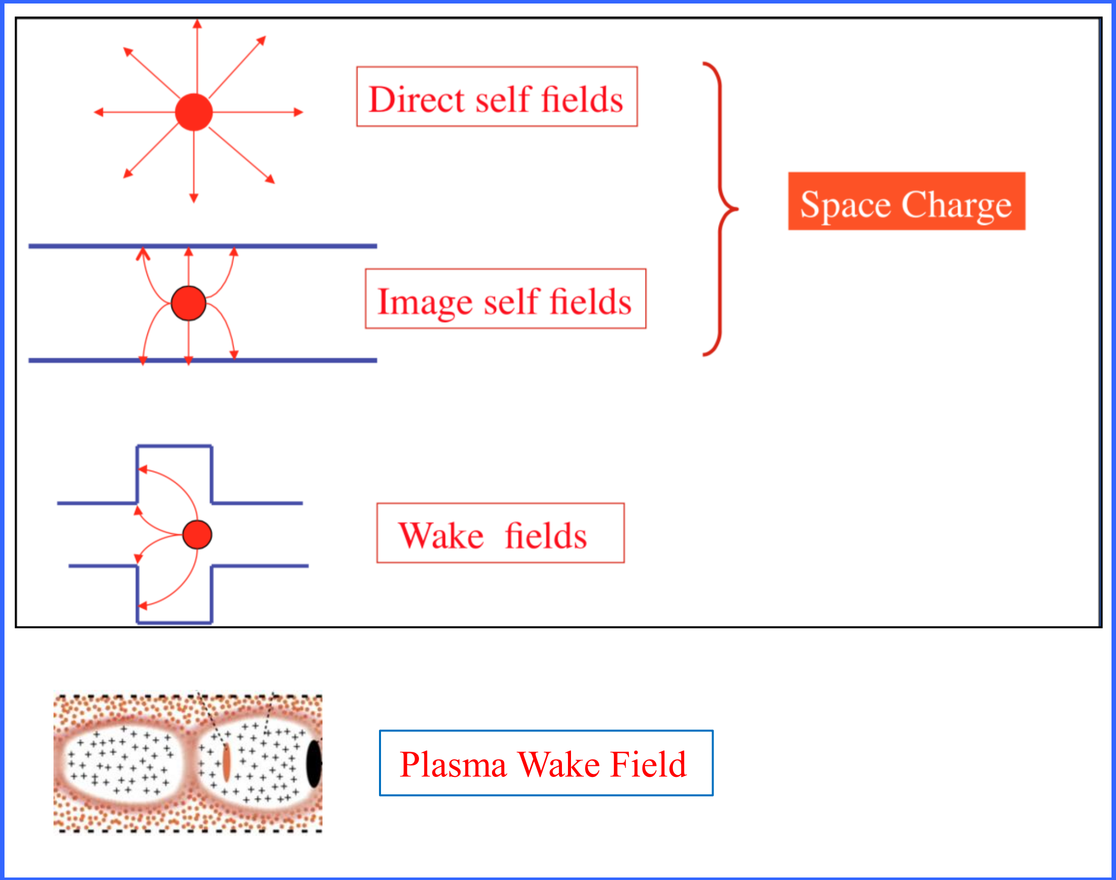
Space Charge in Linear Machines

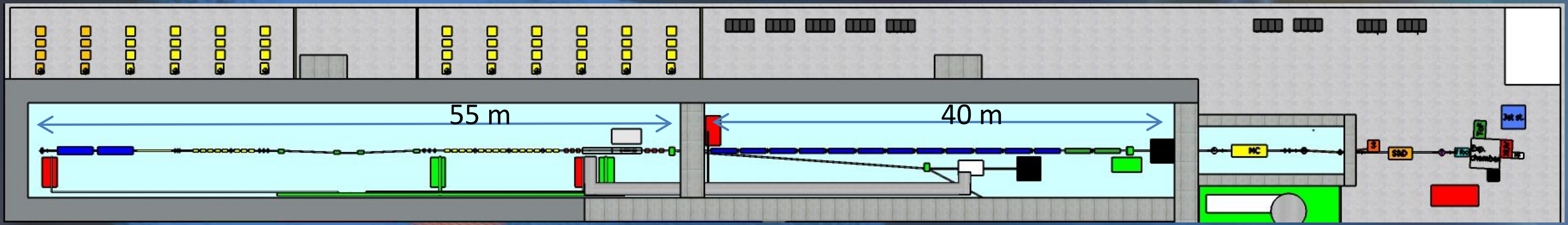
Massimo.Ferrario@LNF.INFN.IT



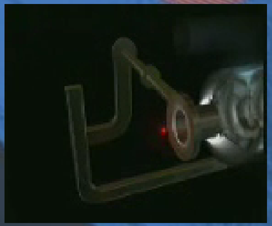
The CERN Accelerator School

Spa – November 14 - 2024

Different Regimes of Beam Propagation



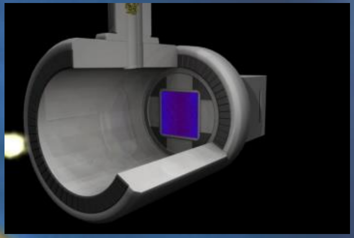
Space charge dominated



Emittance dominated (Wake Fields)

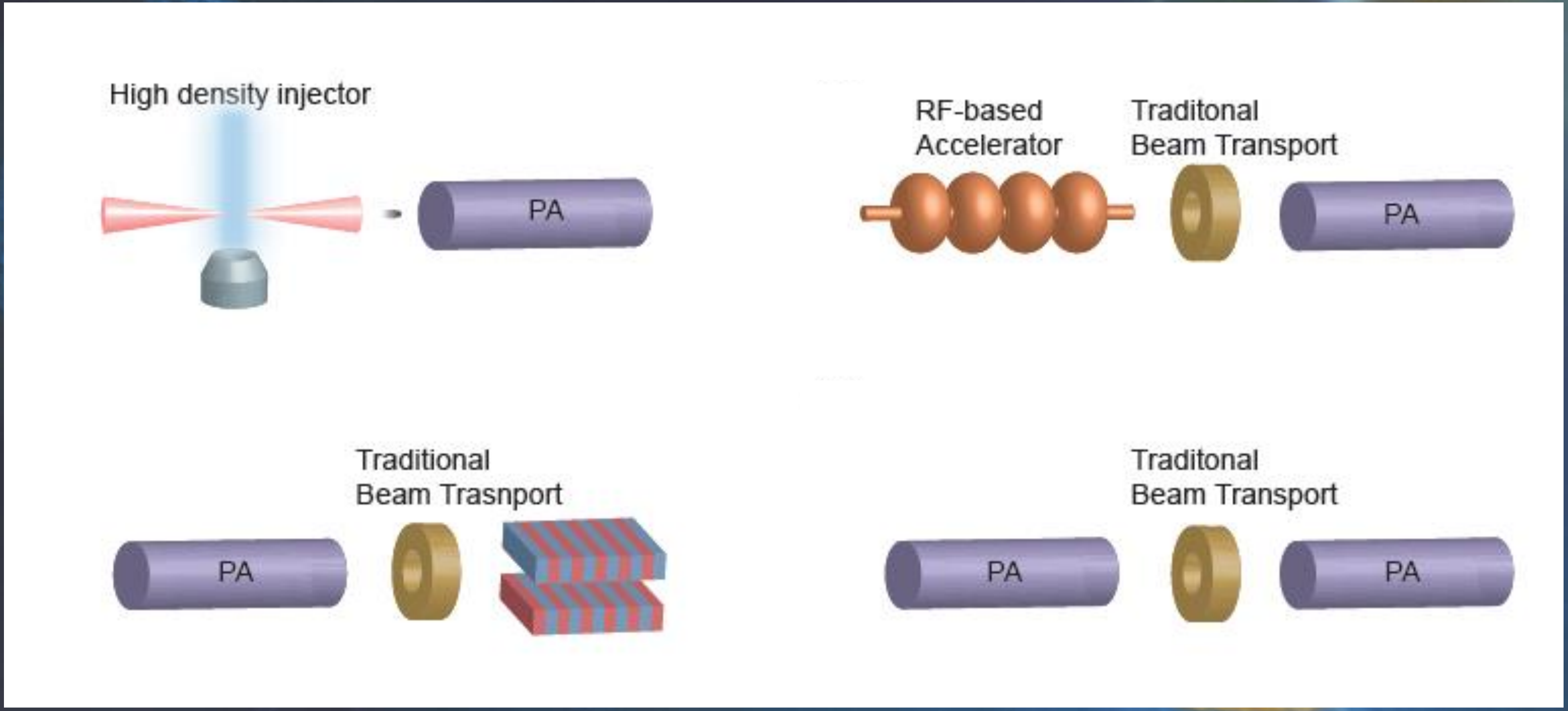
Plasma dominated

Radiation dominated



Users dominated

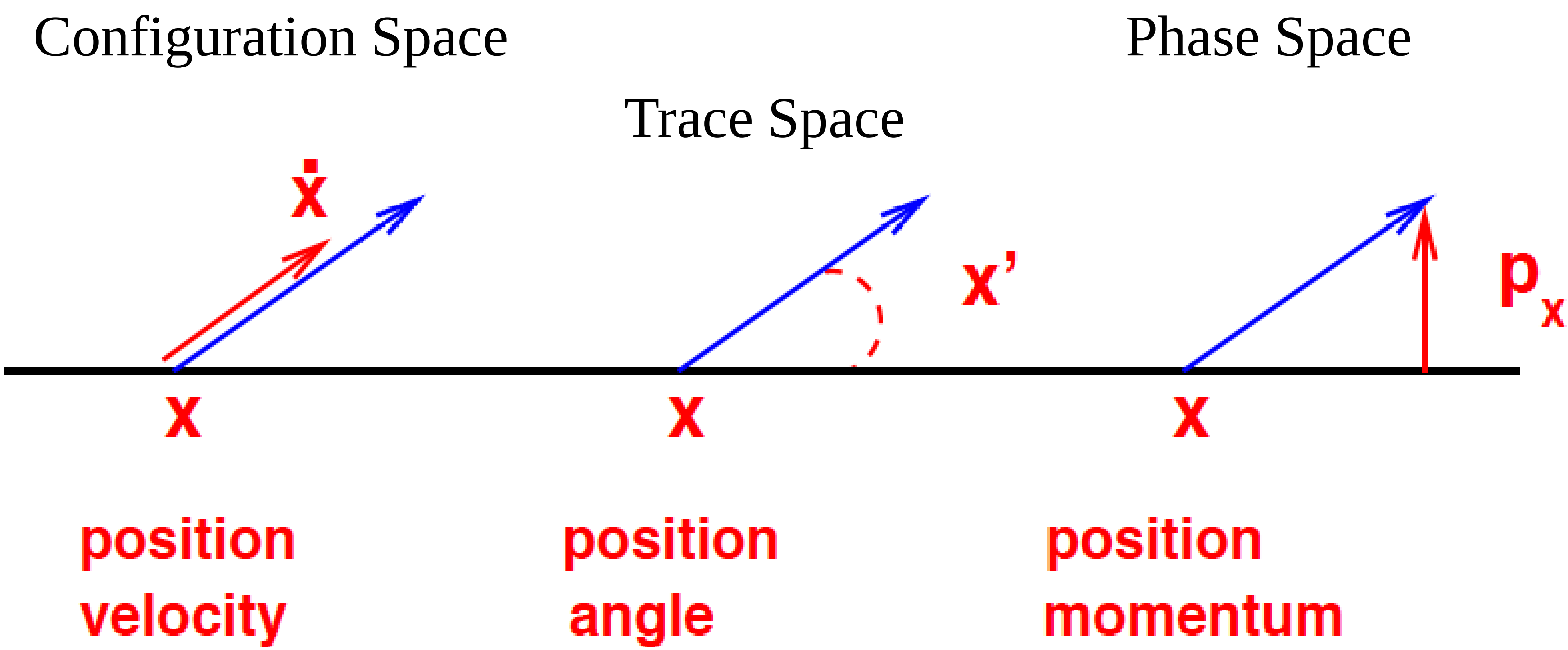
Matching Conditions are fundamental



OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

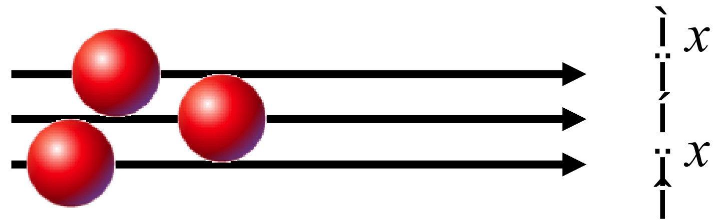
Typical coordinates to describe the particle motion
(6 per particle)



