CERN

CAS Advanced Accelerator Physics

Collective effects

Part 3: Transverse wake fields – impact on machine elements and beam dynamics

Kevin Li and Giovanni Rumolo

Outline

We have **discussed longitudinal wake fields** and impedances and their impact on both the machine as well as the beam.

We have learned about **beam induced heating** and how it is related to the beam power spectrum and the machine impedance.

We have discussed the effects of **potential well distortion** (stable phase and synchrotron tune shifts, bunch lengthening and shortening).

We have seen one example of **longitudinal instabilities** (Microwave).

Part 3: Multiparticle dynamics with wake fields – their different types and impact on transverse beam dynamics

- Transverse wake function and impedance
- Effect on a bunch and transverse „potential well distortion“
- Some examples of beam instabilities

Outline

We will move to the description and the impact of **transverse wake fields**.

We will discuss the **different types** of transverse wake fields, outline how they can be implemented numerically and then investigate **their impact on beam dynamics**.

We will see some **examples of transverse instabilities** such as the transverse mode coupling instability (TMCI) or headtail instabilities.

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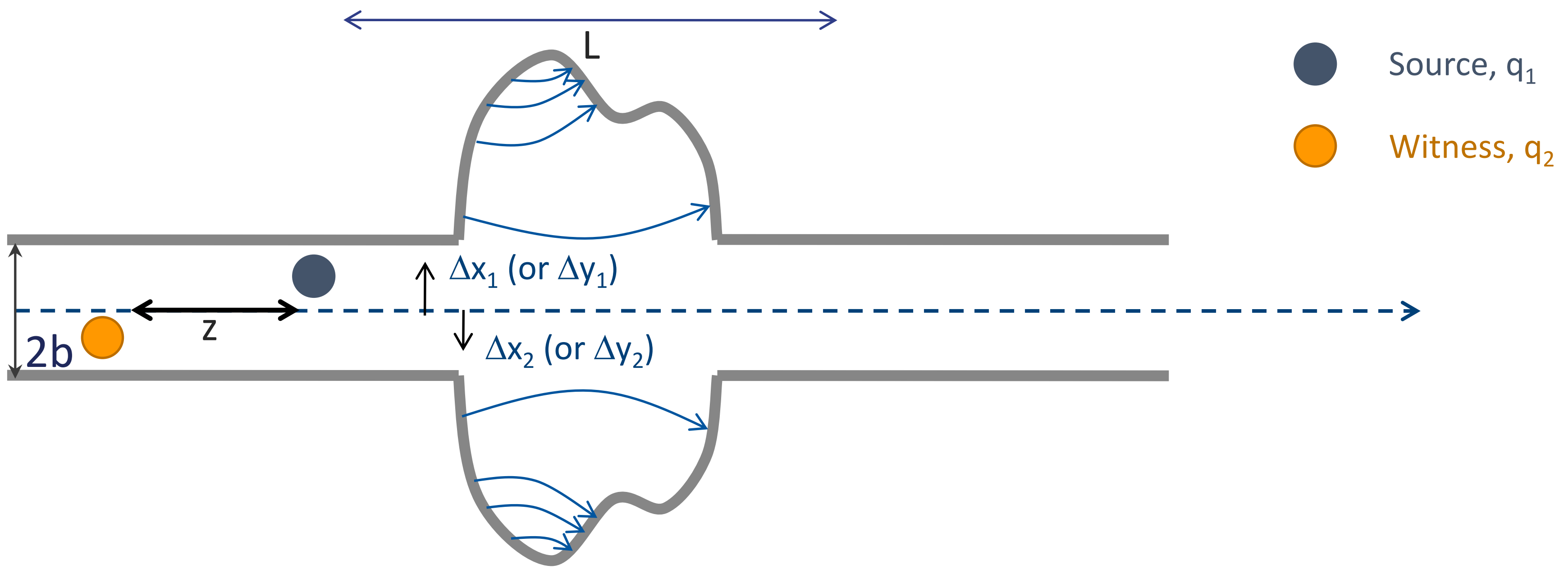
Recap: wake functions in general



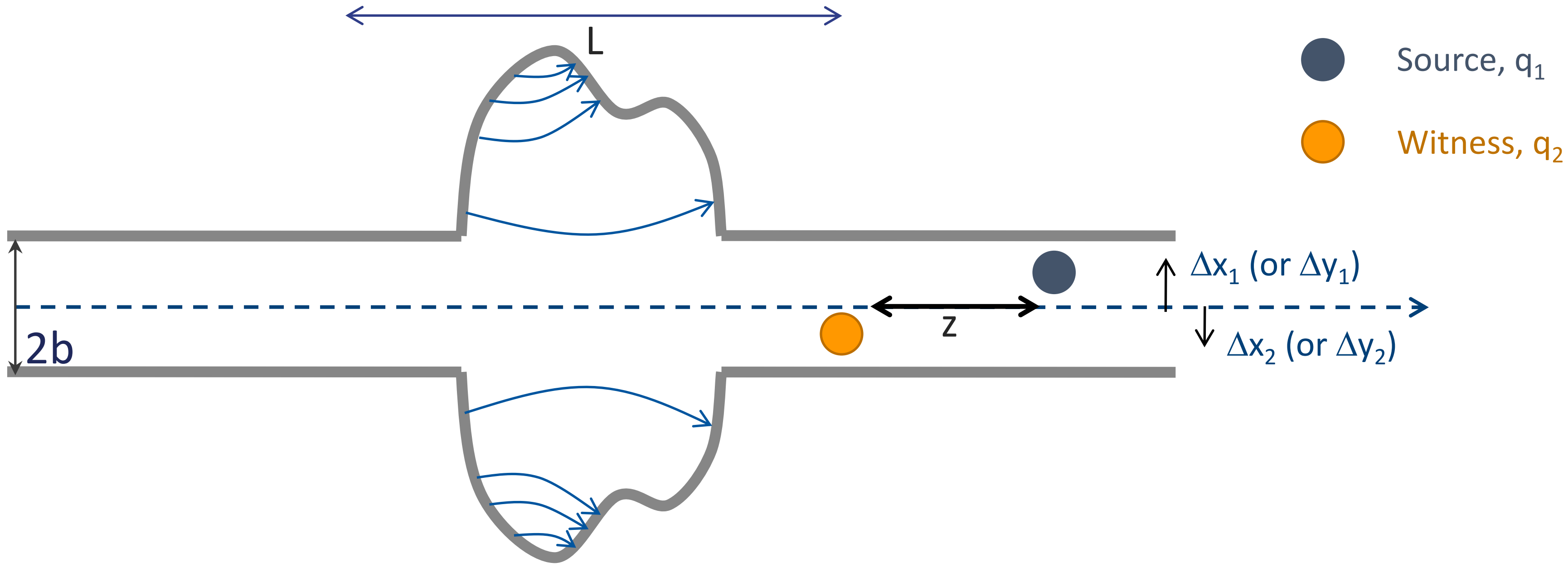
Definition as the **integrated force** felt by a witness charge following a source charge ('energy kick'):

- In general, for two point-like particles, we have

$$\Delta E_2 = \int F(x_1, x_2, z, s)\, ds = -q_1 q_2\, \boldsymbol{w(x_1, x_2, z)}$$

w is typically expanded in the transverse offsets of source and witness particles. This yields the different types of wake fields (dipole, quadrupole, coupling wakes)

