SPACE CHARGE IN CIRCULAR MACHINES

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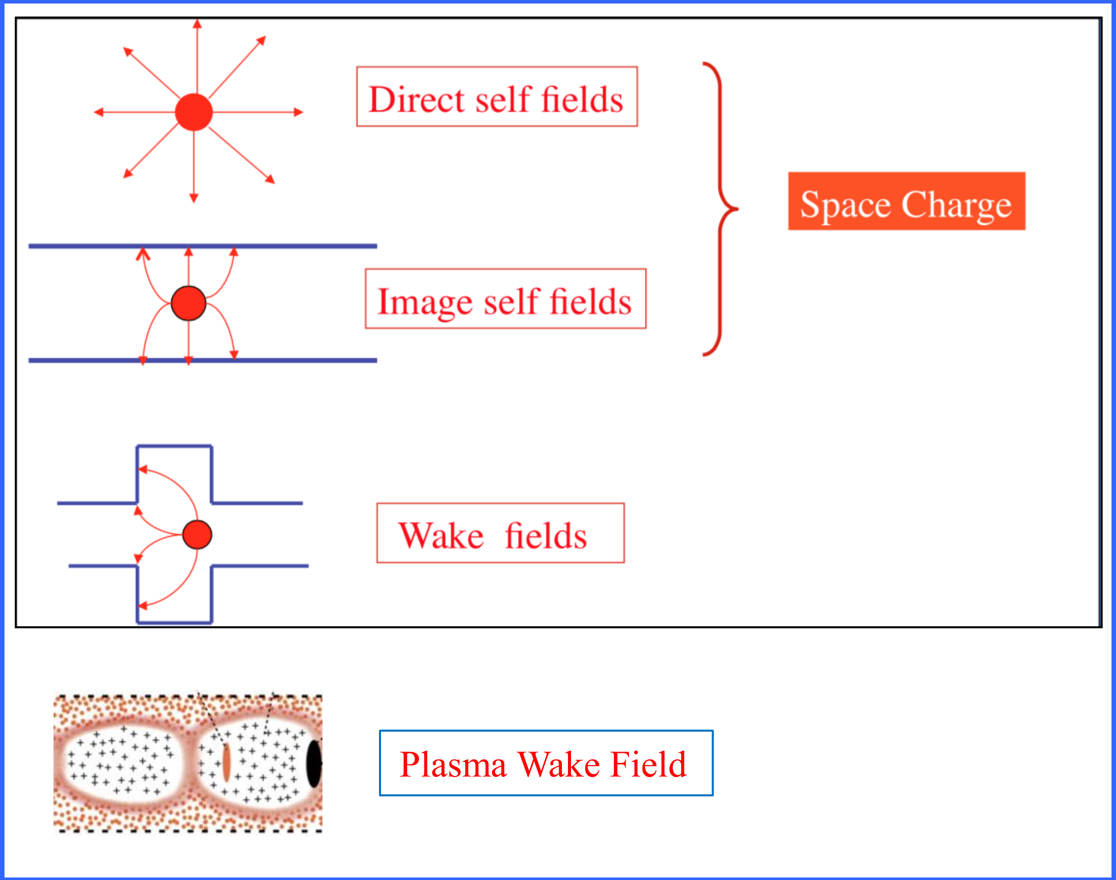


The CERN Accelerator School

Spa – November 14 - 2024

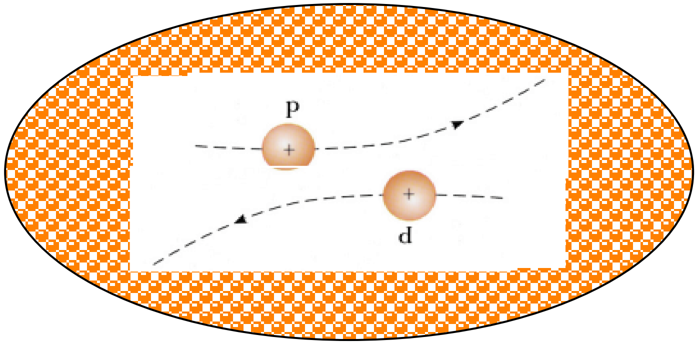
OUTLINE

- Direct Space Charge Effects
  - Intra-beam scattering IBS
- Image Charge Effects
  - Image self fields
  - Space charge effects in Storage Rings

The lifetimes of the beams in circular machines are much longer than in linear devices