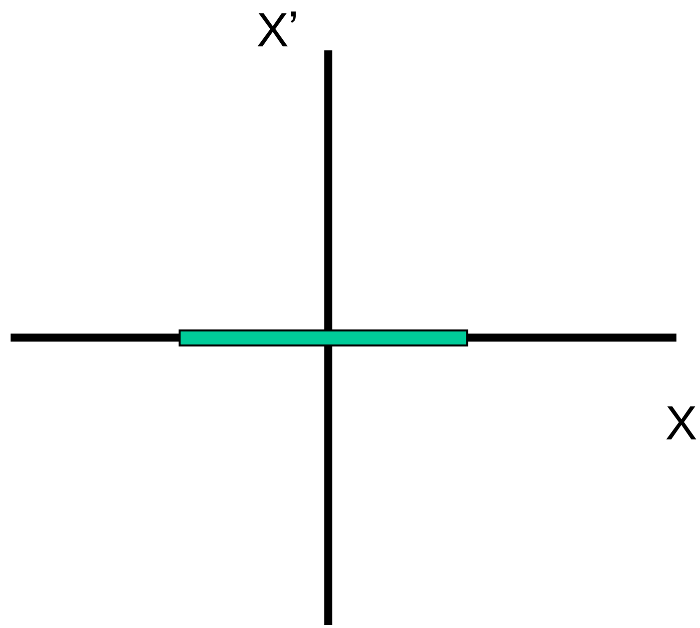
$$\begin{cases} x \\ x' = \frac{dx}{dz} = \frac{p_x}{p_z} \end{cases}$$

$$p_z = \gamma m_o v_z = \beta_z \gamma m_o c$$

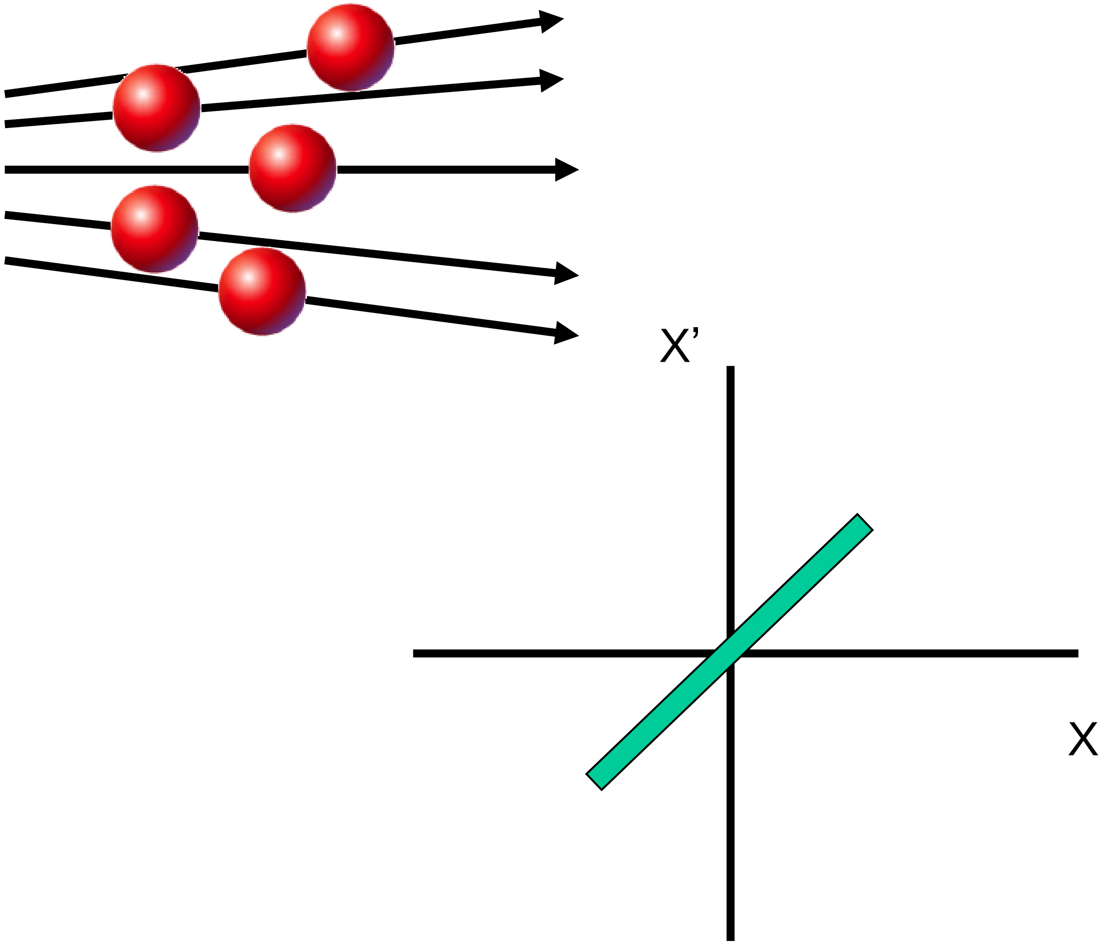
Trace space of an ideal laminar beam



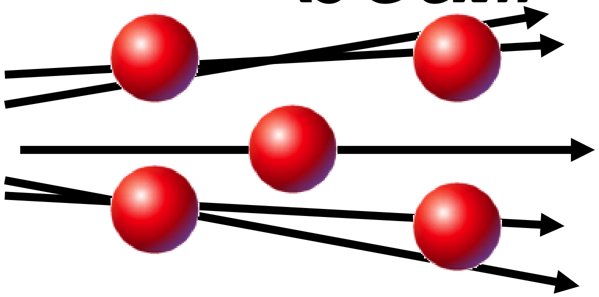
$$x' = \frac{dx}{dz} = \frac{p_x}{p_z} \qquad p_x \ll p_z$$

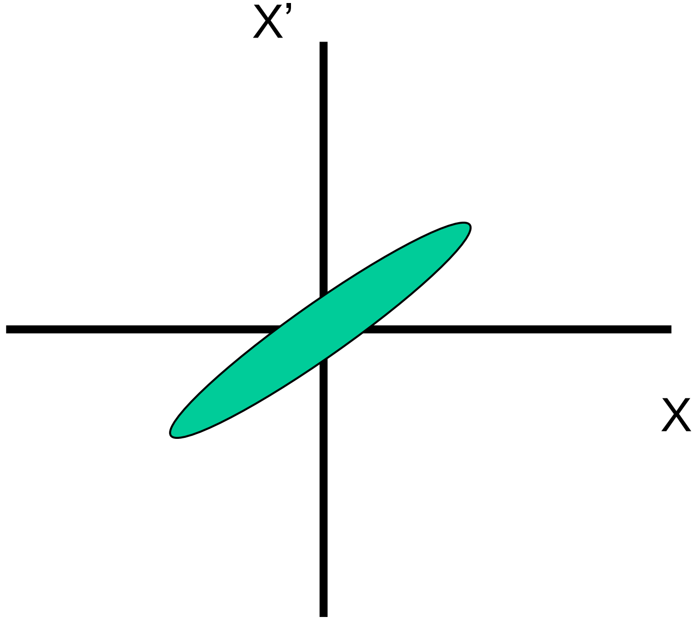


Trace space of a laminar beam



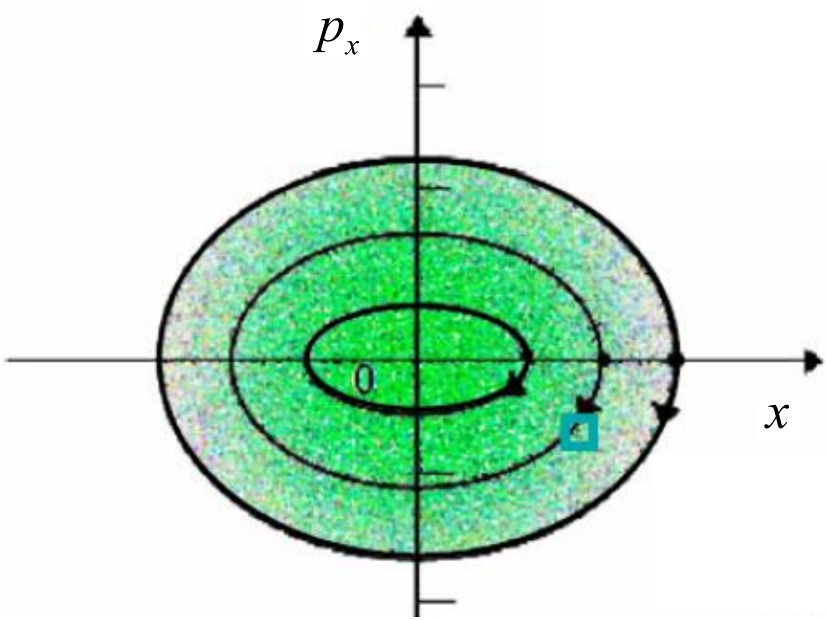
Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x' trace space or x-p_x phase space.

$$\ddot{x} + k^2 x = 0$$



$$H = \frac{1}{2m}[p_x^2 + m^2\omega^2 x^2]$$

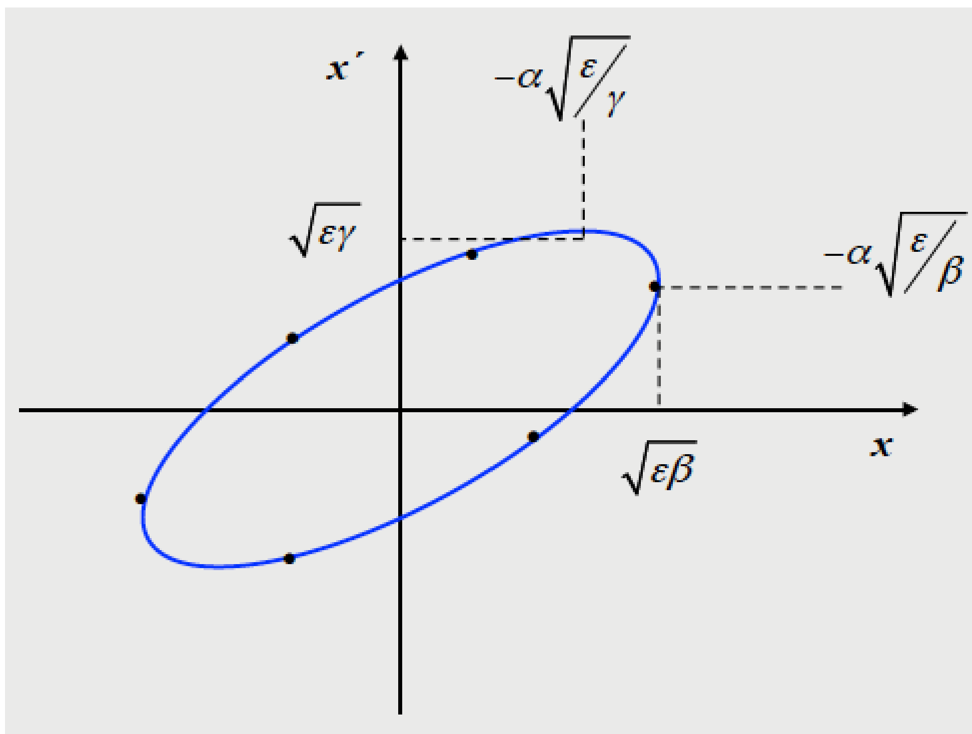
$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

Geometric emittance: ε_g

Ellipse equation: $$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$$

Twiss parameters: $$\beta\gamma - \alpha^2 = 1 \qquad \beta' = -2\alpha$$

Ellipse area: $$A = \pi\varepsilon_g$$



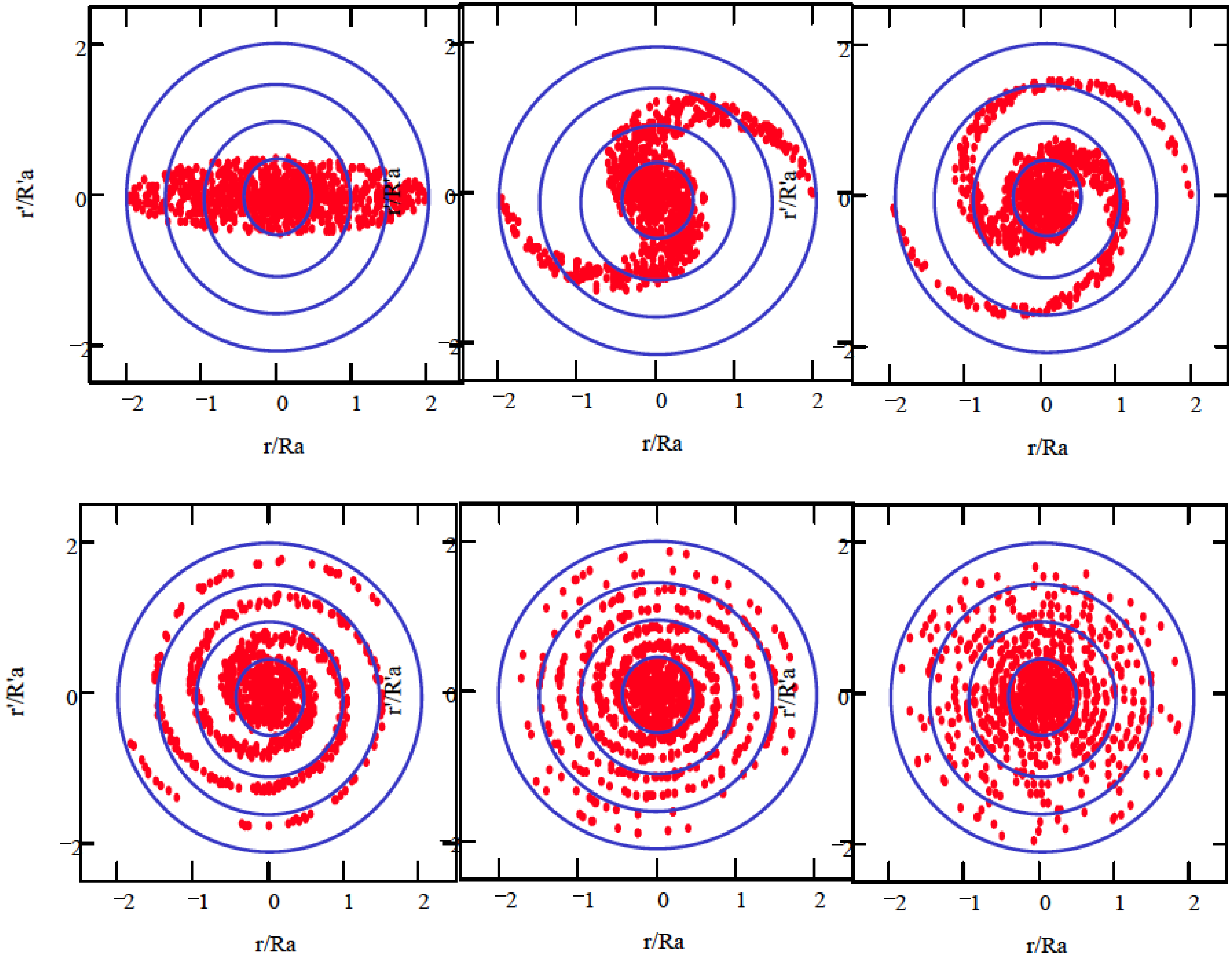
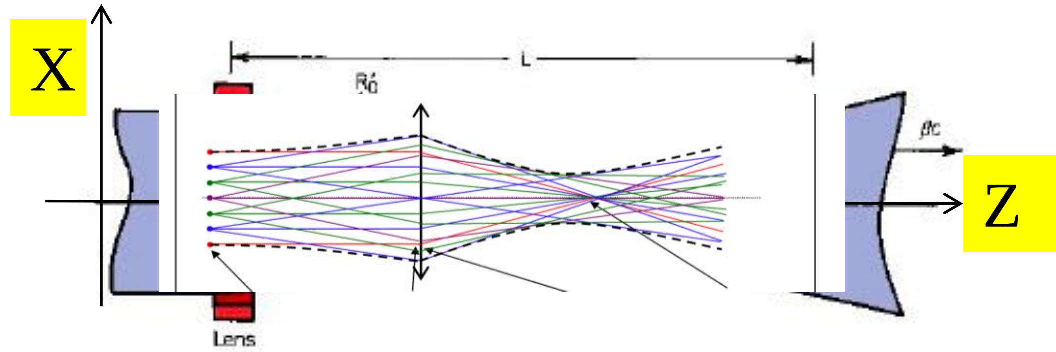
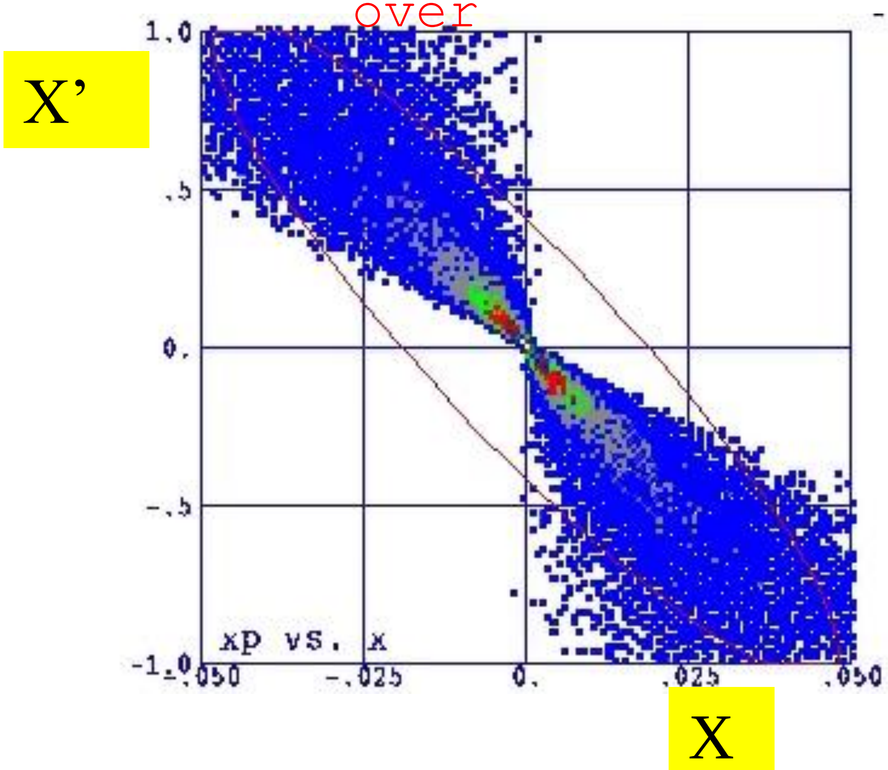


