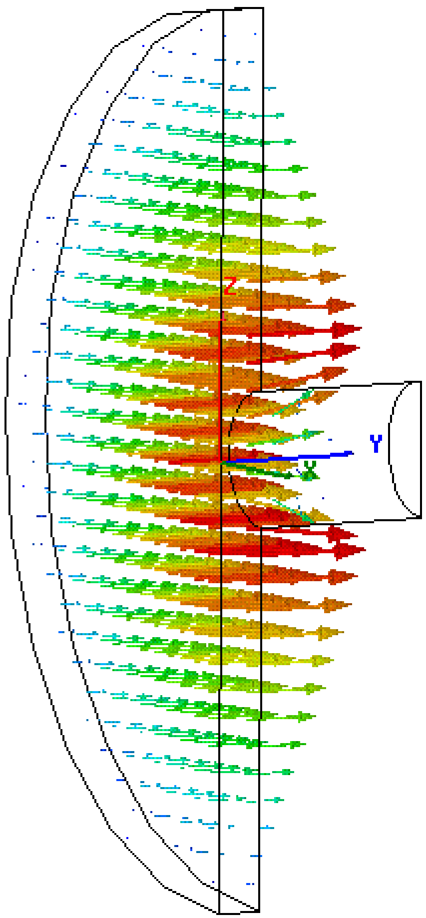
$$E_z = J_0(kr)\, e^{i\omega t}$$

$$H_\theta = -\frac{i}{Z_0} J_1(kr)\, e^{i\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \qquad \lambda = 2.62a \qquad Z_0 = 377\Omega$$

The Pill Box Cavity

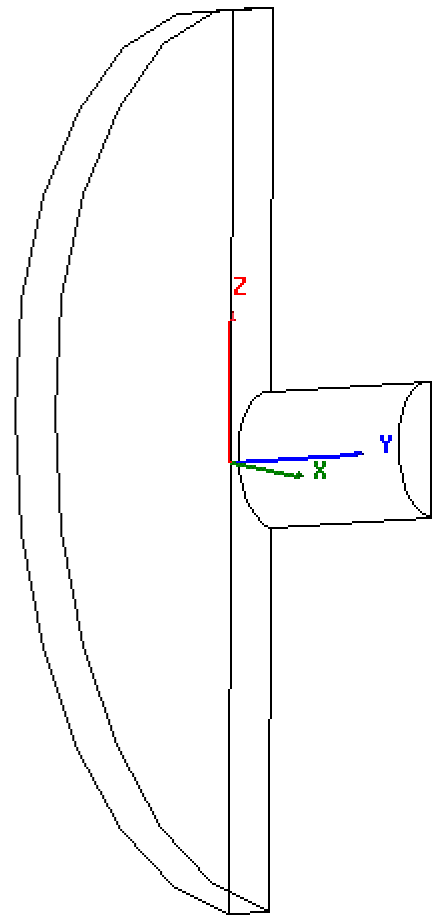
One needs a hole for the beam pipe – circular waveguide below cutoff



electric field

TM_{010}-mode

(only 1/4 shown)



magnetic field

Transit time factor

The accelerating field varies during the passage of the particle
=> particle does not always see maximum field => effective acceleration smaller

Transit time factor defined as:

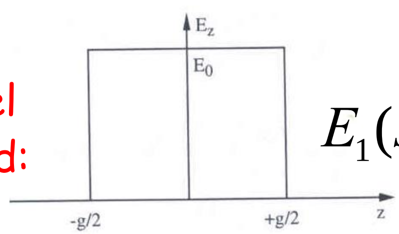
$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \to \infty)}$$

In the general case, the transit time factor is:

$$T_a = \frac{\left| \int_{-\infty}^{+\infty} E_1(s,r) \cos\left(\omega_{RF} \frac{s}{v}\right) \mathrm{d}s \right|}{\int_{-\infty}^{+\infty} E_1(s,r)\, \mathrm{d}s}$$

for $E(s,r,t) = E_1(s,r) \cdot E_2(t)$

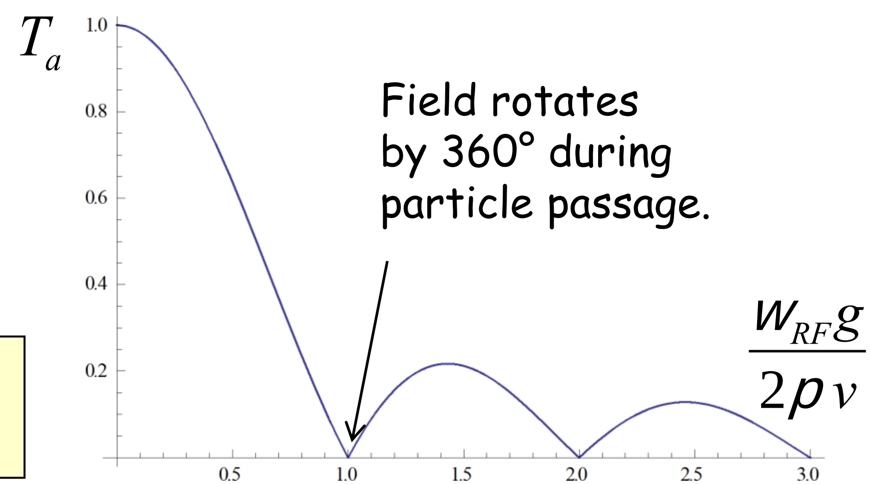
Simple model uniform field:



$$E_1(s,r) = \frac{V_{RF}}{g}$$

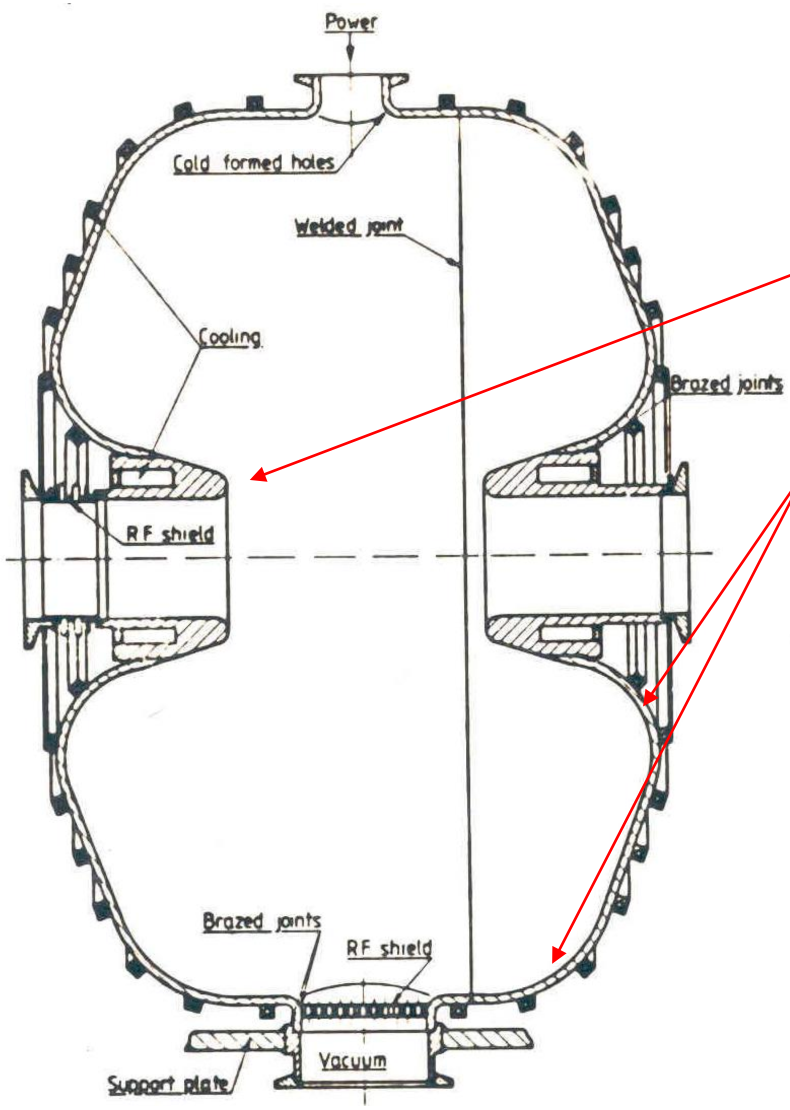
follows:

$$T_a = \left| \sin\frac{\omega_{RF}\, g}{2v} \middle/ \frac{\omega_{RF}\, g}{2v} \right|$$



$0 < T_a < 1$, $T_a \to 1$ for $g \to 0$, smaller ω_{RF}
Important for low velocities (ions)

The Pill Box Cavity (2)



The design of a cavity can be sophisticated in order to improve its performances:

- A nose cone can be introduced in order to concentrate the electric field around the axis

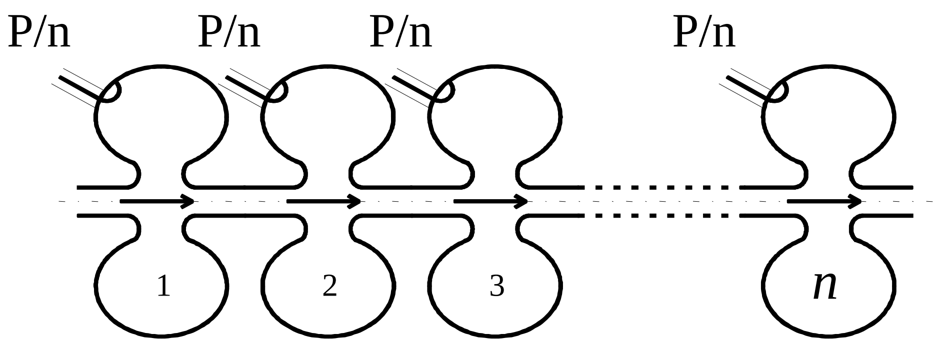
- Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses.
It also prevents from multipactoring effects (e- emission and acceleration).

A good cavity efficiently transforms the RF power into accelerating voltage.

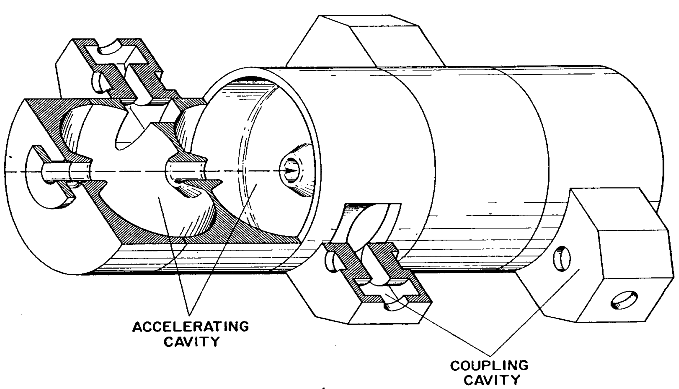
Simulation codes allow precise calculation of the properties.

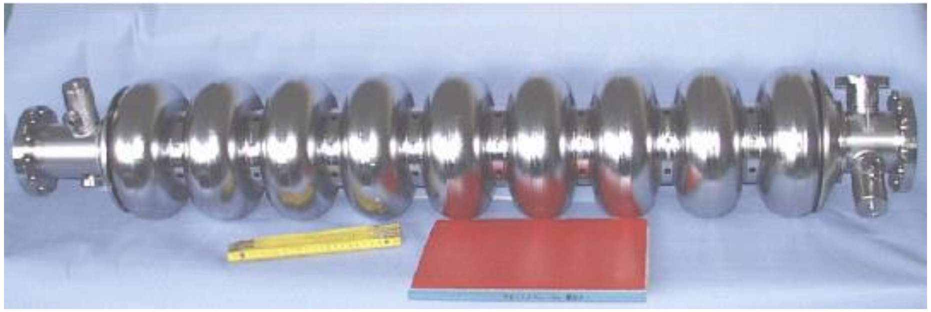
Multi-Cell Cavities

Acceleration of one cavity limited => distribute power over several cells
Each cavity receives P/n
Since the field is proportional √P, you get $\sum E_i \propto n\sqrt{P/n} = \sqrt{n}E_0$



Instead of distributing the power from the amplifier, one might as well couple the cavities, such that the power automatically distributes, or have a cavity with many gaps (e.g. drift tube linac).





Multi-Cell Cavities - Modes

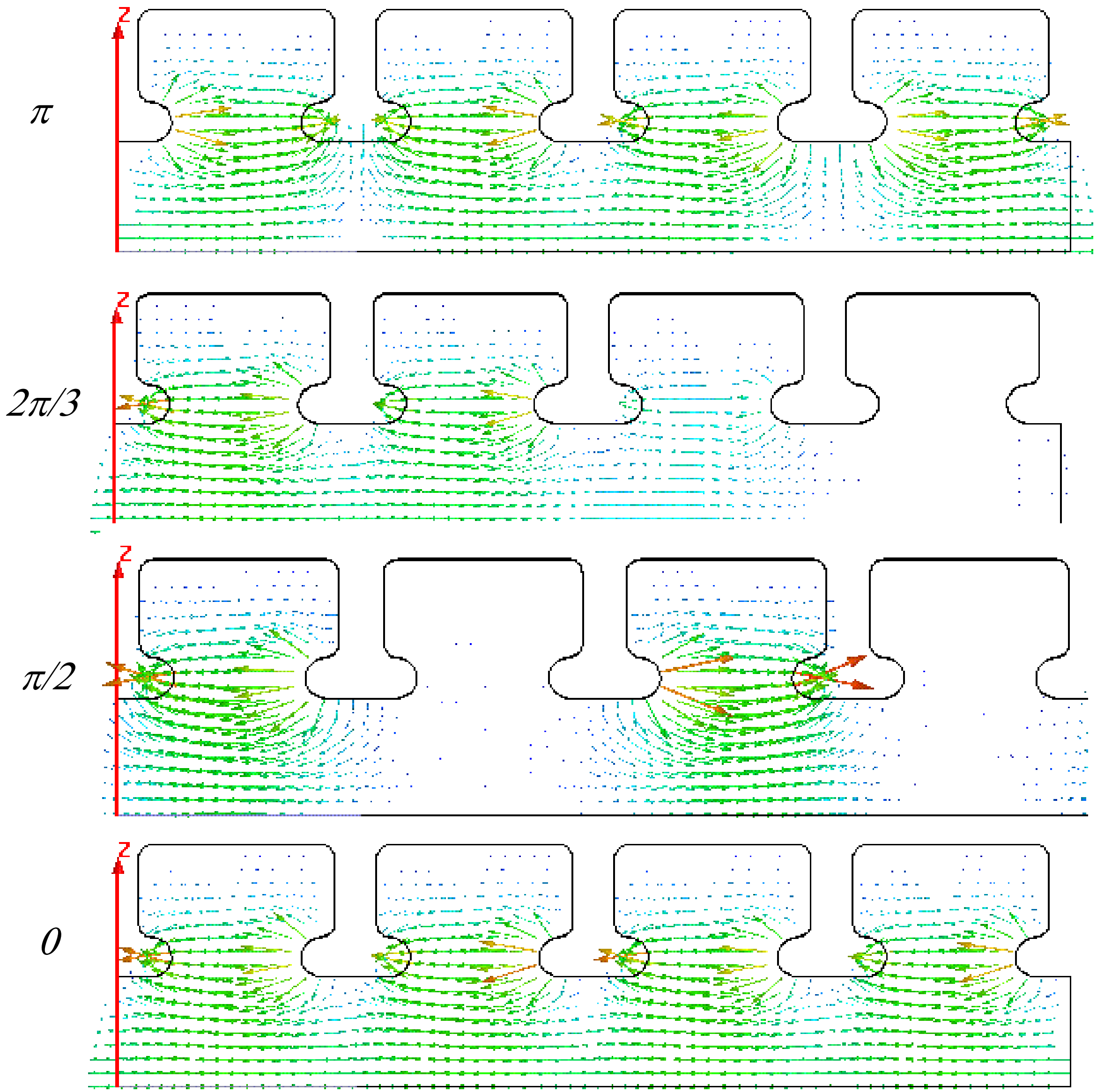
The phase relation between gaps is important!

Coupled harmonic oscillator

=> Modes, named after the phase difference between adjacent cells.

=> Different synchronism conditions for the cell length L and relativistic β

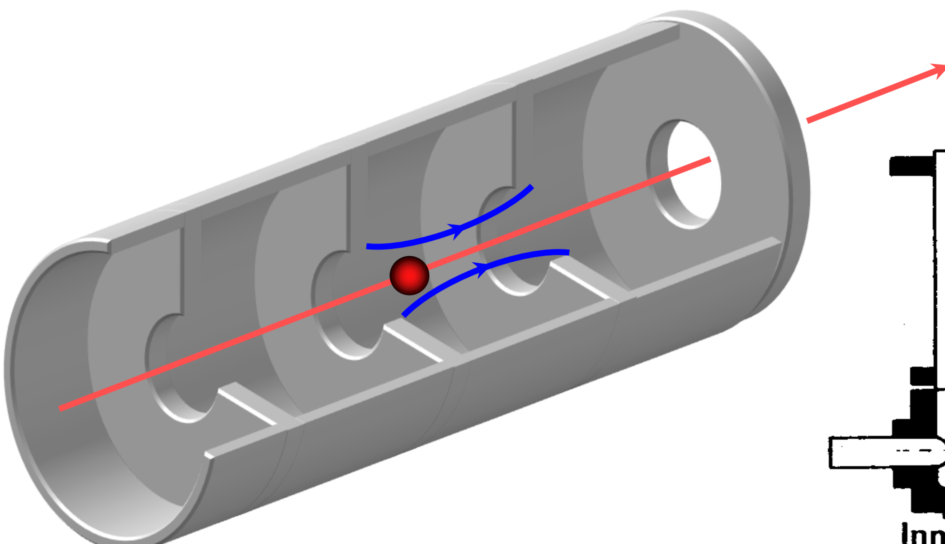
Mode	L
0 (2π)	βλ
π/2	βλ/4
2π/3	βλ/3
π	βλ/2

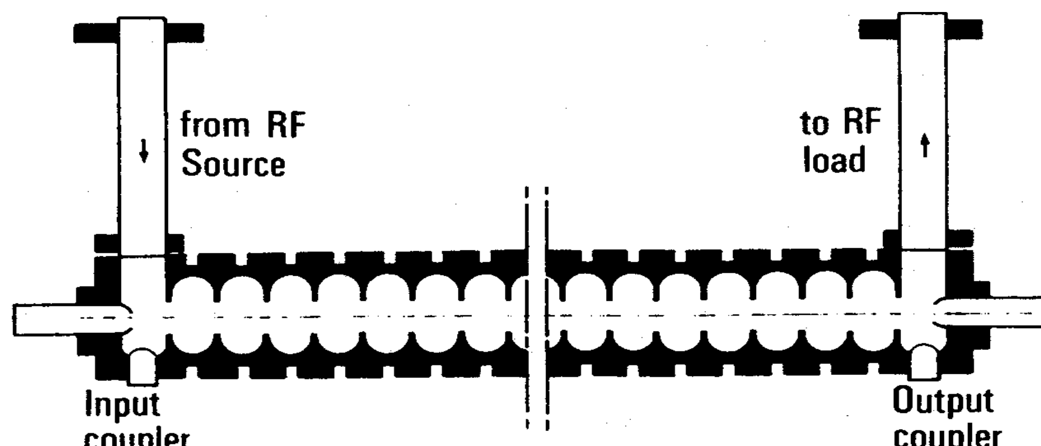


Disc-Loaded Traveling-Wave Structures

When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

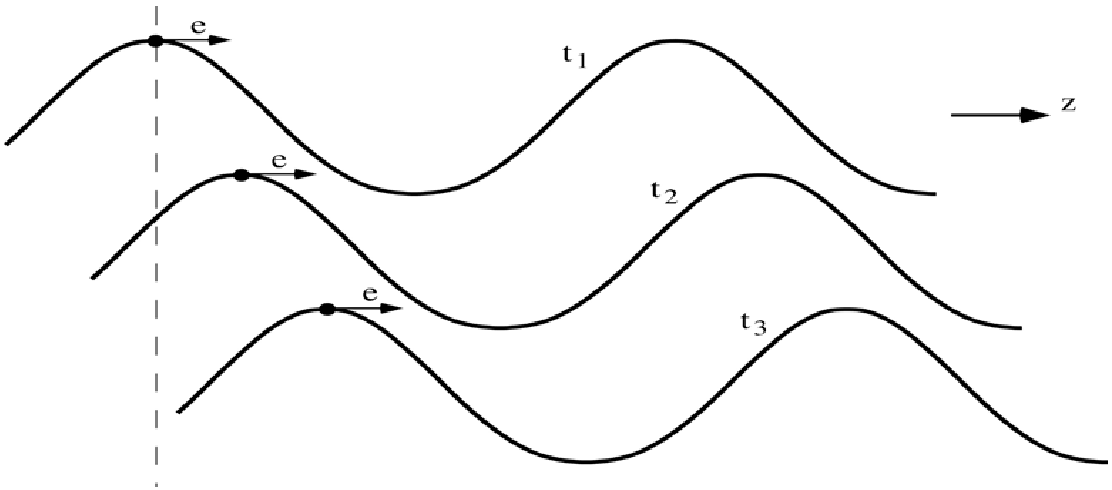
Next came the idea of suppressing the drift tubes using traveling waves. However, to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.





solution: slow wave guide with irises ==> iris loaded structure

The Traveling Wave Case



$$E_z = E_0 \cos\left(\omega_{RF} t - kz\right)$$

$$k = \frac{\omega_{RF}}{v_\varphi} \qquad \text{wave number}$$

$$z = v(t - t_0)$$

v_φ = phase velocity
v = particle velocity

The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

If synchronism satisfied: $v = v_\varphi$ and $E_z = E_0 \cos\phi_0$

where Φ_0 is the RF phase seen by the particle.

Cavity Parameters: Quality Factor Q

The total energy stored is $$W = \iiint_{cavity} \left(\frac{\varepsilon}{2}\left|\vec{E}\right|^2 + \frac{\mu}{2}\left|\vec{H}\right|^2\right) dV.$$

- Quality Factor Q (caused by wall losses) defined as

$$Q_0 = \frac{\omega_0 W}{P_{loss}}$$

Ratio of stored energy W and dissipated power P_{loss} on the walls in one RF cycle

The Q factor determines the maximum energy the cavity can fill to with a given input power.
Larger Q => less power needed to sustain stored energy.

The Q factor is 2π times the number of rf cycles it takes to dissipate the energy stored in the cavity (down by 1/e).

- function of the geometry and the surface resistance of the material:
  - superconducting (niobium) : Q= 10^{10}
  - normal conducting (copper) : Q=10^4

Important Parameters of Accelerating Cavities

- Accelerating voltage V_{acc}

$$V_{acc} = \int_{-\infty}^{\infty} E_z e^{-i\frac{\omega z}{\beta c}} dz$$

Measure of the acceleration

- R upon Q

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}$$

Relationship between acceleration V_{acc} and stored energy W

independent from material!

Attention: Different definitions are used!

- Shunt Impedance R

$$R = \frac{|V_{acc}|^2}{2P_{loss}}$$

Relationship between acceleration V_{acc} and wall losses P_{loss}

depends on
- material
- cavity mode
- geometry

Important Parameters of Accelerating Cavities (cont.)

- Fill Time t_F

  - **standing wave cavities**:

$$P_{loss} = -\frac{dW}{dt} = \frac{\omega_0}{Q} W$$

Exponential decay of the stored energy W due to losses

$$t_F = \frac{2Q}{\omega_0}$$

time for the field to decrease by 1/e after the cavity has been filled
measure of how fast the stored energy is dissipated on the wall

Several fill times needed to fill the cavity!

  - **travelling wave cavities**:

time needed for the electromagnetic energy to fill the cavity of length L

$$t_F = \int_0^L \frac{dz}{v_g(z)}$$

v_g: velocity at which the energy propagates through the cavity

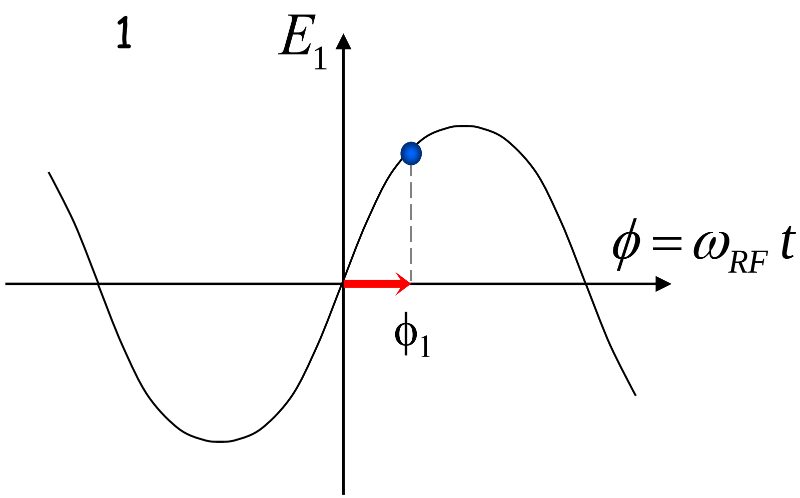
Cavity is completely filled after 1 fill time!