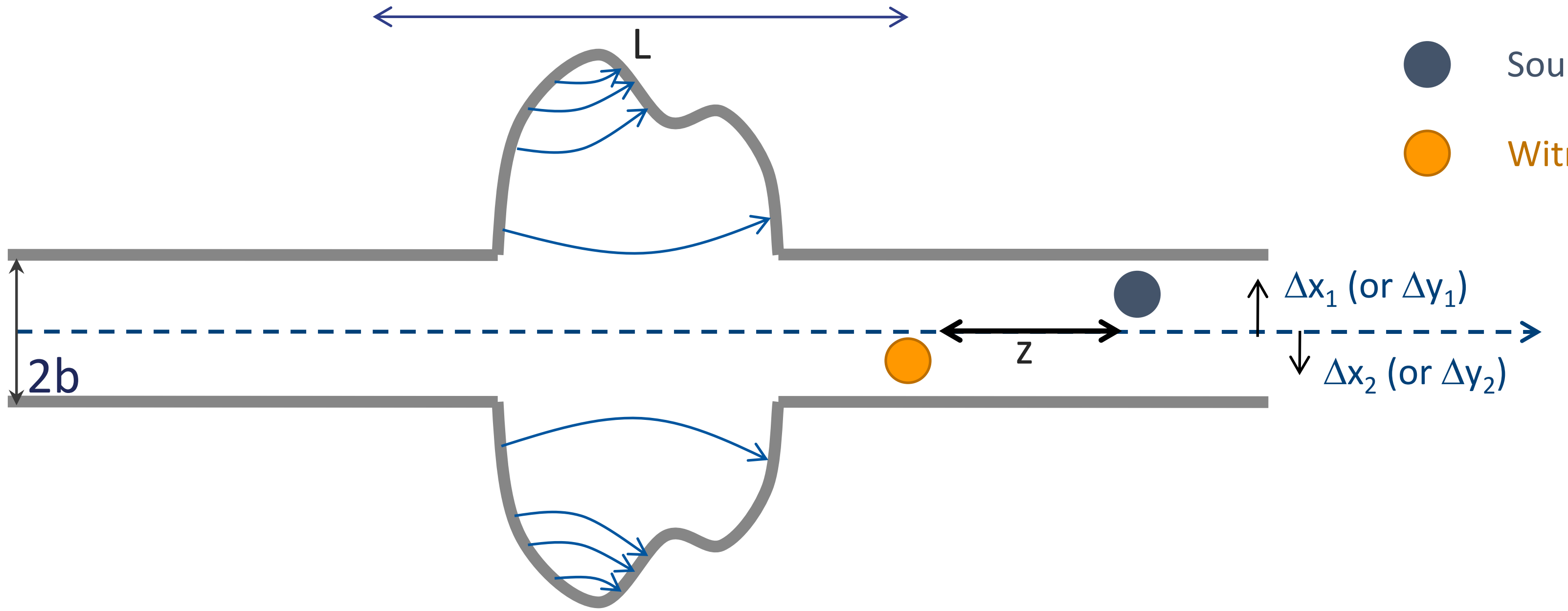
Transverse wake functions



- Transverse wake fields

$$\beta c\,\Delta p_{x2} = \int F_x(x_1, x_2, z, s)\, ds$$

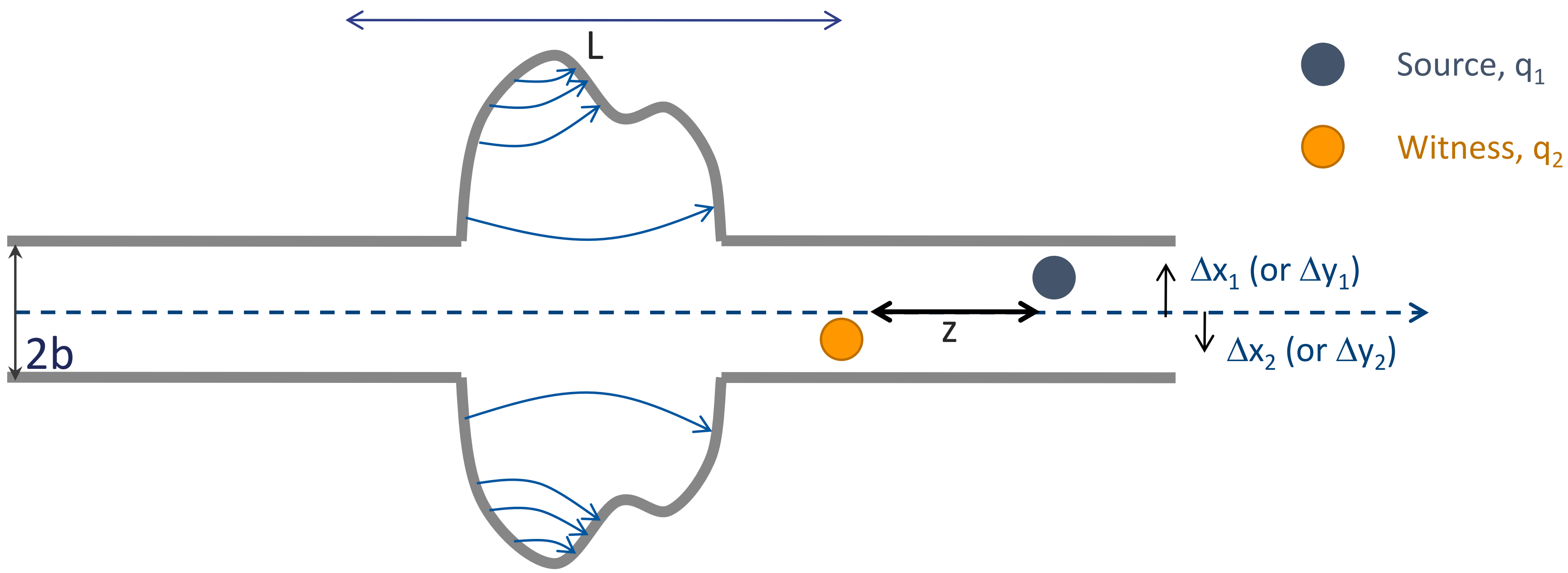
Transverse wake functions



- Transverse wake fields

$$\beta c\,\Delta p_{x2} = \int F_x(x_1, x_2, z, s)\,ds = -q_1 q_2 \left(W_{C_x}(z) + W_{Dx}(z)\,\Delta x_1 + W_{Q_x}(z)\,\Delta x_2\right)$$

Transverse wake functions



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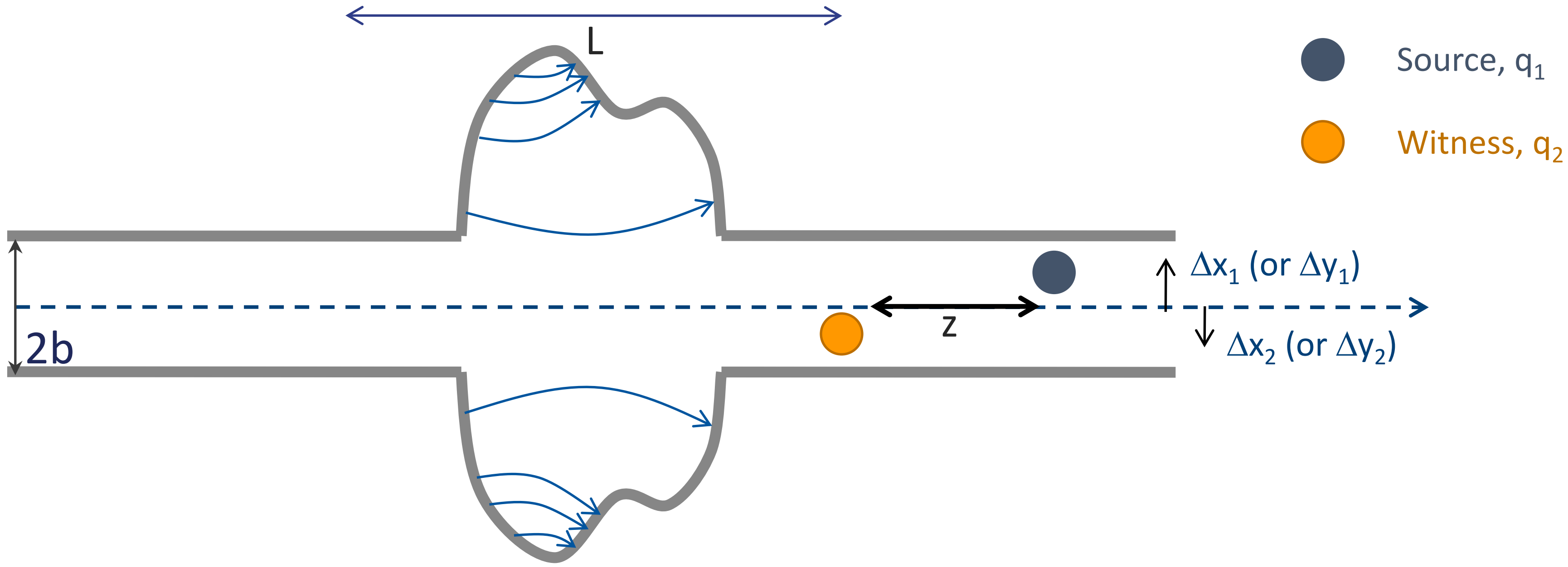
$$\longrightarrow \frac{\Delta p_{x2}}{p_0} = \Delta x_2'$$

Transverse deflecting kick of the witness particle from transverse wakes

Transverse wake functions



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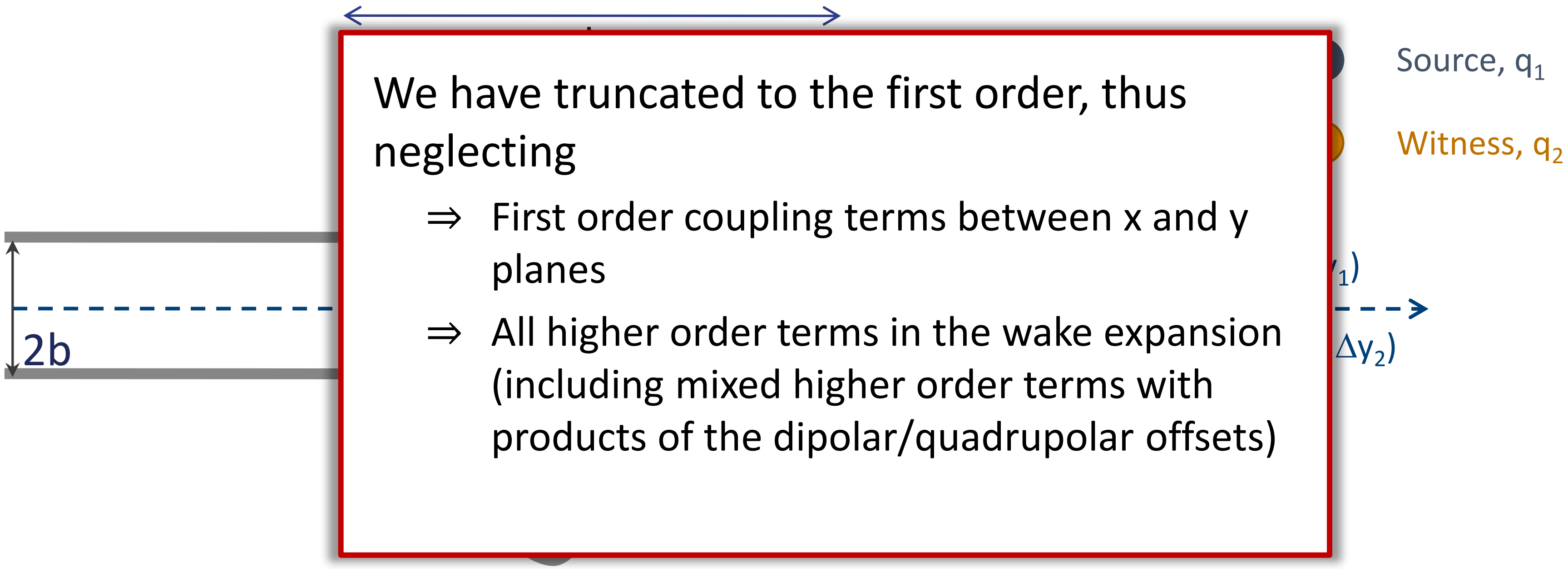
Zeroth order for asymmetric structures
→ Orbit offset