Fig. 17: Filamentation of mismatched beam in non-linear force

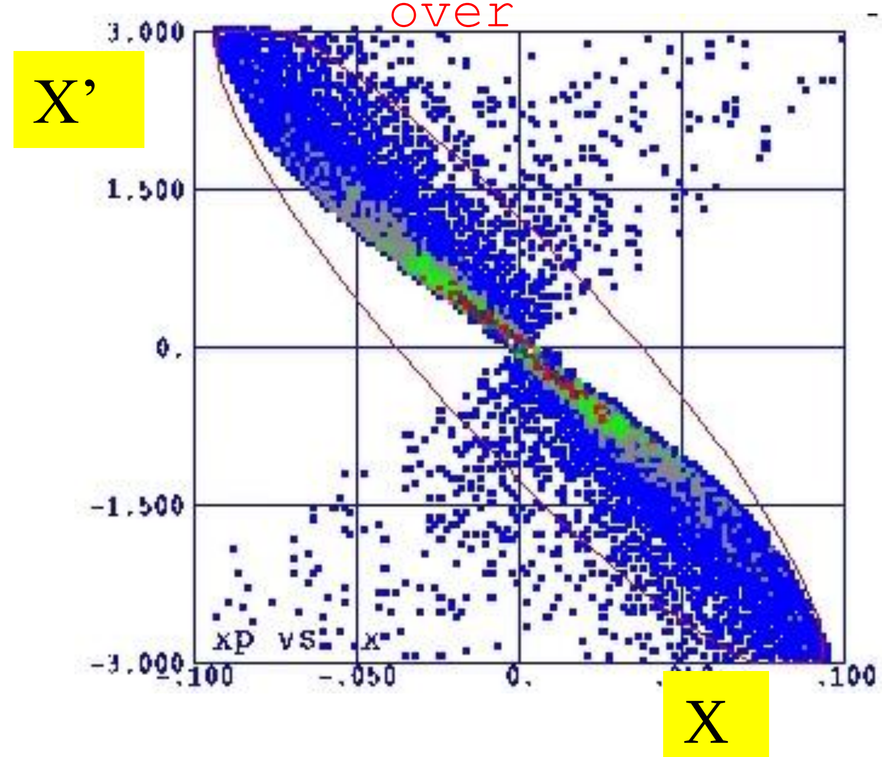
Phase space evolution



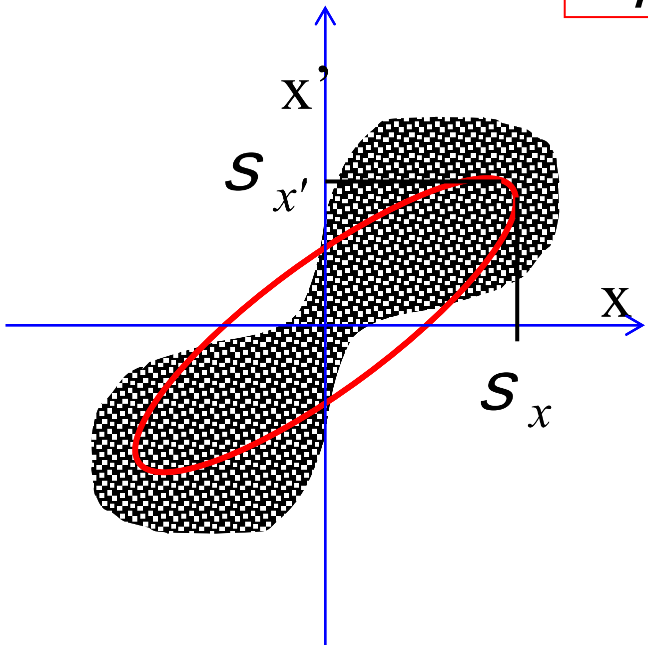
No space charge => cross over



With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1$$

$$f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\beta' = -2\alpha$

it follows: $$\alpha = -\frac{1}{2\varepsilon_{rms}}\frac{d}{dz}\langle x^2 \rangle = -\frac{\langle x x' \rangle}{\varepsilon_{rms}} = -\frac{\sigma_{xx'}}{\varepsilon_{rms}}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

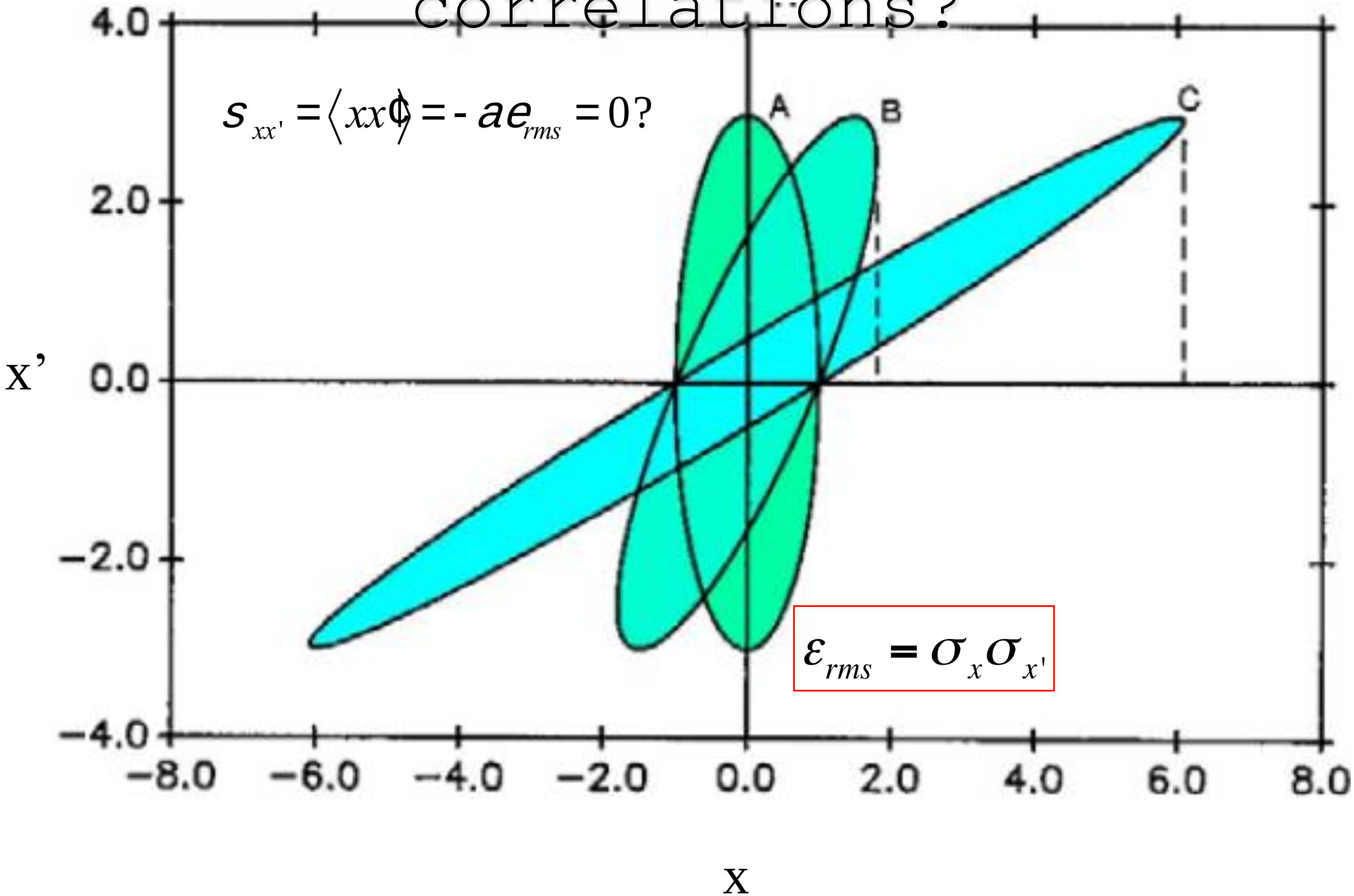
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)}$$

$$x' = \frac{p_x}{p_z}$$

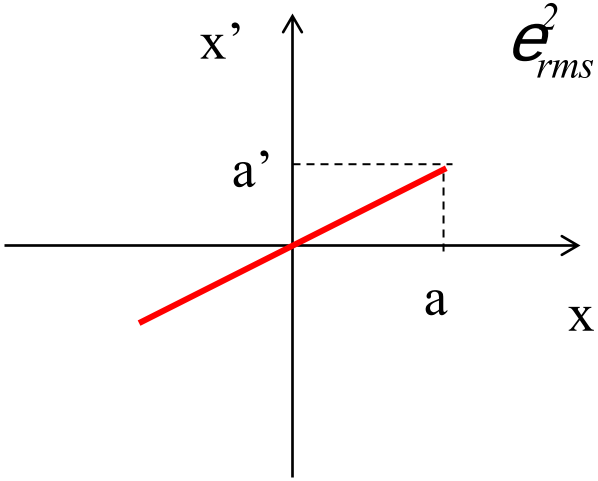
Which distribution has no correlations?

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha\varepsilon_{rms} = 0?$$

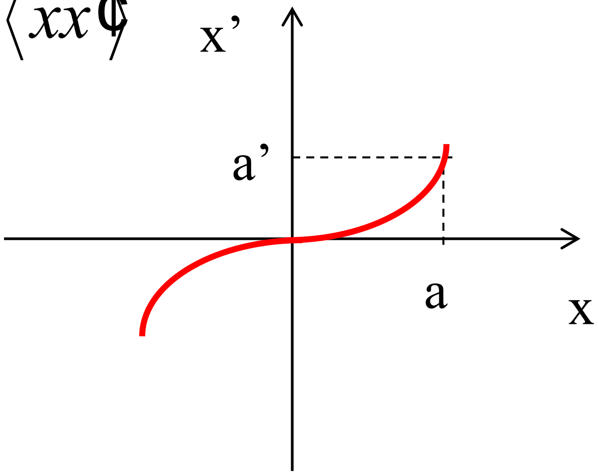


$$\varepsilon_{rms} = \sigma_x \sigma_{x'}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When n = 1 ==> $\varepsilon_{rms} = 0$

When n ≠ 1 ==> $\varepsilon_{rms} \neq 0$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c \beta\gamma x'$ $\quad p_z \approx p$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)} \approx \langle\beta\gamma\rangle\varepsilon_{rms}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings.

But rms emittance is not Liouvillian!