Common Phase Conventions

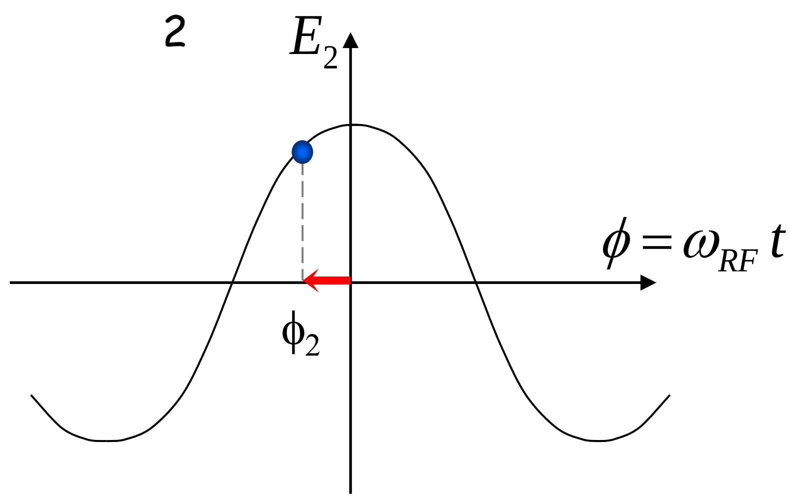
1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_1(t) = E_0 \sin\left(w_{RF}\, t\right)$$



$$E_2(t) = E_0 \cos\left(w_{RF}\, t\right)$$

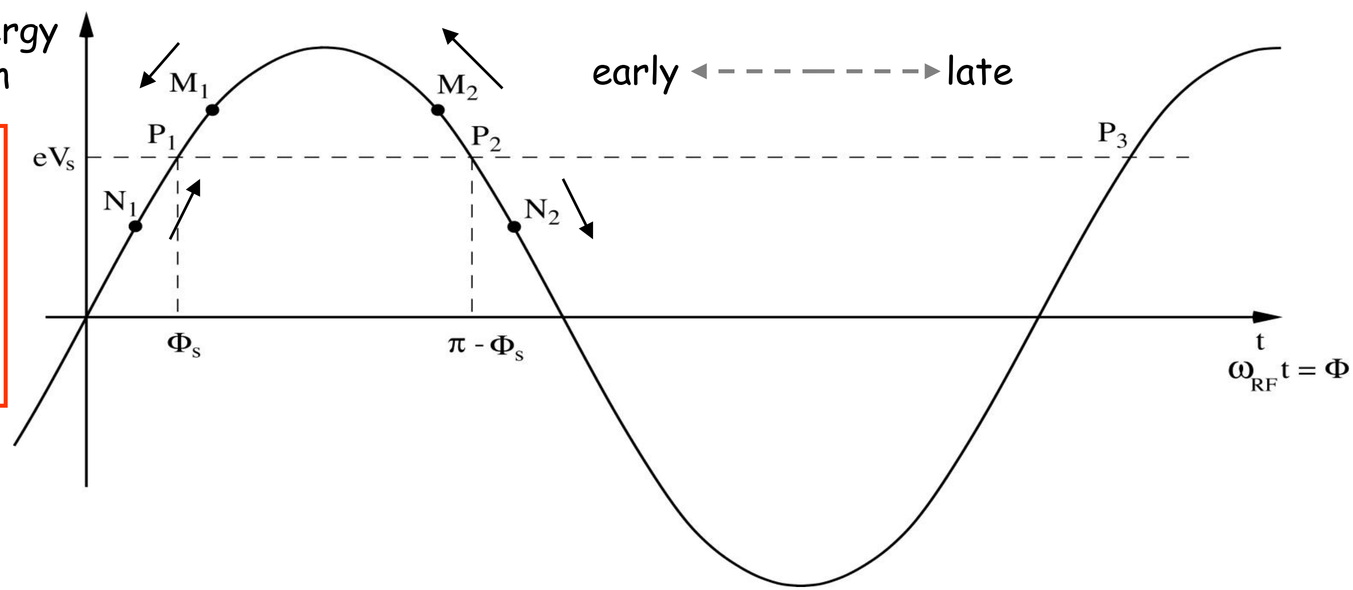
3. I will stick to **convention 1** in the following to avoid confusion

Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$eV_s = e\hat{V}\sin\Phi_s$ **is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.**

For a 2π mode, the electric field is the same in all gaps at any given time.



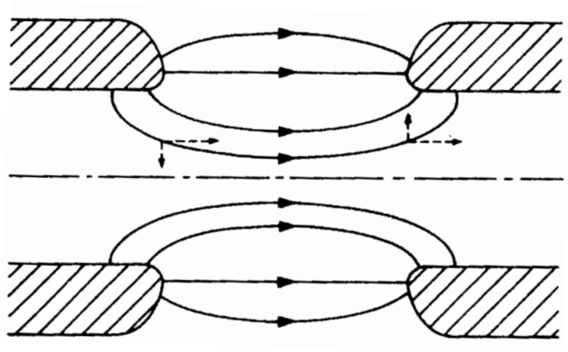
If an energy increase is transferred into a velocity increase =>

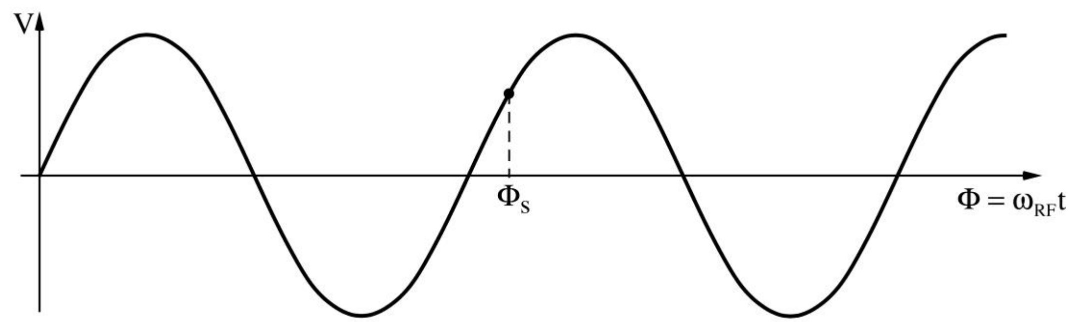
M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability





The divergence of the field is zero according to Maxwell :

$$\nabla \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} = - \frac{\partial E_z}{\partial z}$$

Transverse fields

- focusing at the entrance and
- defocusing at the exit of the cavity.

Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

External focusing (solenoid, quadrupole) is then necessary

Energy-phase Oscillations (Small Amplitude) (1)

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin f_s$$

- Rate of energy gain for a non-synchronous particle, expressed in **reduced variables**, $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$:

$$\frac{dw}{dz} = eE_0\left[\sin\left(\phi_s + \varphi\right) - \sin \phi_s\right] \approx eE_0 \cos \phi_s.\varphi \quad \left(small\ \varphi\right)$$

- Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}\left(v - v_s\right)$$

Leads finally to:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Energy-phase Oscillations (Small Amplitude) (2)

Combining the two 1st order equations into a 2nd order equation gives the equation of a harmonic oscillator:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Slower for higher energy!

Stable harmonic oscillations imply: $W_s^2 > 0 \quad \text{and real}$

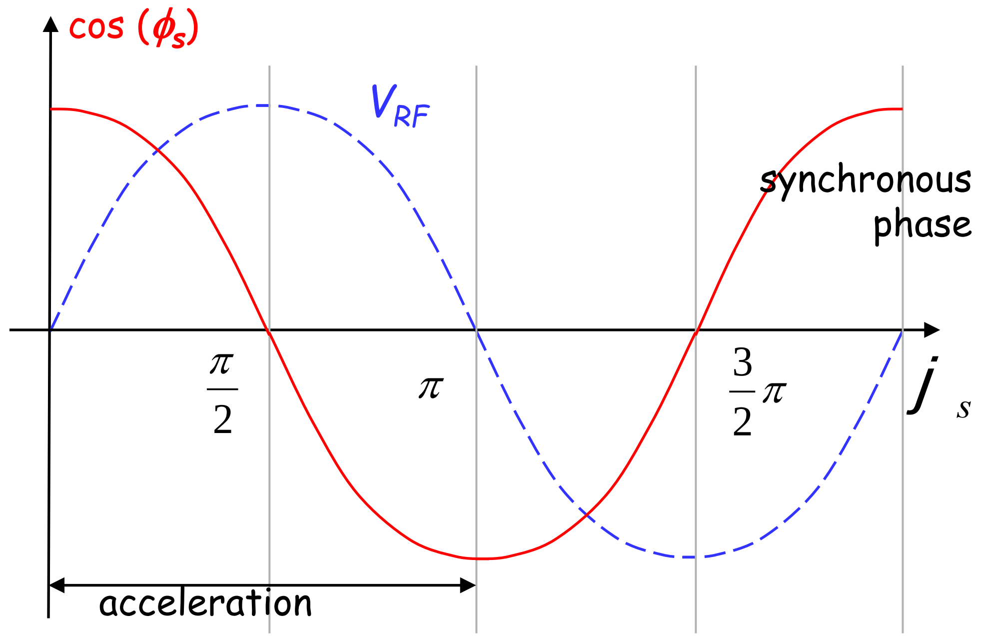
hence: $\cos\phi_s > 0$

And since acceleration also means:

$$\sin\phi_s > 0$$

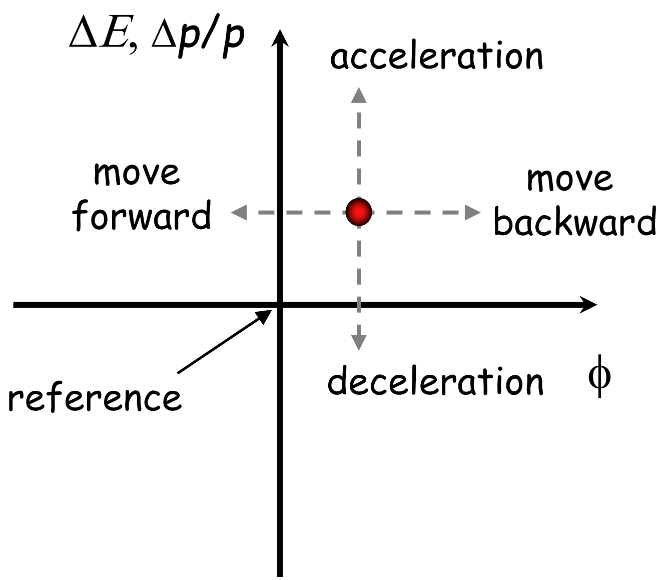
You finally get the result for the stable phase range:

$$0 < \phi_s < \frac{\pi}{2}$$

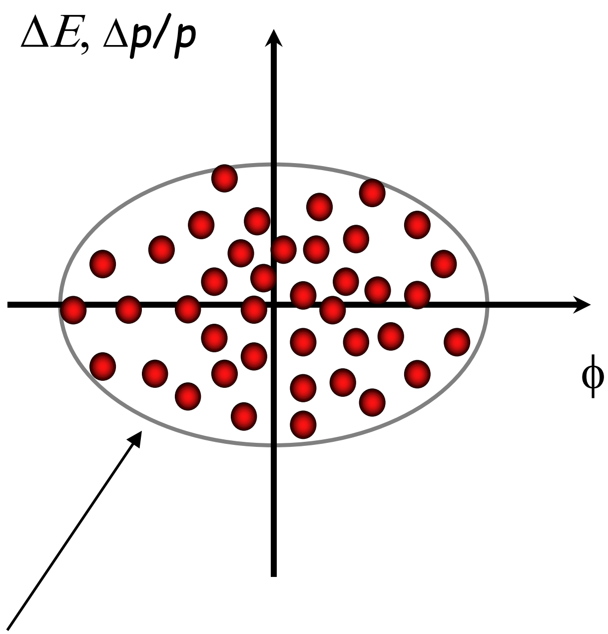


Longitudinal phase space

The energy – phase oscillations can be drawn in phase space:



The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

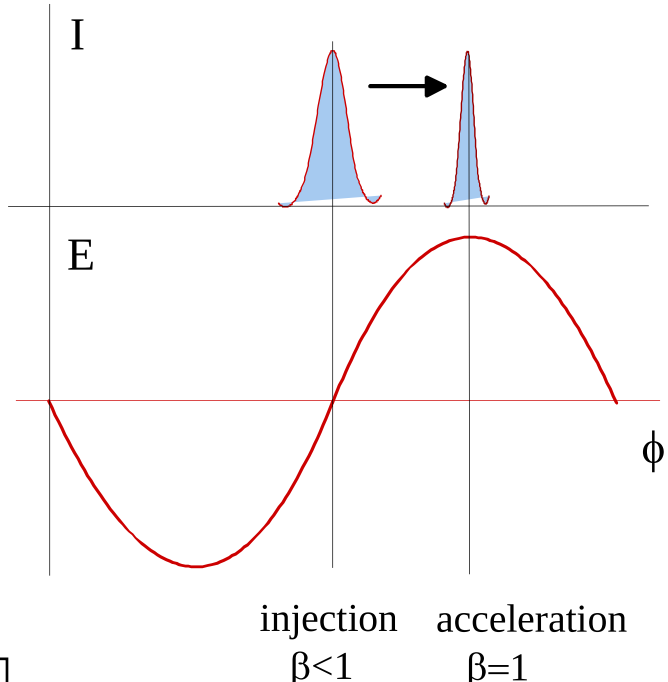
Longitudinal Dynamics - Electrons

At relativistic velocity phase oscillations stop – the bunch is frozen longitudinally.
=> Acceleration can be at the crest of the RF for maximum energy gain.

Electrons injected into a TW structure designed for v=c:

$$\cos\varphi = \cos\varphi_0 + \frac{2\pi}{\lambda_g}\frac{mc^2}{qE_0}\left[\sqrt{\frac{1-\beta}{1+\beta}} - \sqrt{\frac{1-\beta_0}{1+\beta_0}}\right]$$

- at v=c remain at the injection phase.
- at v<c will move from injection phase φ_0 to an asymptotic phase φ, which depends on gradient E_0 and β_0 at injection.



The beam can be injected with an offset in phase, to reach the crest of the wave at β=1

Capture condition, relating gradient E_0 and β_0:

$$E_0 \geq \frac{2\pi}{\lambda_g}\frac{mc^2}{q}\left[\sqrt{\frac{1-\beta_0}{1+\beta_0}}\right]$$

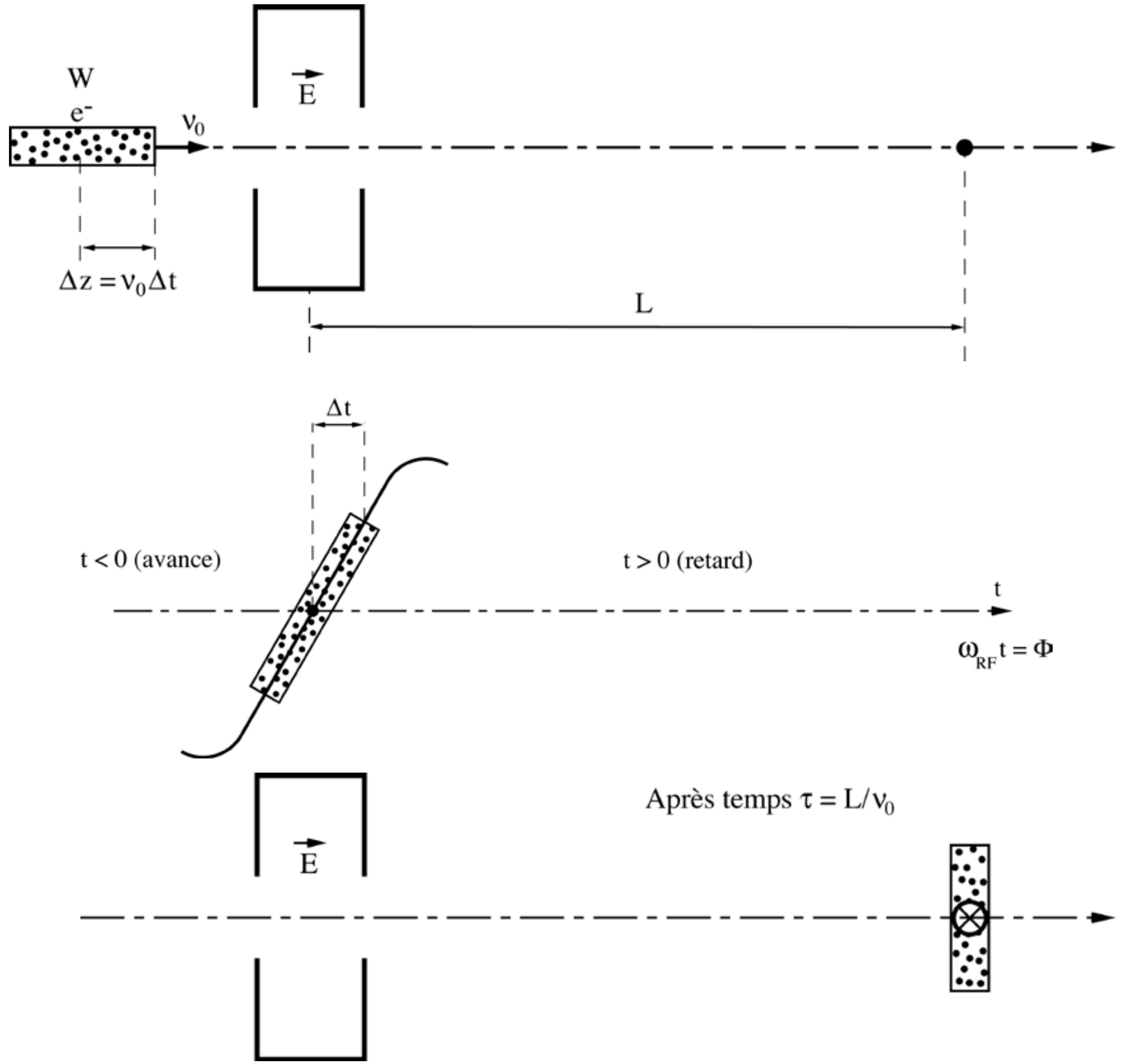
Example: λ=10cm → W_{in}=150 keV for E_0=8 MV/m.

In high current linacs, a bunching and pre-acceleration section up to 4-10 MeV prepares the injection in the TW structure (that occurs already on the crest)

Bunching with a Pre-buncher

A long bunch coming from the gun enters an RF cavity.
The reference particle is the one which has no velocity change. The others get accelerated or decelerated, so the bunch gets an energy and velocity modulation.

After a distance L bunch gets shorter: bunching effect.
This short bunch can now be captured more efficiently by a TW structure (v_ϕ=c).

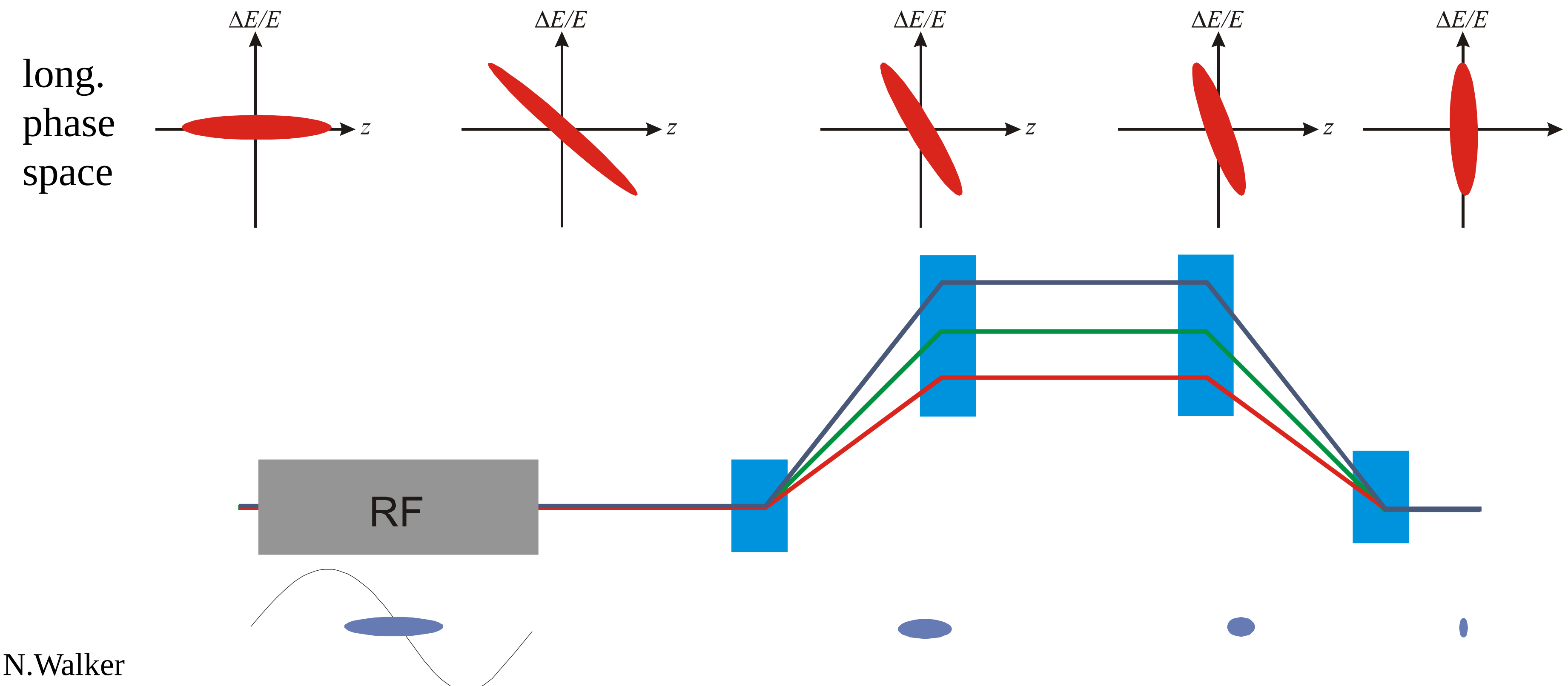


Bunch compression

At ultra-relativistic energies (γ>> 1) the longitudinal motion is frozen.
For linear e+/e- colliders, you need very short bunches (few 100-50μm).