Dipole wakes –
depends on **source particle**

Quadrupole wakes –
depends on **witness particle**

Transverse wake functions



We have truncated to the first order, thus neglecting

⇒ First order coupling terms between x and y planes

⇒ All higher order terms in the wake expansion (including mixed higher order terms with products of the dipolar/quadrupolar offsets)

- Transverse wake fields

$$\beta c\,\Delta p_{x2} = \int F_x(x_1, x_2, z, s)\,ds = -q_1 q_2 \left(\boxed{W_{C_x}(z)} + \boxed{W_{Dx}(z)}\Delta x_1 + \boxed{W_{Q_x}(z)}\Delta x_2\right)$$

Zeroth order for asymmetric structures → Orbit offset

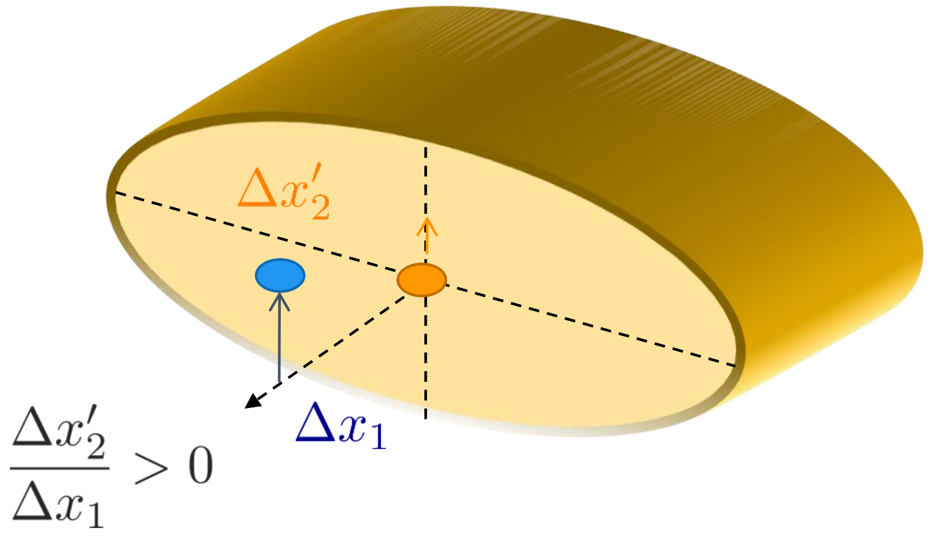
Dipole wakes – depends on **source particle**

Quadrupole wakes – depends on **witness particle**

Transverse dipolar wake function (driving)

$$W_{D_x}(z) = -\frac{\beta^2 E_0}{q_1 q_2}\frac{\Delta x'_2}{\Delta x_1} \xrightarrow{z\to 0} W_{D_x=0}(0) = 0$$

- The value of the transverse dipolar wake function **in z=0 vanishes** because source and witness particles are traveling parallel and they can only – mutually – interact through space charge, which is not included in this framework
- ***W_{Dx}(0–)<0*** since trailing particles are **deflected toward the source particle** (Δx_1 and $\Delta x'_2$ have the same sign)



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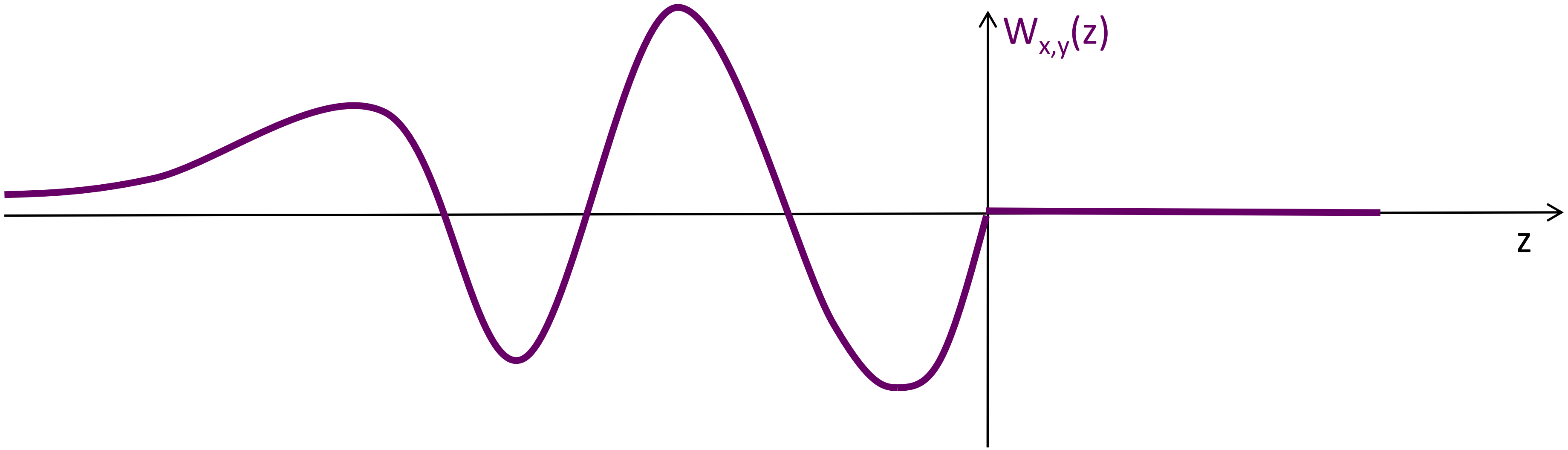
- ***$W_{Dx}(z)$*** has a discontinuous derivative in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation



Transverse quadrupolar wake function (detuning)

$$W_{Q_x}(z) = -\frac{\beta^2 E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2} \quad \xrightarrow{z\to 0} \quad W_{Q_x=0}(0) = 0$$

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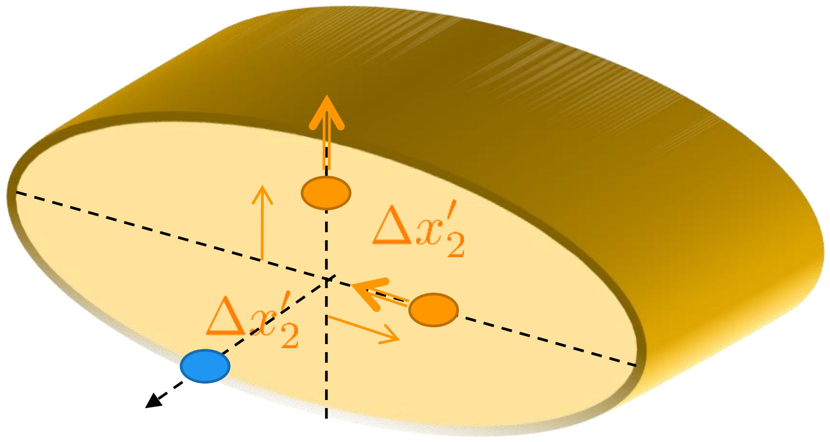
- ***$W_{Qx}(0-)<0$*** can be of either sign since trailing particles can be **either attracted or deflected yet further off axis** (depending on geometry and boundary conditions)



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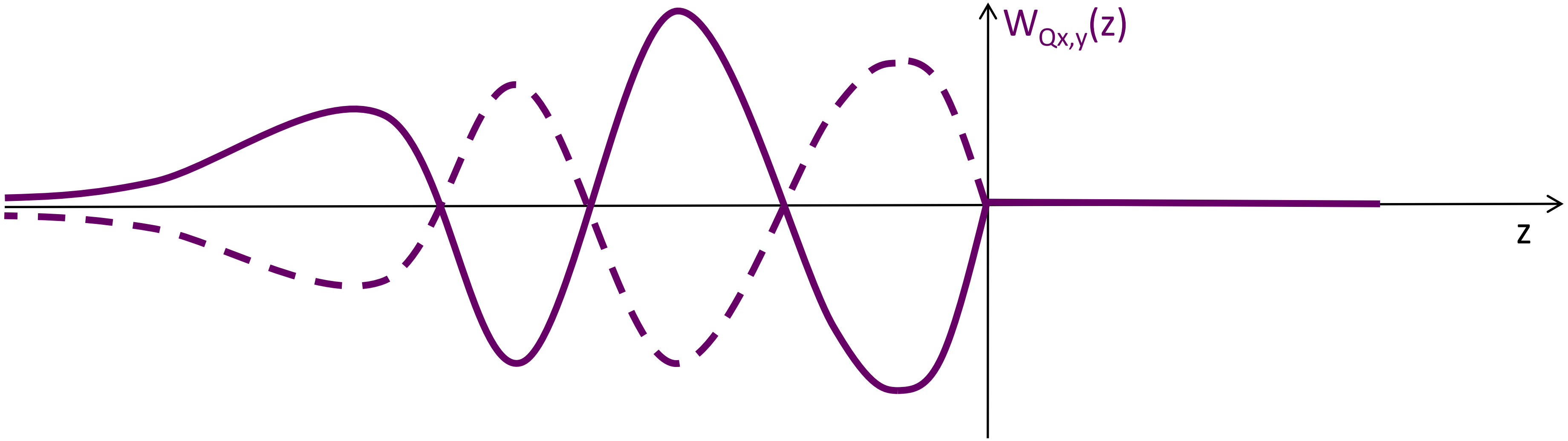


Transverse impedance

$$W_{D_x}(z) = -\frac{\beta^2 E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_1} \qquad W_{Q_x}(z) = -\frac{\beta^2 E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2}$$

- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
  → Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**
  → This is the definition of **transverse beam coupling impedance** of the element under study

Dipolar (or driving)

$$\boxed{Z_{D_x}(\omega)} = i\int_{-\infty}^{\infty} W_{D_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

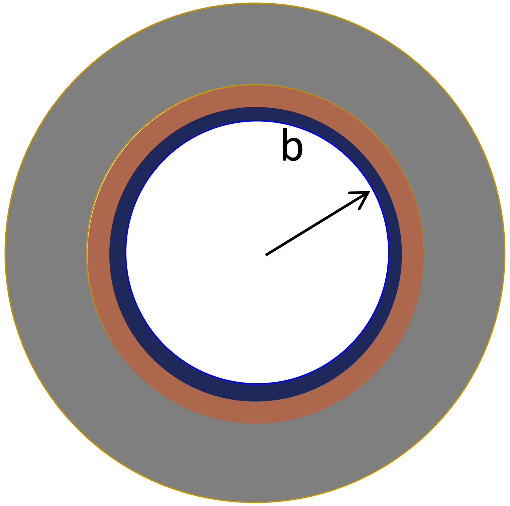
Quadrupolar (or detuning)

$$\boxed{Z_{Q_x}(\omega)} = i\int_{-\infty}^{\infty} W_{Q_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

$[\Omega/\mathrm{m}]$

Examples of transverse wakes/impedances

- **Analytical or semi-analytical** approach, when geometry is simple (or simplified)
  - Same as in the longitudinal plane in terms of approach



→ An example: **axisymmetric beam chamber** with several layers with different EM properties

$$\nabla \times \vec{E} = -i\omega \vec{B} \qquad \nabla \cdot \vec{E} = \frac{\tilde{\rho}}{\epsilon_0 \epsilon_1(\omega)}$$

$$\nabla \times \vec{B} = \mu_0 \mu_1(\omega) \vec{J} + i\omega \frac{\mu_1(\omega)\epsilon_1(\omega)}{c^2} \vec{E}$$

$$\nabla \cdot \vec{B} = 0$$

\+ Boundary conditions

$$\tilde{\rho}(r,\theta,s,\omega) = \frac{q_1}{r_1 v} \delta(r - r_1) \delta_P(\theta) \exp\left(-\frac{i\omega s}{v}\right)$$

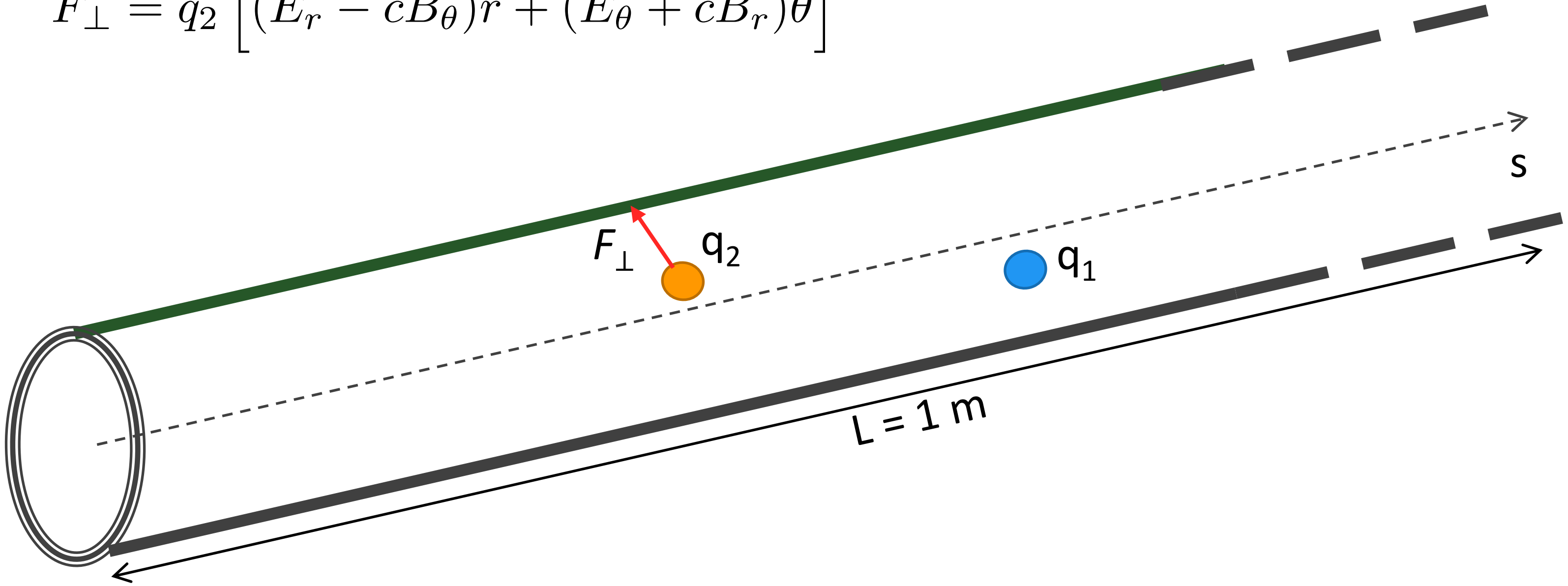
$$\vec{J}(r,\theta,s,\omega) = \tilde{\rho}(r,\theta,s,\omega)\vec{v}$$

Examples of transverse wakes/impedances

- **Analytical or semi-analytical** approach, when geometry is simple (or simplified)
  - Same as in the longitudinal plane in terms of approach
  - But we have to calculate the transverse force from an (offset) source to an (offset) witness

→ We are interested in the transverse force on a test charge q_2 following the source q_1 at a distance z (wake per unit length of chamber)

$$F_\perp = q_2 \left[(E_r - cB_\theta)\hat{r} + (E_\theta + cB_r)\hat{\theta} \right]$$



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$$F_\perp = q_2 \left[\underbrace{(E_r - cB_\theta)}_{F_r = \frac{iq_2 v}{\omega} \frac{\partial E_s}{\partial r}} \hat{r} + \underbrace{(E_\theta + cB_r)}_{F_\theta = \frac{iq_2 v}{\omega r} \frac{\partial E_s}{\partial \theta}} \hat{\theta} \right]$$

$$F_r = \frac{iq_2 v}{\omega} \frac{\partial E_s}{\partial r} \quad F_\theta = \frac{iq_2 v}{\omega r} \frac{\partial E_s}{\partial \theta}$$

Same as for the longitudinal plane

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial s^2} + \frac{\omega^2}{c^2} \epsilon_1(\omega) \mu_1(\omega) \right] E_s =$$

$$= \frac{1}{\epsilon_0 \epsilon_1(\omega)} \frac{\partial \tilde{\rho}}{\partial s} + i\omega \mu_0 \mu_1(\omega) \tilde{\rho} v$$

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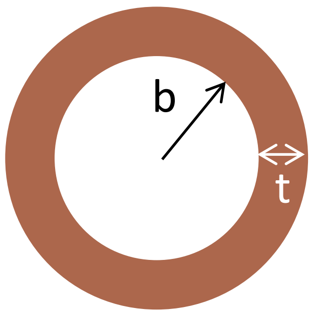
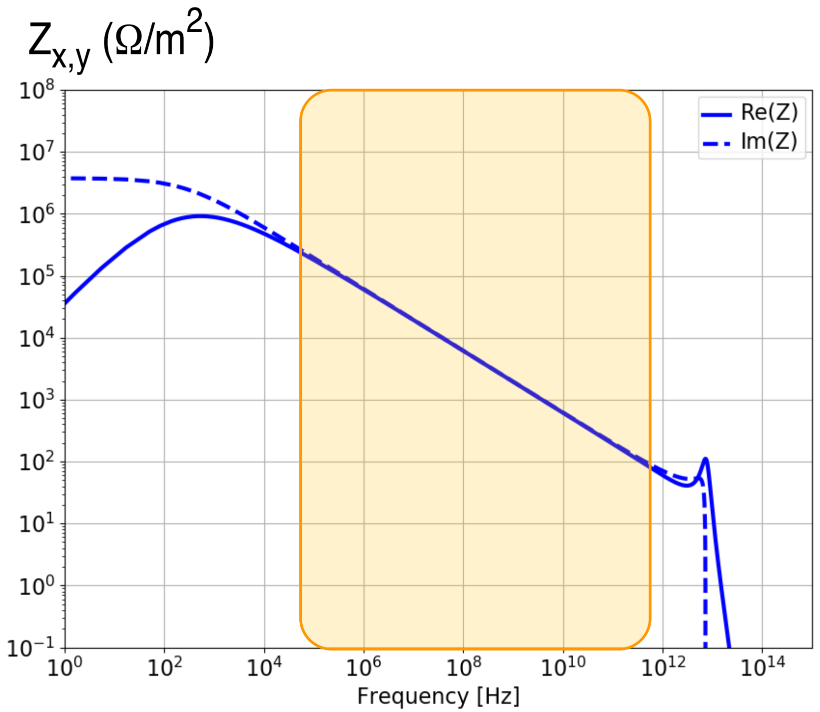
  - We just need E_s also to characterize the transverse wake function



→ An example: a 1 m long Cu pipe with radius b=2 cm and thickness t = 4 mm in vacuum



- Highlighted region shows the typical $\omega^{-1/2}$ scaling
- Scaling with respect to b:
  - Transverse impedance $\sim b^{-3}$