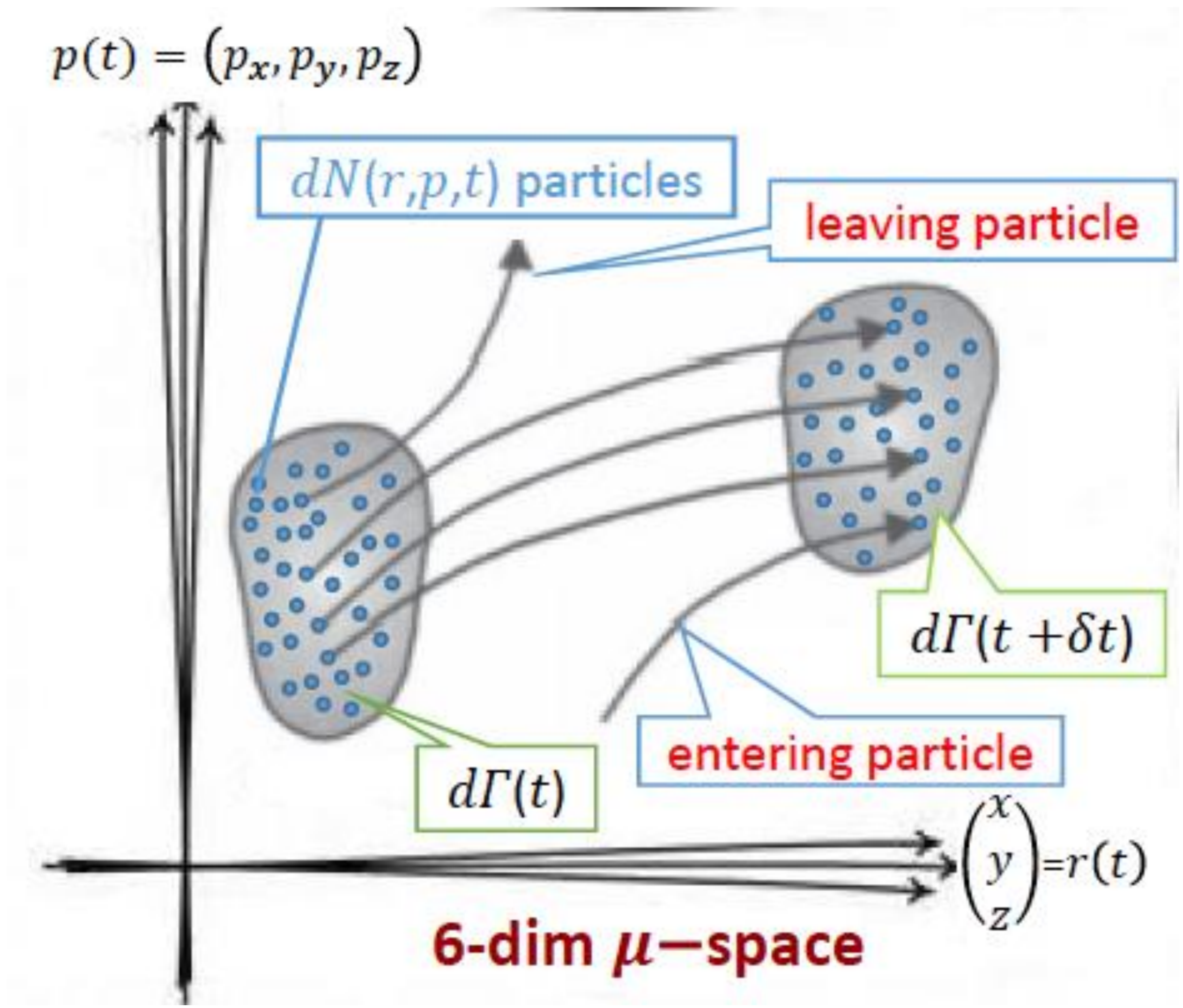
Collisional regime

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



1) multiple small-angle scattering events **Intra-Beam Scattering (IBS)**
2) large-angle single scattering events **Touschek Effect**

Liouville theorem does not hold anymore under Collisions => non Conservative forces involved



Beam Thermodynamics

Definition of beam temperature in analogy with kinetic theory of gases :

transverse

$$k_B T_{beam,x} = \gamma m_o \langle v_x^2 \rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms,x}}{\beta_x}$$

longitudinal

$$k_B T_{beam,z} = \gamma^3 m_o \langle \Delta v_z^2 \rangle = \frac{\beta^2 c^2}{\gamma} m_o \left\langle \left(\frac{\Delta p}{p} \right)^2 \right\rangle$$

$$\Delta v_z = \frac{\beta c}{\gamma^2} \frac{\Delta p}{p}$$

In a Circular machine when a particle accelerates above transition energy it becomes slower and behaves like a particle with negative mass:

$$k_B T_{beam,z} = m^* \langle \Delta v_z^2 \rangle = -\frac{\gamma m_o}{\eta} \langle \Delta v_z^2 \rangle$$

$$\eta = \alpha - \frac{1}{\gamma_o^2} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_o^2}$$

η Slip Factor
α Momentum Compaction
γ_t Transition Energy

Conservation Law

Let us first consider the ideal machine with a smooth-focusing lattice below transition and negligible dispersion.

The total thermal energy per particle in a smooth linear beam channel is conserved, for a beam with constant energy (γ_0 = const)

$$k_B T_x + k_B T_y + k_B T_{\parallel} = \text{const}$$

Coulomb collisions drive the beam toward an isotropic thermal equilibrium, in which case the three temperatures would be the same:

$$k_B T_x = k_B T_y = k_B T_{\parallel} = k_B T_{\text{eq}}$$

We can put the conservation law into the form:

$$\frac{\varepsilon_{rms,x}}{\beta_x} + \frac{\varepsilon_{rms,y}}{\beta_y} + \frac{1}{\gamma_o^2}\left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle = const$$

in a circular machine we must replace $1/\gamma_o$ by:

$$-\eta = \frac{1}{\gamma_o^2} - \alpha = \frac{1}{\gamma_o^2} - \frac{1}{\gamma_t^2}$$

$$\frac{\varepsilon_{rms,x}}{\beta_x} + \frac{\varepsilon_{rms,y}}{\beta_y} - \eta\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle = const$$

This relationship is the invariant for intra-beam scattering derived in 1974 by Piwinski. For a circular machine the behavior of the system depends on the sign of η i.e. whether it is below transition ($\gamma_0 < \gamma_t$) or above ($\gamma_0 > \gamma_t$).

➔ below transition $\eta < 0$ ➔ thermal equilibrium can be reached.

➔ above transition $\eta > 0$ ➔ thermal equilibrium is not possible.

An increase in momentum spread must be balanced by a corresponding increase in the transverse emittances to maintain the "conservation law"

For instance, in the LHC at 7 TeV, although $\gamma = 7461 \gg \gamma_t \approx 53.8$ ($\eta_t \approx 3.4\times10^{-4}$), the undesirable growth of the bunch emittances caused by IBS is counterbalanced by the *synchrotron radiation damping* effect.

The emittance growth rate for intra-beam scattering in high-energy circular machines defined a⁻⁻

$$\frac{1}{\tau_j(\epsilon)} = \frac{1}{\tilde{\epsilon}_j}\frac{d\tilde{\epsilon}_j}{dt}$$

can be written in the relativistic form

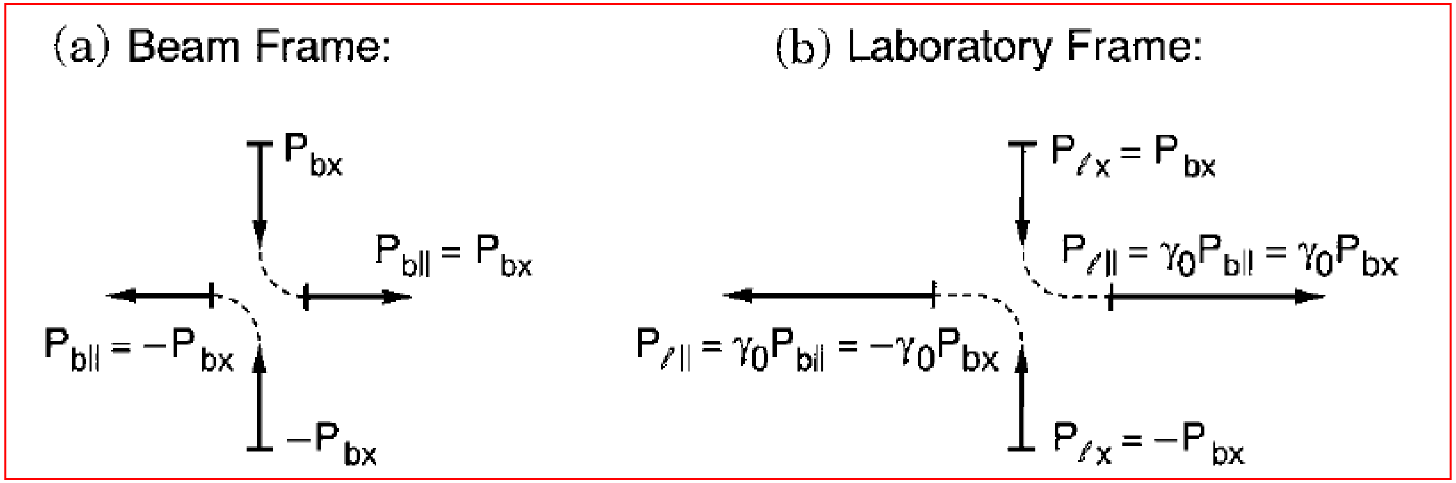
$$\frac{1}{\tau_j} = \frac{1}{\tau_0}\langle H_j \rangle = \frac{\pi^2 c r_c^2 m^3 N \ln \Lambda}{\gamma_0 \Gamma}\langle H_j \rangle$$