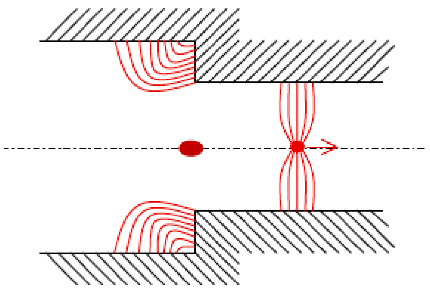
Solution: introduce **energy/time correlation** + a magnetic **chicane.**

Increases energy spread in the bunch => chromatic effects
=> compress at low energy before further acceleration to reduce relative $\Delta E/E$



Longitudinal Wake Fields - Beamloading

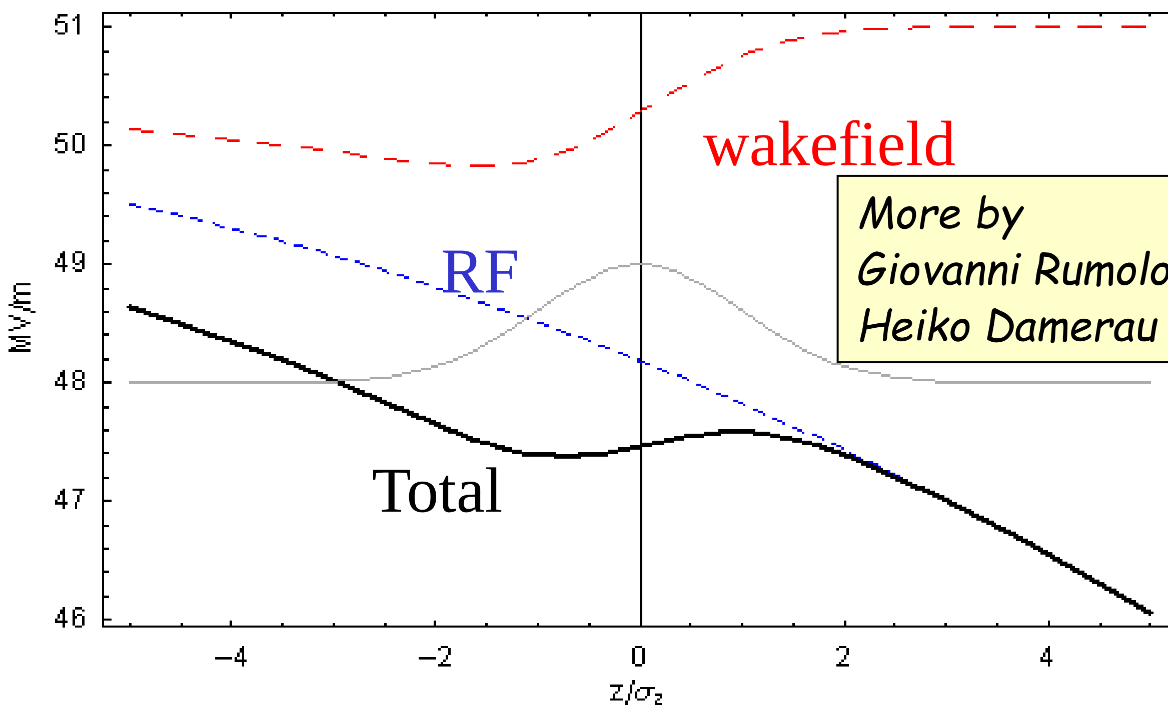
Beam induces wake fields in cavities (in general when chamber profile changing)
⇒ decreasing RF field in cavities
(beam absorbs RF power when accelerated)



Particles within a bunch see a decreasing field
⇒ energy gain different within the single bunch

Locating bunch off-crest
at the best RF phase
minimises energy spread

Example: Energy gain
along the bunch
in the NLC linac (TW):

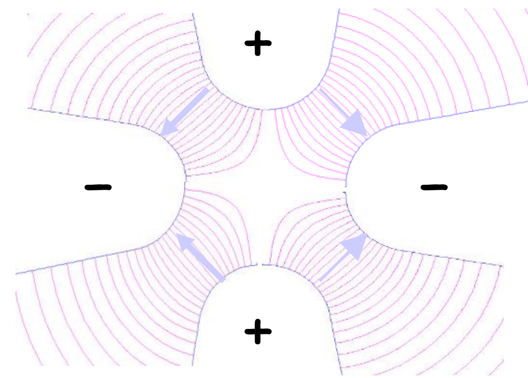


More by
Giovanni Rumolo
Heiko Damerau

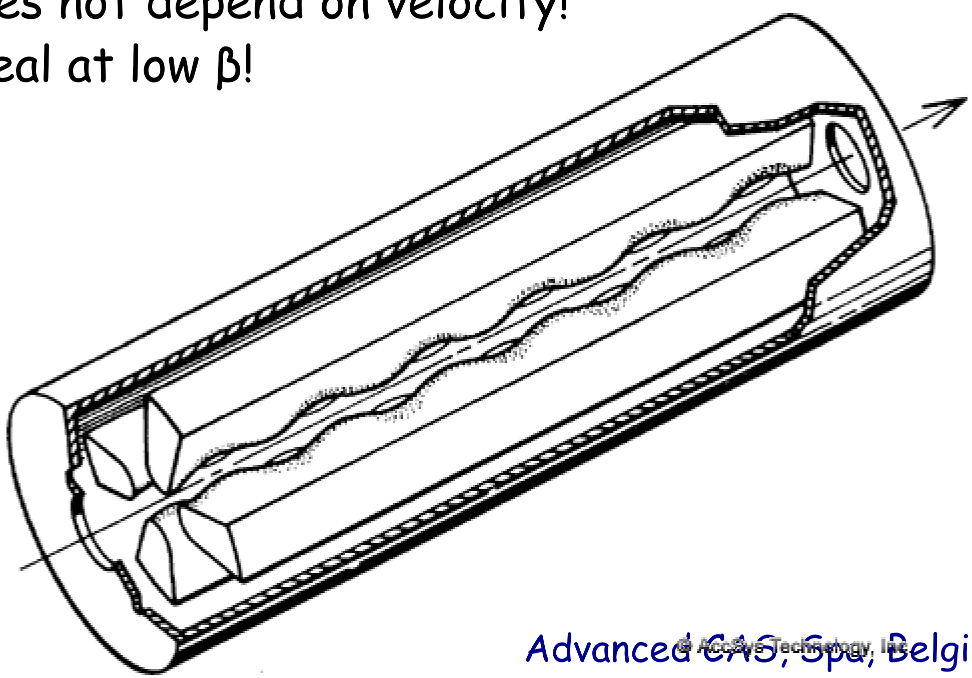
The Radio-Frequency Quadrupole - RFQ

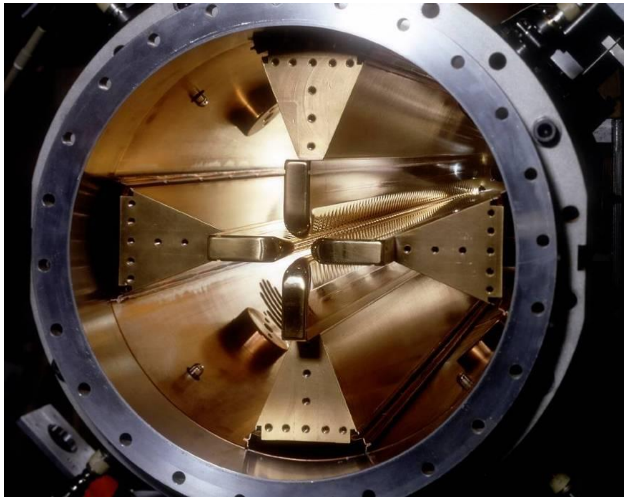
Initial acceleration difficult for protons and ions at low energy (space charge, low β ⇒ short cell dimensions, bunching needed)

RFQ = Electric quadrupole
focusing channel + bunching + acceleration



Alternating electric quadrupole field gives transverse focusing like magnetic focusing channel.
Does not depend on velocity!
Ideal at low β!



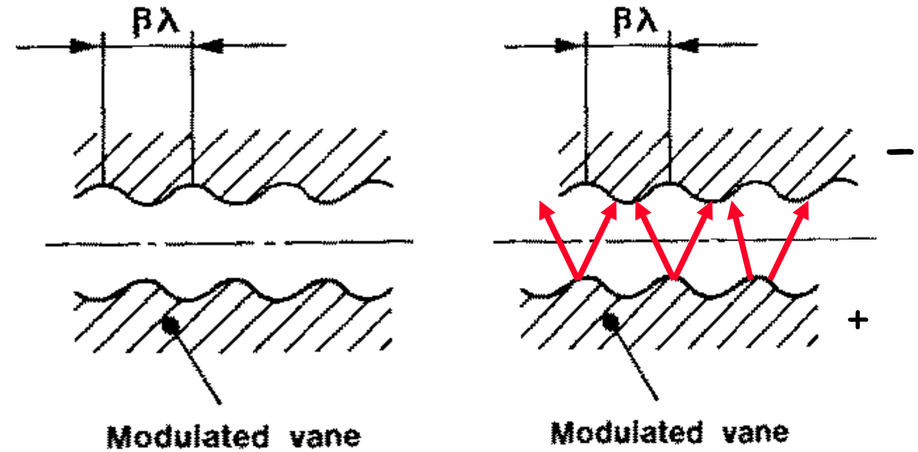


The Radio-Frequency Quadrupole - RFQ

The vanes have a longitudinal modulation with period = $\beta\lambda$

$\rightarrow$ this creates a longitudinal component of the electric field.

The modulation corresponds exactly to a series of RF gaps and can provide acceleration.

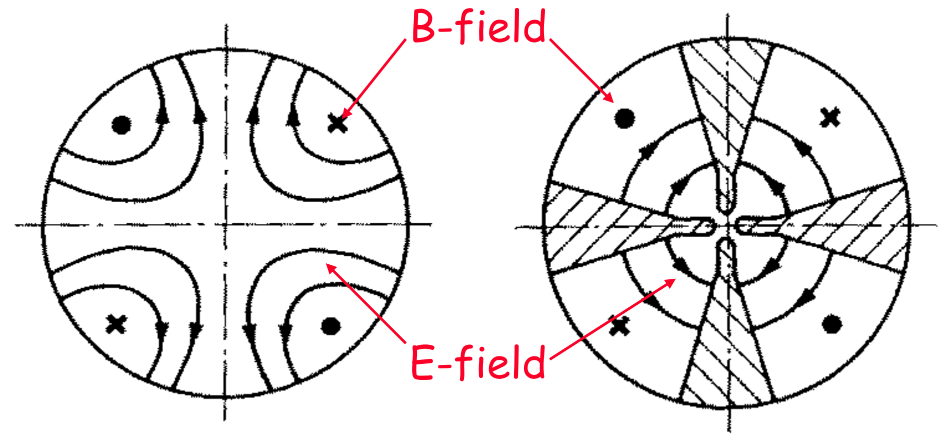


Opposite vanes (180°) Adjacent vanes (90°)

RF Field excitation:

An empty cylindrical cavity can be excited on different modes.

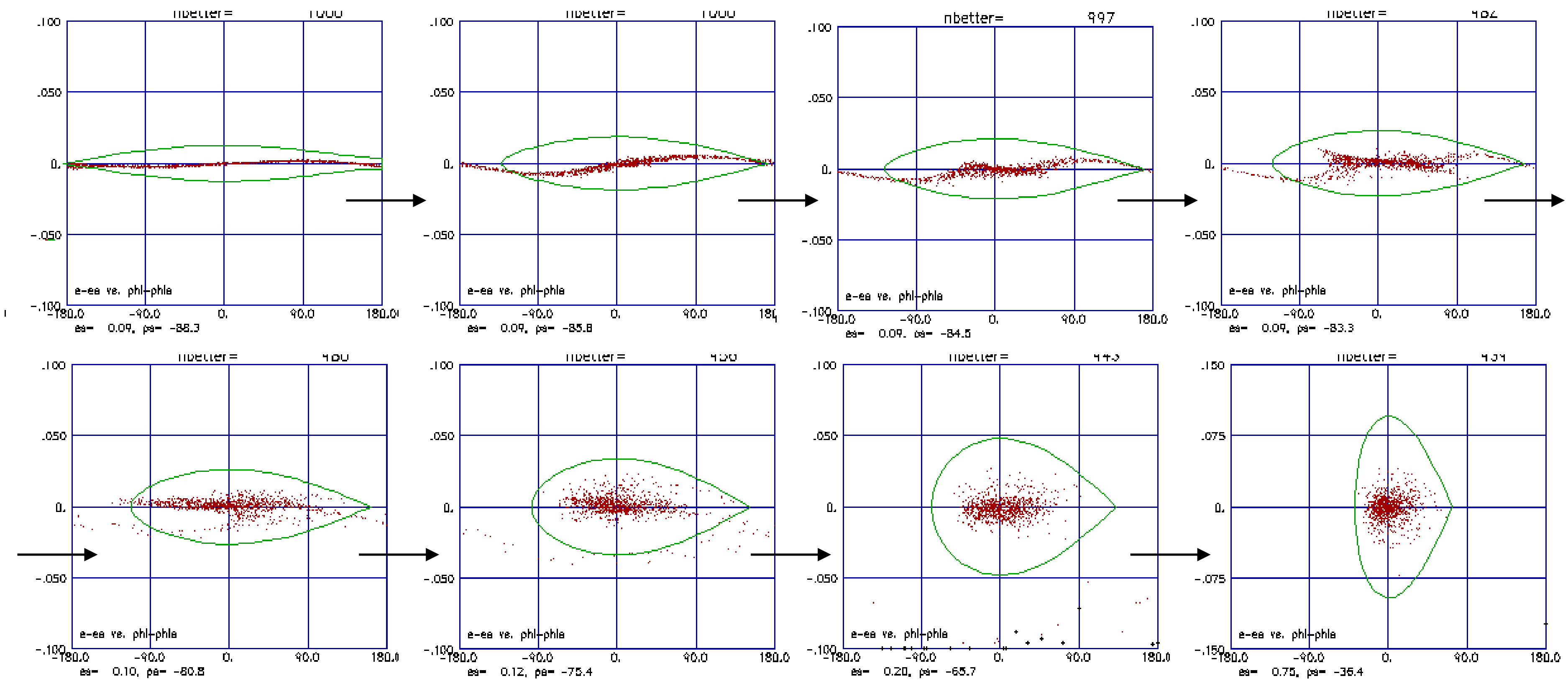
Some of these modes have only transverse electric field (the TE modes), and one uses in particular the "quadrupole" mode, the TE_{210}.



Empty cavity; mode TE_{21} Cavity with vanes

RFQ Design + Longitudinal Phase Space

RFQ design: The modulation period can be slightly adjusted to change the phase of the beam inside the RFQ cells, and the amplitude of the modulation can be changed to change the accelerating gradient
→ start with some bunching cells, progressively bunch the beam (adiabatic bunching channel), and only in the last cells accelerate.



Longitudinal beam profile of a proton beam along the CERN RFQ2

Summary up to here…

- Acceleration by electric fields, static fields limited
  => time-varying fields
- Synchronous condition needs to be fulfilled for acceleration
- Particles perform oscillation around synchronous phase
- visualize oscillations in phase space

- Electrons are quickly relativistic, speed does not change
  use traveling wave structures for acceleration
- Protons and ions
  - RFQ for bunching and first acceleration
  - need changing structure geometry

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

(neglecting transit time factor)

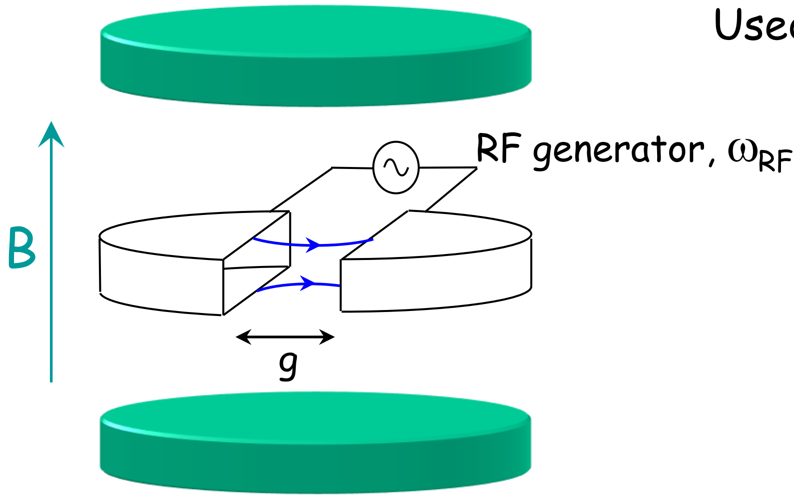
The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Circular accelerators

Cyclotron

Synchrotron

Circular accelerators: Cyclotron



Used for protons, ions

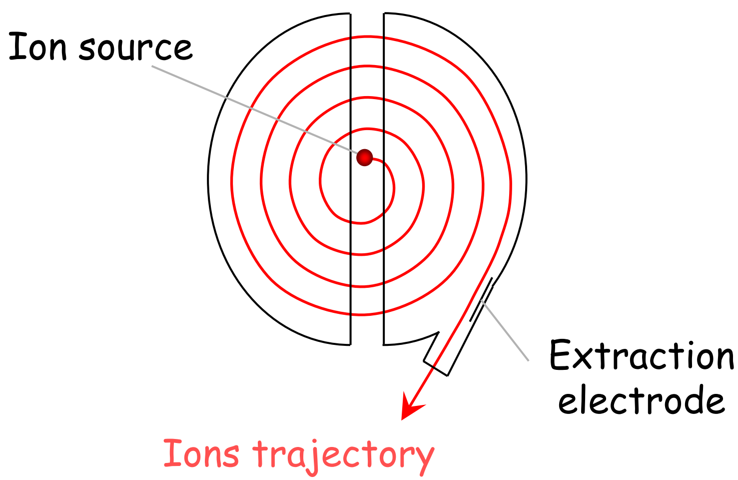
B = constant

ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$

$$2\pi\ \rho = v_s\ T_{RF}$$



Cyclotron frequency $\omega = \dfrac{q\,B}{m_0\,\gamma}$

1. γ increases with the energy
   ⇒ no exact synchronism
2. if **v << c** ⇒ $\gamma \cong 1$

Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php

Cyclotron / Synchrocyclotron

TRIUMF 520 MeV cyclotron Vancouver - Canada

CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma\ \omega_{RF}$ = constant

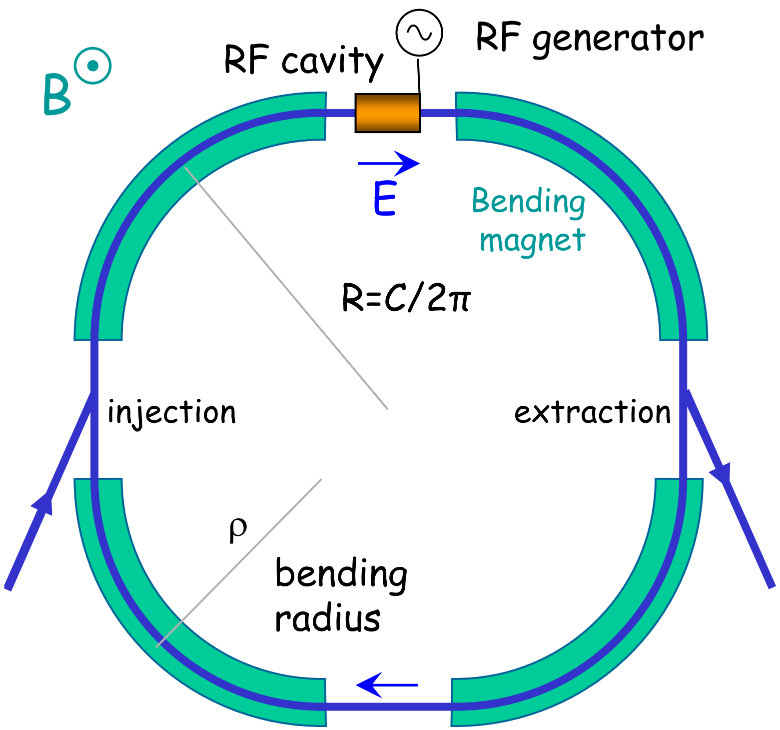
ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\ B}{m_0\ \gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition

$$T_s = h\,T_{RF}$$

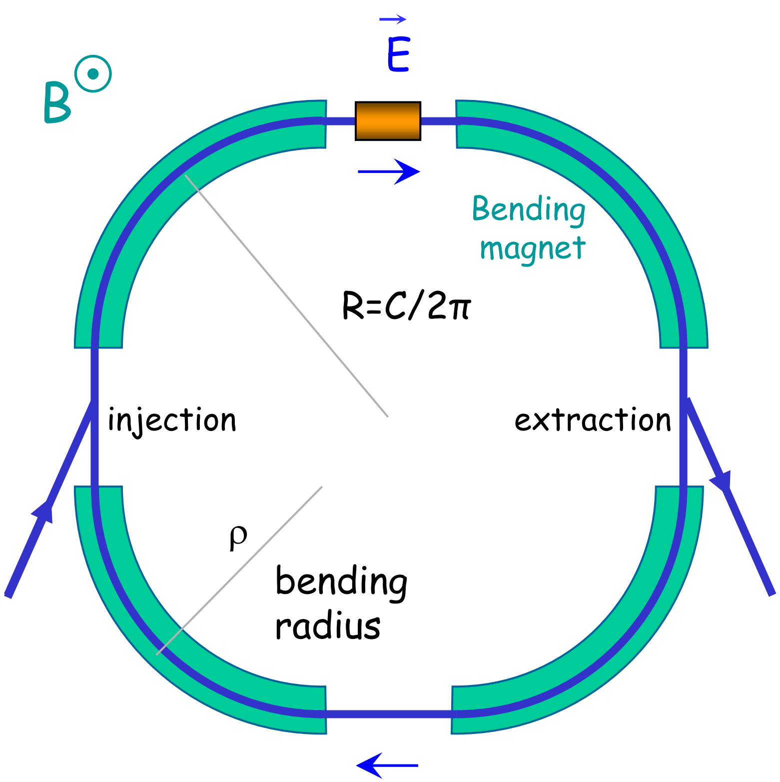
$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer,
harmonic number:
number of RF cycles per revolution

h is the maximum number of bunches in the synchrotron.
Normally less bunches due to gaps for kickers, collision constraints,...

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$q\hat{V}\sin\phi \longrightarrow$ **Energy gain per turn**

$f = f_s \longrightarrow$ **Synchronous particle**

$\omega_{RF} = h\omega \longrightarrow$ **RF synchronism (h - harmonic number)**

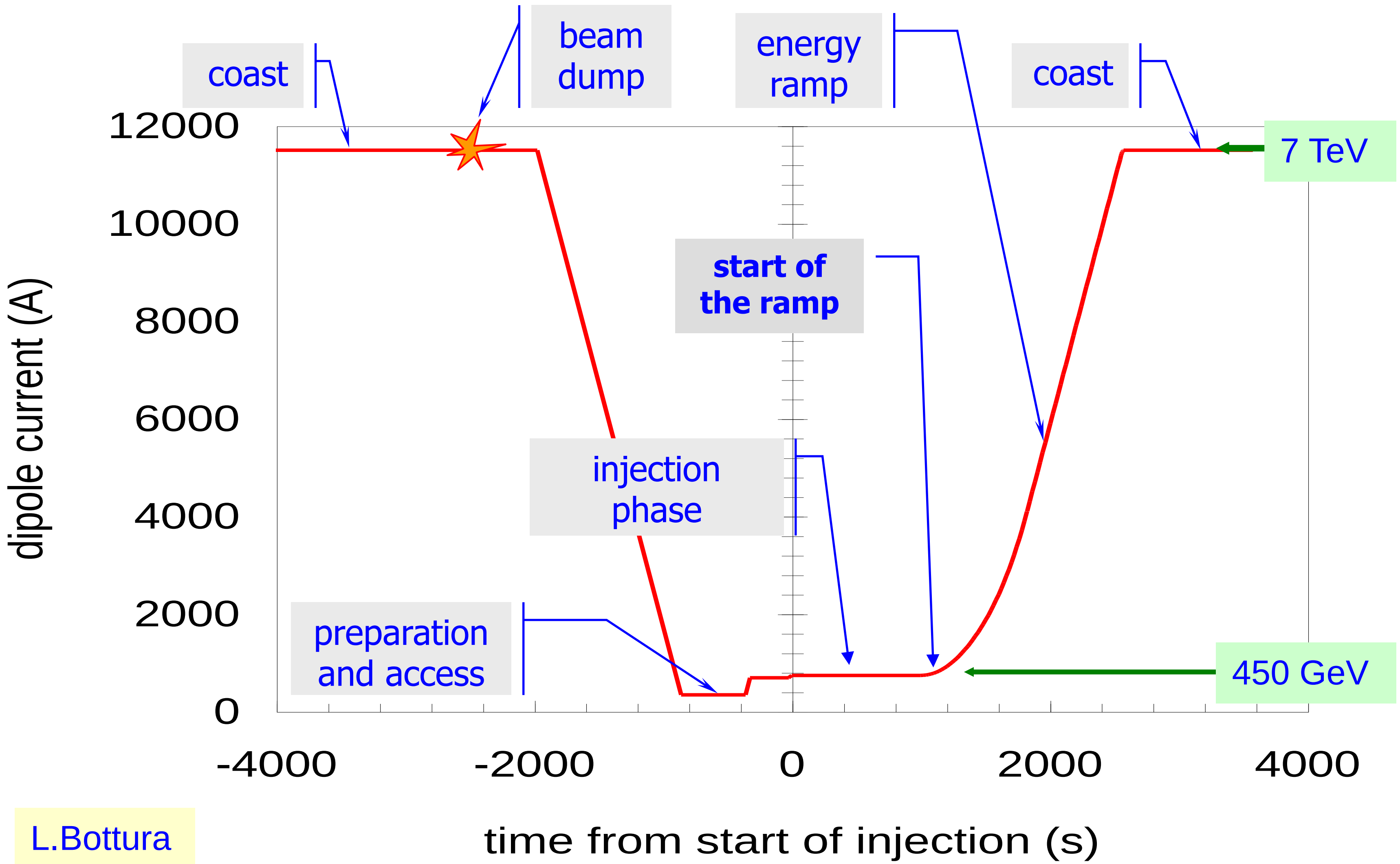
$r = cte \quad R = cte \longrightarrow$ **Constant orbit**

$B\rho = {}^{p}/{}_{q}$ Þ $B \longrightarrow$ **Variable magnetic field**

If v≈c, ω hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = qB\rho \underset{\rho\ \text{const.}}{\Rightarrow} \frac{dp}{dt} = q\dot{B}\rho \quad \Rightarrow \quad (\Delta p)_{turn} = q\dot{B}\rho T_r = \frac{2\pi q\rho R\dot{B}}{v}$$