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Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2 \rangle = \frac{1}{2\sigma_x} 2\langle xx' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle + \langle xx'' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

Envelope Equation with Linear Focusing

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\epsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\epsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

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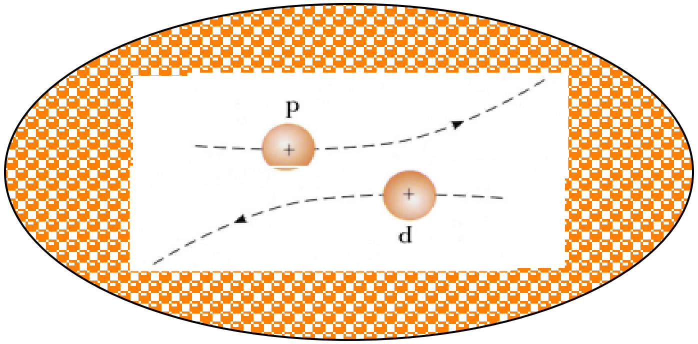
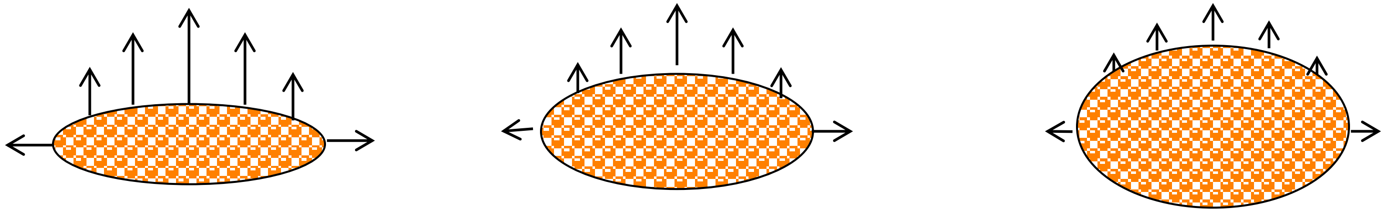
Space Charge: what does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

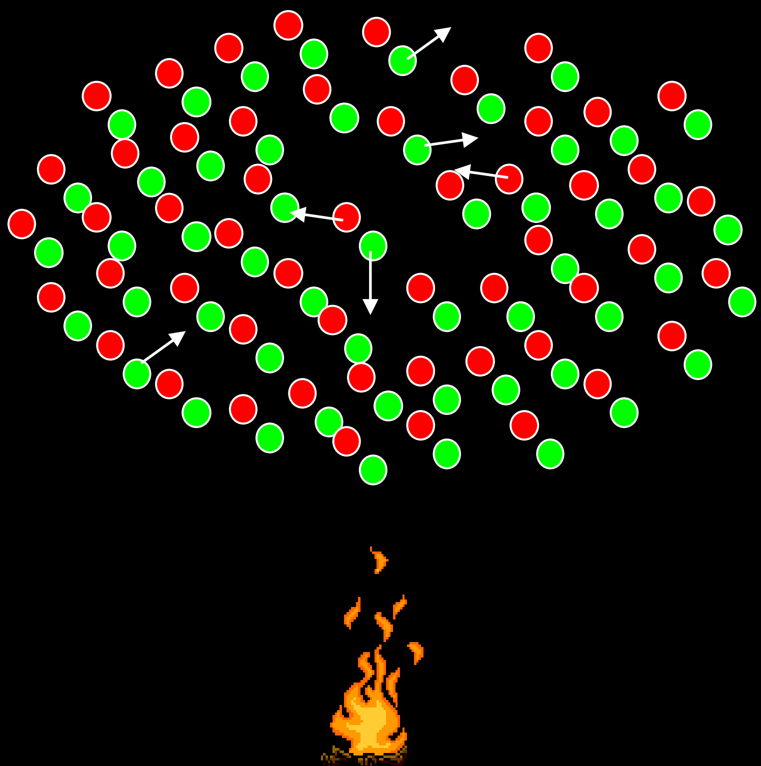


2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**

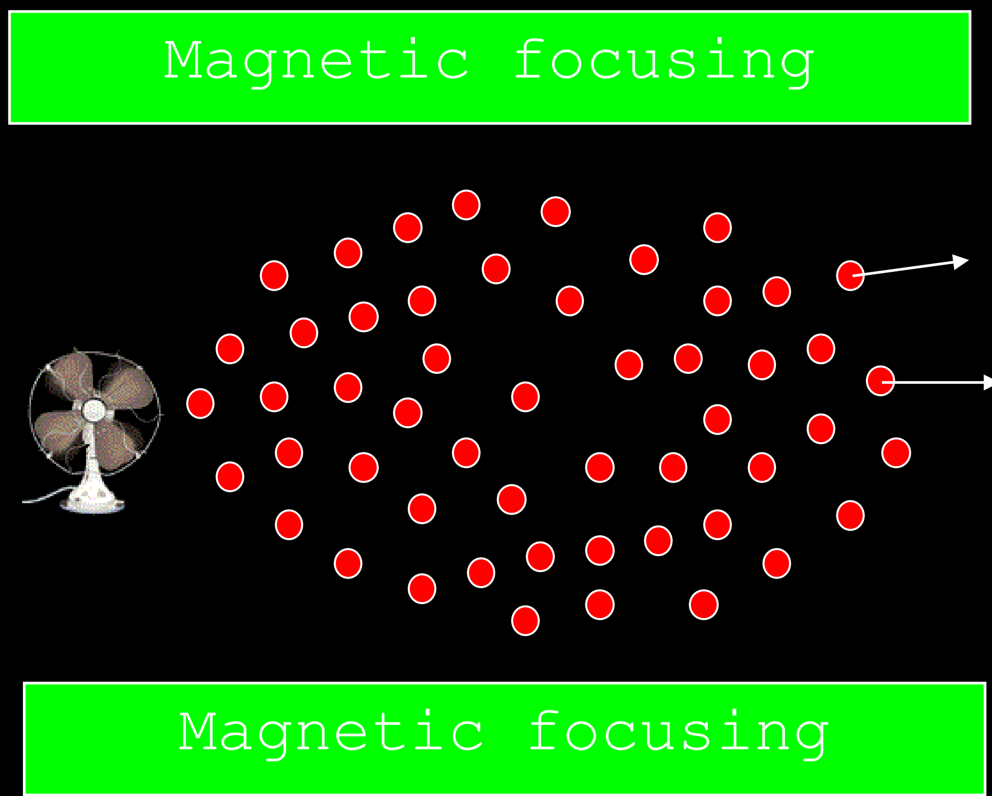


Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma



A measure for the relative importance of collisional versus collective effects is the

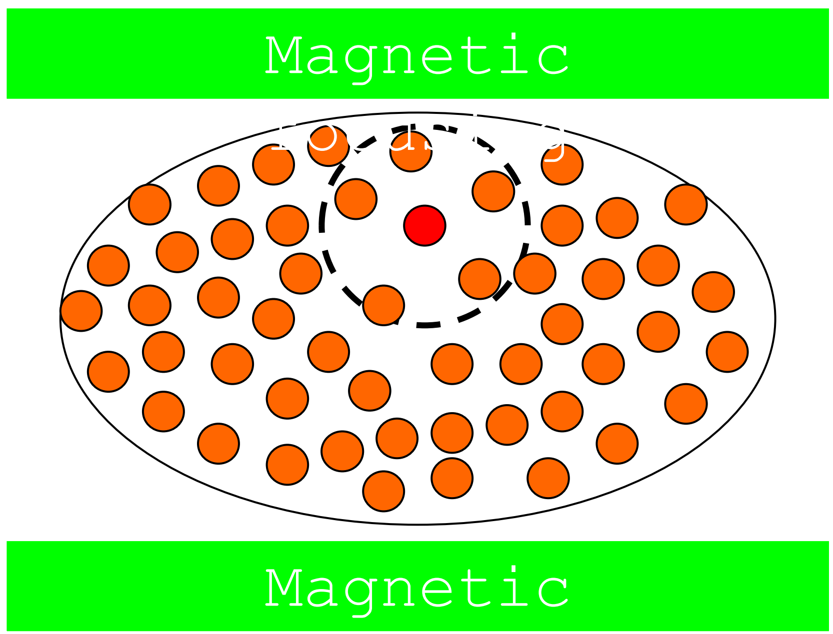
Debye Length λ_D

Let consider a **non-neutralized** system of **identical charged particles**

We wish to calculate the effective potential of a test charged particle surrounded by other particles that are statistically distributed.

$$F(\vec{r}) = \frac{C}{r}$$

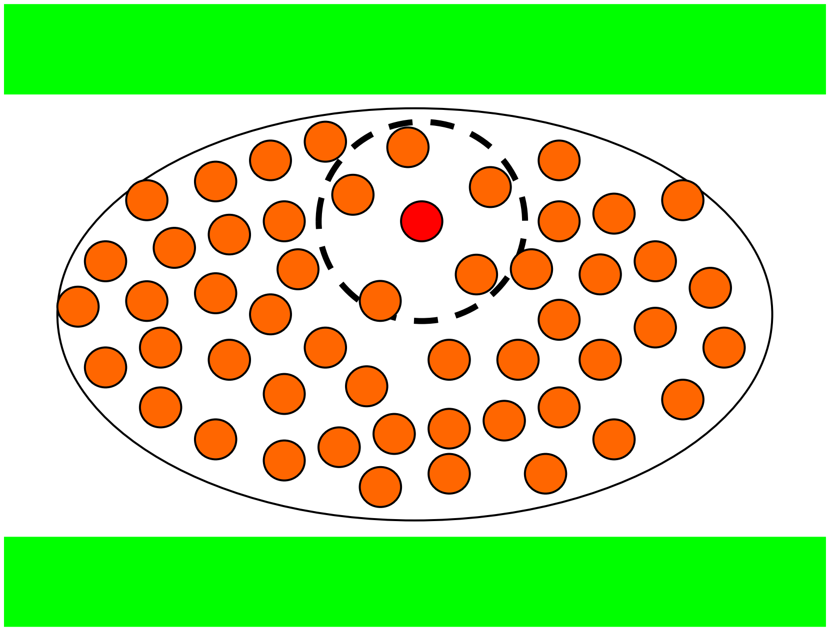
$$C = \frac{e}{4\pi\epsilon_o}$$



$$F_D(\vec{r}) = ?$$

The plasma responds to an external charge by rearranging the charge distribution around it.
This response is governed by the Boltzmann distribution for the density of particles at thermal equilibrium

The effective potential of a test charge can be defined as the sum of the potential of the single particle δ and a "perturbed" term Δn.



From Poisson Equation:

$$\nabla^2 \Phi_D(\vec{r}) = \frac{e}{\varepsilon_o}\delta(\vec{r}) + \frac{e}{\varepsilon_o}\Delta n(\vec{r})$$

$$\Delta n = n e^{-e\Phi_D/k_B T} - n \approx -\frac{ne}{k_B T}\Phi_D$$

$$\nabla^2 \Phi_D(\vec{r}) + \lambda_D \Phi_D(\vec{r}) = \frac{e}{\varepsilon_o}\delta(\vec{r})$$

$$\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$

$$\boxed{\Phi_D(\vec{r}) = \frac{C}{r}e^{-r/\lambda_D}}$$

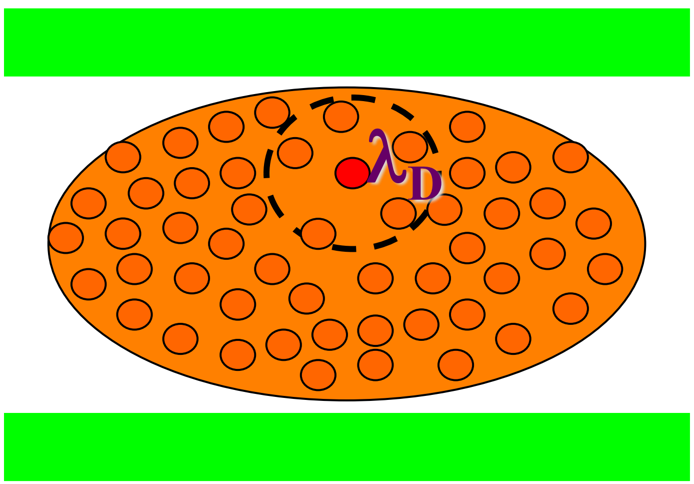
N => total number of particles
n => particle density (N/V)
k_B=> Boltzman constant
T => Temperature
k_B T => average kinetic energy of the particles

The Debye length indicates the distance over which charge imbalances are neutralized by the collective behavior of the plasma.

the effective interaction range of the test particle is limited to the **Debye length**

The charges sourrounding the test particles have a screening effect

$$\Phi_D(\vec{r}) = \frac{C}{r} e^{-r/\lambda_D} \quad \Rightarrow \quad \begin{cases} \Phi_D(\vec{r}) \approx \Phi(\vec{r}) & \text{for} \quad r << \lambda_D \\ \Phi_D(\vec{r}) << \Phi(\vec{r}) & \text{for} \quad r \geq \lambda_D \end{cases}$$



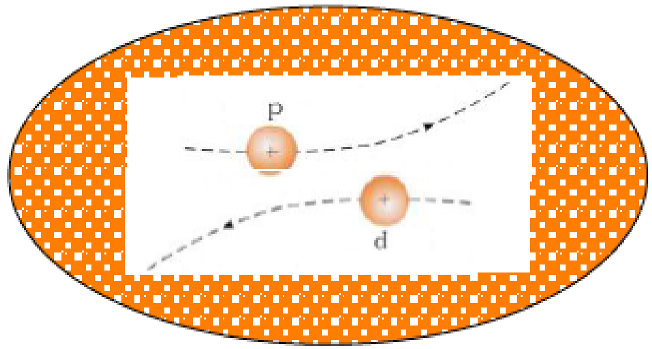
$$\Phi_{SC}(\vec{r}) >> \Phi_D(\vec{r})$$

Smooth functions for the charge and field distributions can be used as long as the Debye length remains small compared to the particle bunch size

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

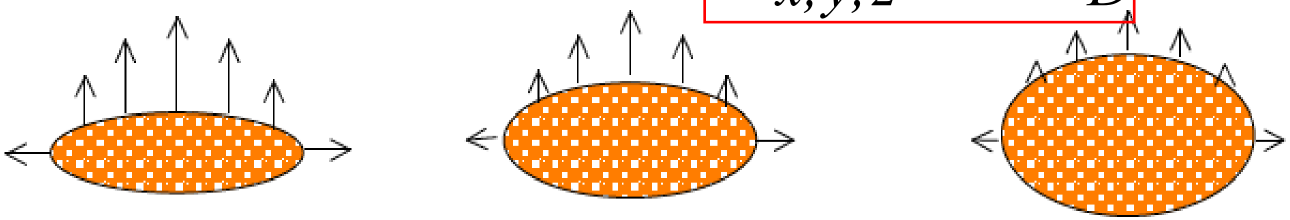
$$\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$