where N is the total number of particles, Γ the six-dimensional phase-space volume occupied by N, and where the function H_j depends on γ_0, the emittances $\tilde{\epsilon}_x$, $\tilde{\epsilon}_y$, $\tilde{\epsilon}_z$, and the lattice parameters $\hat{\beta}_x$, D_e, $\hat{\beta}'_x$, D'_e, and $\hat{\beta}_y$. The function H_j is averaged over a lattice period and the subscript j denotes the three orthogonal directions (i.e., $j =$ horizontal (x), vertical (y), and longitudinal (s)].

$$\Gamma_b = (2\pi)^3 \frac{P_0^3}{c^3}\tilde{\epsilon}_x\tilde{\epsilon}_y\tilde{\epsilon}_z = (2\pi)^3(\beta_0\gamma_0)^3 m^3 \tilde{\epsilon}_x\tilde{\epsilon}_y\tilde{\epsilon}_z$$

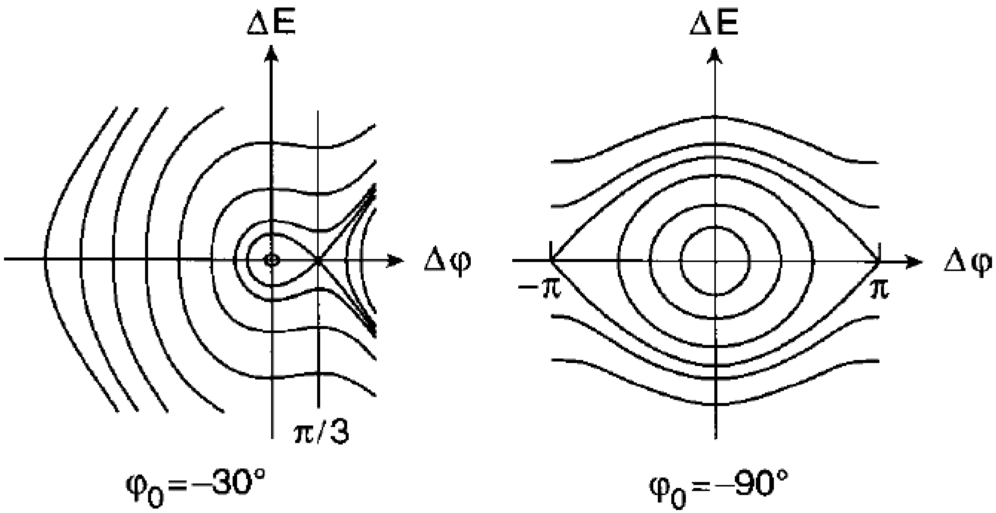
Touschek Effect

In a relativistic storage ring, Coulomb collisions lead to a momentum transfer from the transverse into the longitudinal direction that is amplified by the Lorentz factor γ_o



While the total momentum in the collision is preserved, the two particles emerge from this collision with opposite longitudinal momentum components that are larger by the factor γ_o than the original transverse momentum component before the collision.

If the longitudinal momentum acquired in such a collision is greater than the momentum acceptance of the rf bucket that keeps the beam longitudinally bunched, the two particles involved in such a collisions will be lost.



$$\Delta E_{\max} = -\Delta E_{\min} = 2\left[\beta_0^3\gamma_0^3\frac{\lambda}{2\pi}mc^2qE_m(\varphi_0\cos\varphi_0 - \sin\varphi_0)\right]^{1/2}$$

The net result is that the lifetime of the stored beam is reduced.

Proceedings of the CAS-CERN Accelerator School: Intensity Limitations in Particle Beams, Geneva, Switzerland, 2–11 November 2015,
edited by W. Herr, CERN Yellow Reports: School Proceedings, Vol. 3/2017, CERN-2017-006-SP (CERN, Geneva, 2017)