With

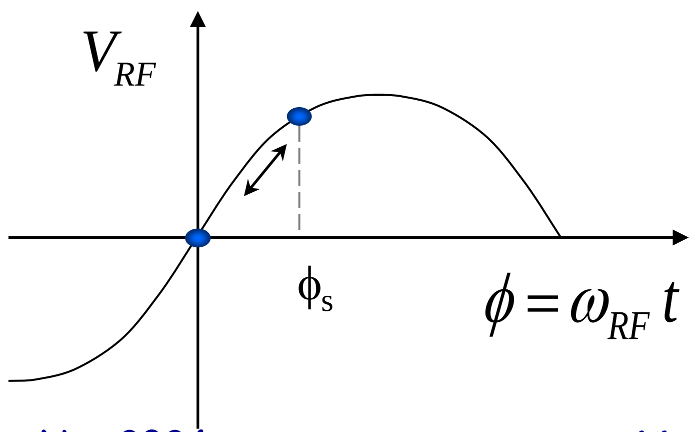
$$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi q\rho R\dot{B} = q\hat{V}\sin\phi_s$$

Synchronous phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- **The synchronous phase depends on**
  - **the change of the magnetic field**
  - **and the RF voltage**



The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

Hence: $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{qc^2}{E_s(t)}\frac{\rho}{R_s}B(t)$$ (using $p(t) = qB(t)\rho, E = mc^2$)

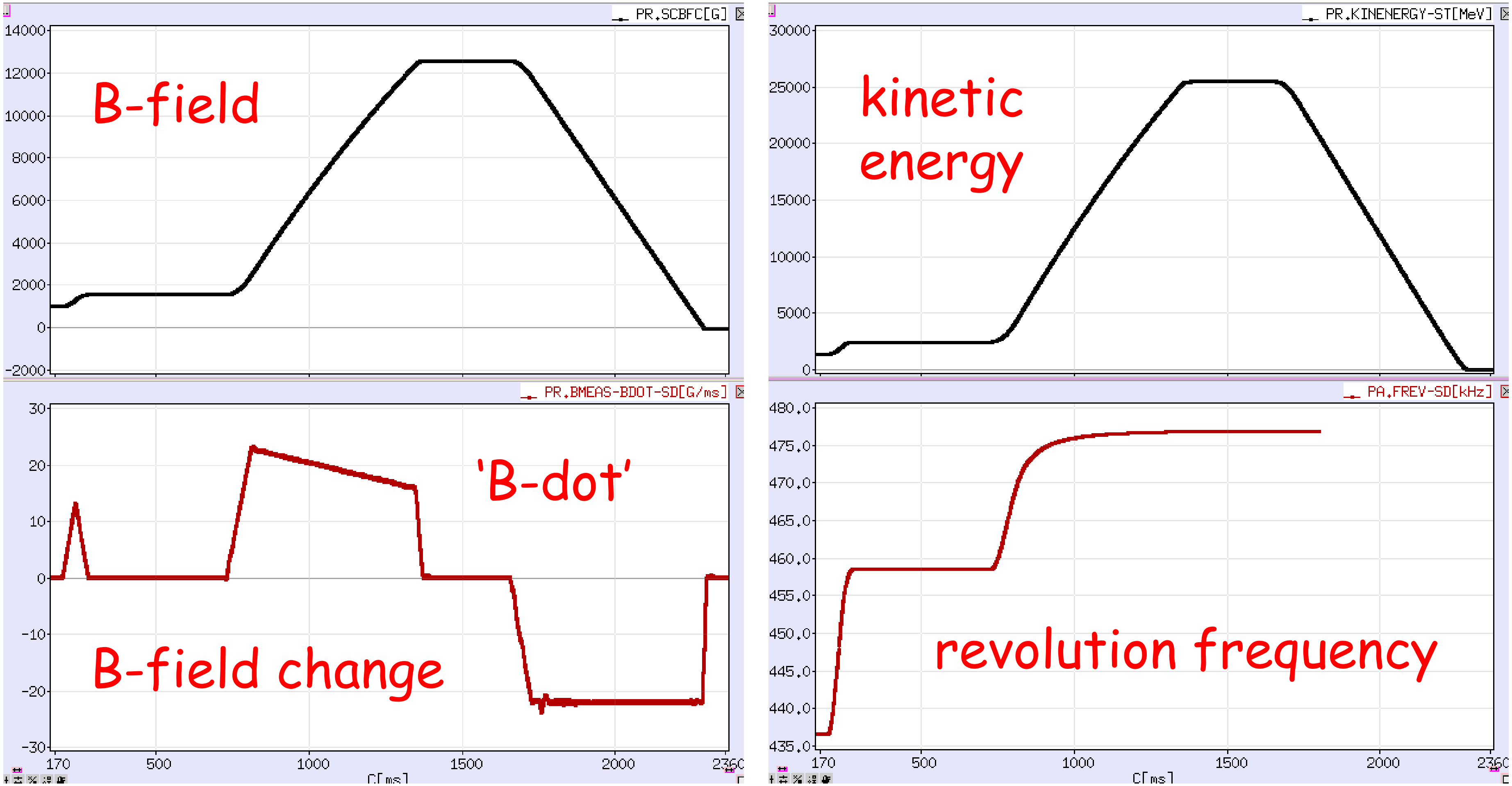
Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\boxed{\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/qc\rho)^2 + B(t)^2}\right\}^{1/2}}$$

RF frequency program during acceleration determined by B-field !

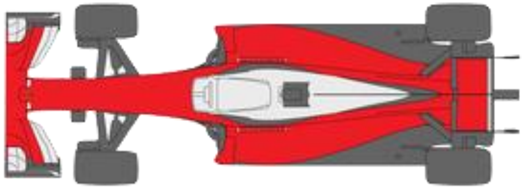
Example: PS - Field / Frequency change

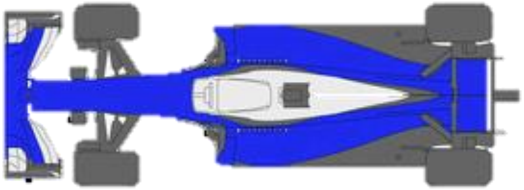
During the energy ramping, the B-field and the revolution frequency increase



time (ms) →   time (ms) →

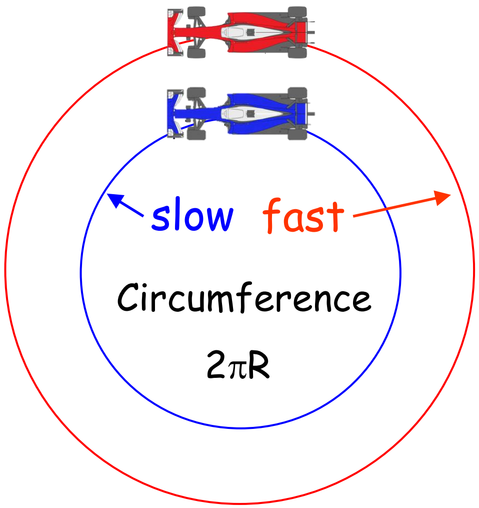
Overtaking in a Formula 1 Race





Overtaking in a Formula 1 Race

Overtaking in a Formula 1 Race



A F1 car wants to overtake another car! It will have a

- a different track length due to a 'dispersion orbit'
- and a different velocity.

$$T = \frac{L}{v} = \frac{2\pi R}{v} \quad \text{and} \quad f_r = \frac{1}{T} = \frac{v}{2\pi R}$$

$$=> \frac{\Delta T}{T} = \frac{\Delta R}{R} - \frac{\Delta v}{v}$$

The winner depends on the relative change in speed compared to the relative change in track length!

v=speed of the car

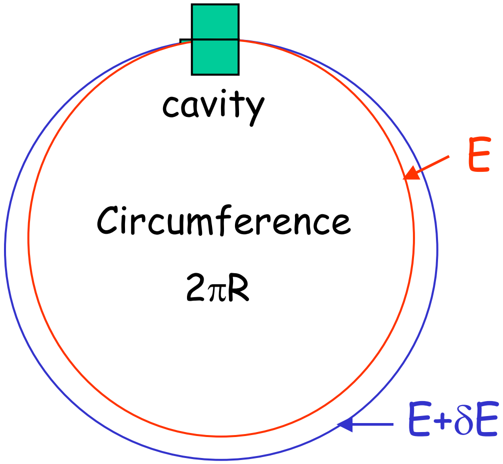
R=track physical radius

T=revolution period

f_r=revolution frequency

If the relative change in speed is larger than the relative change in track length => the red car will win!

Overtaking in a Synchrotron



A particle with a momentum deviation will have a

- dispersion orbit and a different orbit length
- a different velocity.

As a result of both effects the revolution time changes with a "slip factor η":

$$\eta = \frac{dT/T}{dp/p}$$

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

Note: you also find η defined with a minus sign!

Effect from orbit defined by Momentum compaction factor:

$$\alpha_c = \frac{dL/_L}{dp/_p}$$

Property of the beam optics:
(derivation see appendix)

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

Dispersion Effects – Revolution Period

The two effects of the orbit length and the particle velocity change the revolution period as:

$$T = \frac{L}{\beta c} \quad \Rightarrow \quad \frac{dT}{T} = \frac{dL}{L} - \frac{d\beta}{\beta} = \alpha_c \frac{dp}{p} - \frac{d\beta}{\beta}$$

definition of momentum compaction factor

$$\boxed{\frac{dT}{T} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{dp}{p}}$$

$$p = mv = \beta\gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2}\frac{d\beta}{\beta}$$

Slip factor: $\eta = \alpha_c - \dfrac{1}{\gamma^2}$ or $\eta = \dfrac{1}{\gamma_t^2} - \dfrac{1}{\gamma^2}$ with $\gamma_t = \dfrac{1}{\sqrt{\alpha_c}}$

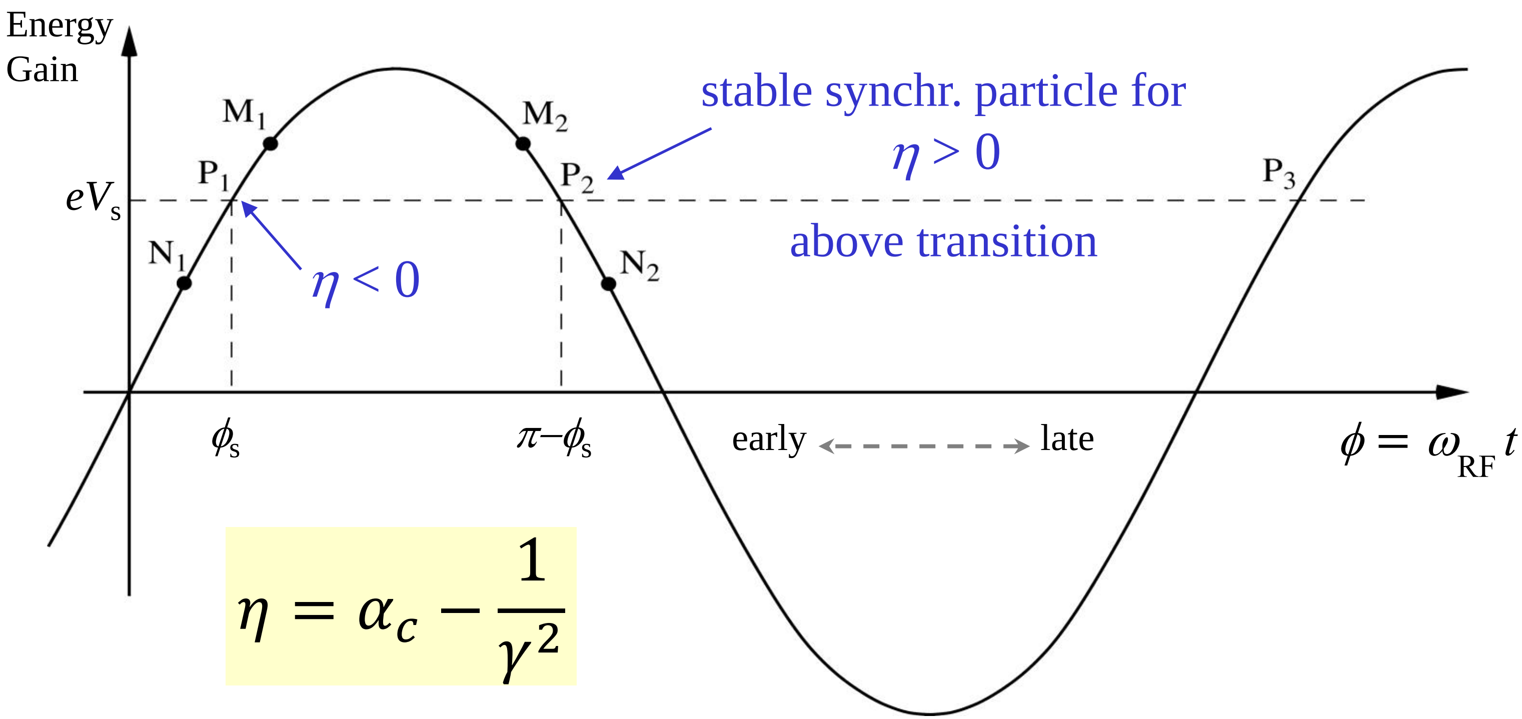
Note: you also find η defined with a minus sign!

At transition energy, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta < 0$) a higher revolution frequency (increase in velocity dominates) while

- above transition ($\eta > 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

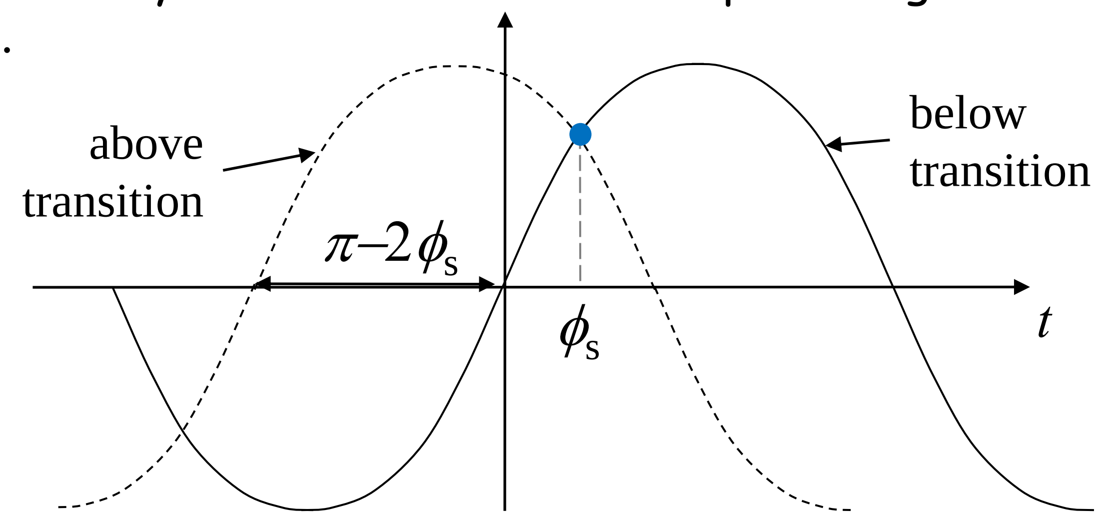


$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.



$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \boxed{\gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x}$$

In the PS: γ_t is at ~6 GeV
In the SPS: γ_t= 22.8, injection at γ=27.7
=> no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

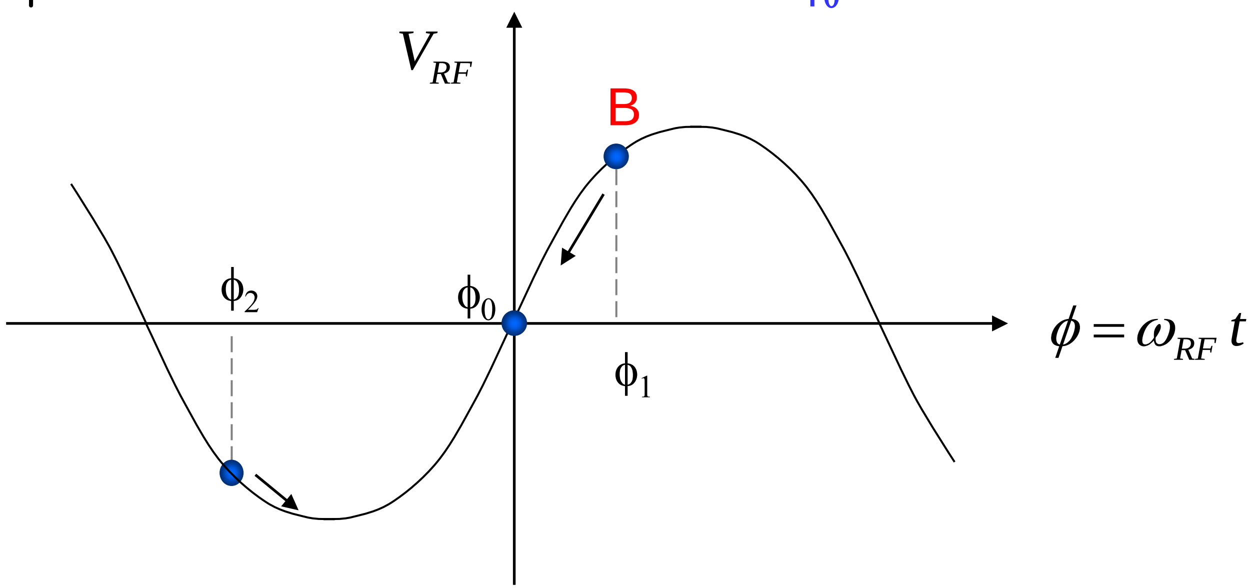
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_{tr}$

The phase of the synchronous particle must therefore be $\phi_0 = 0$.

ϕ_1
- The particle B is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0

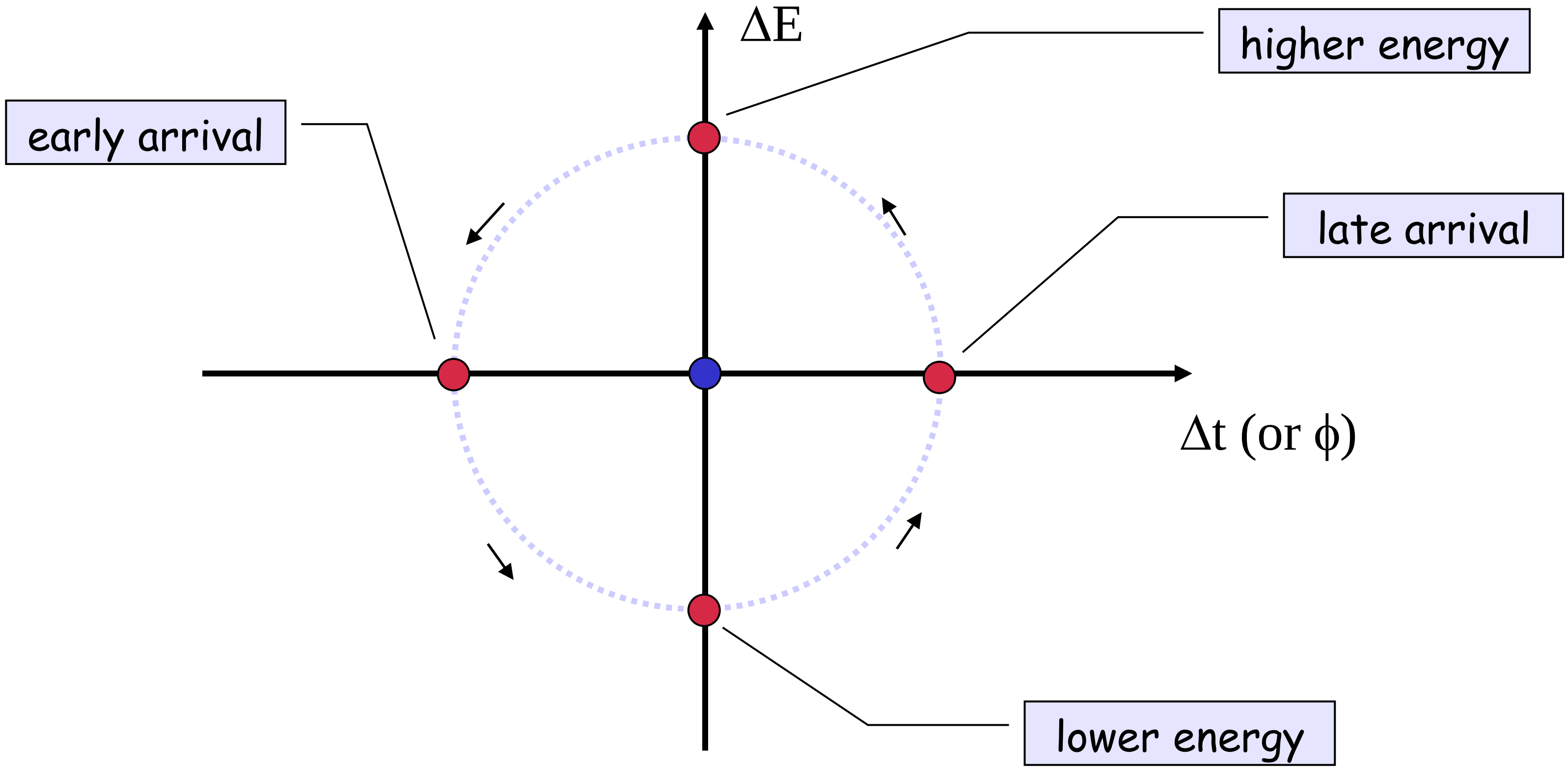


ϕ_2
- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

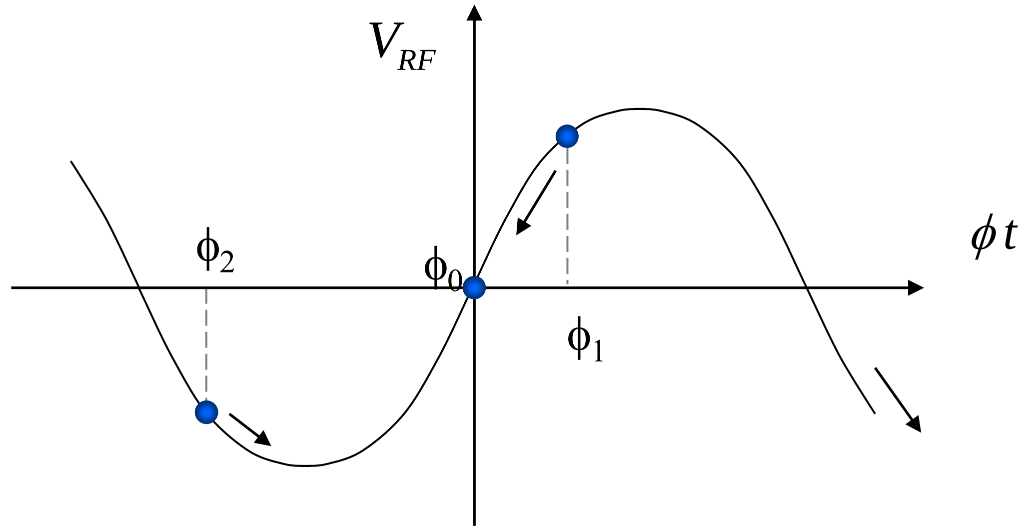
Longitudinal Phase Space Motion

Particle B performs a synchrotron oscillation around the synchronous particle A

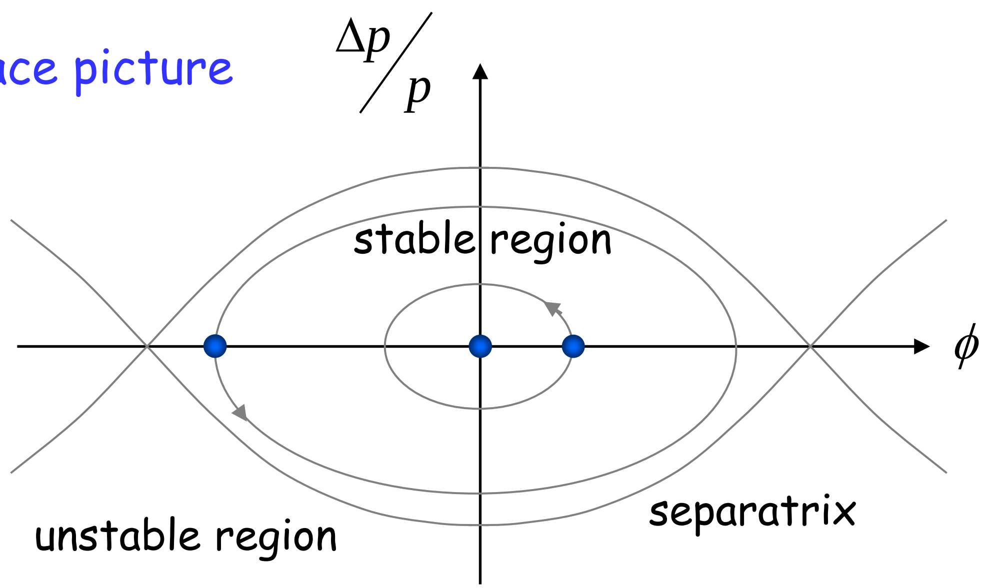
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations – No acceleration