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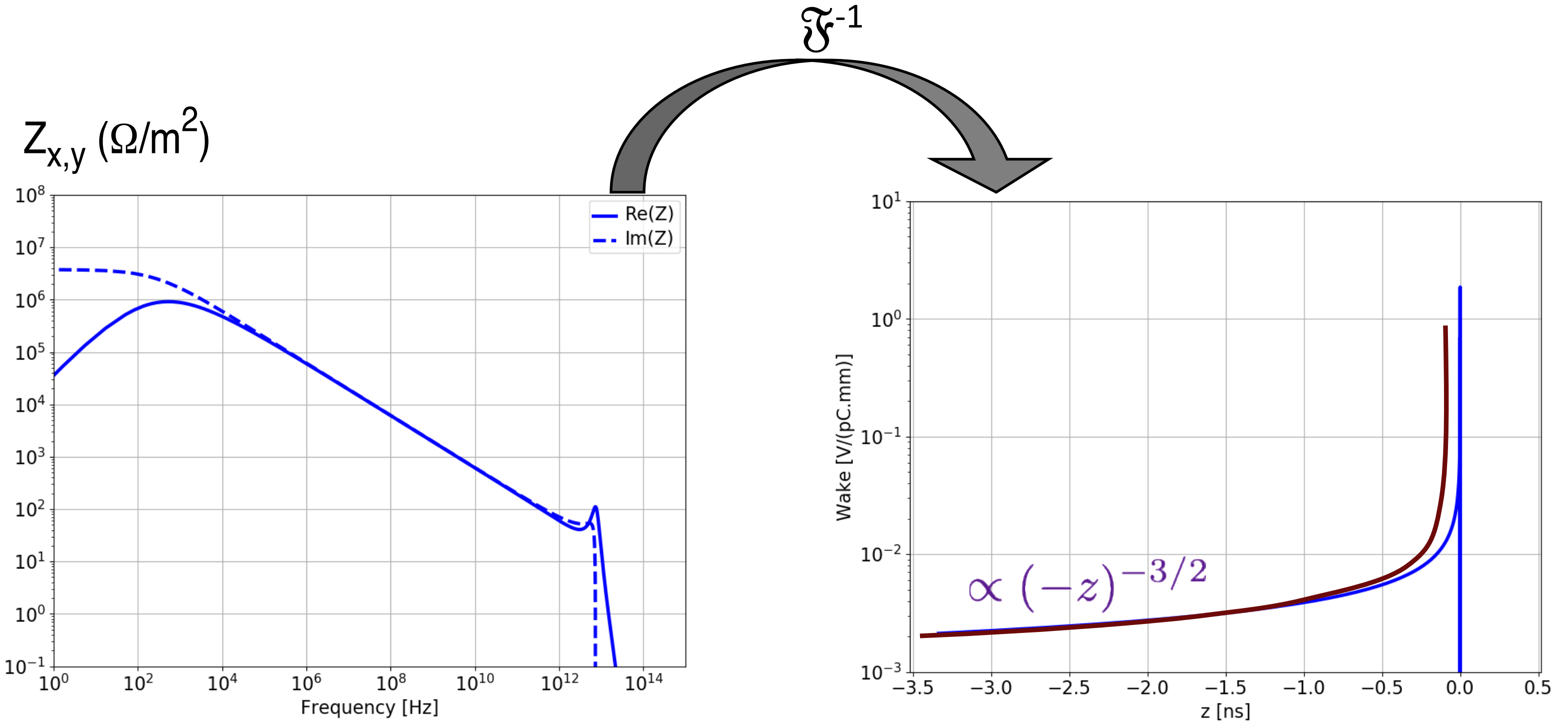

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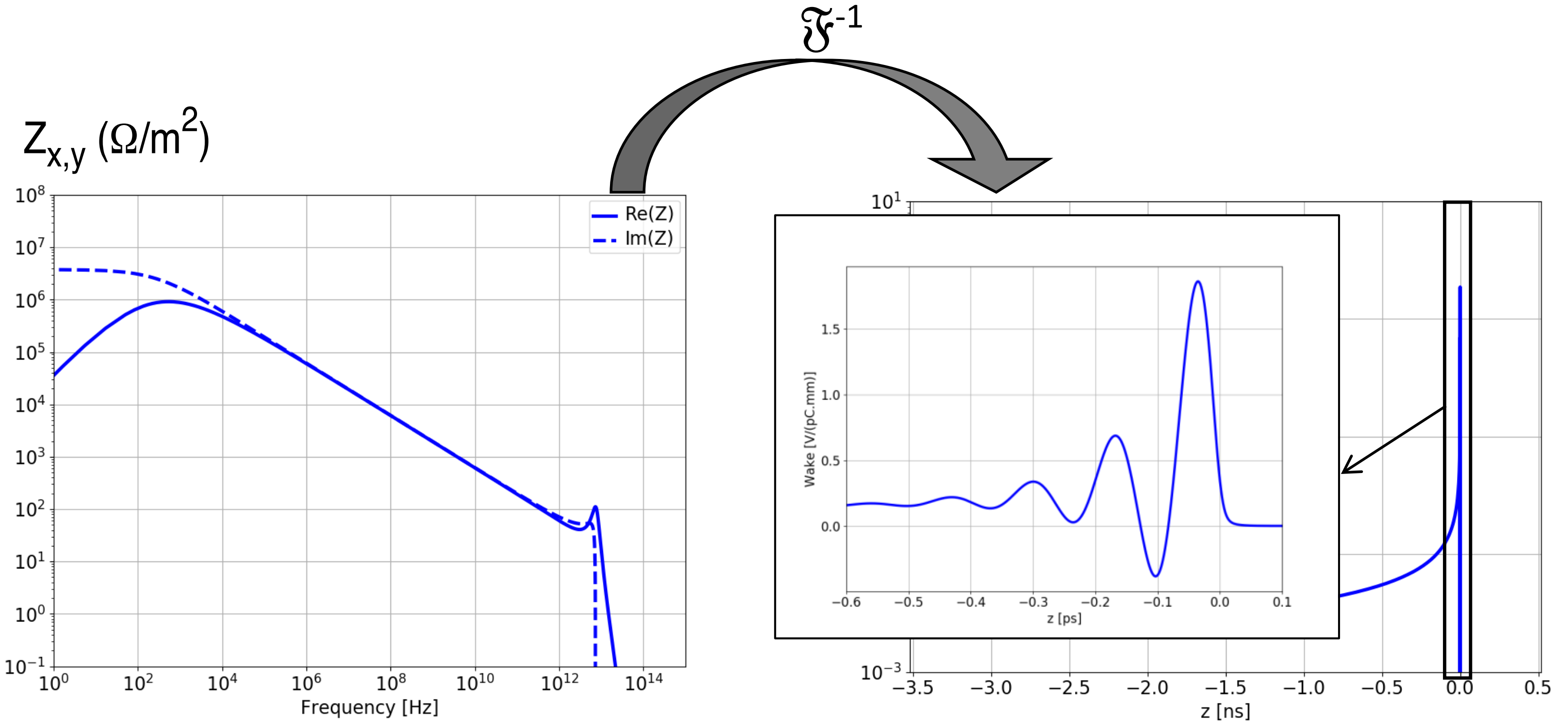
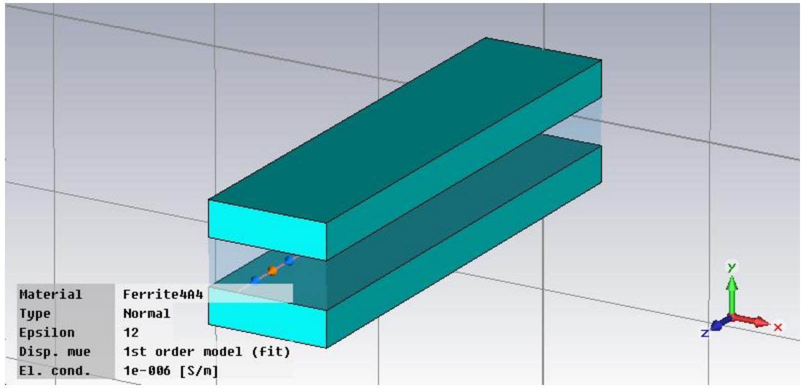
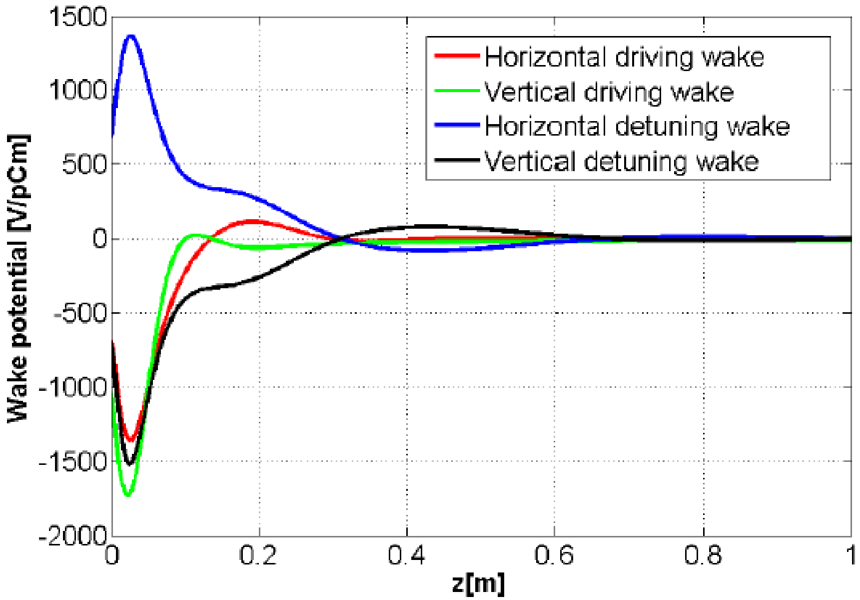