Intrabeam Scattering: Anatomy of the Theory

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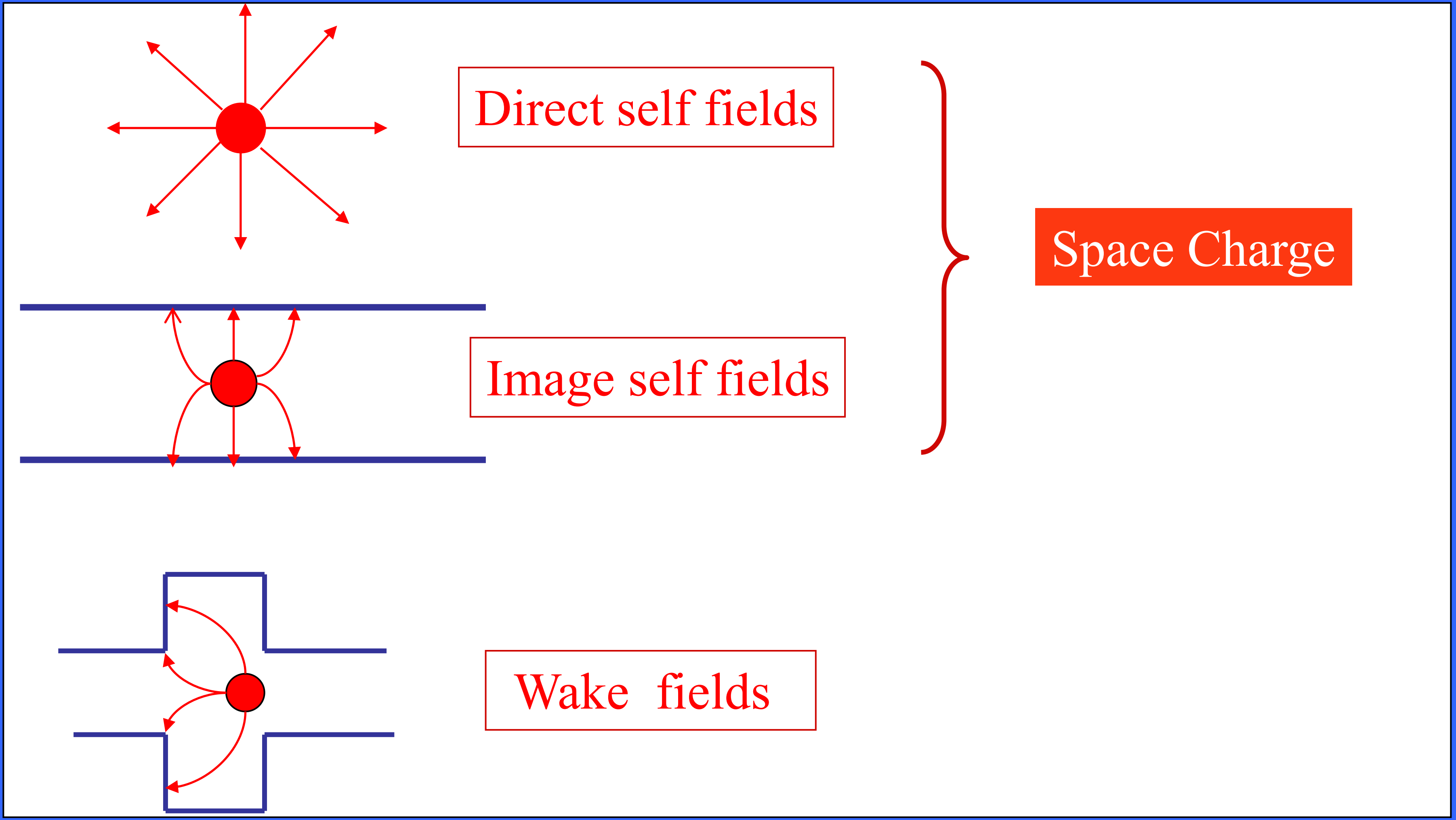
OUTLINE

- Direct Space Charge Effects
  - The rms emittance concept
  - rms envelope equation
  - Space charge forces
  - Beam (Plasma) emittance oscillations
- Image Charge Effects
  - Image self fields
  - Space charge effects in Storage Rings

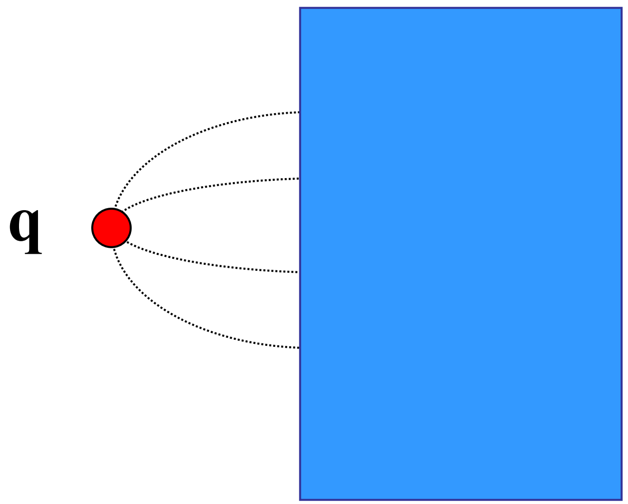
IMAGE SELF FIELDS

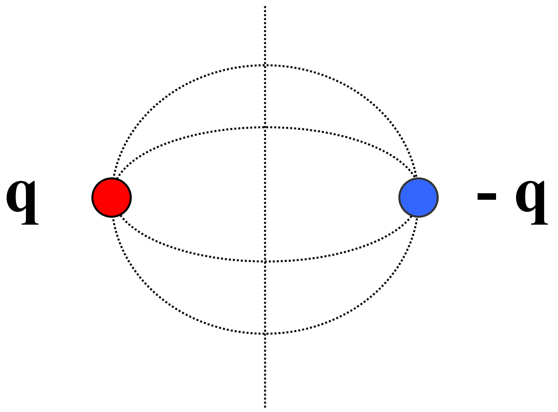


Static Fields: conducting screens

Let us consider a point charge q close to a conducting screen.

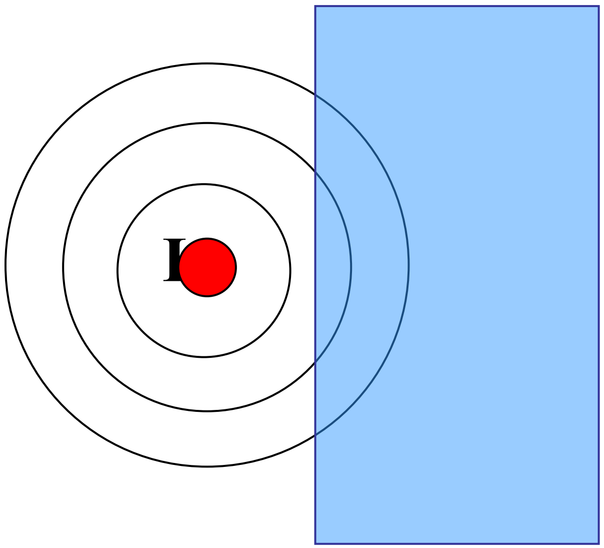
The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen



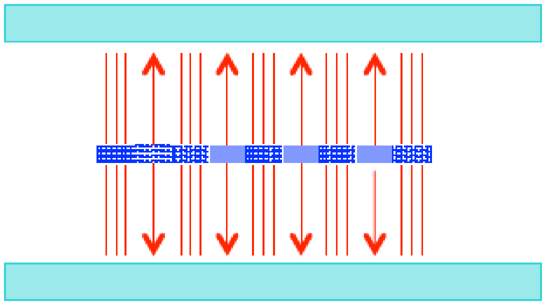


A constant current in the free space produces circular magnetic field

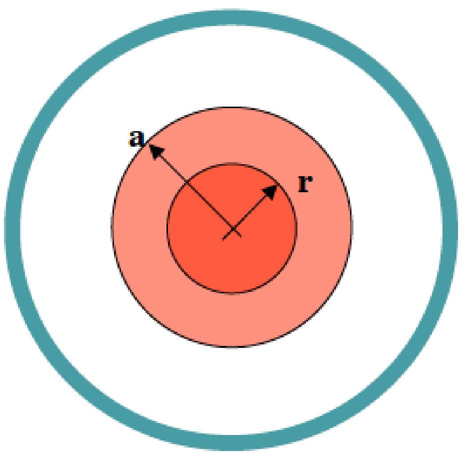
If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.