Phase space picture



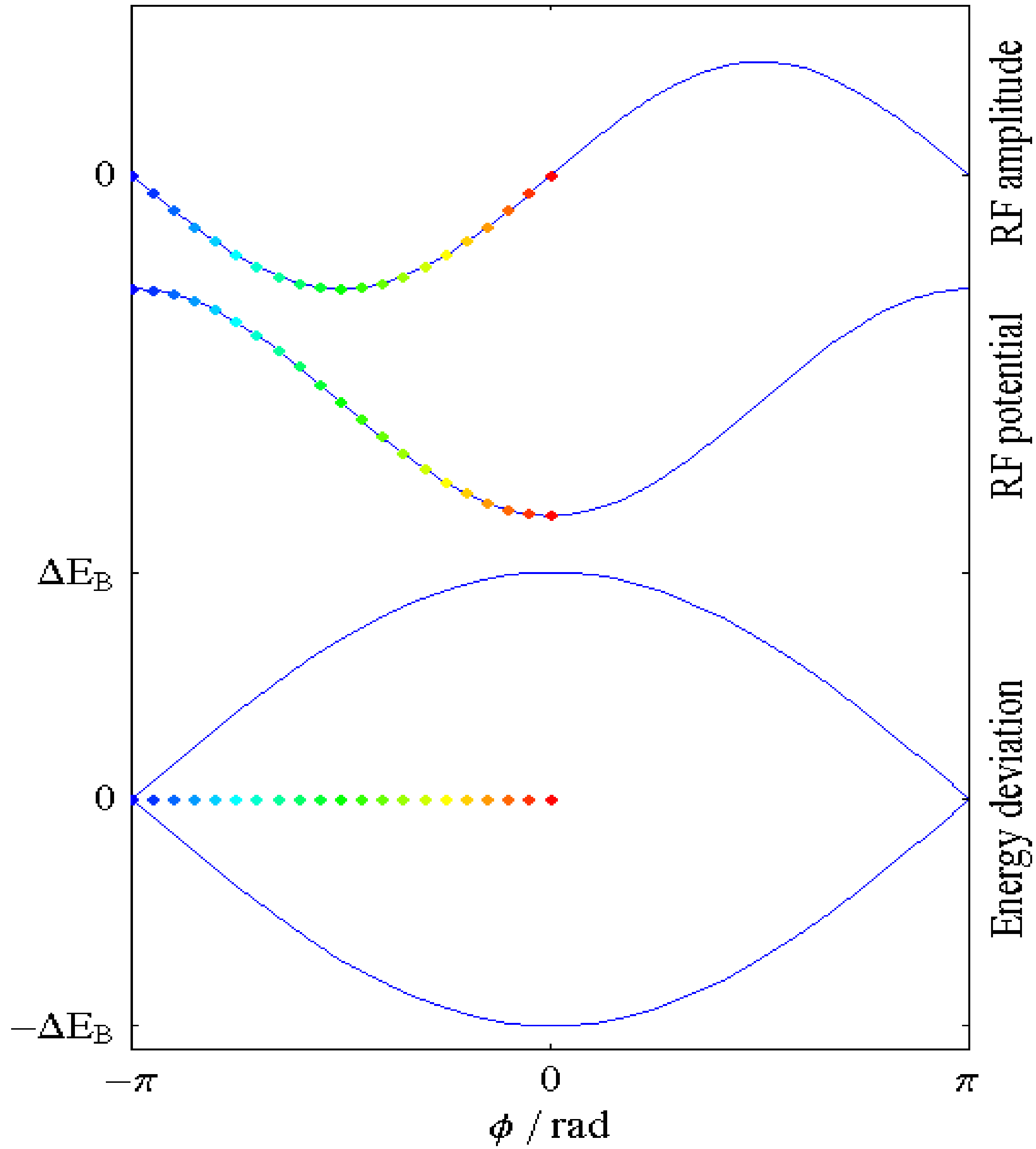
Synchrotron motion in phase space

Remark:
Synchrotron frequency much smaller than betatron frequency.

The restoring force is non-linear.

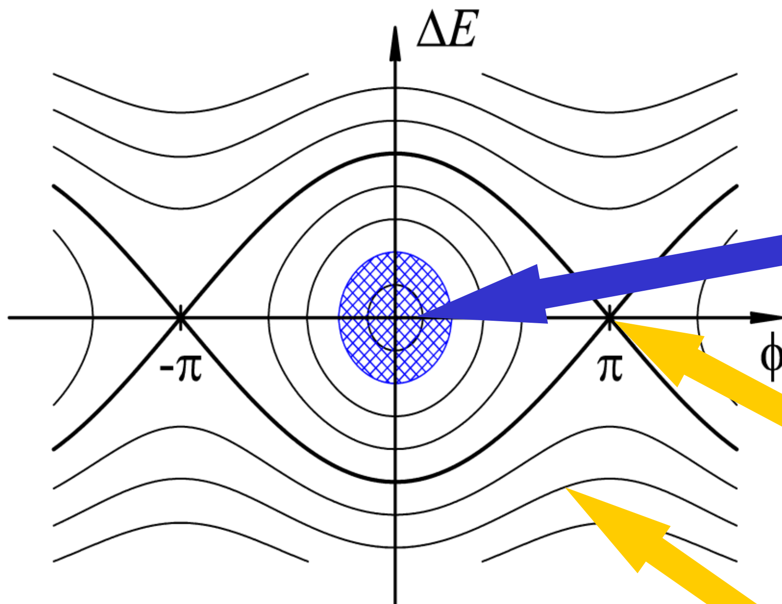
⇒ speed of motion depends on position in phase-space

(here shown for a stationary bucket)



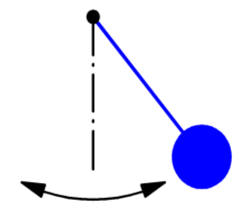
Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)



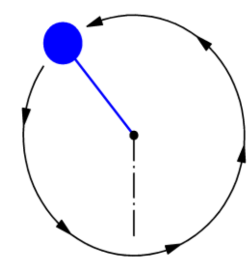
Dynamics of a particle
Non-linear, conservative oscillator → e.g. pendulum

Particle **inside the separatrix:**



Particle at the **unstable fix-point**

Particle **outside the separatrix:**

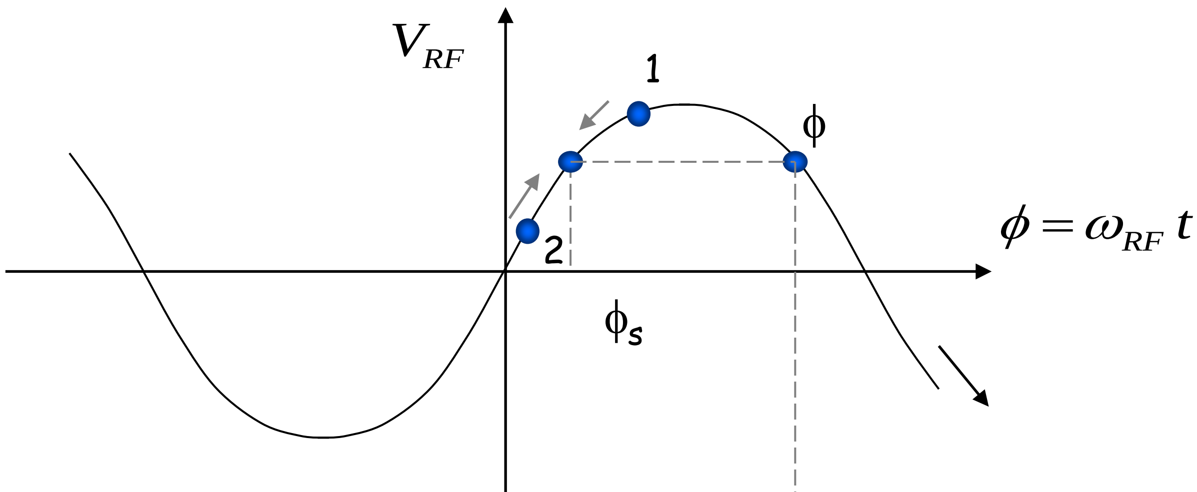


Bucket area:
area enclosed by the separatrix
=> **longitudinal Acceptance** [eVs]

The area covered by particles is the longitudinal emittance.

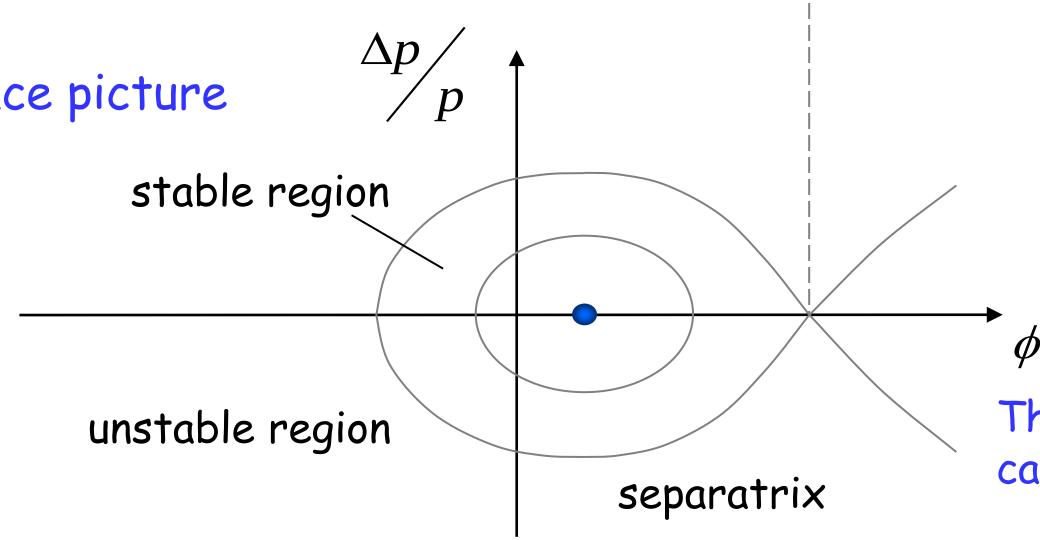
Synchrotron oscillations (with acceleration)

Case with acceleration *B increasing* $\gamma < \gamma_{tr}$



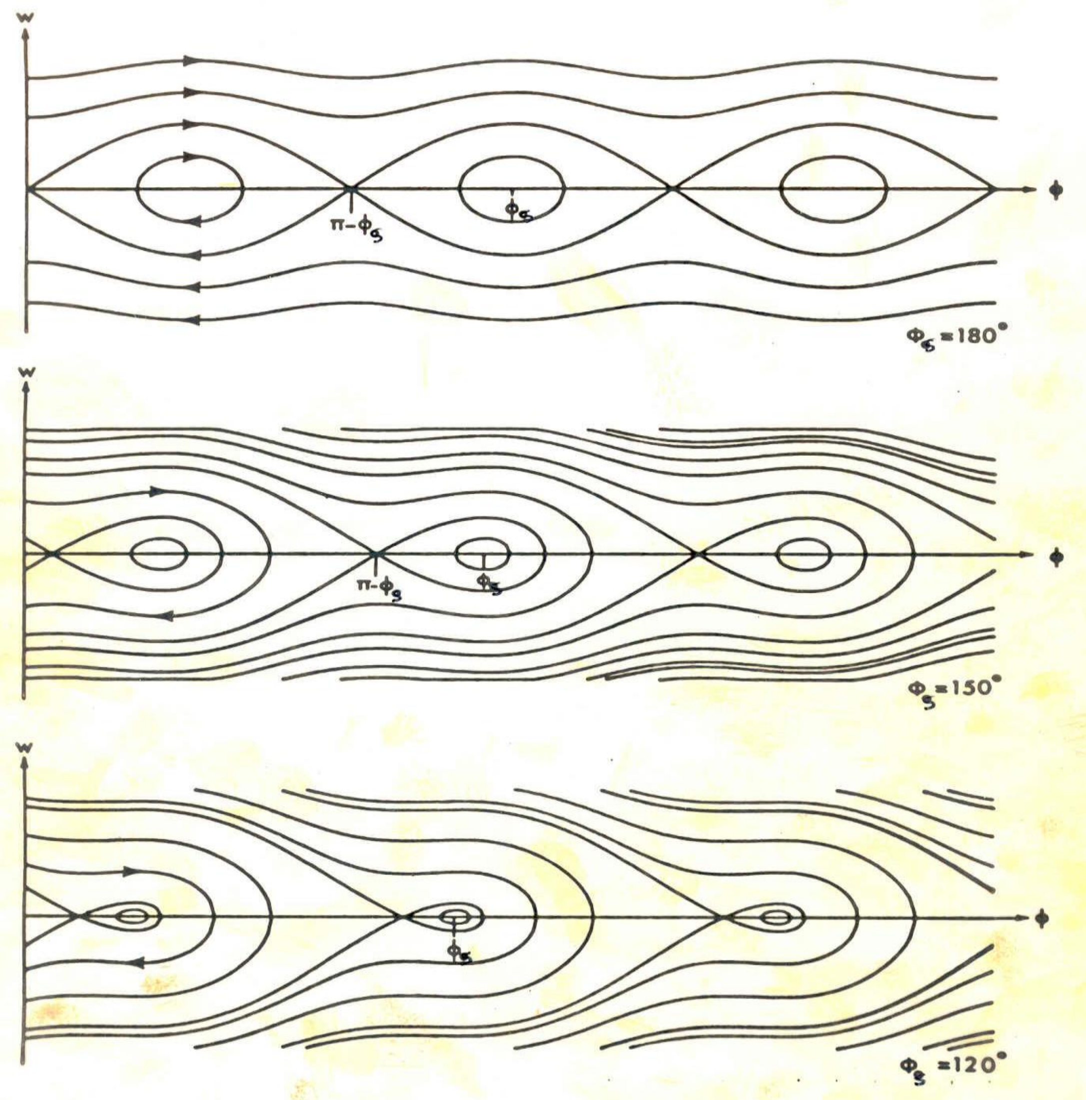
$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture



The symmetry of the case B = const. is lost

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

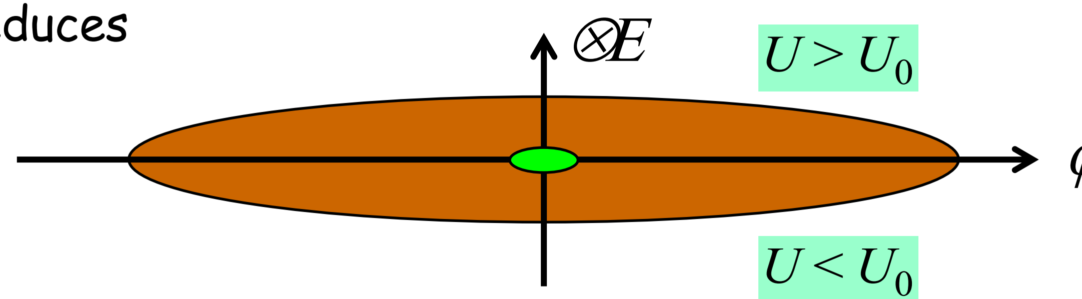
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

$$U_0 = \frac{4}{3}\pi\frac{r_e}{\left(m_0c^2\right)^3}\frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

=> The synchrotron motion is damped toward an equilibrium bunch length and energy spread.

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_0, it is sufficient to follow other particles with respect to that particle.

So let's introduce the following **reduced variables**:

particle RF phase :	$\Delta\phi = \phi - \phi_s$
particle momentum :	$\Delta p = p - p_0$
particle energy :	$\Delta E = E - E_0$
angular frequency :	$\Delta\omega = \omega - \omega_0$
azimuth orbital angle:	$\Delta\theta = \theta - \theta_s$

Look at difference from synchronous particle

Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

$$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0}\dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = q\,\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{-p_0 R}{h\eta\omega_0}\frac{d\phi}{dt}\right] + \frac{q\,\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

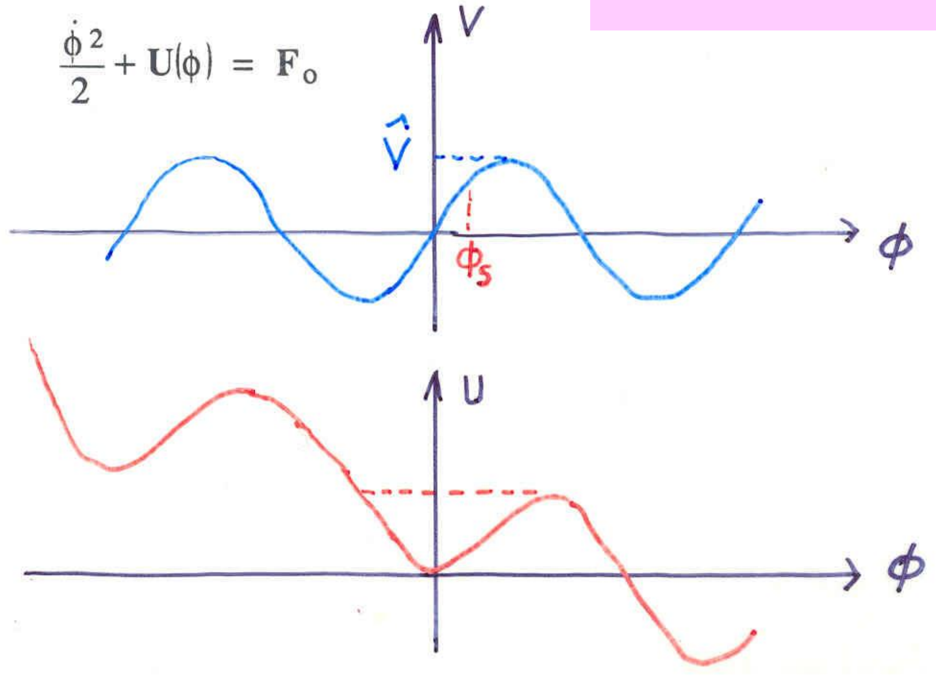
We will study some cases in the following…

Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^{\phi} F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi \sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_0} \longrightarrow \begin{aligned} \frac{d\phi}{dt} &= \frac{h\eta\omega_0}{p_0 R} W \\ \frac{dW}{dt} &= \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

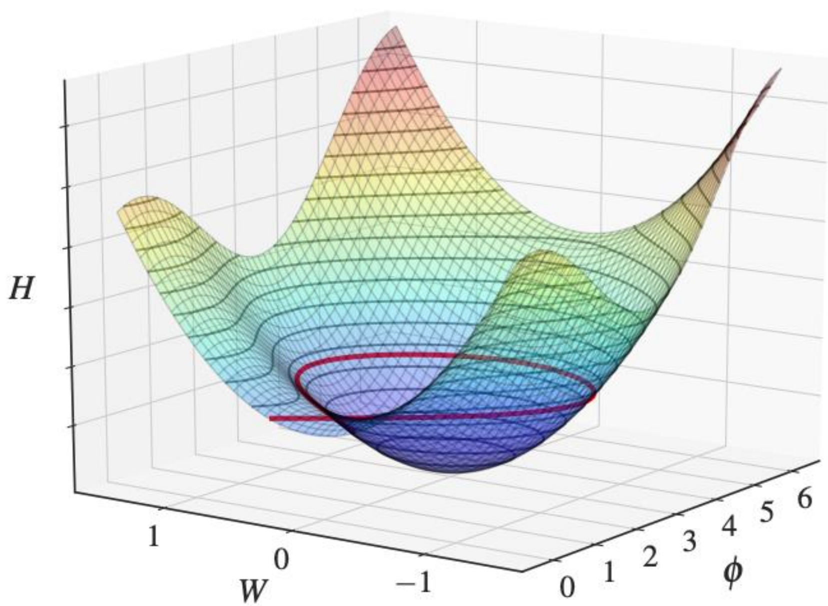
$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = \frac{1}{2}\frac{h\eta\omega_0}{p_0 R}W^2 + \frac{e\hat{V}}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

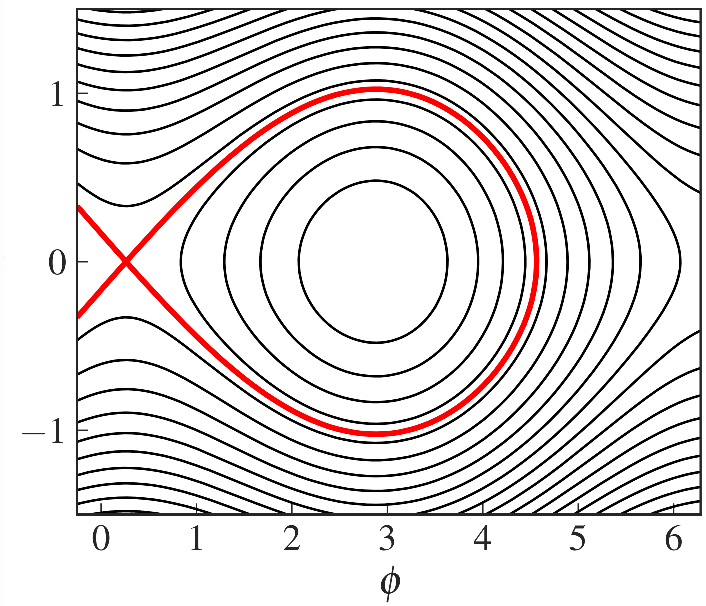
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of H (φ,W)



Contours of H (φ,W)



Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, …)

Small Amplitude Oscillations

Let's assume constant parameters R, p_0, ω_0 and η:

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0$$

with

$$\Omega_s^2=\frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0}\cos\phi_s$$

Consider now small phase deviations from the reference particle:

$$\sin\phi-\sin\phi_s=\sin\left(\phi_s+\Delta\phi\right)-\sin\phi_s\cong\cos\phi_s\Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$

where Ω_s is the synchrotron angular frequency.