Examples of transverse wakes/impedances

- **Numerical approach**
  - Same as in the longitudinal plane
  - Use numerical codes to solve Maxwell's equations numerically
  - E.g. CST Particle Studio – provides driving and detuning wakes in the two planes by offsetting source and witness, respectively

→ An example: A (simplified) kicker made of two ferrite plates





- Wake is generated for a finite length excitation
- Note than $W_{Qx}(z) = -W_{Qy}(z)$ → general property from Maxwell's equations

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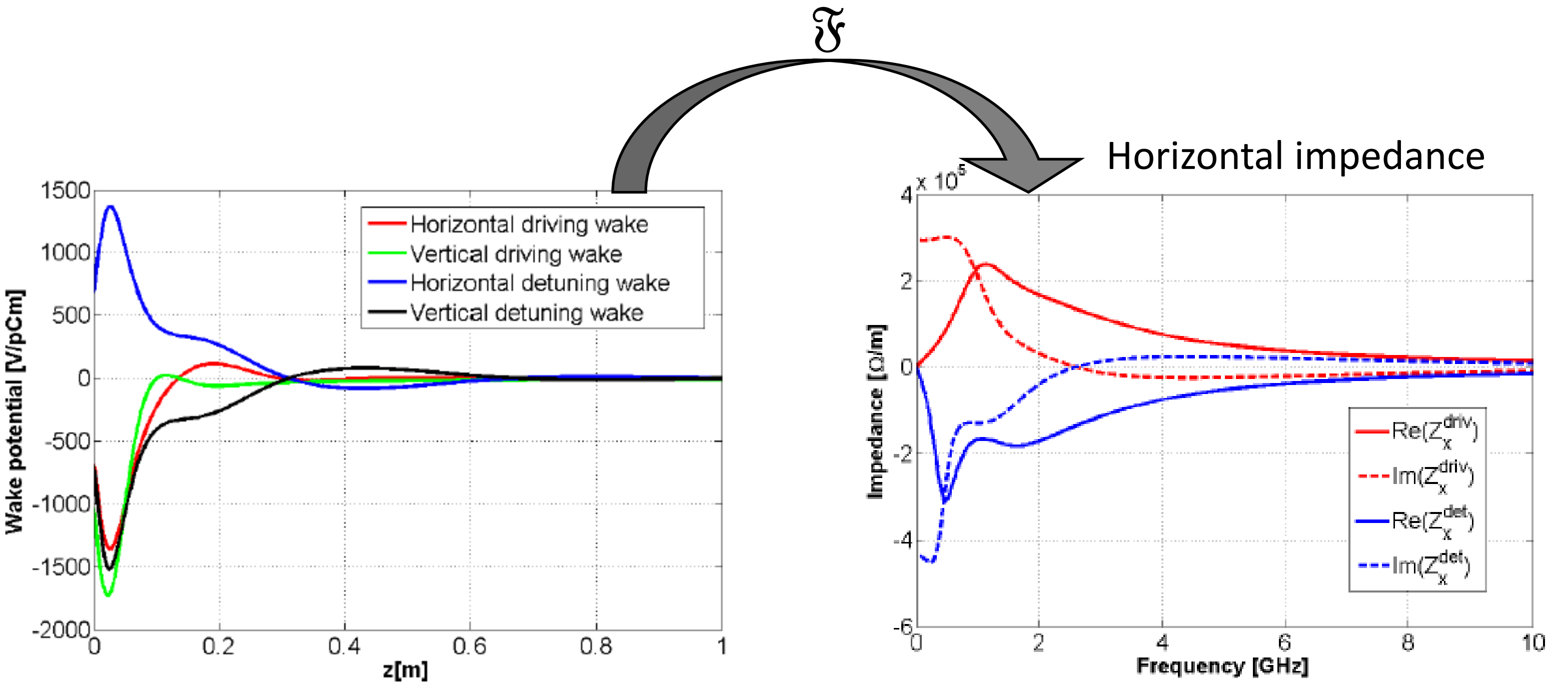

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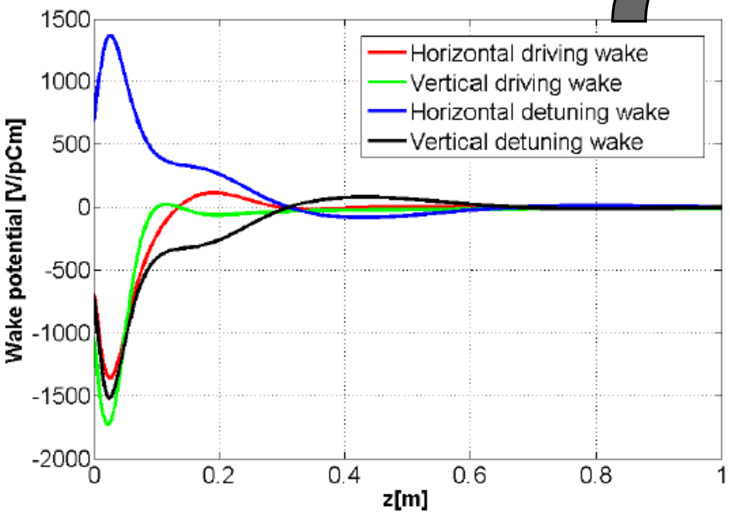

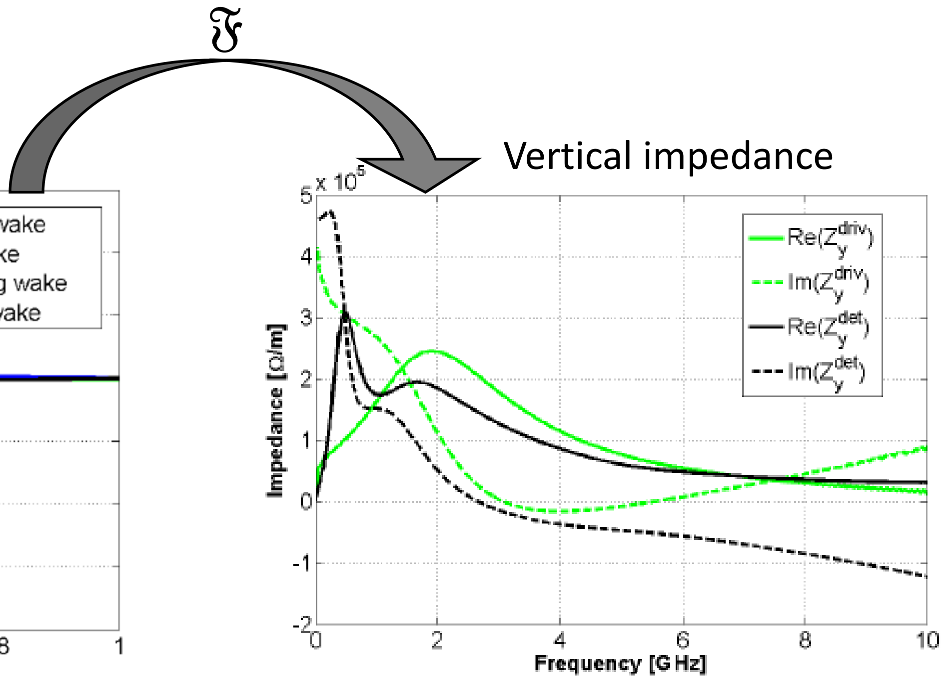

Examples of transverse wakes/impedances

- **Numerical approach**
  - Same as in the longitudinal plane
  - As in the longitudinal case, sometimes it is useful to approximate the impedance with one or more resonators (e.g. one broad band resonator in the case of the kicker)

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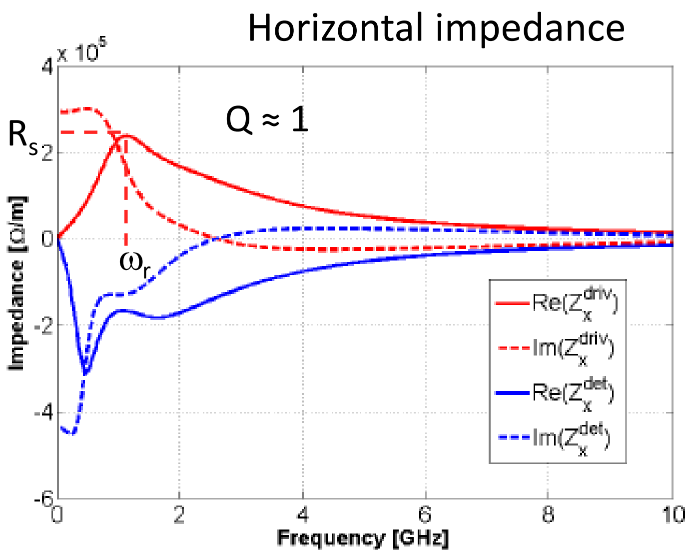

$$Z_{x,y}^{\mathrm{Res}}(\omega) = \frac{\omega_r}{\omega} \frac{R_{s(x,y)}}{1 + iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

$$W_{x,y}^{\mathrm{Res}}(z) = \begin{cases} \frac{R_{s(x,y)}\omega_r^2}{Q\bar{\omega}} \exp\left(\frac{\alpha_t z}{c}\right) \sin\left(\frac{\bar{\omega} z}{c}\right) & \text{if} \quad z < 0 \\ 0 \quad \text{if} \quad z \geq 0 \end{cases}$$

$$\alpha_t = \frac{\omega_r}{2Q} \qquad \bar{\omega} = \sqrt{\omega_r^2 - \alpha_t^2}$$

Signpost

We have seen the **definition of transverse wake fields** and how they can be classified into constant, dipolar and quadrupolar wake fields.

We have discussed how to calculate the **transverse wakes and impedances.**

We will now look into how the impact of wake fields onto charged particle beams can be **modeled numerically** to prepare for investigating the different types of coherent instabilities further along.