$$s_{x,y,z} << l_D$$

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

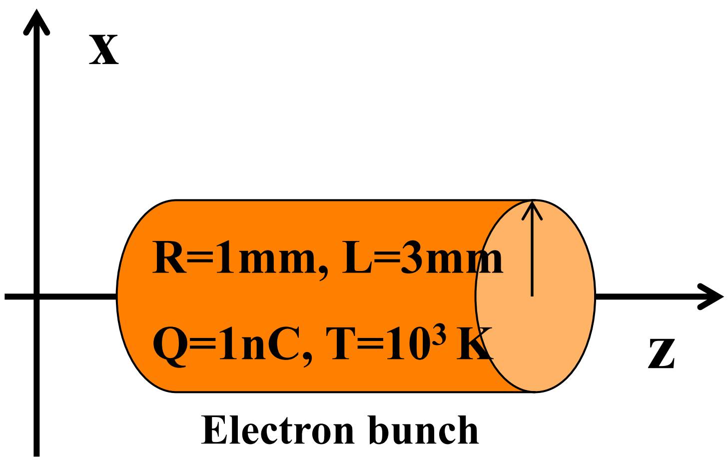
$$s_{x,y,z} >> l_D$$



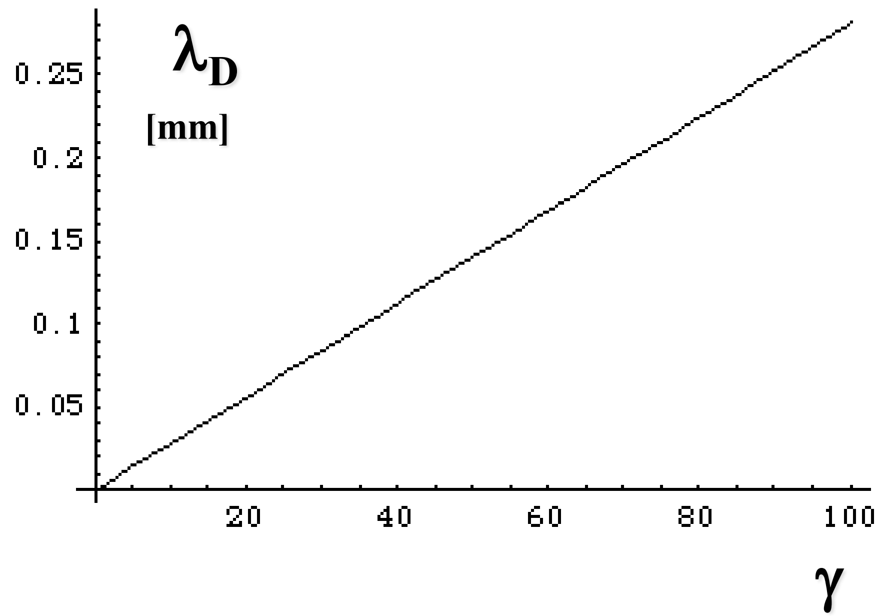
In a charged particle beam moving at a longitudinal relativistic velocity, assuming that the random transverse motion in the beam is non-relativistic, the Debye length has the following form:

$$\lambda_D = \sqrt{\frac{\varepsilon_o \gamma^2 k_B T}{e^2 n}}$$

$$\langle v_x \rangle = \sqrt{\frac{k_B T}{\gamma n}} << c$$

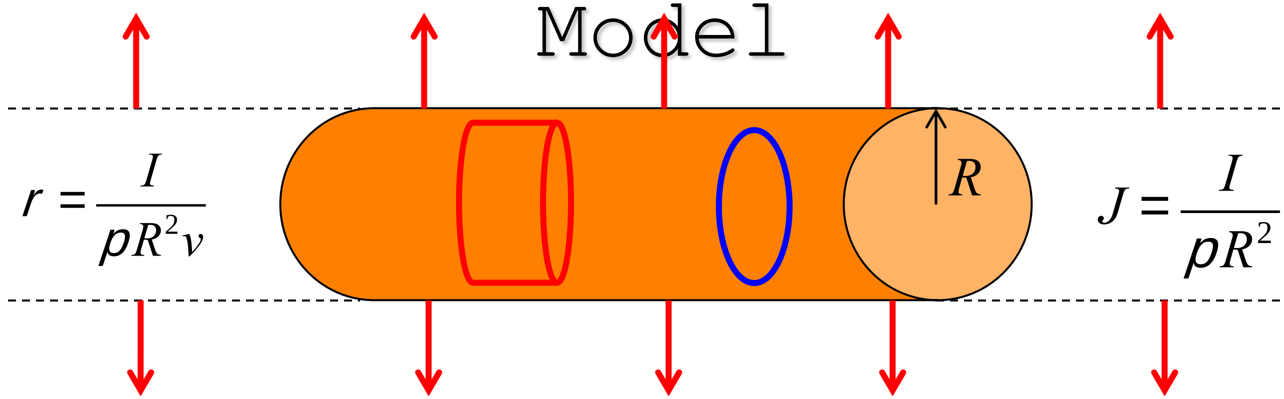


$$v_z \approx c$$



Continuous Uniform Cylindrical Beam Model

$$\rho = \frac{I}{\pi R^2 v}$$



$$J = \frac{I}{\pi R^2}$$

Gauss's law

$$\oint \varepsilon_o E \cdot dS = \oint \rho \, dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o R^2 v} r \quad \text{for} \quad r \leq R$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad \text{for} \quad r > R$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere's law

$$\oint B \cdot dl = \mu_o \oint J \cdot dS$$

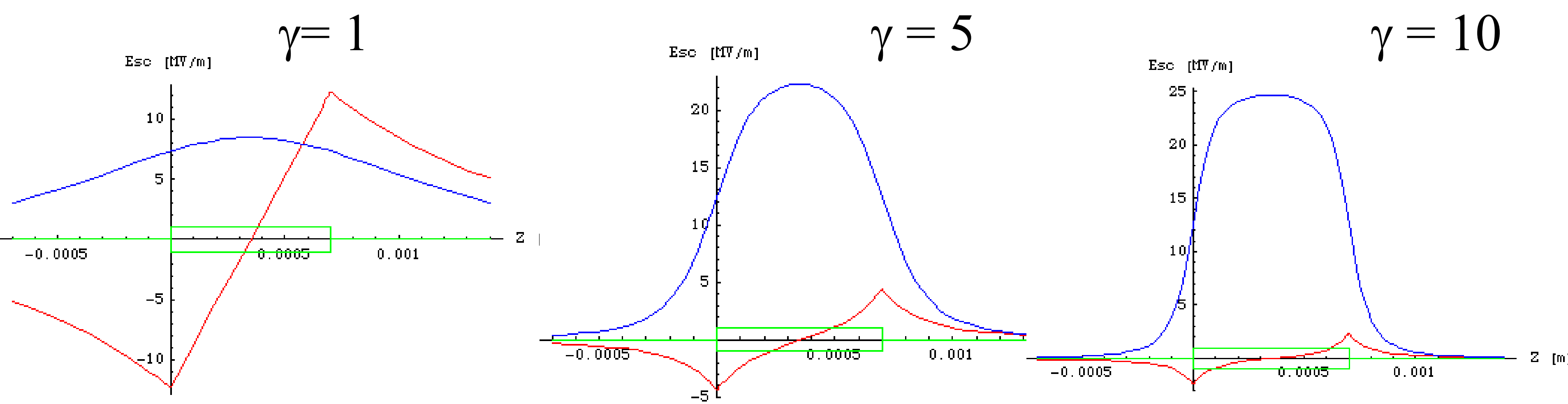
$$B_\theta = \mu_o \frac{Ir}{2\pi R^2} \quad \text{for} \quad r \leq R$$

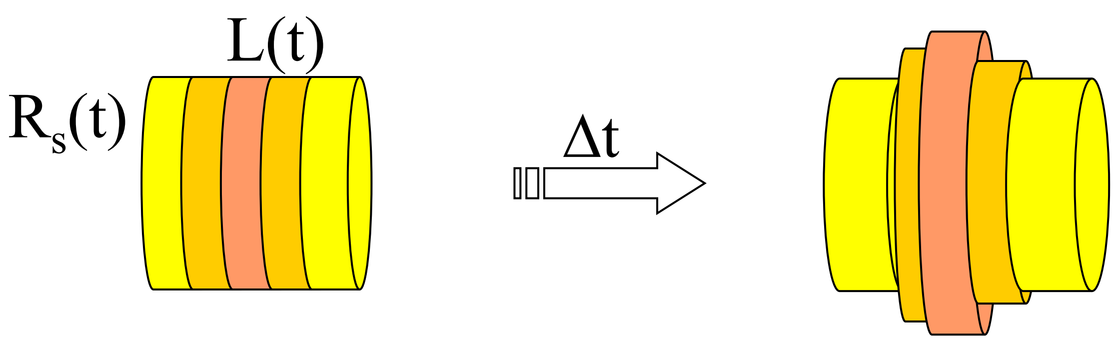
$$B_\theta = \mu_o \frac{I}{2\pi r} \quad \text{for} \quad r > R$$

Bunched Uniform Cylindrical Beam Model

$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$





$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

Lorentz Force

$$F_r = e\left(E_r - \beta c B_\theta\right) = e\left(1 - \beta^2\right) E_r = \frac{eE_r}{\gamma^2}$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2 \varepsilon_0 R^2 \beta c} g(s,\gamma)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect. Using $R=2\sigma_x$ for a uniform distribution:

$$F_x = \frac{eIx}{8\pi\gamma^2 \varepsilon_0 \sigma_x^2 \beta c} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x' \qquad p = const.$$

$$\frac{d}{dt}\left(p x'\right) = \beta c \frac{d}{dz}\left(p\,x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$F_x = \frac{eIx}{8\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g\left(s,\gamma\right)$$

$$x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2} x$$

$$k_{sc} = \frac{2I}{I_A} g\left(s,\gamma\right)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e}$$

Now we can calculate the term $\langle x x'' \rangle$ that enters in the envelope equation

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle x x'' \rangle}{\sigma_x}$$

$$\langle x x'' \rangle = \frac{k_{sc}}{\sigma_x^2}\langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

(emittance term crossed out)

ρ>>1 Laminar Beam

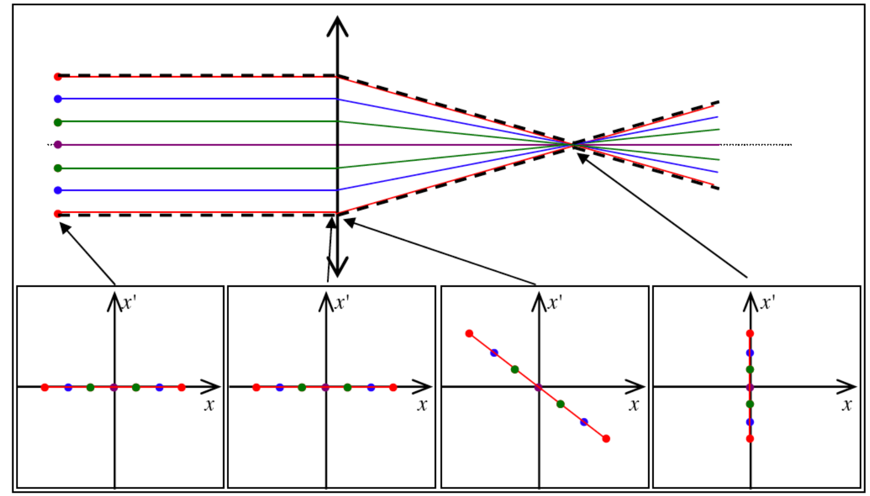


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

(space charge term crossed out)

ρ<<1 Thermal Beam

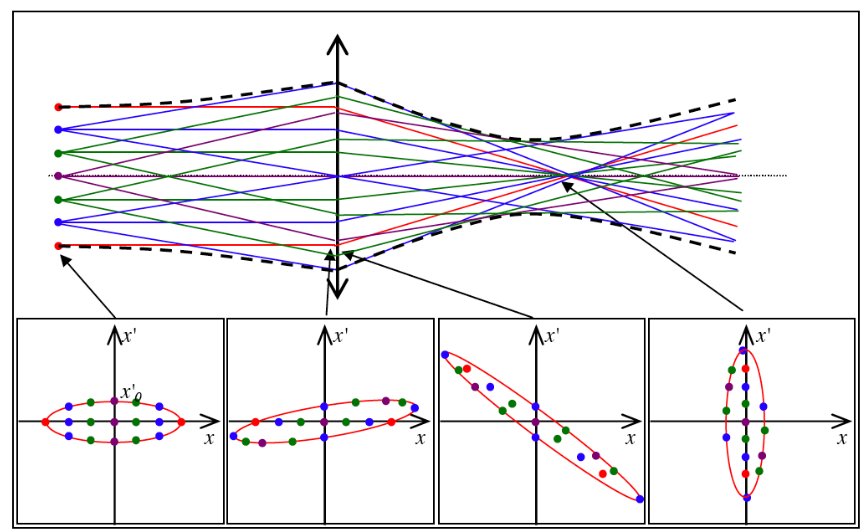


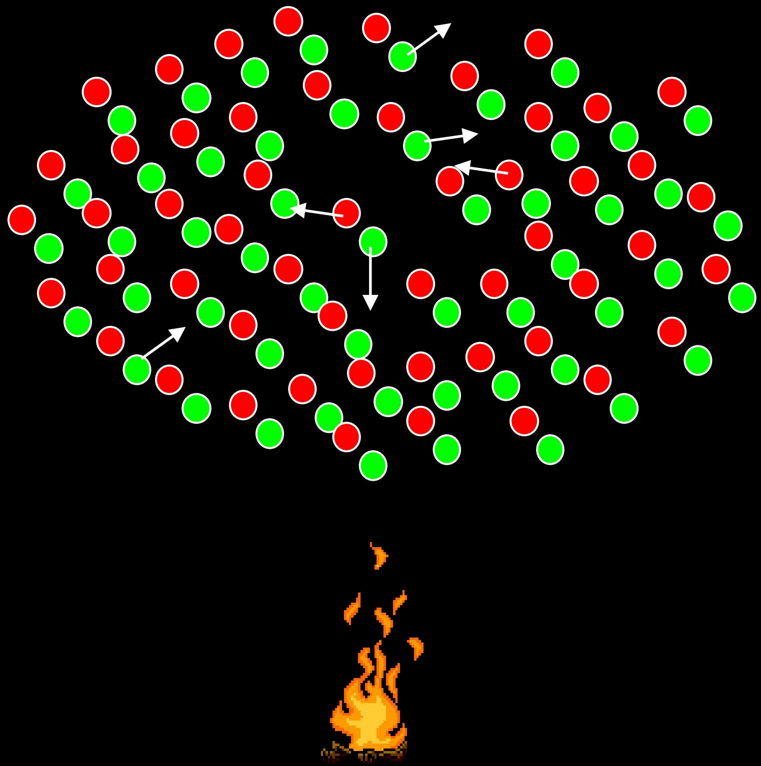
Fig. 11: Particle trajectories in non-zero emittance beam