The **synchrotron tune ν_s** is the number of synchrotron oscillations per revolution:

$$\nu_s=\Omega_s/\omega_0$$

Typical values are <<1, as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order 10^{-3}
- electron storage rings of the order 10^{-1}

Stability condition for ϕ_s

$$\boxed{\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0}\cos\phi_s} \Leftrightarrow \quad \Omega_s^2 = \omega_0^2\frac{-q\hat{V}_{RF}\eta h}{2\pi\beta^2 E}\cos\phi_s$$

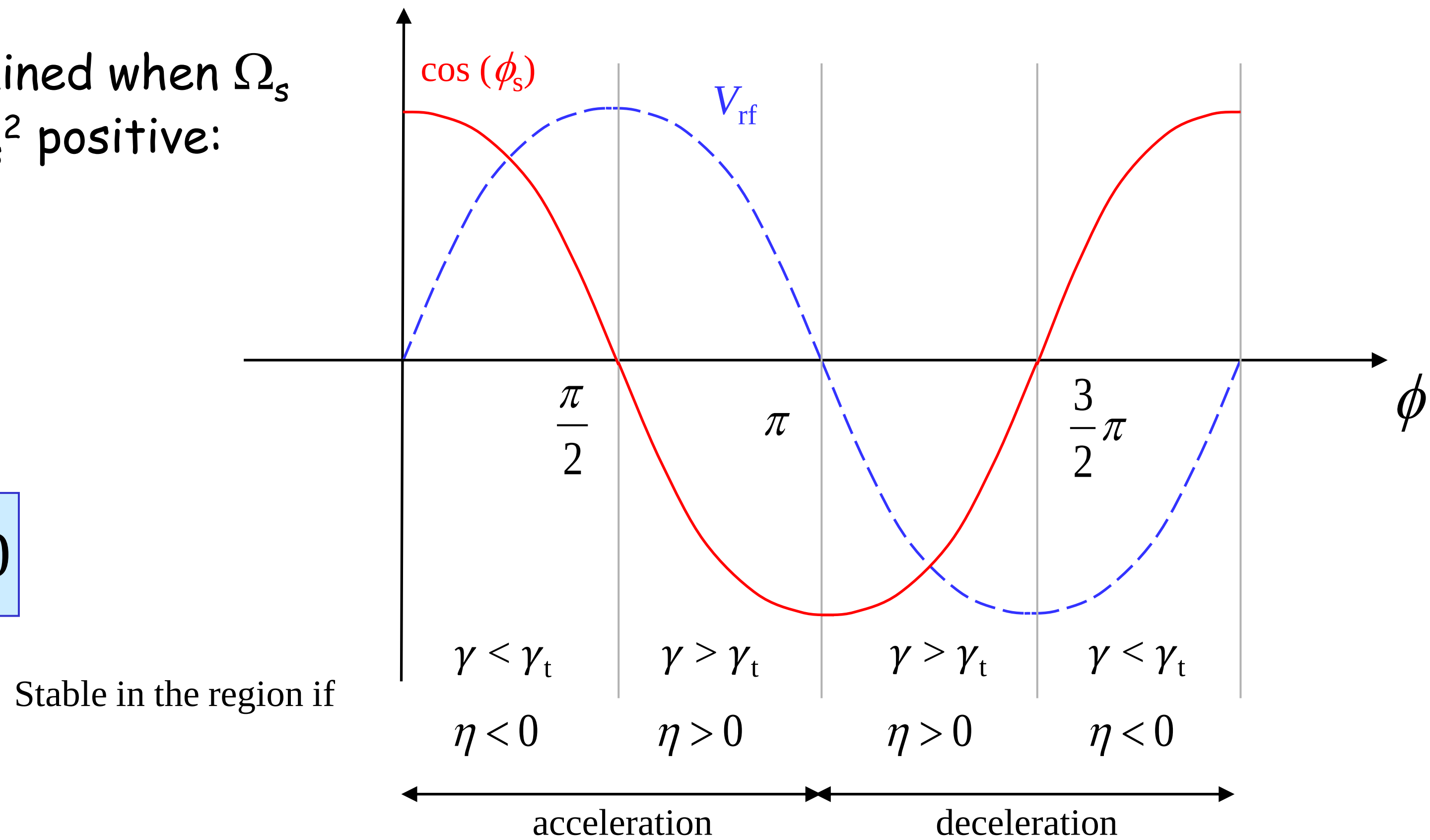
with $Rp = \dfrac{\beta^2 E}{\omega}$

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

$$\Updownarrow$$

$$\boxed{\eta\cos\phi_s < 0}$$



Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

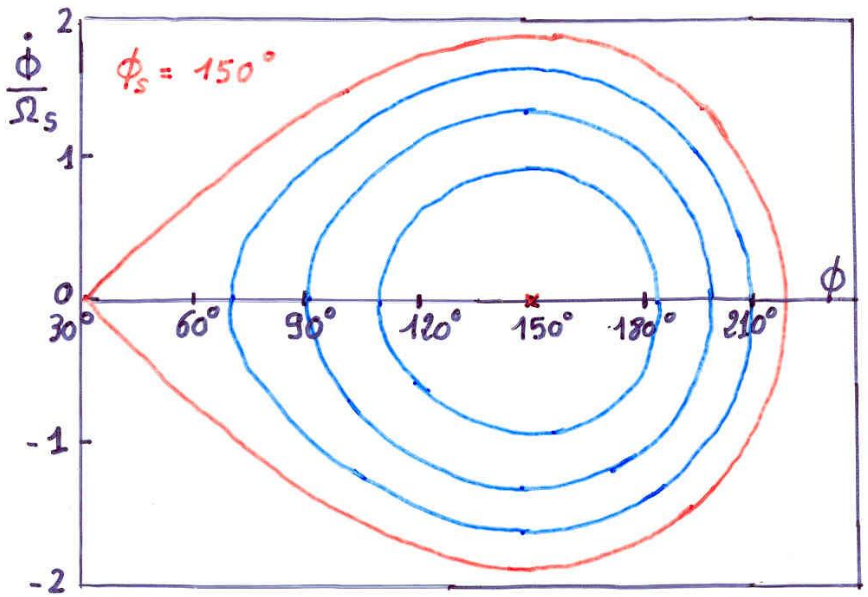
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

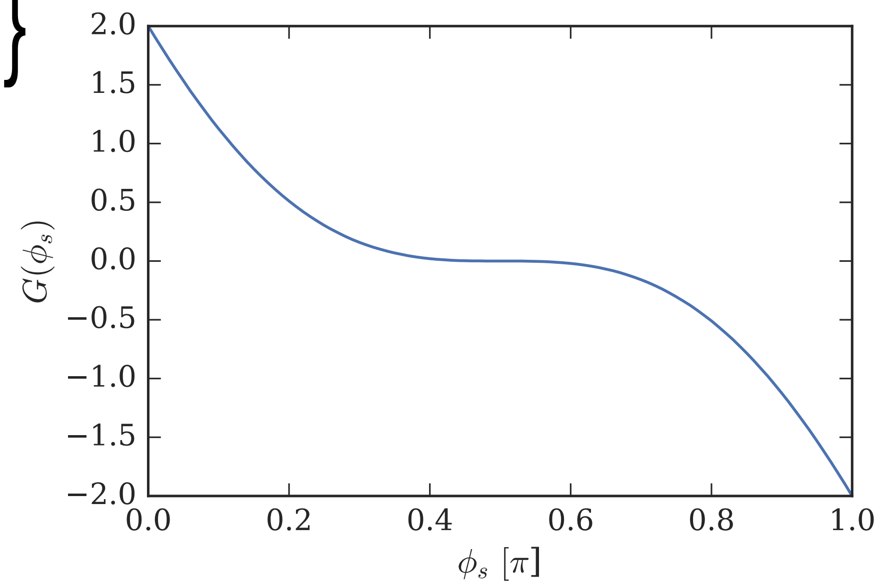
From the equation of motion, it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi\right) \tan\phi_s \right\}$$

That translates into an energy acceptance:

$$\boxed{\left(\frac{\Delta E}{E_s}\right)_{max} = \pm\beta \sqrt{\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)}}$$

$$G(\phi_s) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$



This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

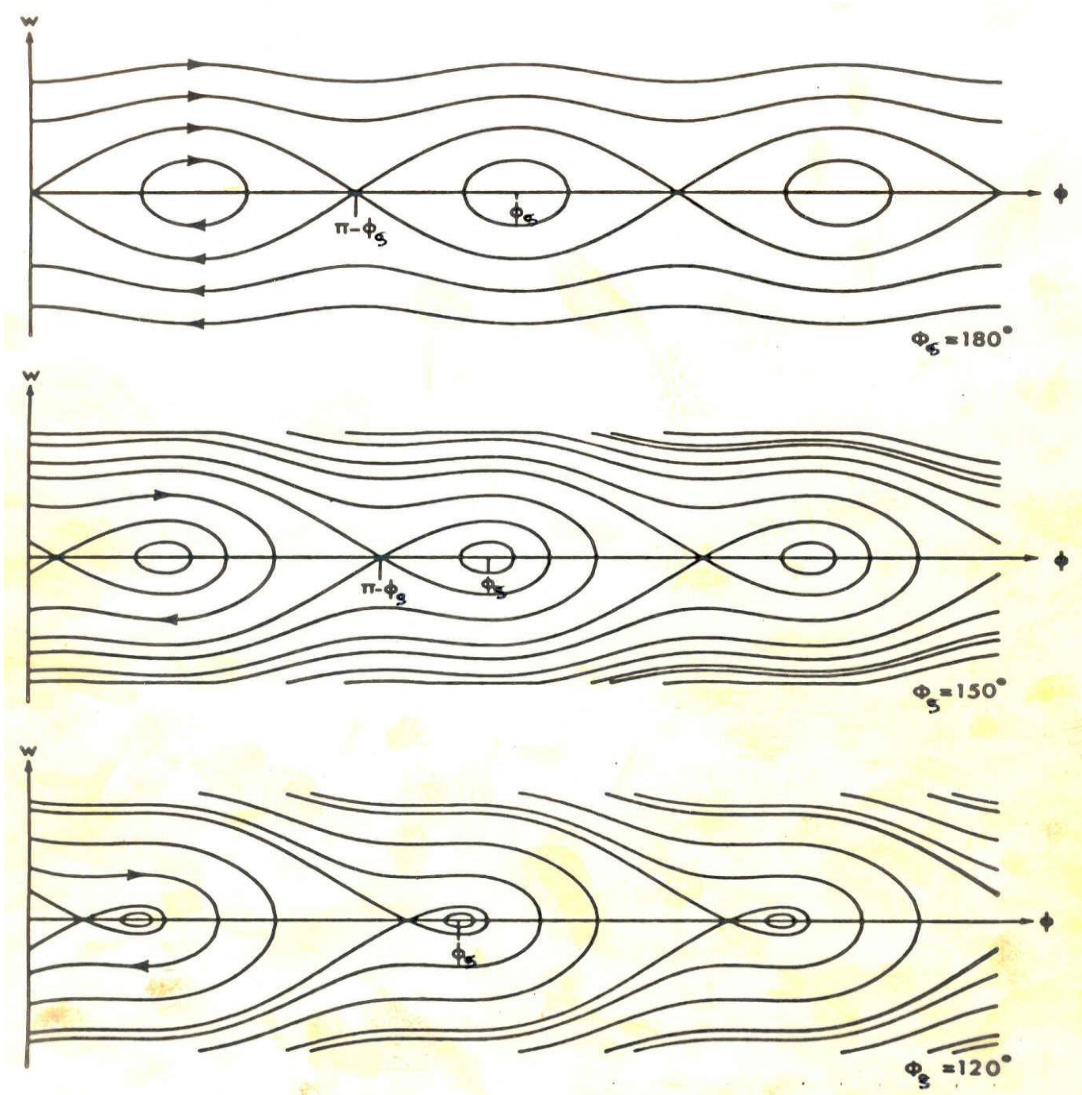
It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

Need a higher RF voltage for higher acceptance.

For the same RF voltage it is smaller for higher harmonics h.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

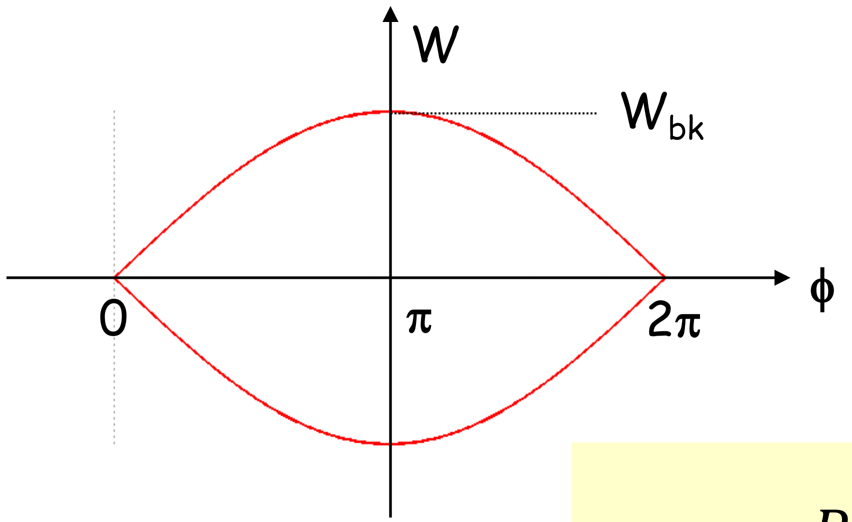
Stationnary Bucket - Separatrix

This is the case sinϕ_s=0 (no acceleration) which means ϕ_s=0 or π . The equation of the separatrix for ϕ_s= π (above transition) becomes:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2$$

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W = \frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

with C=2πR

$$W = \pm\frac{R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h\eta}}\sin\frac{\phi}{2} = \pm W_{bk}\sin\frac{\phi}{2}$$

Bucket height - bucket area

Setting $\phi=\pi$ in the previous equation gives the height of the stationary bucket:

$$W_{bk} = \frac{R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

The bucket area is: $$A_{bk} = 2\int_0^{2\pi} W\, d\phi$$

Since: $$\int_0^{2\pi} \sin\frac{\phi}{2}\, d\phi = 4$$

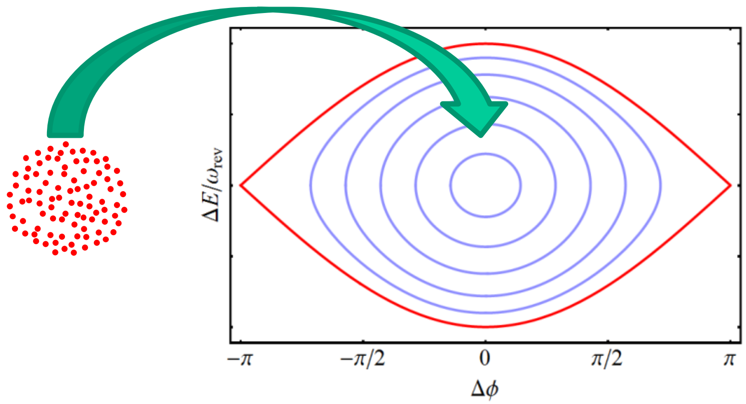
one gets: $$A_{bk} = 8\, W_{bk} = \frac{8R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}} \longrightarrow W_{bk} = \frac{A_{bk}}{8}$$

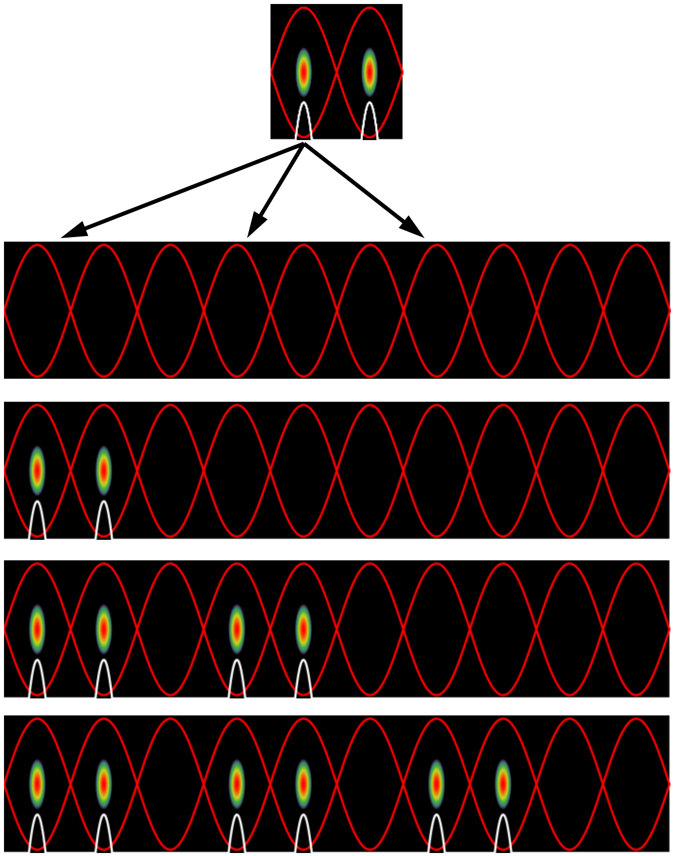
For an accelerating bucket, this area gets reduced by a factor depending on Φ_s:

$$\alpha(\phi_s) \approx \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving





Advantages:

→ Particles always subject to longitudinal focusing

→ No need for RF capture of de-bunched beam in receiving accelerator

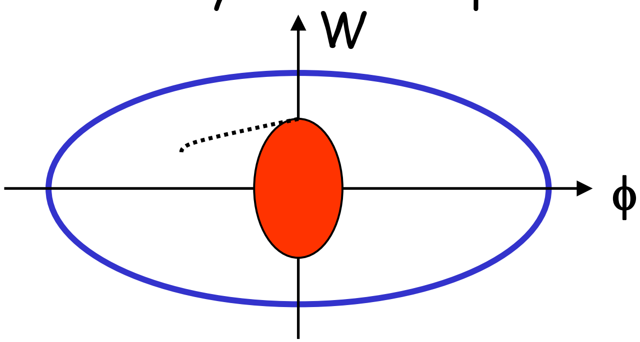
→ No particles at unstable fixed point

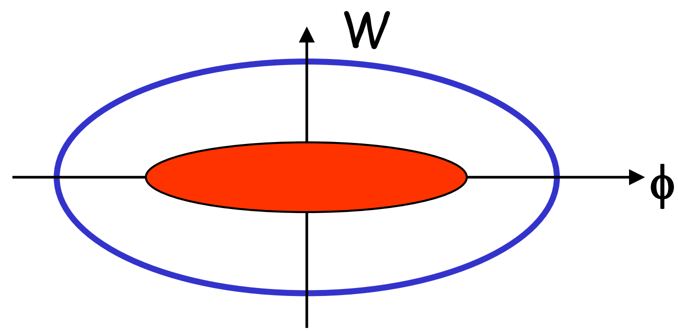
→ Time structure of beam preserved during transfer

Bunch Transfer - Effect of a Mismatch

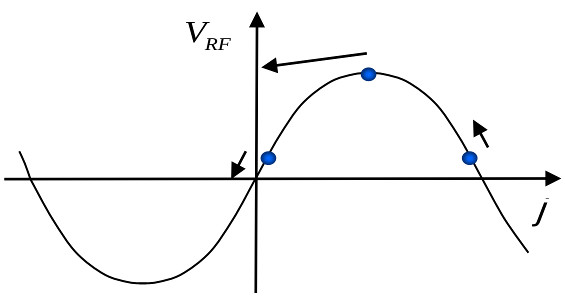
When you transfer the bunch from one RF system to another, the shape of the phase space and the bunch need to match.

Mismatch example: Injected bunch: short length and large energy spread after 1/4 synchrotron period: longer bunch with a smaller energy spread.

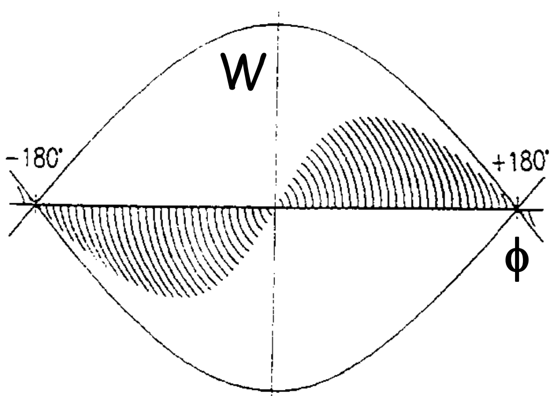




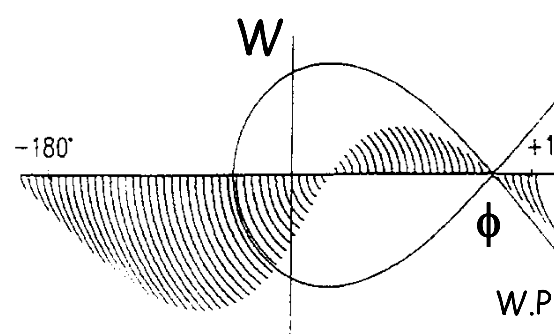
For larger amplitudes, the angular phase space motion is slower (1/8 period shown below) => can lead to filamentation and emittance growth



restoring force is non-linear



stationary bucket



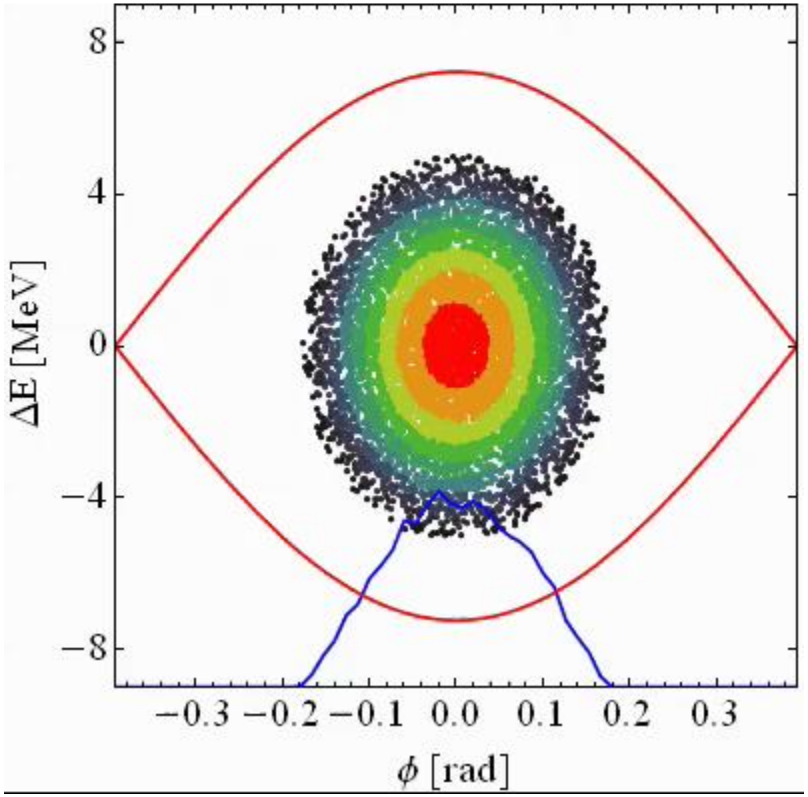
W.Pirkl

accelerating bucket

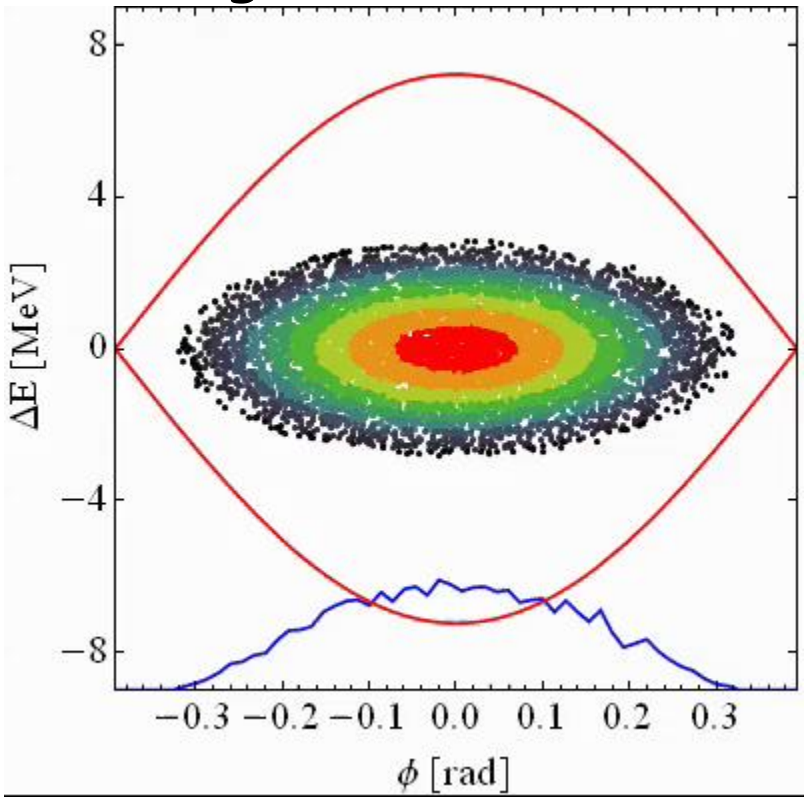
Effect of a Mismatch (2)

- Long. emittance is only preserved for correct RF voltage

Matched case



Longitudinal mismatch



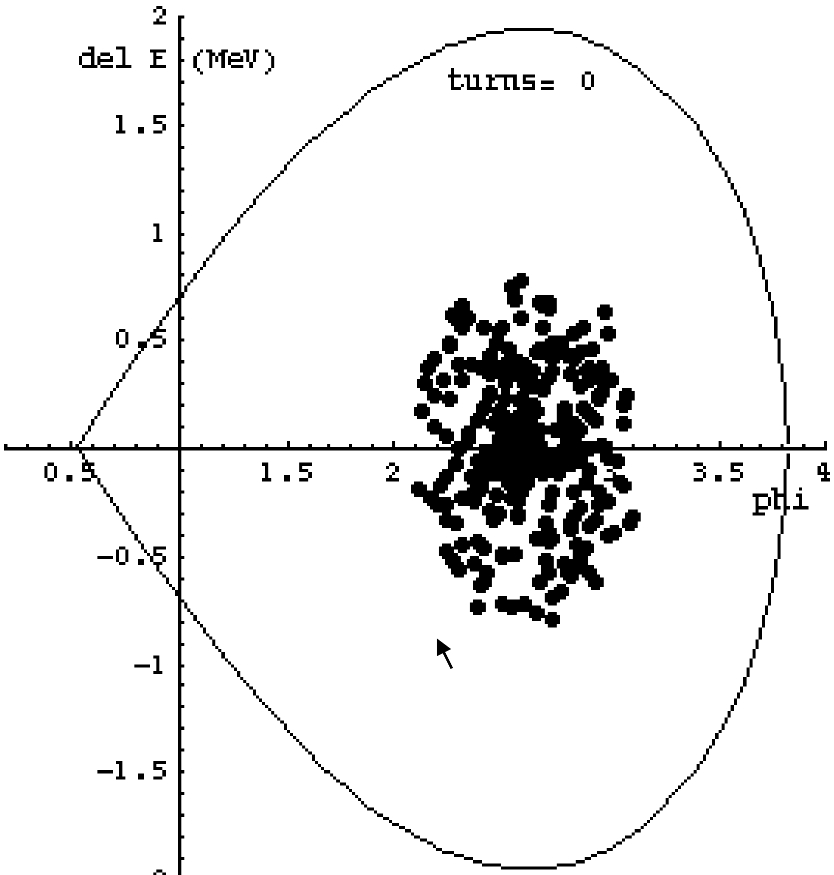
→ Bunch is fine, longitudinal emittance remains constant