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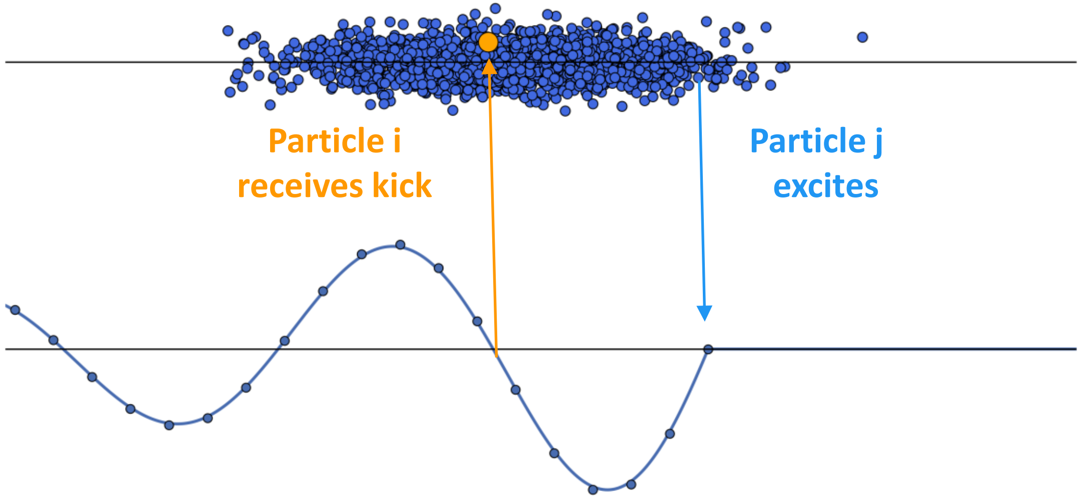
Transverse impedance kick

- Single traversal of a bunch through an impedance source

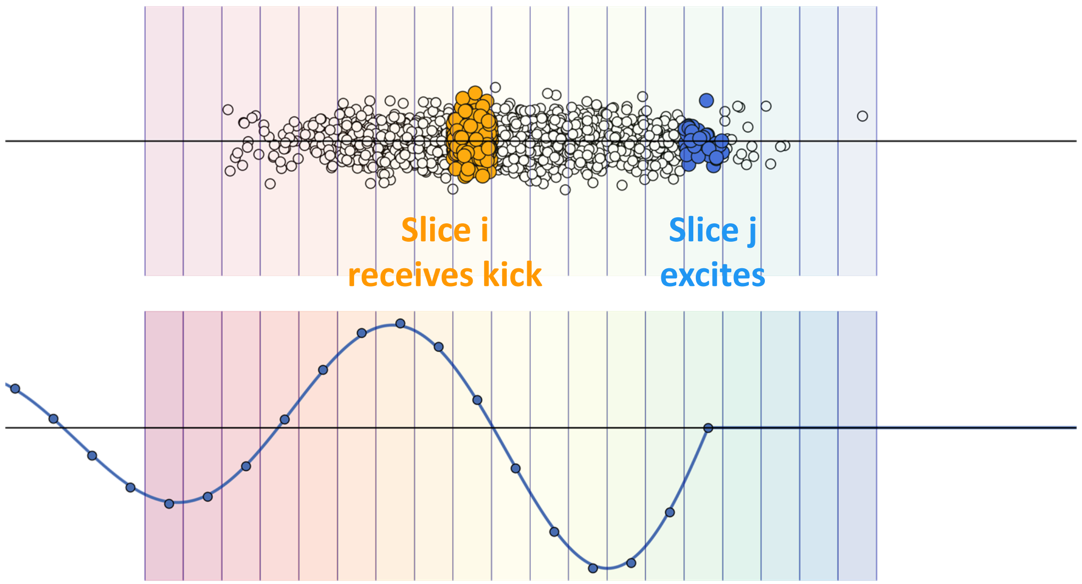


$$\Delta x'_i = -\frac{e^2}{\beta^2 E_0} \begin{cases} W_{Cx}(z_i - z_j) \\ x_j W_{Dx}(z_i - z_j) \\ x_i W_{Qx}(z_i - z_j) \end{cases}$$

$$\langle \Delta x'_{\text{bunch}} \rangle = -\frac{e^2}{N_b \beta^2 E_0} \sum_{j=1}^{N_b} \sum_{i=1}^{N_b} \begin{cases} W_{Cx}(z_i - z_j) \\ x_j W_{Dx}(z_i - z_j) \\ x_i W_{Qx}(z_i - z_j) \end{cases}$$

Transverse impedance kick

- Single traversal of a bunch through an impedance source
  - Let’s neglect the quadrupolar wake in first instance → equal kicks on particles in slice i



$$\Delta x'_{ij} = -\frac{e^2}{\beta^2 E_0} N[j]\left[W_{Cx}[(i-j)\Delta z] + \langle x\rangle[j] W_{Dx}[(i-j)\Delta z]\right]$$

$$\Delta x'_{i} = -\frac{e^2}{\beta^2 E_0} \sum_{j=0}^{i} N[j]\left[W_{Cx}[(i-j)\Delta z] + \langle x\rangle[j] W_{Dx}[(i-j)\Delta z]\right]$$

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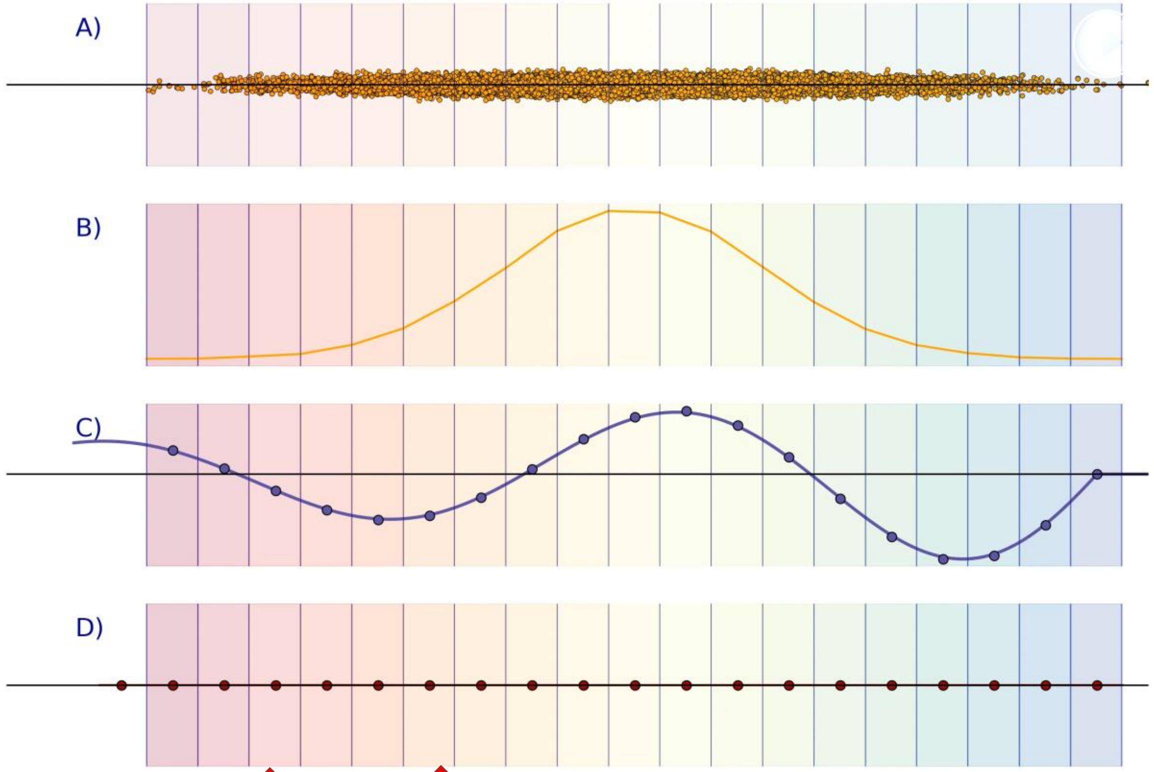

$$\Delta x_i' = -\frac{e^2}{\beta^2 E_0} \sum_{j=0}^{i} N[j] \left[W_{Cx}[(i-j)\Delta z] + \cancel{\langle x \rangle [j] W_{Dx}[(i-j)\Delta z]} \right]$$

as $\langle x \rangle (z) = 0$

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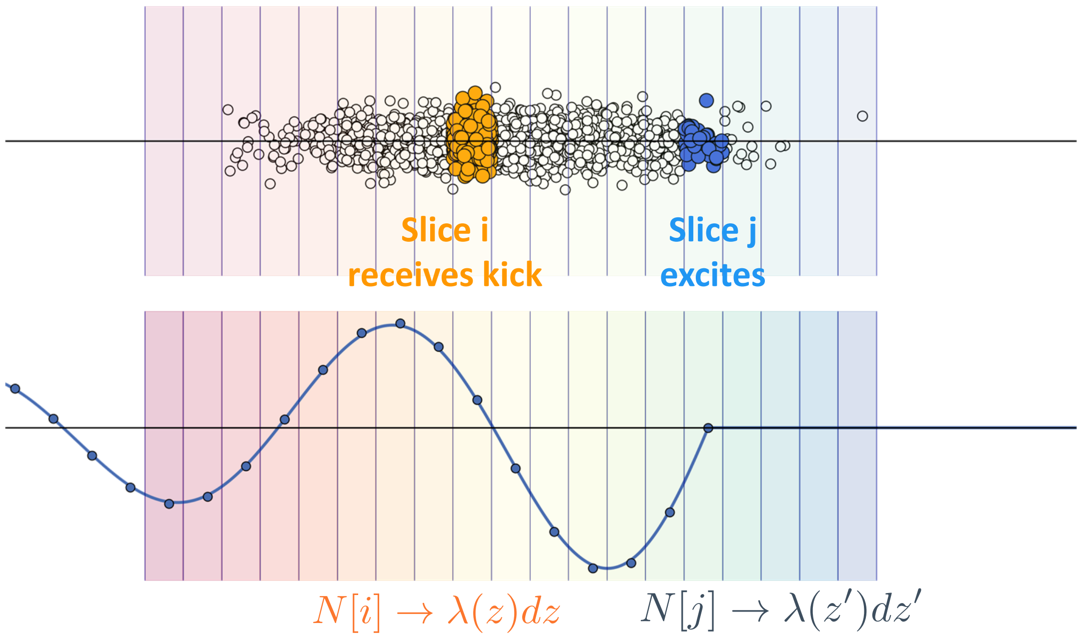


$$\langle \Delta x' \rangle_{\text{bunch}} = \frac{1}{N_b} \sum_{i=1}^{N_{\text{slices}}} N[i] \Delta x_i$$

$$\langle \Delta x' \rangle_{\text{bunch}} = -\frac{e^2}{N_b \beta^2 E_0} \int \lambda(z) dz \int \underbrace{\lambda(z') \left[W_{Cx}(z - z')\right.}_{} + \underbrace{\left.\langle x \rangle (z') W_{Dx}(z - z') \right]}_{} dz'$$