Circular Perfectly Conducting Pipe (Beam at Center)



In the case of cylindrical charge distribution, and $\gamma \rightarrow \infty$, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws.



$$\lambda(r) = \lambda_0 \left(\frac{r}{a}\right)^2 ; \int_S E_r (2\pi r) \Delta z = \frac{\lambda(r) \Delta z}{\varepsilon_0}$$

$$E_r = \frac{\lambda(r)}{2\pi\varepsilon_0 r}; \quad \mathrm{B}_\theta = \frac{\beta}{c} E_r$$

$$E_r(\boldsymbol{r}) = \frac{\lambda_0}{2\pi\,\varepsilon_0} \frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda_0 \beta}{2\pi\varepsilon_0 c} \frac{r}{a^2}$$

$$F_\perp(r) = e(E_r - \beta c\, B_\theta) = \frac{e}{\gamma^2} E_r$$

there is a cancellation of the electric and magnetic forces

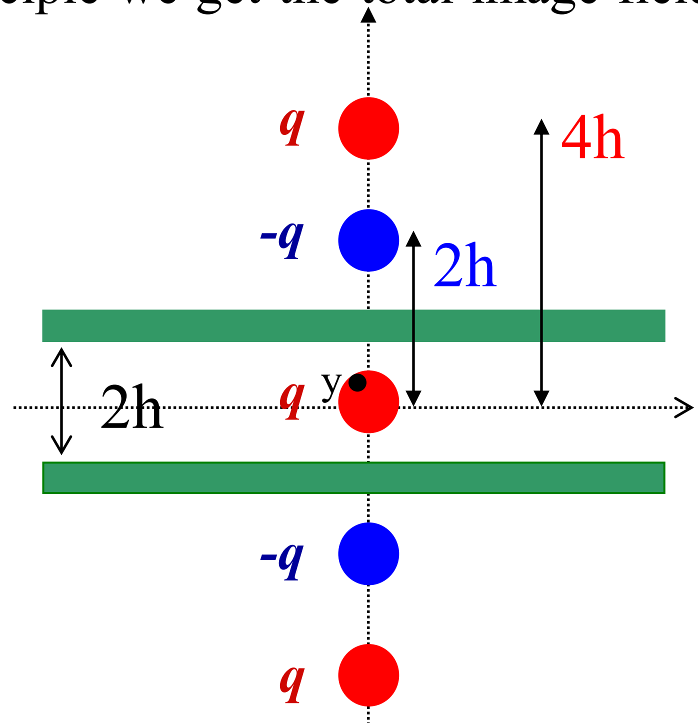
Parallel Plates (beam at center)

In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle we get the total image field at a position *y* inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\epsilon_o}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y}-\frac{1}{2nh-y}\right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\epsilon_o}\sum_{n=1}^{\infty}(-1)^n\frac{-2y}{(2nh)^2-y^2} \cong \frac{\lambda(z)}{4\pi\epsilon_o h^2}\frac{\pi^2}{12}y$$

Where we have assumed: $h>>a>y$.



For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. We do not need "image currents "As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component, notice the opposite sign:

$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\varepsilon_o h^2}\frac{\pi^2}{12}x$$

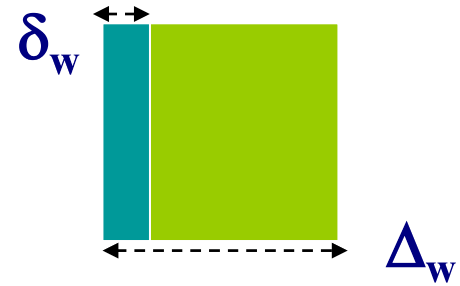
Including also the direct space charge force, we get:

$$\begin{cases} F_x(z,x) = \dfrac{e\lambda(z)x}{\pi\varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right) \\ F_y(z,x) = \dfrac{e\lambda(z)y}{\pi\varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right) \end{cases}$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

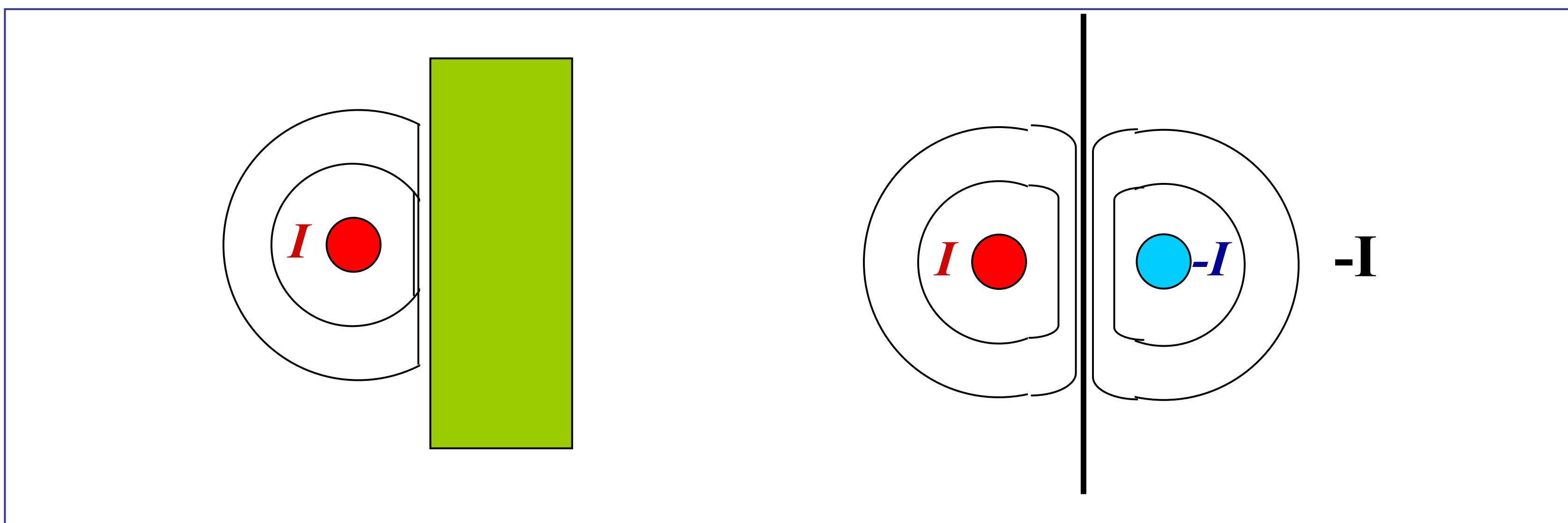
Time-varying fields

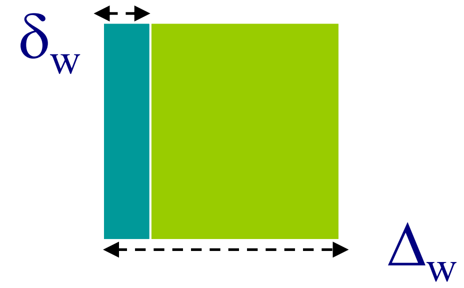
It is necessary to compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.



$$\delta_w \cong \sqrt{\frac{2}{\omega\sigma\mu}}$$

If the fields penetrate and pass through the material, we are practically in the static boundary conditions case. Conversely, if the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.