→ Dilution of bunch results in increase of long. emittance

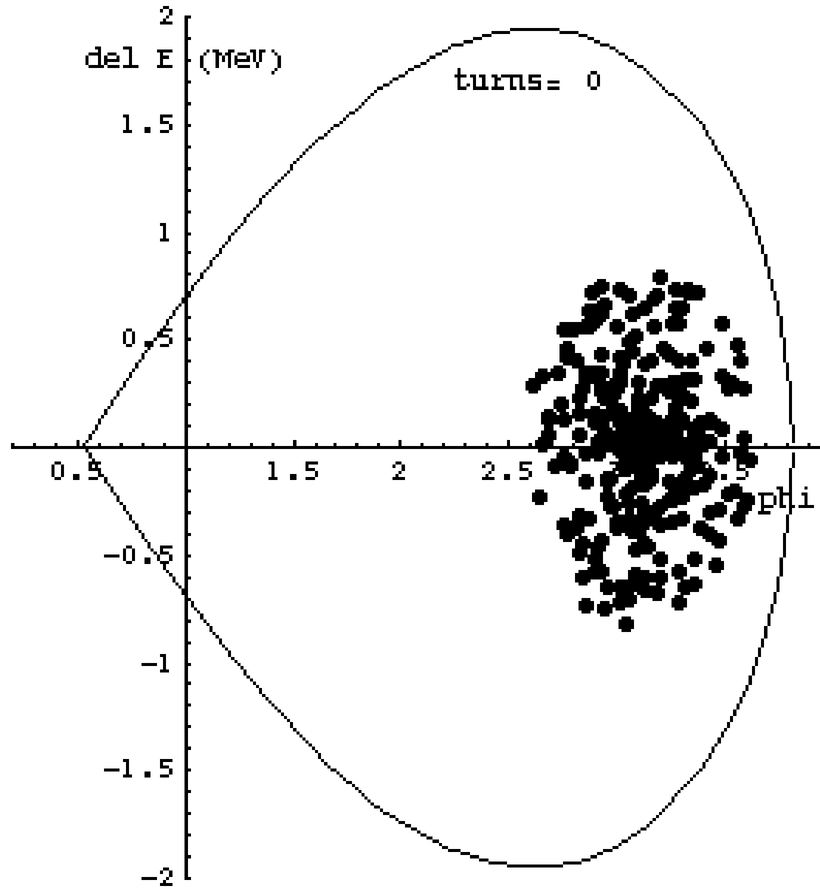
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



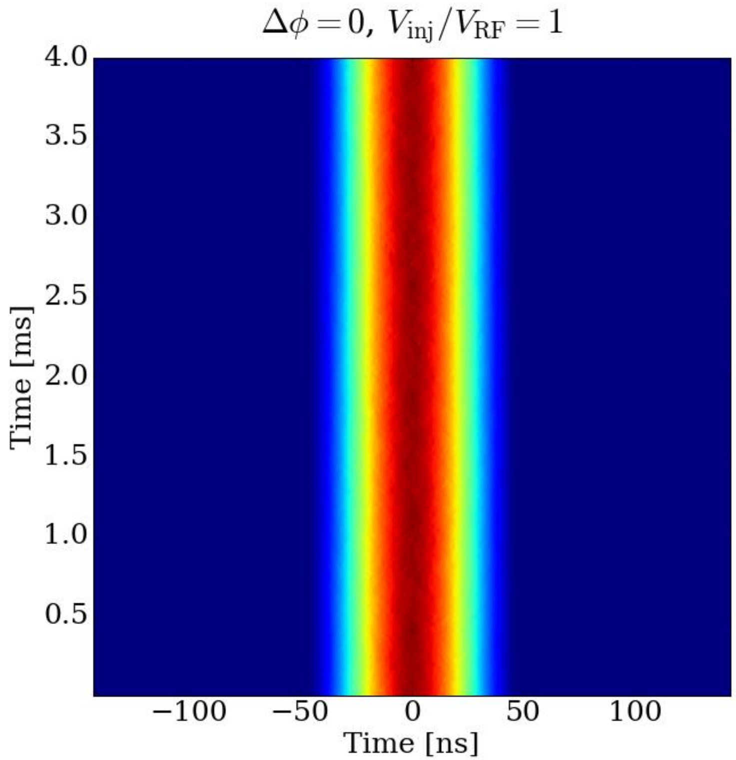
matched beam



mismatched beam – **phase error**

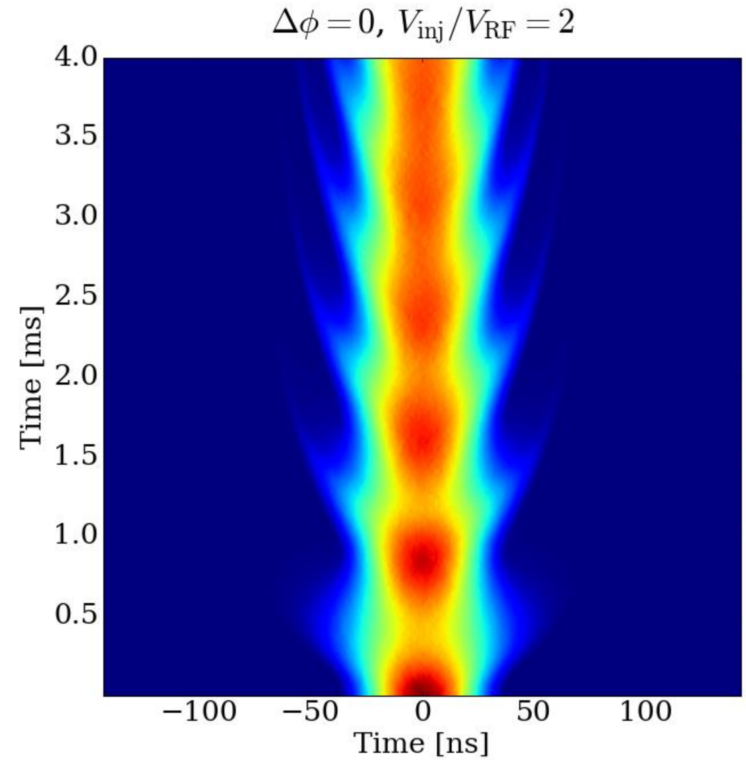
Longitudinal matching – Beam profile

Matched case



→ Bunch is fine, longitudinal emittance remains constant

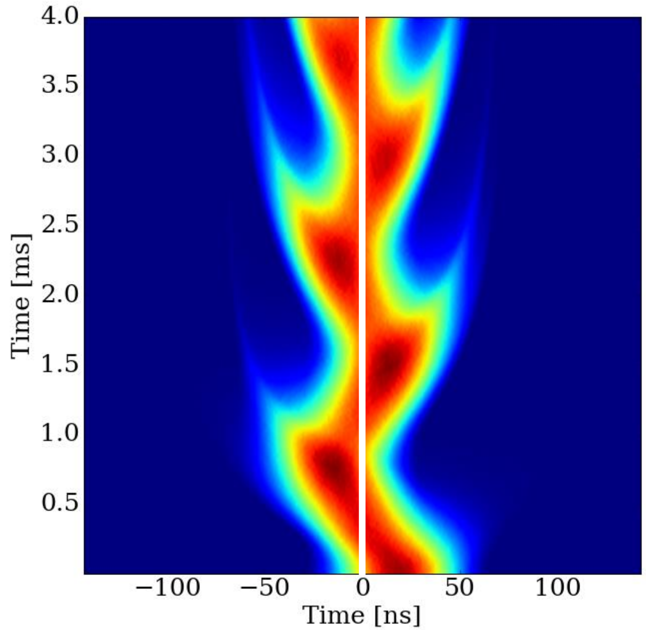
Longitudinal mismatch

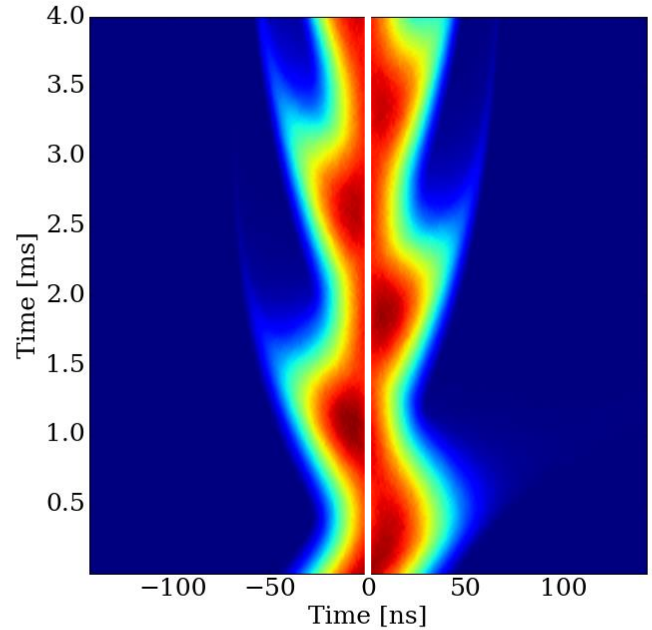


→ Dilution of bunch results in increase of long. emittance

Matching quiz!

- Find the difference!





→ -45° phase error at injection

→ Can be easily corrected by bucket phase

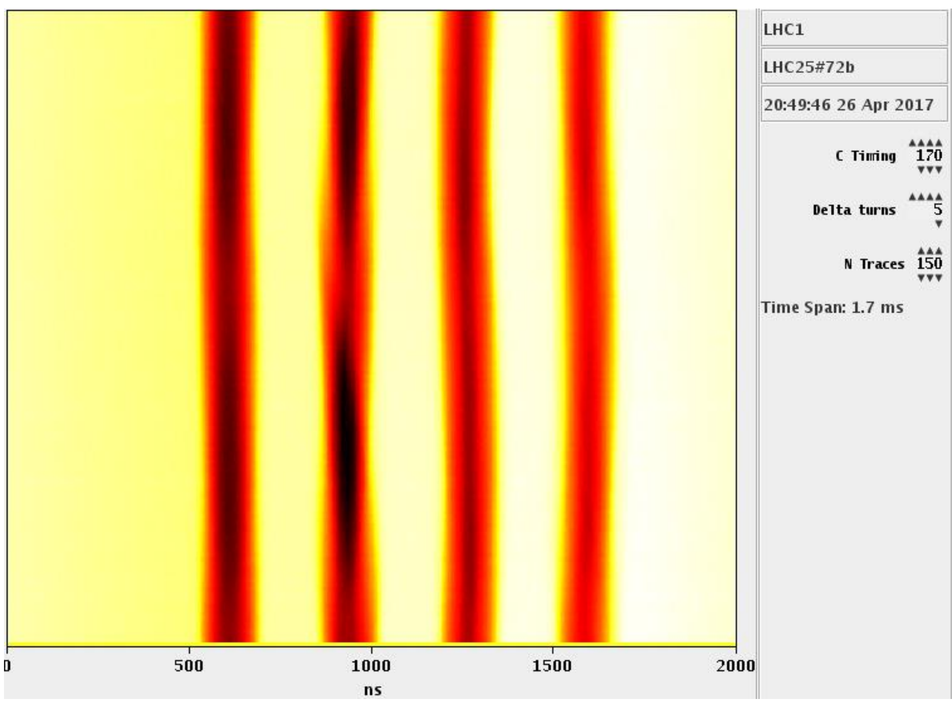
→ Equivalent energy error

→ Phase does not help: requires beam energy change

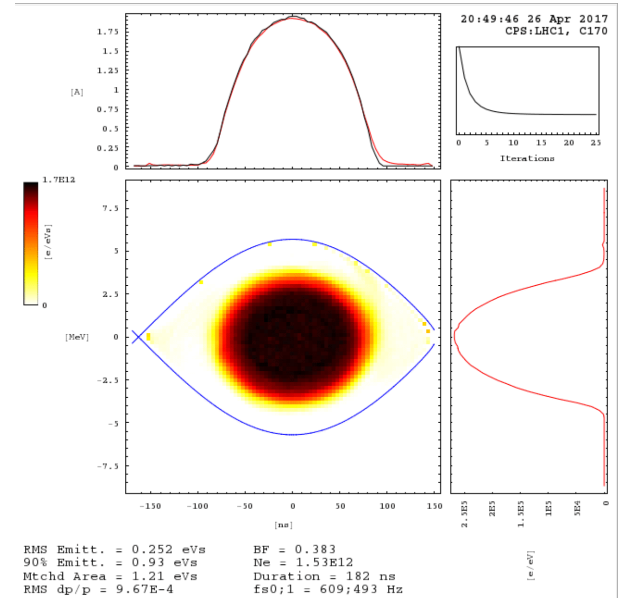
Phase Space Tomography

We can reconstruct the phase space distribution of the beam.

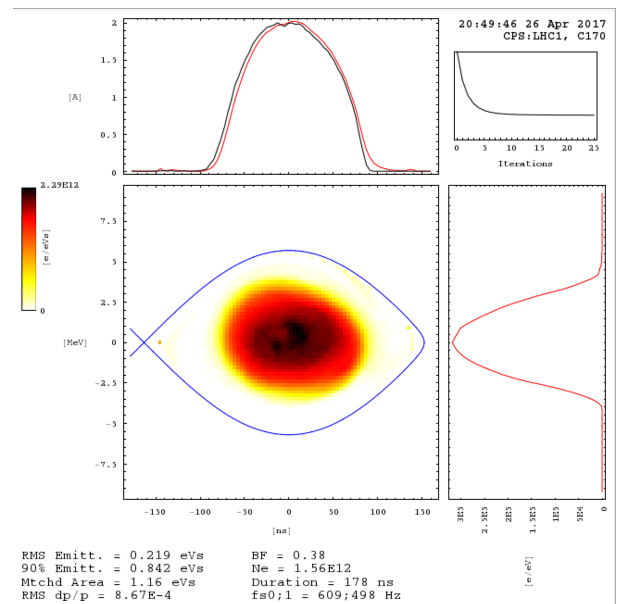
- Longitudinal bunch profiles over a number of turns
- Parameters determining Ω_s



1st bunch



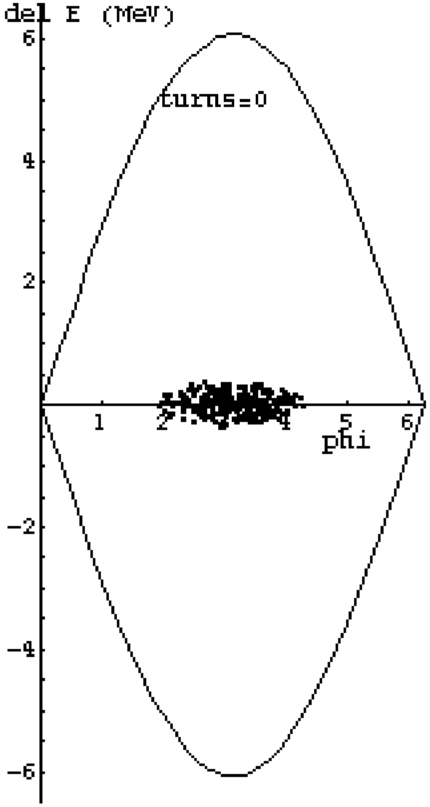
2nd bunch

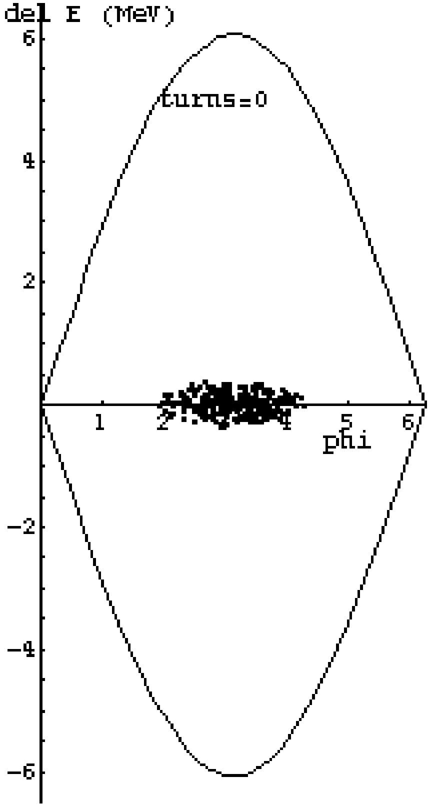


Bunch Rotation

Phase space motion can be used to make short bunches.

Start with a long bunch and extract or recapture when it's short.

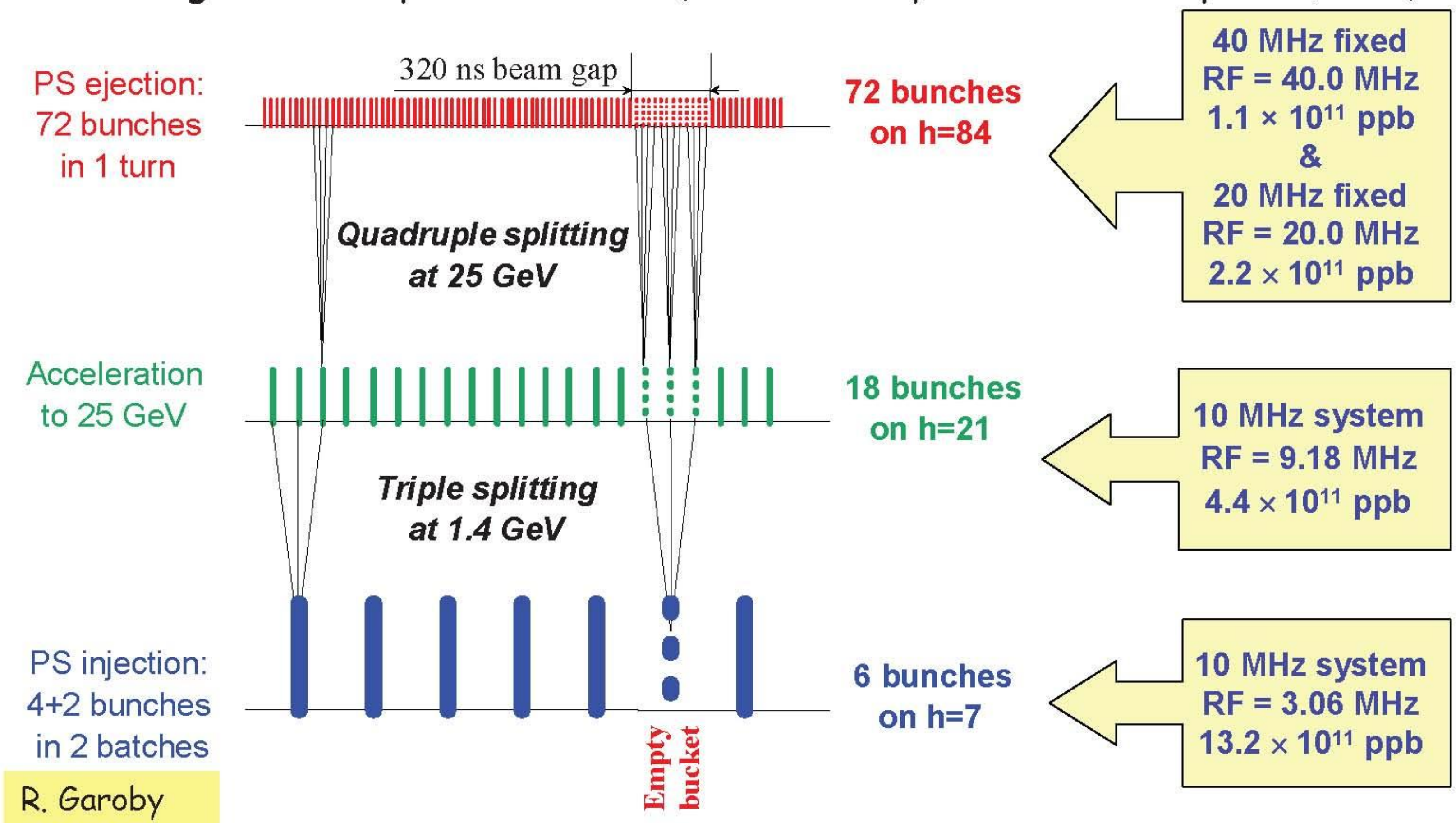




initial beam

Generating a 25ns LHC Bunch Train in the PS

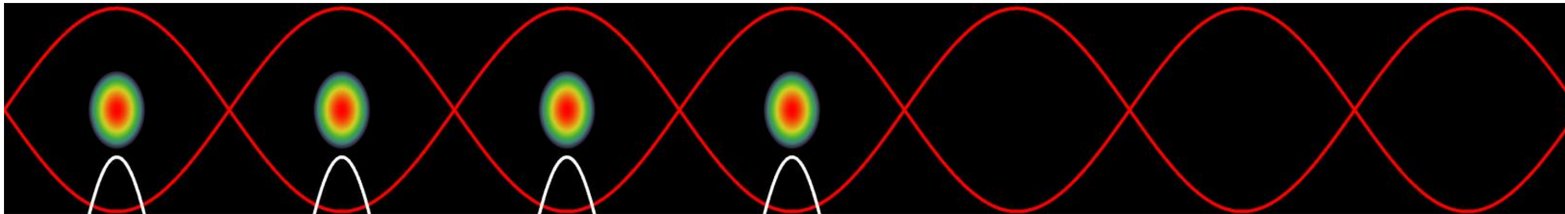
- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



Use double splitting at 25 GeV to generate 50ns bunch trains instead

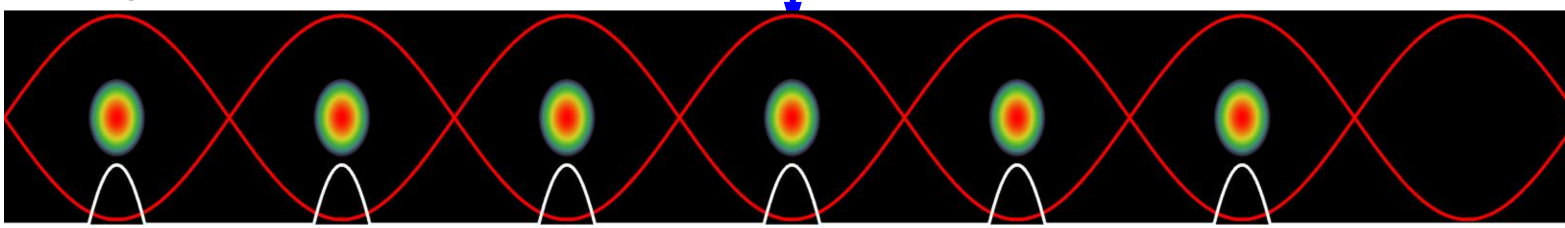
Production of the LHC 25 ns beam

1. Inject four bunches **~ 180 ns, 1.3 eVs**

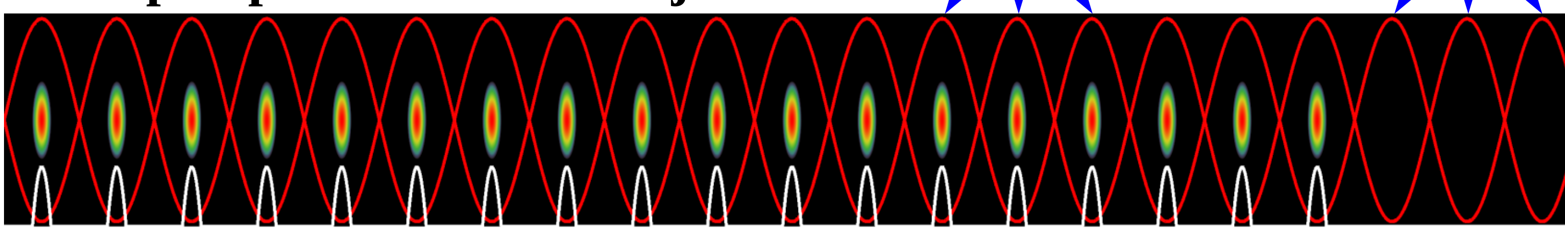


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

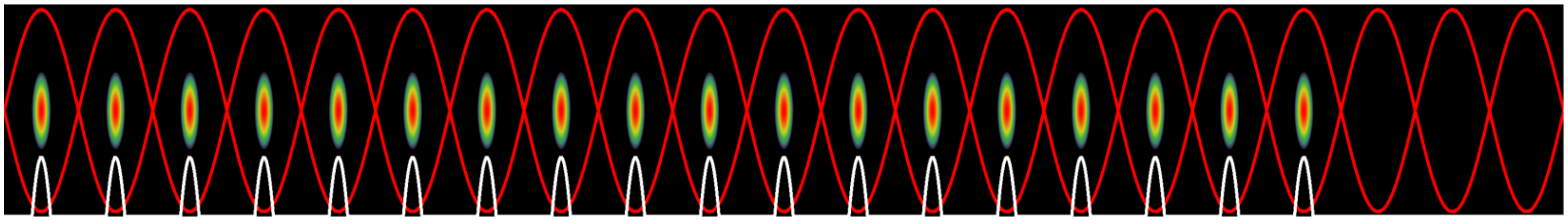


~ 0.7 eVs

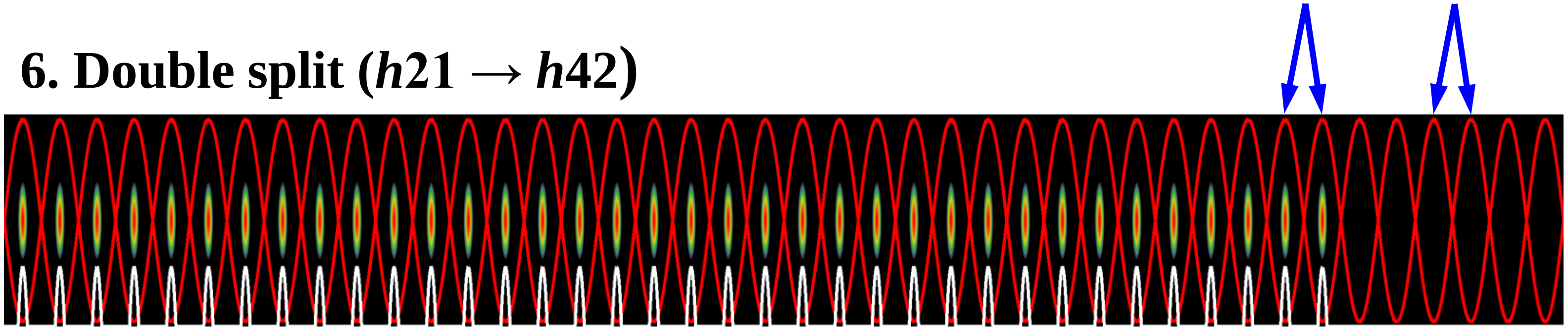
4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