OUTLINE

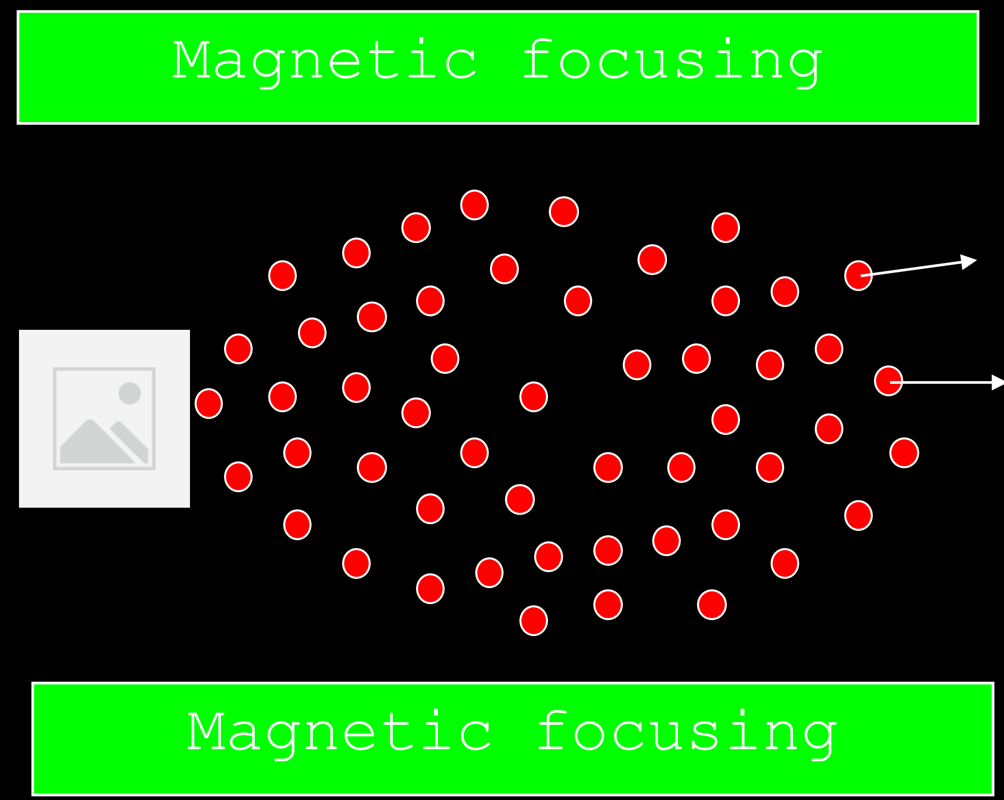
- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



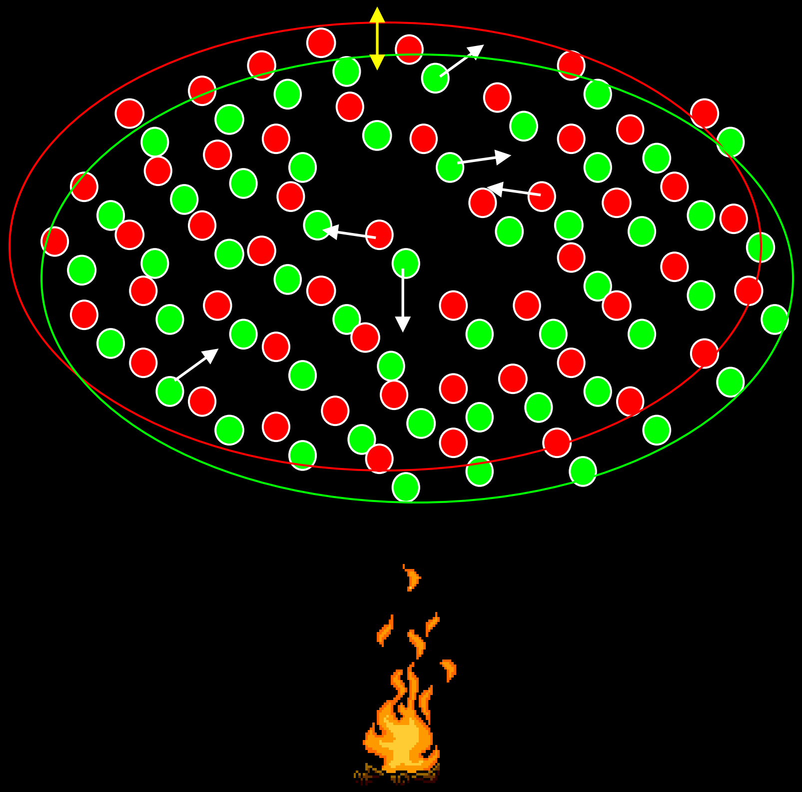
Single Component Cold Relativistic Plasma



Surface charge density

$$\sigma = e\,n\,\delta x$$

δx



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p^{\,2}\,\delta x$$

Plasma frequency

$$\omega_p^{\,2} = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

$$\delta x = (\delta x)_0\,\cos\,(\omega_p\,t)$$

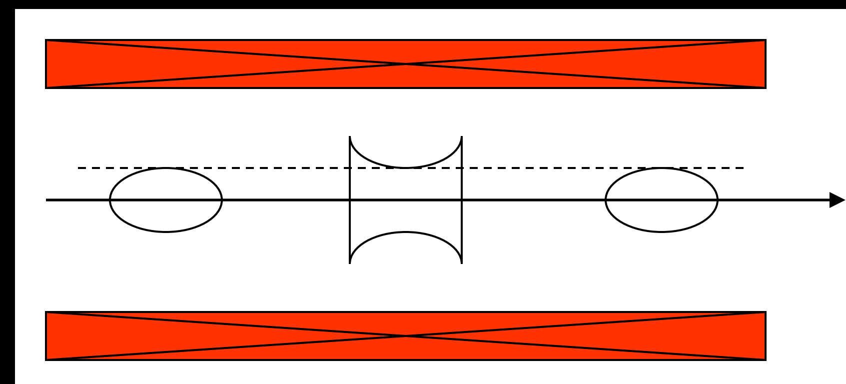
Single Component Relativistic Plasma

$$s'' + k_s^2 s = \frac{k_{sc}(s,g)}{s}$$

$$k_s = \frac{qB}{2mcbg}$$

Equilibrium solution:

$$s_{eq}(s,g) = \frac{\sqrt{k_{sc}(s,g)}}{k_s}$$



Small perturbation:

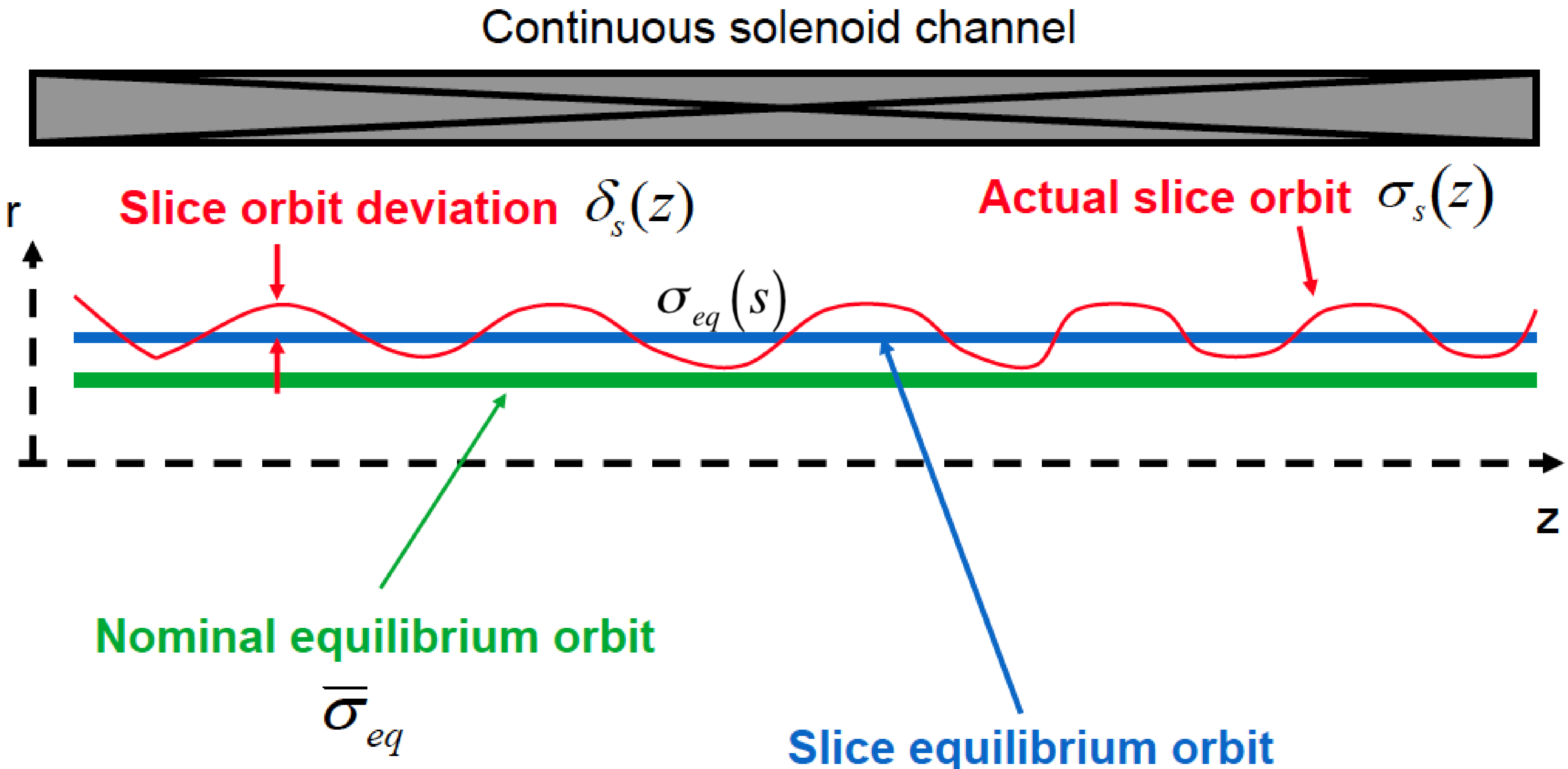
$$s(z) = s_{eq}(s) + ds(s)$$

$$ds''(s) + 2k_s^2 ds(s) = 0$$

$$ds(s) = ds_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$s(s) = s_{eq}(s) + ds_o(s)\cos\left(\sqrt{2}k_s z\right)$$

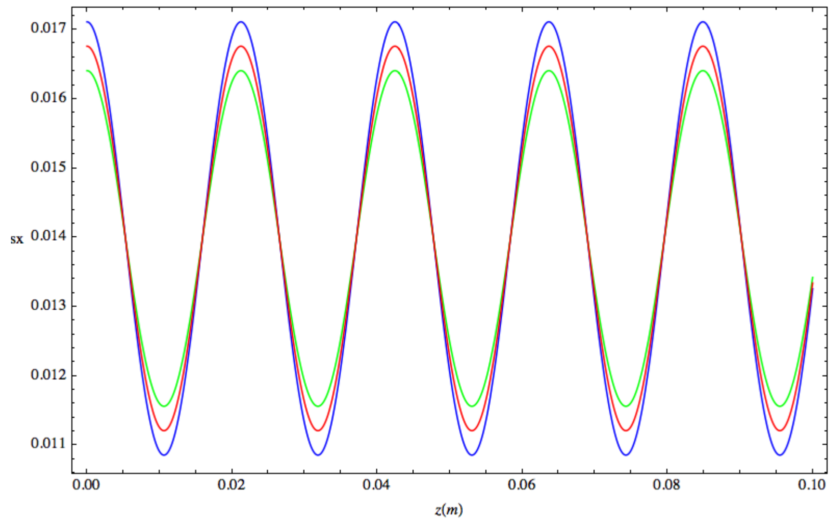


Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

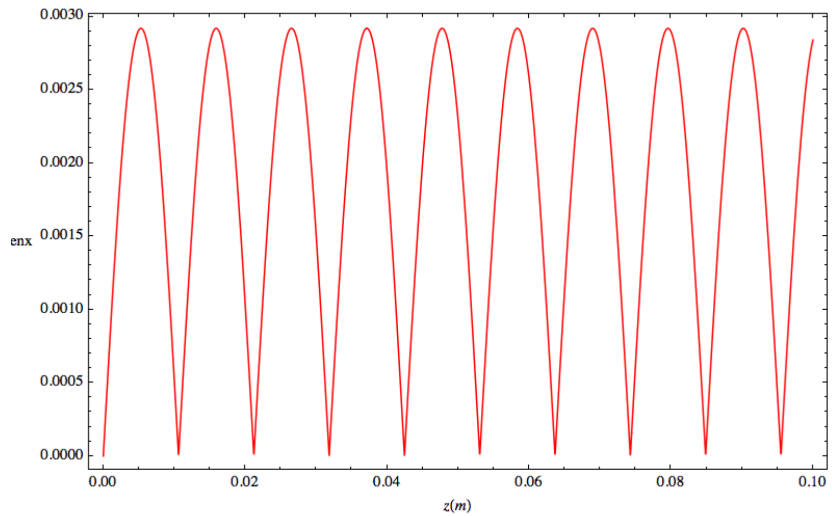
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Envelope oscillations drive Emittance

σ(z)



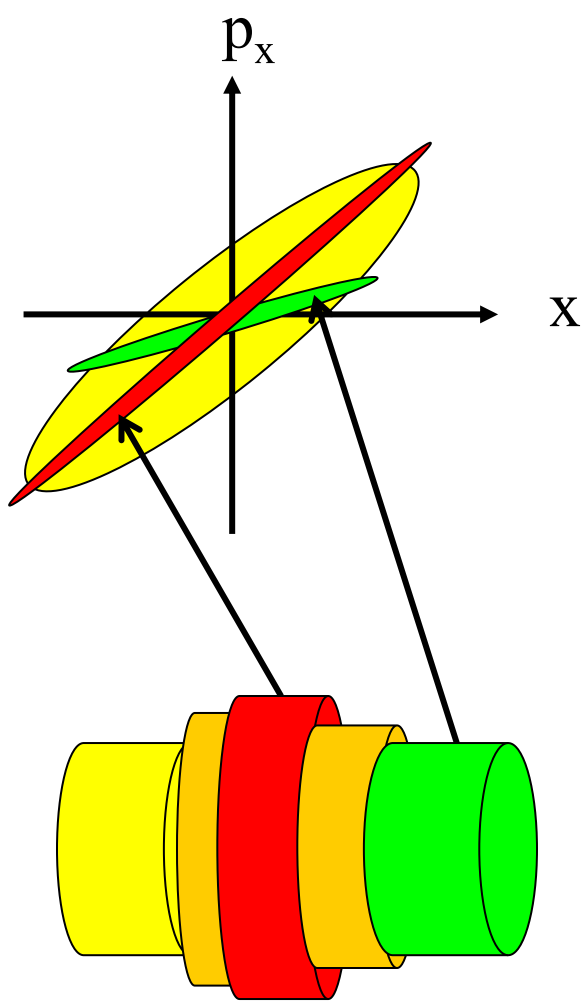
ε(z)



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)} \approx \left| \sin\left(\sqrt{2} k_s z\right) \right|$$