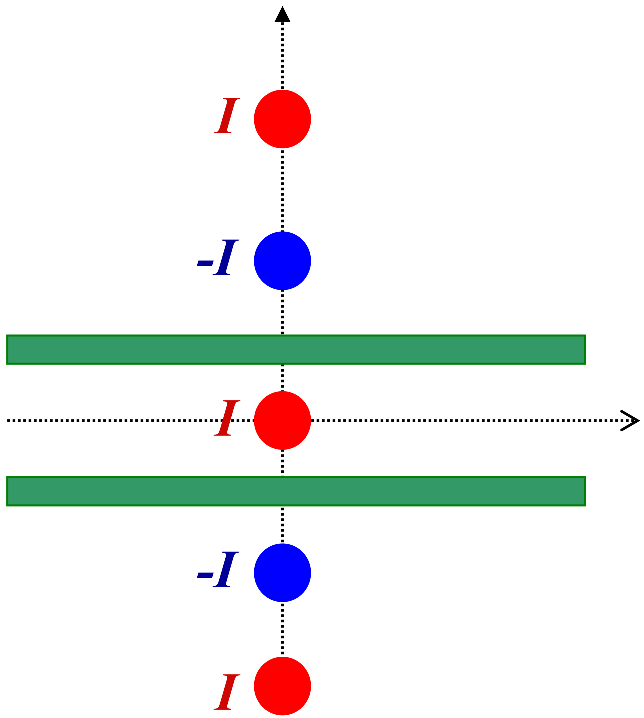
Parallel Plates (Beam at Center) a.c. currents

Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component, I, for which $\delta_w \gg \Delta_w$, and an a.c. component, Î, for which $\delta_w \ll \Delta_w$.

While the d.c. component of the magnetic field does not perceives the presence of the material, its a.c. component is obliged to be tangent at the wall. For a charge density λ we have $I=\lambda v$.

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



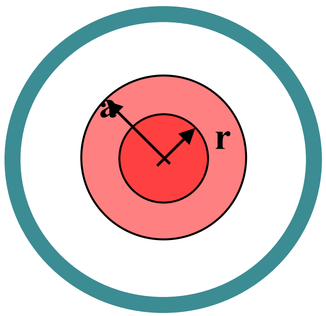
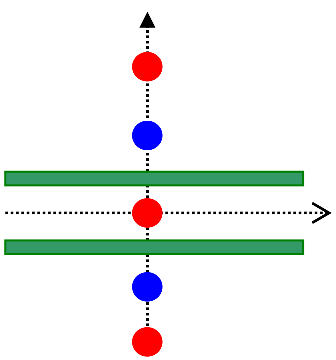
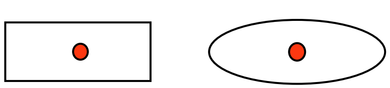
$$\begin{cases} \tilde{E}_y(z,x) = \dfrac{\tilde{\lambda}(z)y}{\pi\varepsilon_o}\dfrac{\pi^2}{48h^2} \\ \tilde{B}_x(z,x) = \dfrac{\beta}{c}\tilde{E}_y(z,x) \end{cases}$$

$$\tilde{F}_y(z,x) = e\left(1-\beta^2\right)E_y = \frac{1}{\gamma^2}\frac{e\tilde{\lambda}(z)y}{\pi\varepsilon_o}\frac{\pi^2}{48h^2}$$

$$\begin{pmatrix} \tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{2\pi\varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} - \dfrac{\pi^2}{24h^2}\right) \\ \tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{2\pi\varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} + \dfrac{\pi^2}{24h^2}\right) \end{pmatrix}$$

There is cancellation of the electric and magnetic forces !!

$$\lambda(z) = \lambda_o + \tilde{\lambda}\cos(k_z z)$$

	D.C.	**A.C.** $(\delta_w \ll \Delta_w)$
	$F_\perp(r) = \dfrac{e}{\gamma^2}\dfrac{\lambda(z)}{2\pi\varepsilon_0}\dfrac{r}{a^2}$	
 	$F_x(z,x) = \dfrac{e\lambda_0 x}{\pi\varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right)$ $F_y(z,x) = \dfrac{e\lambda_0 y}{\pi\varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right)$	$\tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{\pi\varepsilon_0\gamma^2}\left(\dfrac{1}{2a^2} - \dfrac{\pi^2}{48h^2}\right)$ $\tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{\pi\varepsilon_0\gamma^2}\left(\dfrac{1}{2a^2} + \dfrac{\pi^2}{48h^2}\right)$