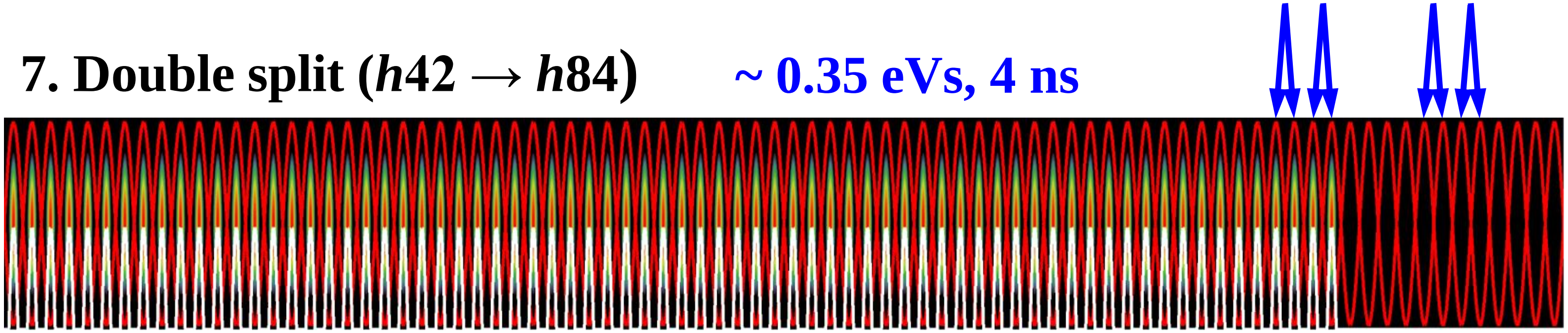
5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs



6. Double split ($h21 \rightarrow h42$)

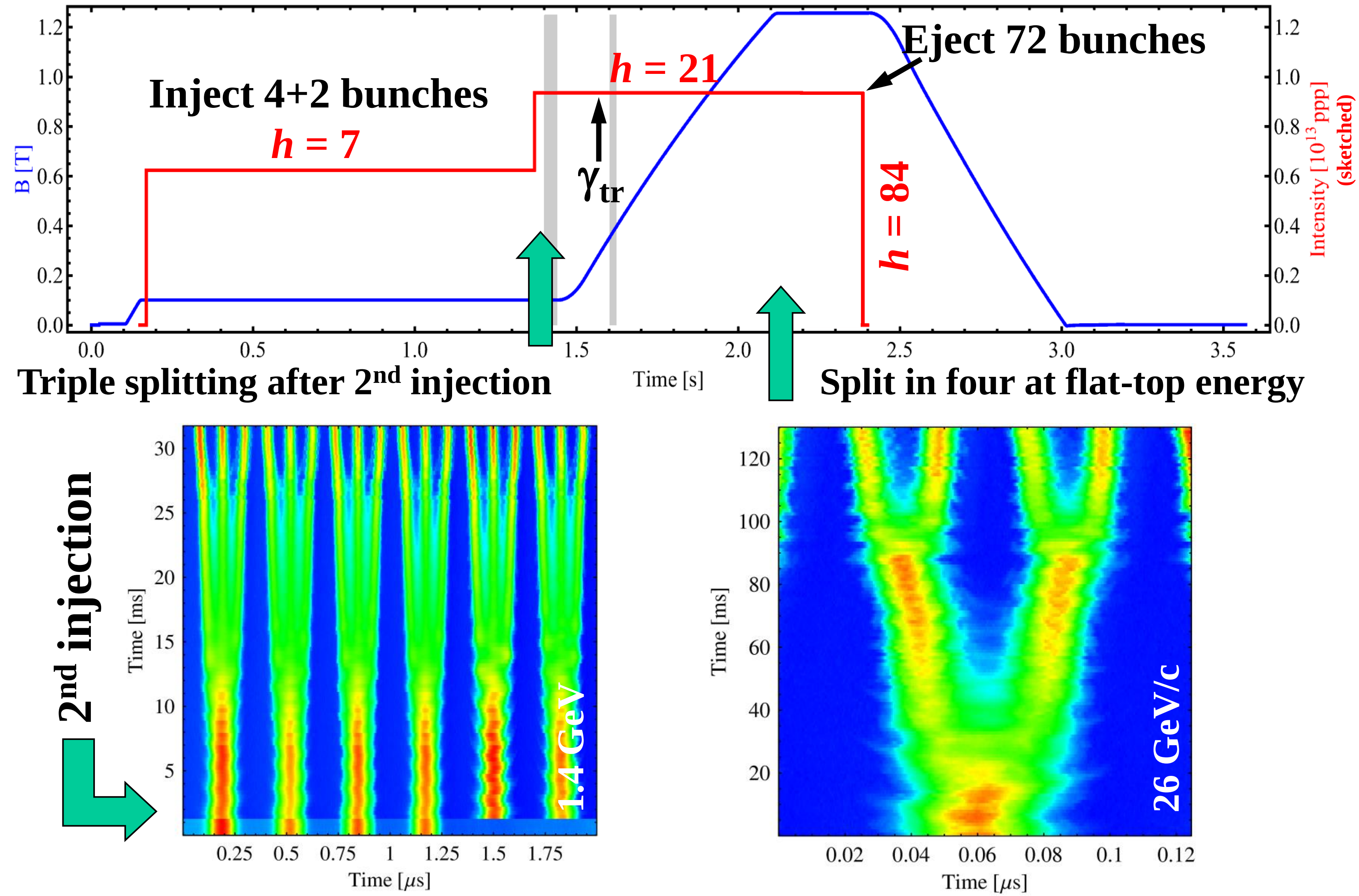


7. Double split ($h42 \rightarrow h84$) **~ 0.35 eVs, 4 ns**



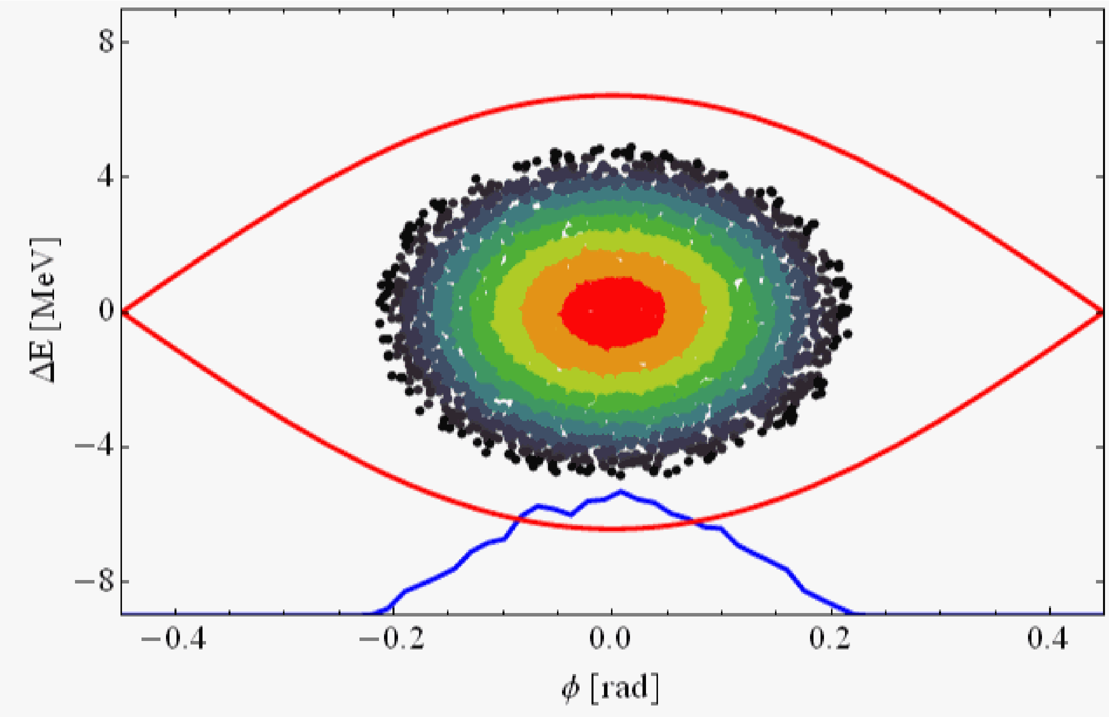
10. Fine synchronization, bunch rotation → Extraction!

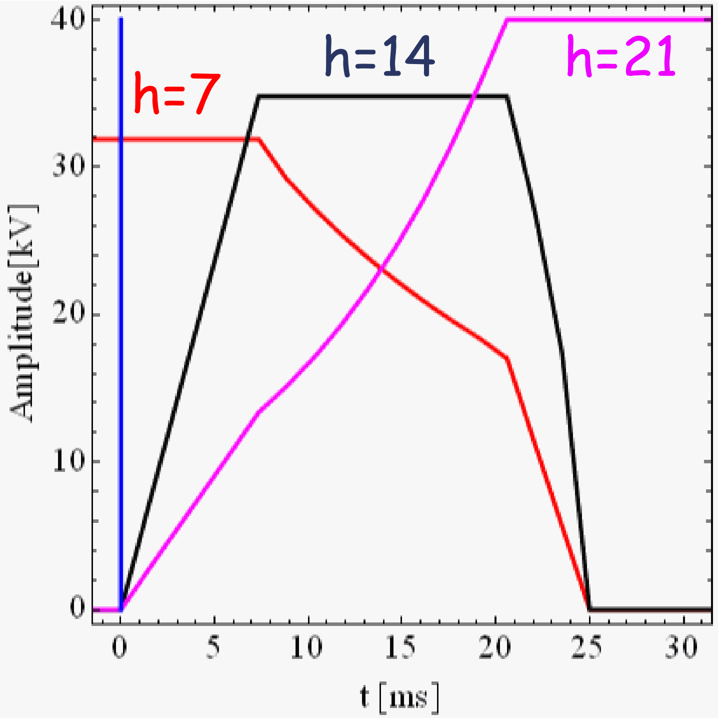
The LHC25 (ns) cycle in the PS



→ Each bunch from the Booster divided by 12 → 6 × 3 × 2 × 2 = 72

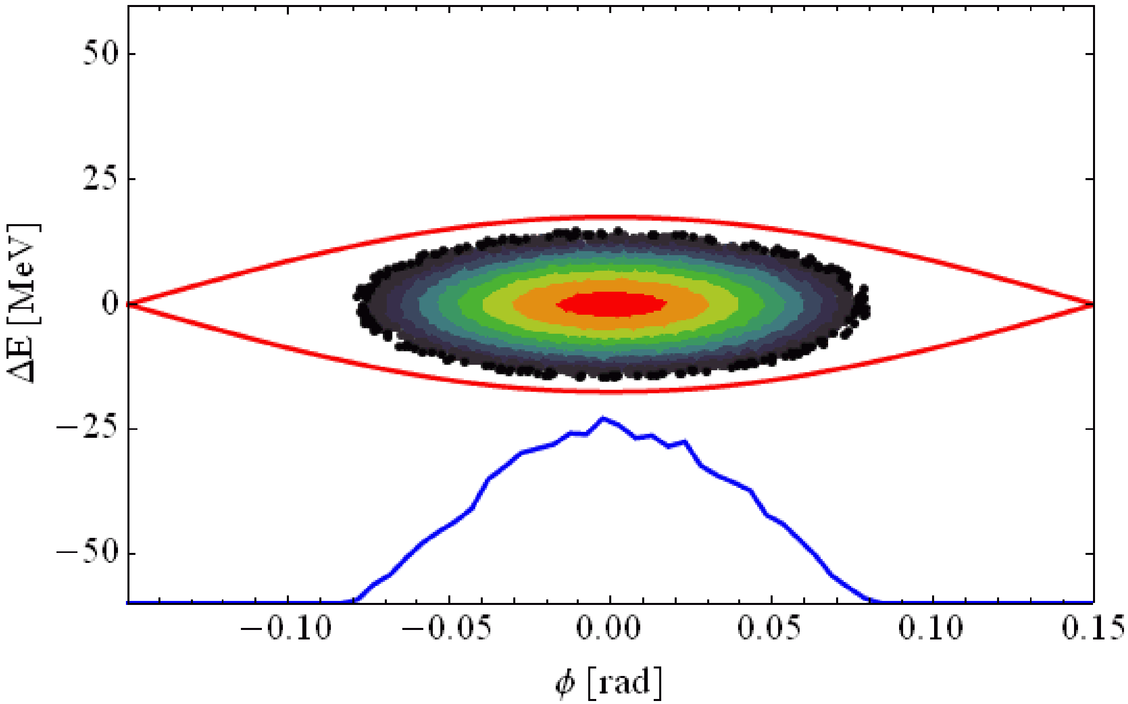
Triple splitting in the PS

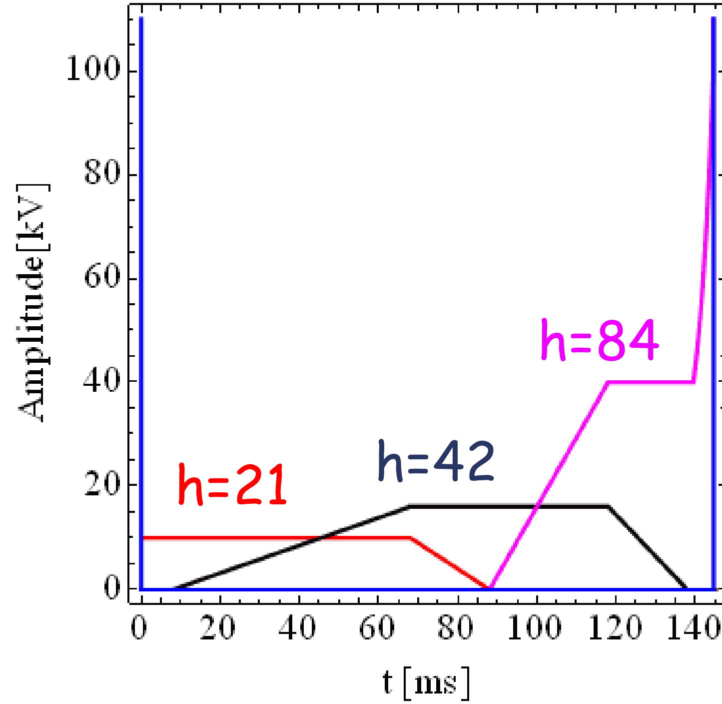




Two times double splitting in the PS

Two times double splitting and bunch rotation:





- Bunch is divided twice using RF systems at
  h = 21/42 (10/20 MHz) and h = 42/84 (20/40 MHz)

- Bunch rotation: first part h84 only + h168 (80 MHz) for final part

Summary


End of our
crash course in
longitudinal dynamics
GIVES YO WINGS
SYNTIUM
intel
www.formula1.com

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In particular: https://arxiv.org/abs/2011.02932
Longitudinal Beam Dynamics in Circular Accelerators

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In particular (hope I don't forget anyone):

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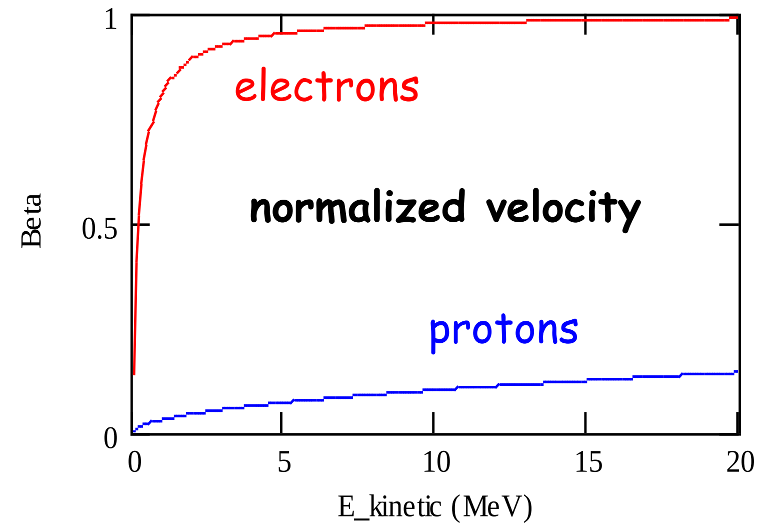
Velocity, Energy and Momentum

normalized velocity $\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$

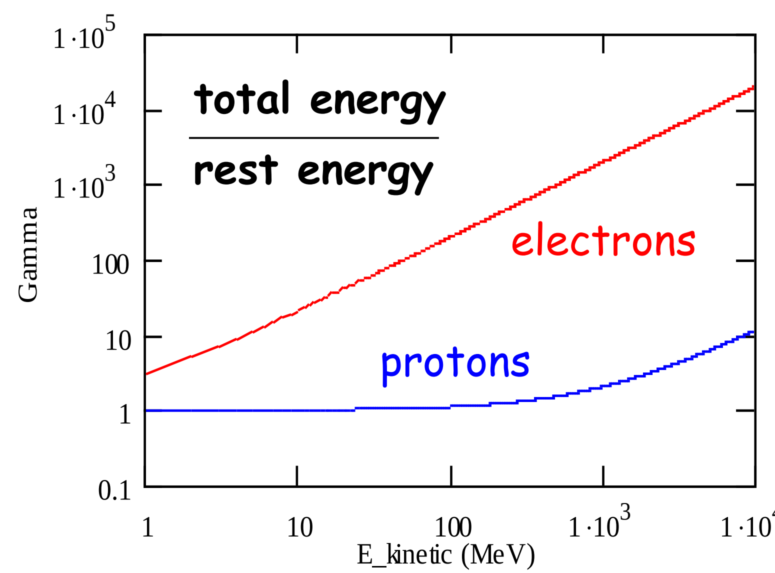
=> electrons almost reach the speed of light very quickly (few MeV range)



$\frac{\text{total energy}}{\text{rest energy}}$

$$E = \gamma m_0 c^2$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



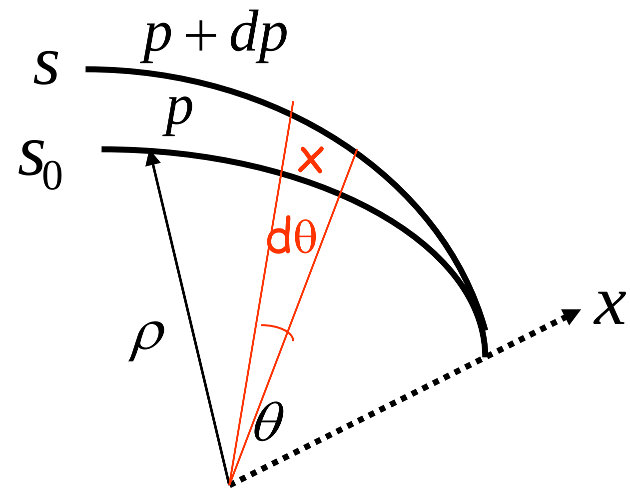
Momentum $p = mv = \frac{E}{c^2}\beta c = \beta \frac{E}{c} = \beta\gamma m_0 c$

Derivation: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{\rho} ds_0 = \oint \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

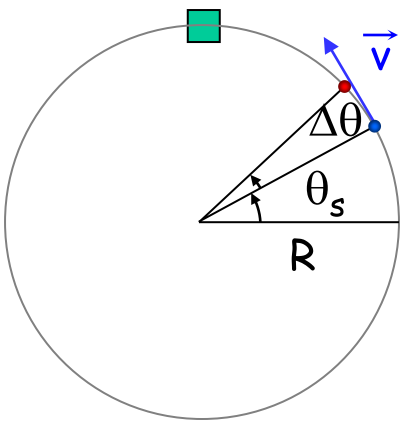
$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad \textit{with} \quad \theta = \int \omega\, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega}{dp}\right)_s$$ and $$E^2 = E_0^2 + p^2c^2$$ $$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets: $$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = q\,\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

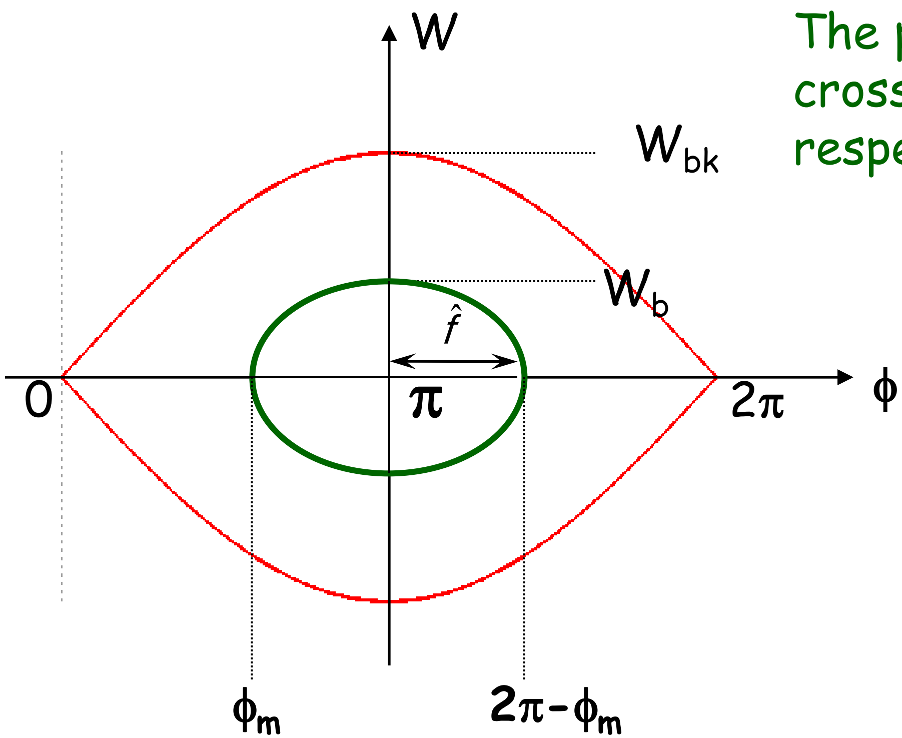
The difference of energy gain with respect to the reference particle $\Delta E = E - E_0$ leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = q\,\hat{V}(\sin\phi - \sin\phi_s)$$

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{\phi_m}{2} - \cos^2\frac{\phi}{2}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{f_m}{2} = W_{bk} \sin\frac{\hat{f}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{DE}{E_s}\right)_b = \left(\frac{DE}{E_s}\right)_{RF} \cos\frac{f_m}{2} = \left(\frac{DE}{E_s}\right)_{RF} \sin\frac{\hat{f}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{f}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{f}^2 - (Df)^2} \quad \longrightarrow \quad \left(\frac{16W}{A_{bk}\hat{f}}\right)^2 + \left(\frac{Df}{\hat{f}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{p}{16} A_{bk} \hat{f}^2$$