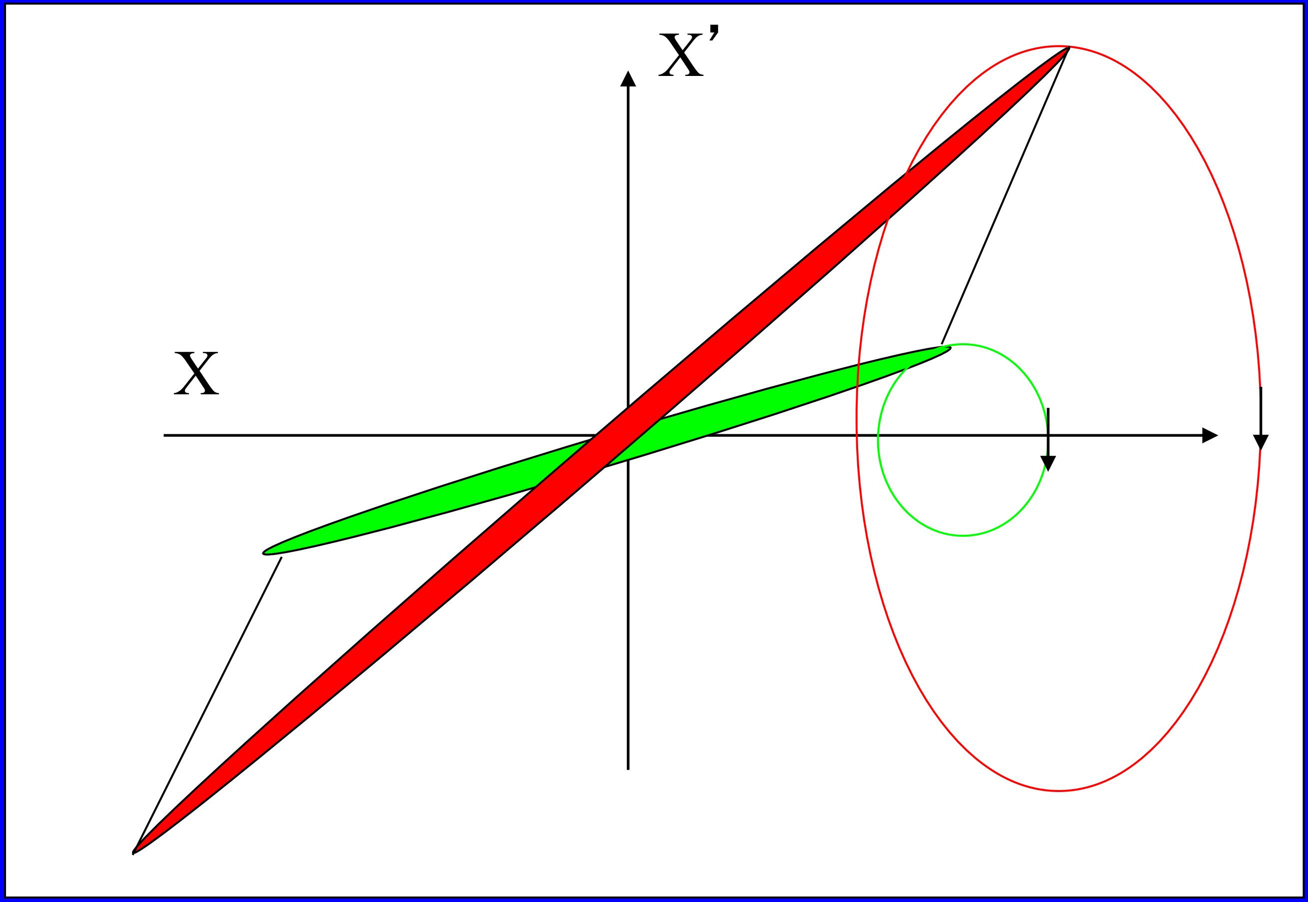
Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

Projected Phase Space



Slice Phase Spaces

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes



OUTLINE

- The rms emittance concept
- rms envelope equation
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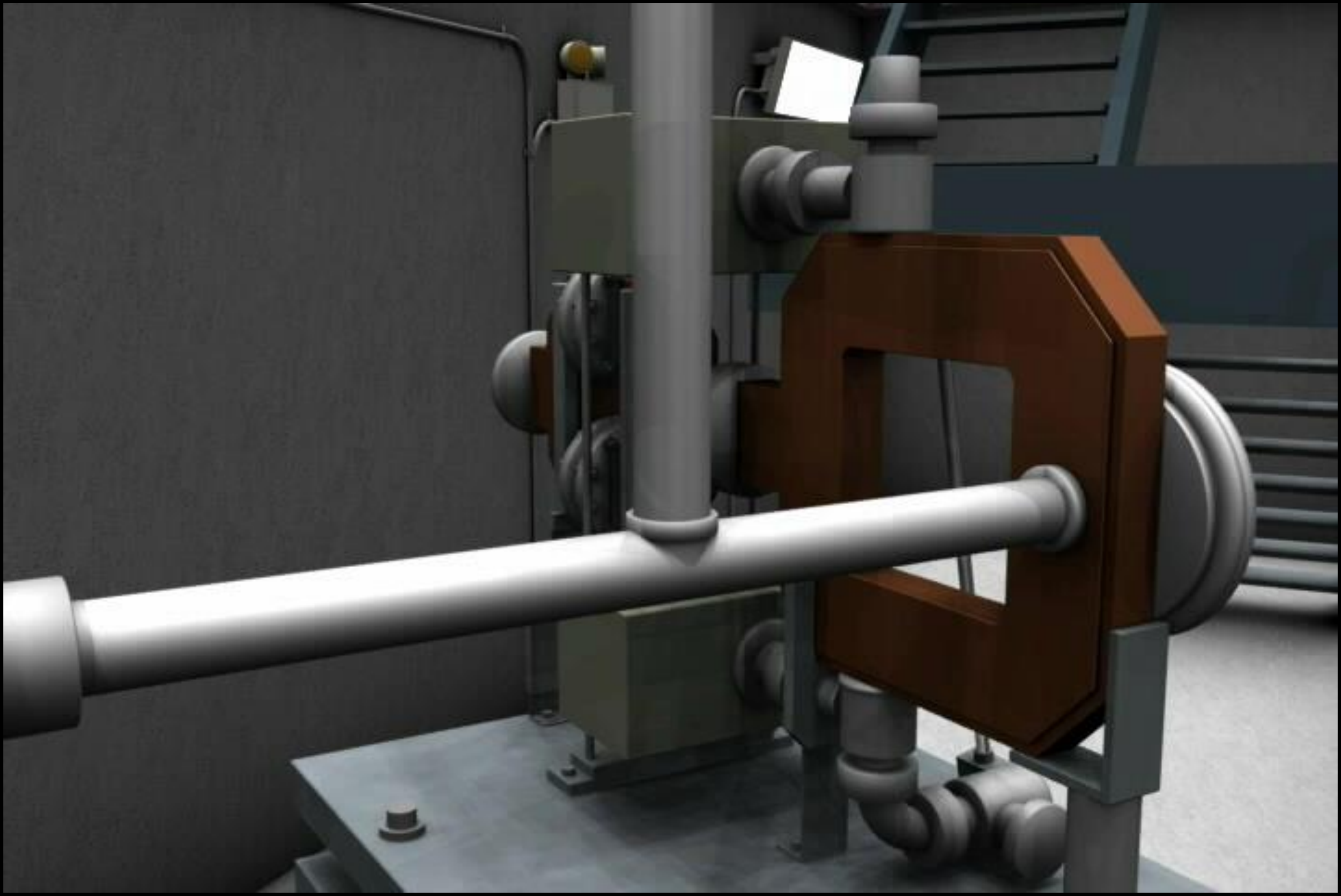
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

High Brightness Photo-Injector



Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}\left(p x'\right) = \beta c \frac{d}{dz}\left(p x'\right) = 0$$

$$p = \beta \gamma m_o c$$

$$x'' + \frac{p'}{p} x' = 0$$

$$x'' = -\frac{\left(\beta \gamma\right)'}{\beta \gamma} x'$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle x x'' \rangle}{\sigma_x}$$

$$\langle x x'' \rangle = -\frac{\left(\beta \gamma\right)'}{\beta \gamma} \langle x x' \rangle = -\frac{\left(\beta \gamma\right)'}{\beta \gamma} \sigma_{xx'} = -\frac{\left(\beta \gamma\right)'}{\beta \gamma} \sigma_x \sigma_x'$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{\left(\beta \gamma\right)'}{\beta \gamma} \sigma_x' + k^2 \sigma_x = \frac{\varepsilon_n^2}{\left(\beta \gamma\right)^2 \sigma_x^3} + \frac{K_{sc}}{\sigma_x}$$

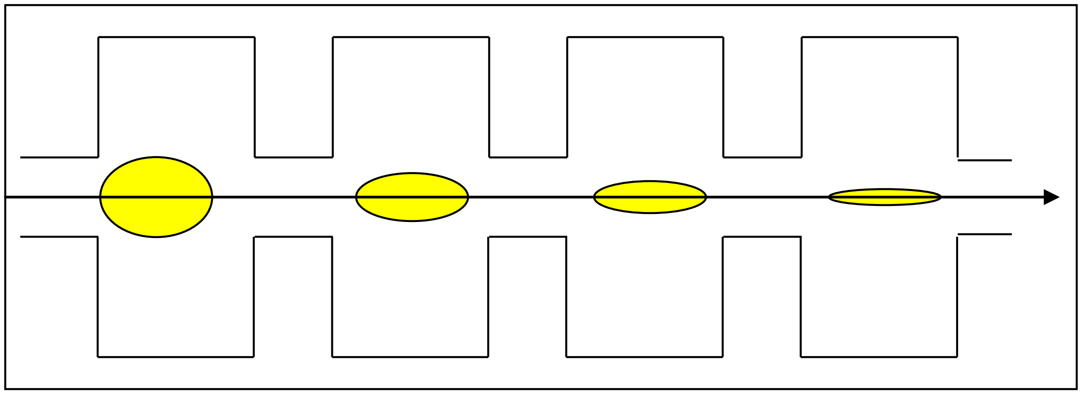
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta \gamma \varepsilon_{rms}$$

Beam subject to strong acceleration



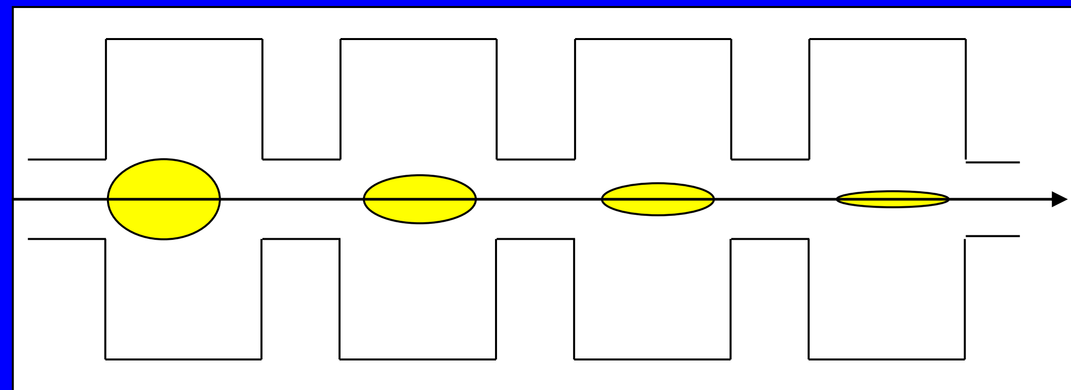
$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_{RF}^2}{\gamma^2}\sigma_x = \frac{\epsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

We must include also the RF focusing force: $k_{RF}^2 = \frac{\gamma'^2}{2}$

$$k_{sc}^o = \frac{2I}{I_A}g(\sigma,\gamma)$$

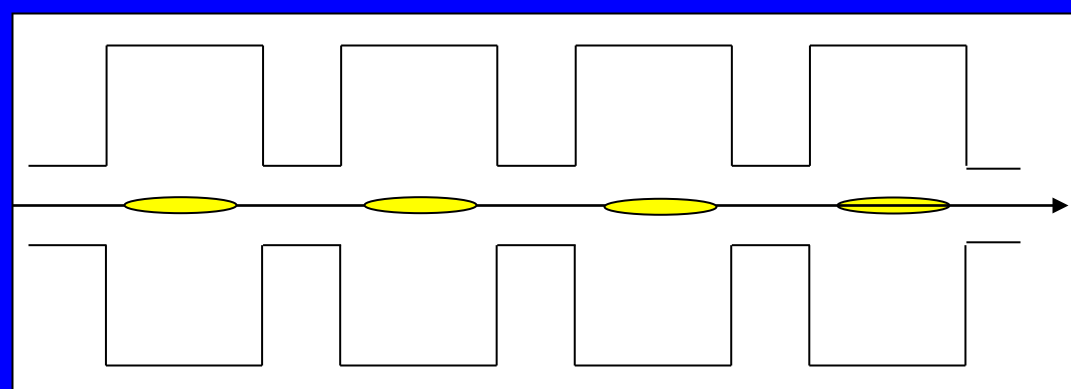
Space charge dominated beam (Laminar)

$$\sigma_{\theta} = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$

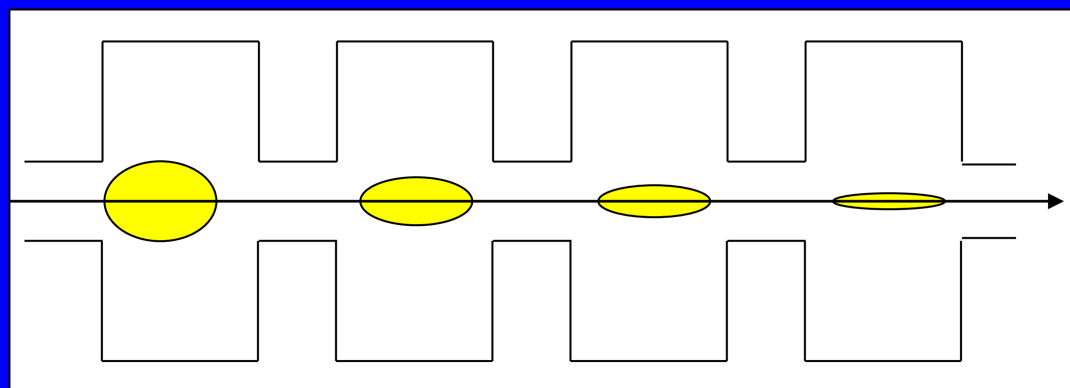


Emittance dominated beam (Thermal)