-L. J. Laslett, LBL Document PUB-6161, 1987, vol III

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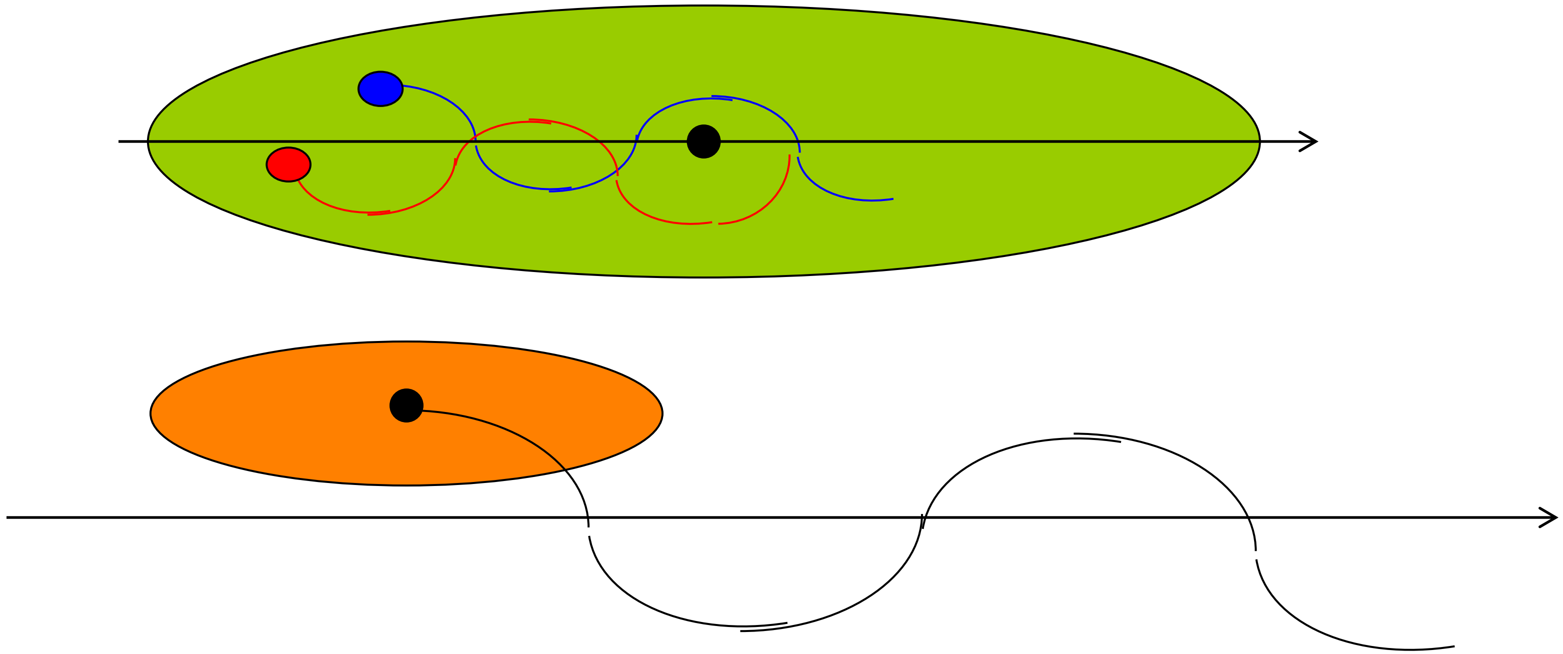
Incoherent and Coherent Transverse Effects

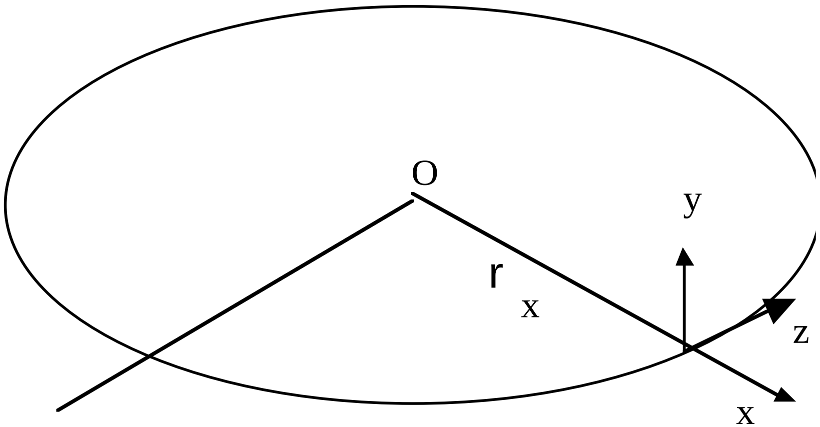
When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass **(coherent),** but change the trajectory of individual charges in the beam **(incoherent).**



These force may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

Self Fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d(m\gamma \mathbf{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})$$

$$\frac{d\mathbf{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

Self Fields and betatron motion

In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

This is the case where the focussing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

Q_x, Betatron tune is the n. of betatron oscillations per turn:

$$Q_x = \frac{2\pi\rho_x}{\lambda_\beta} = \frac{2\pi\rho_x\sqrt{K_x}}{2\pi} = \rho_x\sqrt{K_x}$$

$$x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)$$

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$F_x^{s.c.}(x,z) \cong \left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \qquad\qquad x'' + \left(\frac{Q_{x0}}{\rho_x}\right)^2 x = \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x$$

$$x''(s) + \left[\left(\frac{Q_x}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0}\right] x(s) = 0$$

$$\left[\left(\frac{Q_x}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0}\right] = \frac{(Q_x + \Delta Q_x)^2}{\rho_x^2} \cong \frac{Q_x^2 + 2Q_x \Delta Q_x}{\rho_x^2} \quad \Rightarrow \quad \Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)$$

The shift of betatron wave numbers (tune shift) is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is, in general, function of "z", therefore we have also a tune spread inside the beam. Furthermore, by including higher order terms in the transverse force, we don't have the harmonic oscillator equation any more.

Consequences of the space charge tune shifts

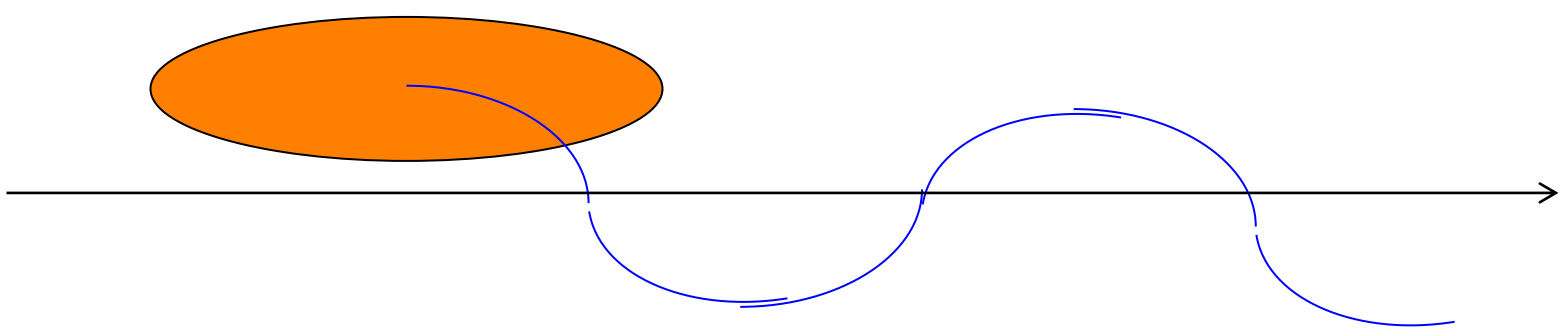
In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable.