$$\sigma_{\epsilon} = \sqrt{\frac{2\epsilon_n}{\gamma'}}$$



$$s_q = \frac{1}{g¢}\sqrt{\frac{2I}{I_A g}}$$



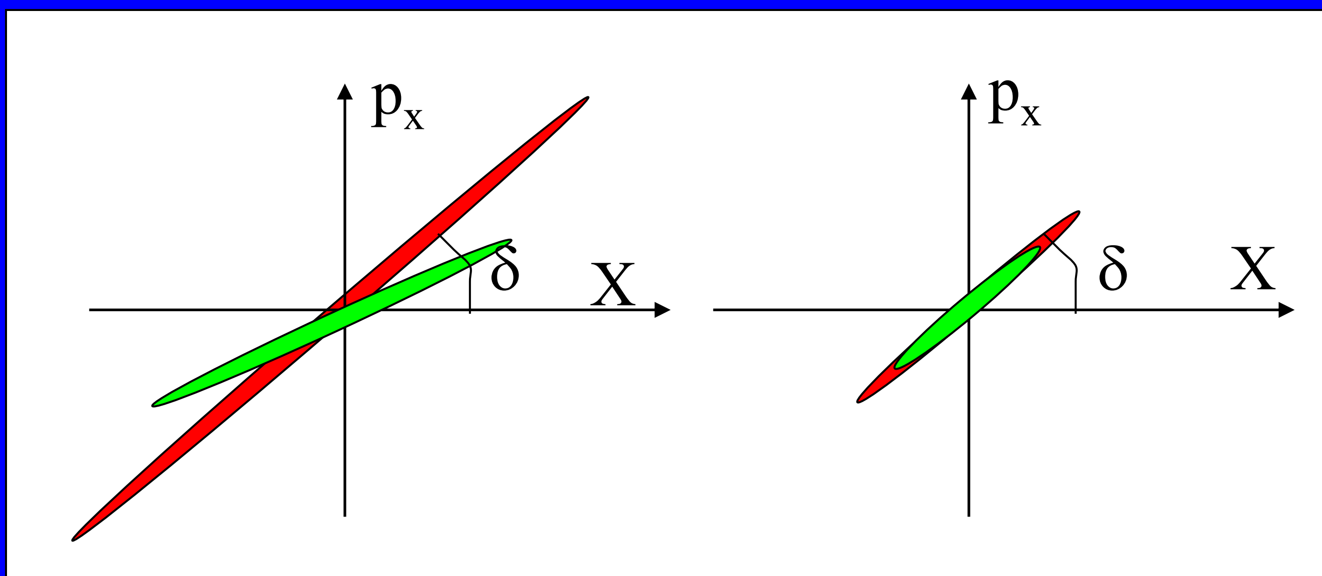
This solution represents a **beam equilibrium mode** that turns out to be the transport mode for achieving minimum emittance at the end of the **emittance correction process**

An important property of the laminar beam

$$s_q = \frac{1}{g¢}\sqrt{\frac{2I}{I_A g}}$$

$$s'_q = -\sqrt{\frac{2I}{I_A g^3}}$$

Constant phase space angle: $d = \frac{g s'_q}{s_q} = -\frac{g¢}{2}$

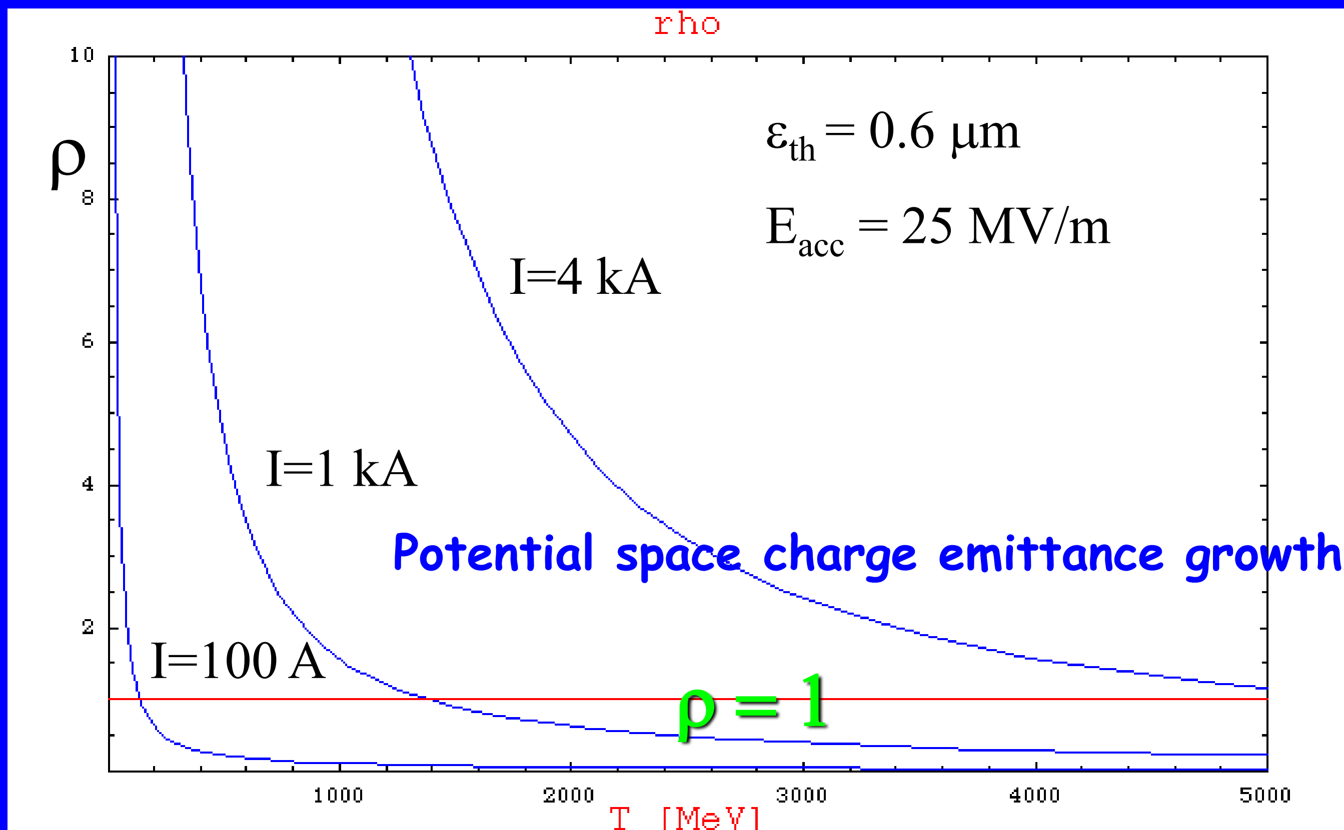


Laminarity parameter

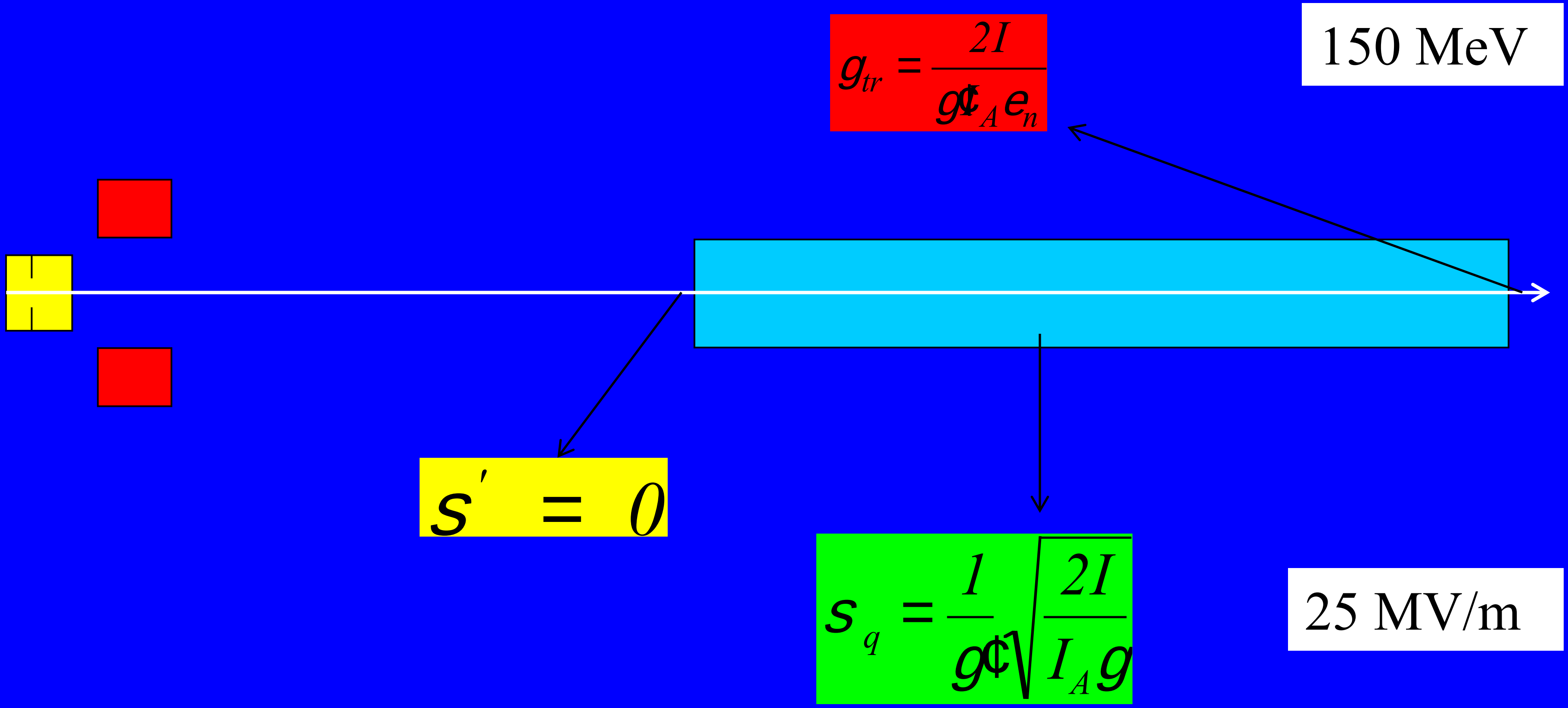
$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \approx \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

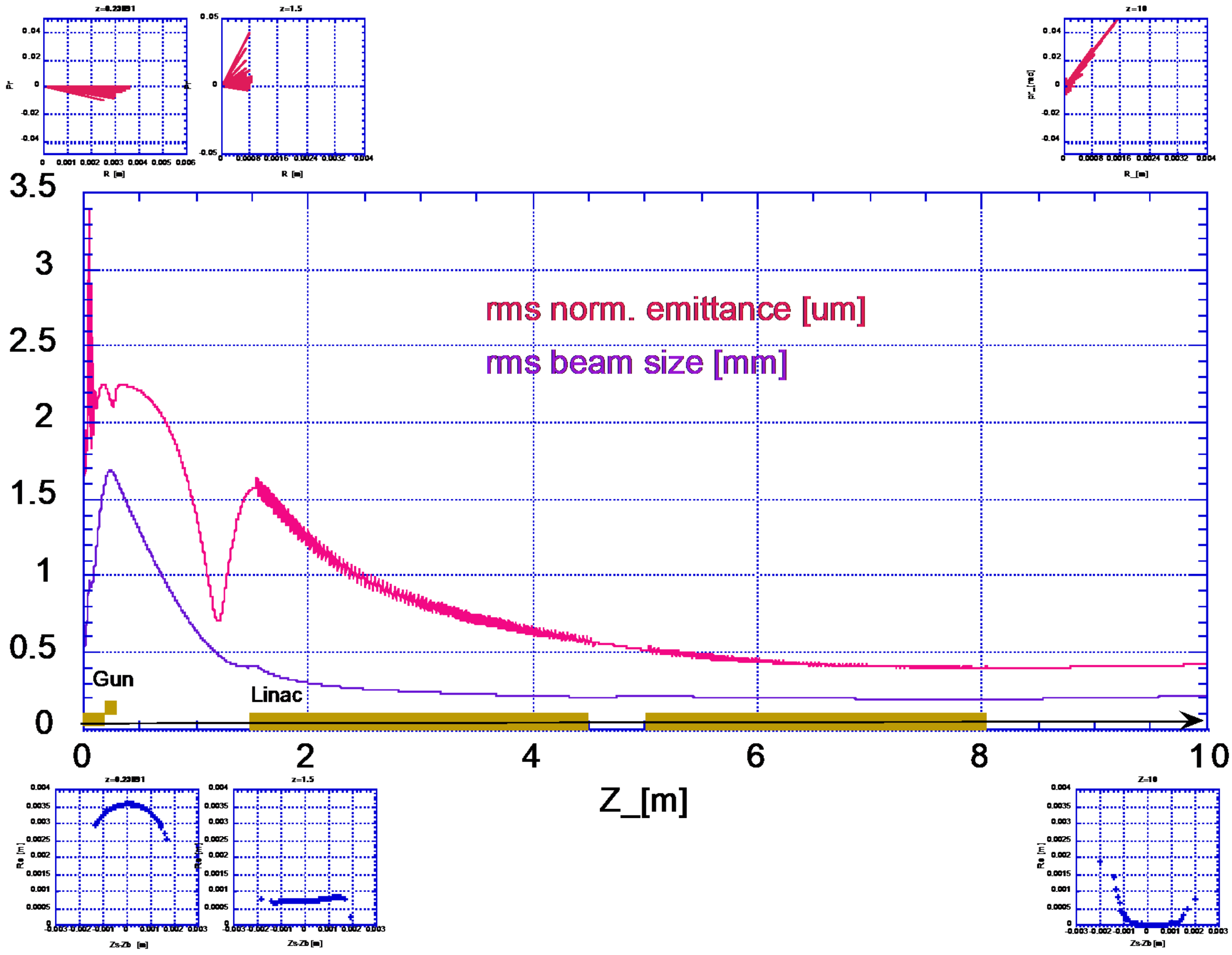
Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



Matching Conditions with a TW Linac





rms norm. emittance [um]
rms beam size [mm]
Gun
Linac
Z_[m]
z=1.5
z=10
R [m]
R_[m]
Zs-Zb [m]

Emittance Compensation for a SC dominated beam: Controlled Damping of Plasma Oscillations

- ε_n oscillations are driven by Space Charge

• propagation close to the laminar solution allows control of ε_n oscillation "phase"

- ε_n sensitive to SC up to the transition energy

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