The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

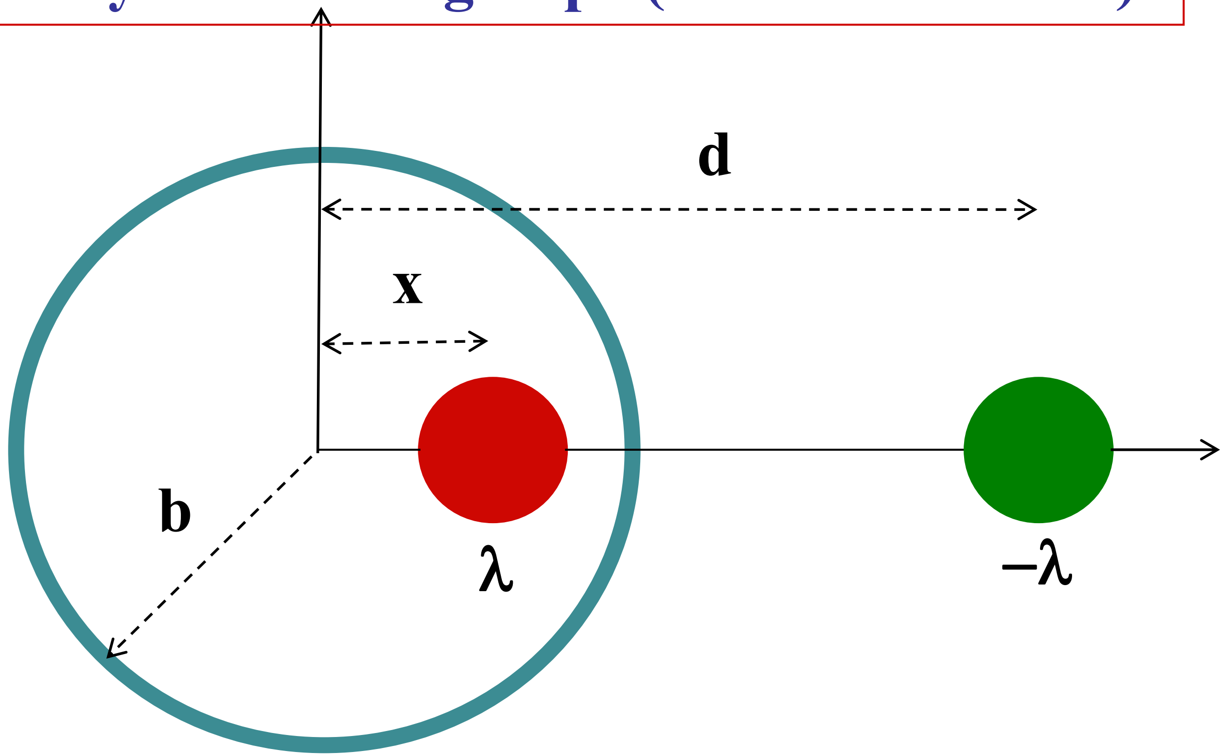
$$\left|\Delta Q_u\right| < 0.3$$

Transverse Coherent Effects

If the beam experiences a transverse deflection kick, it starts to perform betatron oscillations as a whole. The beam, source of the space charge fields moves transversely inside the pipe, while individual particles still continue their incoherent motion around the common coherent trajectory.



Circular Perfectly Conducting Pipe (Beam off Center)



$$d = \frac{b^2}{x}$$

The image charge is at a distance "d" such that the pipe surface is at constant voltage, and pulls the beam away from the center of the pipe.

The effect is defocusing, the horizontal electric image field E and the horizontal force F are:

$$E_{xc}(\boldsymbol{x}) = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d-x} \approx \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d} = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$F_{xc}(r) \approx \frac{e\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$\Delta Q_{xc} = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\left(\frac{\partial F_{xc}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\frac{e\lambda(z)}{2\pi\ \varepsilon_o b^2}$$

$$\Delta Q_{xc} = -\frac{r_e \rho_x^2}{\beta^2 \gamma Q_{x0}}\frac{N}{b^2 l_0}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$

This coherent betatron tune shift, differently from the incoherent one does not depend on the beam size but on the pipe radius and it is inversely proportional to the beam energy.

Space Charge Effects

CERN-2014-009, pp.331-356

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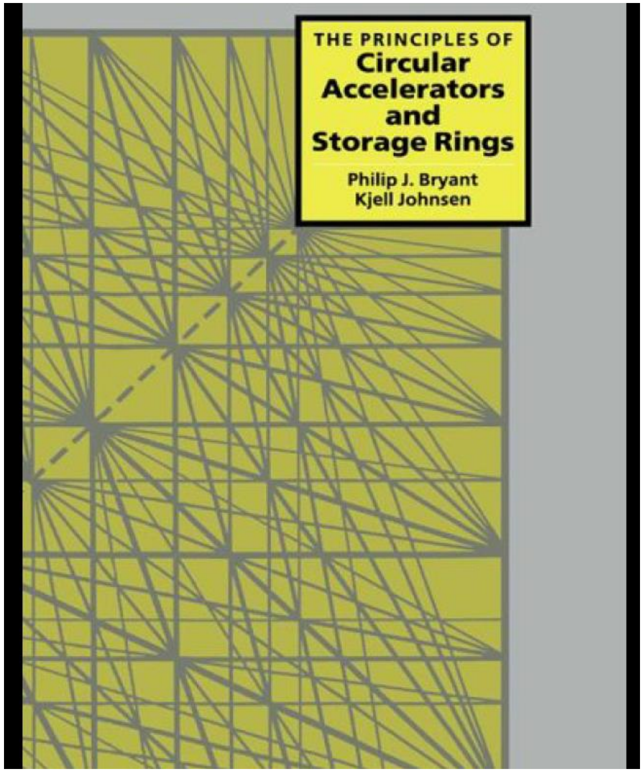
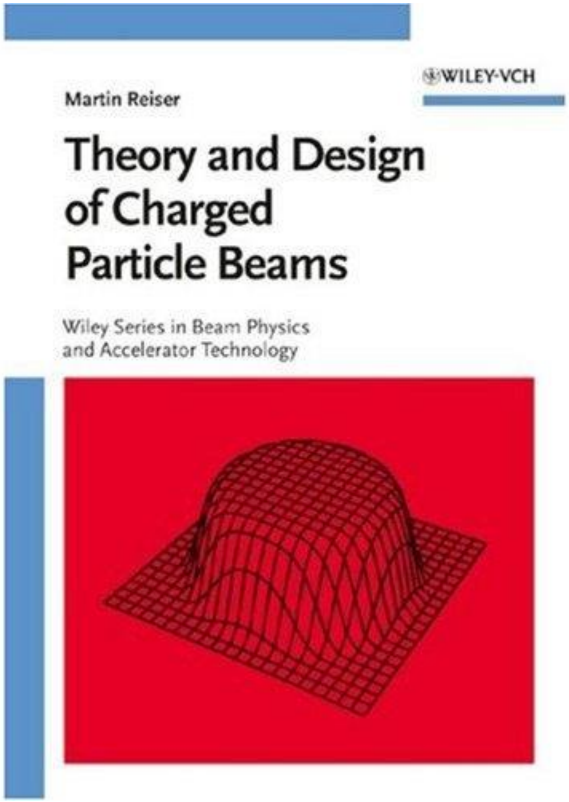
INFN-LNF and University of Rome “La Sapienza”





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