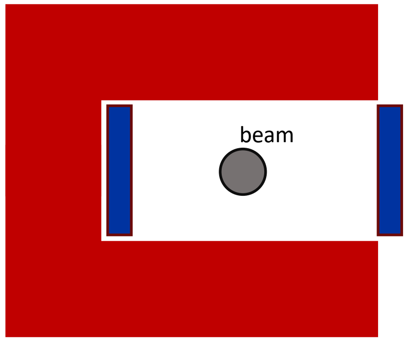
"Constant" kick, dipole type, associated to a closed orbit distortion

Kick dependent on <x>, quadrupole type, associated to a detuning

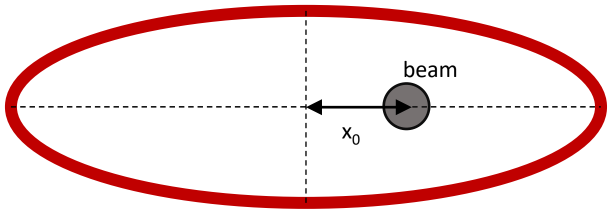
Transverse impedance kick

- Single traversal of a bunch through an impedance source
  - Let’s neglect the quadrupolar wake in first instance → equal kicks on particles in slice i

$$\langle \Delta x' \rangle_{\text{bunch}} = -\frac{e^2}{N_b \beta^2 E_0} \int \lambda(z) dz \int \lambda(z') \left[\cancel{W_{Cx}(z} - z') + \langle x \rangle (z') \cancel{W_{Dx}}(z - z')\right] dz'$$



$$\langle \Delta x' \rangle_{\text{bunch}} = -\frac{e^2 c^2}{N_b E_0} \int_{-\infty}^{\infty} |\hat{\lambda}(\omega)|^2 \text{Im}\left[Z_{Cx}(\omega)\right] d\omega$$

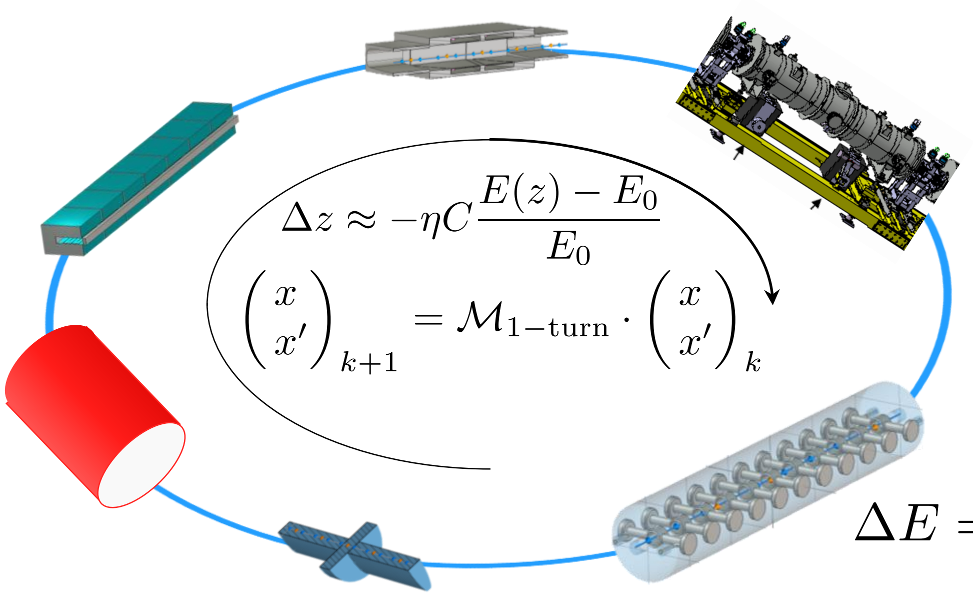


$$\langle \Delta x' \rangle_{\text{bunch}} = -\frac{e^2 c^2 x_0}{N_b E_0} \int_{-\infty}^{\infty} |\hat{\lambda}(\omega)|^2 \text{Im}\left[Z_{Dx}(\omega)\right] d\omega$$

$$\langle \Delta x' \rangle_{\text{bunch}} = -\frac{e^2 c^2 x_0}{N_b E_0} \int_{-\infty}^{\infty} |\hat{\lambda}(\omega)|^2 \left[\text{Im}[Z_{Dx}(\omega)] + \text{Im}[Z_{Qx}(\omega)]\right] d\omega$$

Transverse wakes in beam dynamics

- Same approach as in the longitudinal plane to build the impedance model of a machine
- For simulations, the impedance is lumped in one place and kicks to beam particles are applied once per turn, with linear matrix transport between turns
  - One word of caution: The effect of the transverse impedance results in a combination of a dipole-type and quadrupole-type kick, therefore the beta functions at the real locations of the impedance source has to be taken into account when combining wakes/impedances



$$\Delta z \approx -\eta C \frac{E(z) - E_0}{E_0}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{k+1} = \mathcal{M}_{1-\mathrm{turn}} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_k$$

$$W^{\mathrm{Ring}}_{Cx,Dx,Qx}(z) = \sum_i \frac{\beta_{xi}}{\langle \beta_x \rangle} W^i_{Cx,Dx,Qx}(z)$$

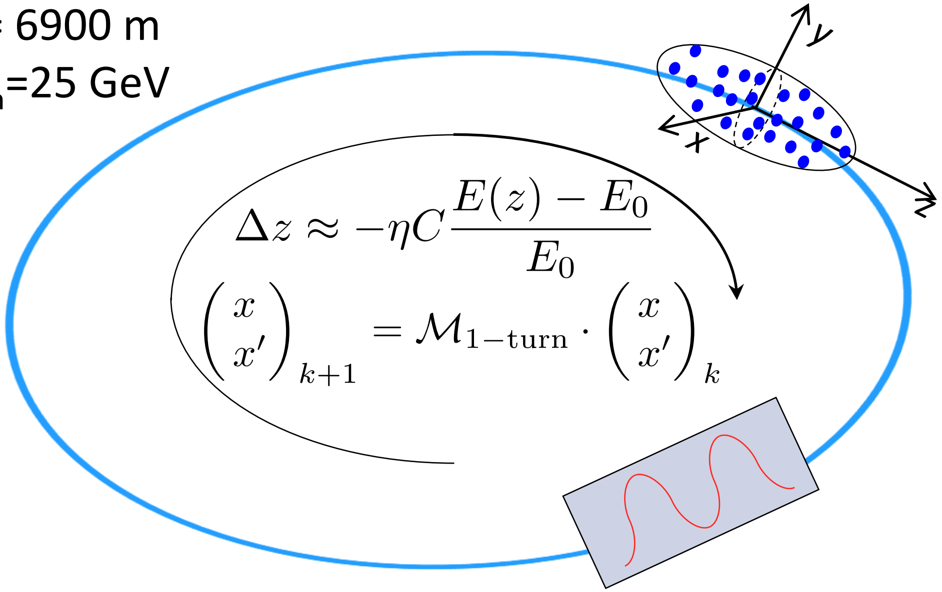
$$Z^{\mathrm{Ring}}_{Cx,Dx,Qx}(\omega) = \sum_i \frac{\beta_{xi}}{\langle \beta_x \rangle} Z^i_{Cx,Dx,Qx}(\omega)$$

$$\Delta E = eV_{\mathrm{rf}}(z)$$

$$\Delta x' = -\frac{e^2}{\beta^2 E_0} \int \lambda(z') \left[W^{\mathrm{Ring}}_{Cx}(z - z') + \langle x \rangle (z') W^{\mathrm{Ring}}_{Dx}(z - z') + x W^{\mathrm{Ring}}_{Qx}(z - z') \right] dz'$$

Effect of a transverse impedance on a bunch

SPS ring
C = 6900 m
E_{kin}=25 GeV



$$\Delta z \approx -\eta C \frac{E(z) - E_0}{E_0}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{k+1} = \mathcal{M}_{1-\mathrm{turn}} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_k$$

Single Gaussian bunch
σ_z = 0.2 m (0.67 ns)

Constant horizontal wake from a kicker (non-axisymmetric)

Two examples:
Frozen synchrotron motion

or

Single RF system
ω_{rf} = 200 MHz
V_{rf}^{max} = 3 MV

$$\Delta x' = -\frac{e^2}{\beta^2 E_0} \int \lambda(z') W_{Cx}^{\mathrm{Kicker}}(z - z') dz'$$

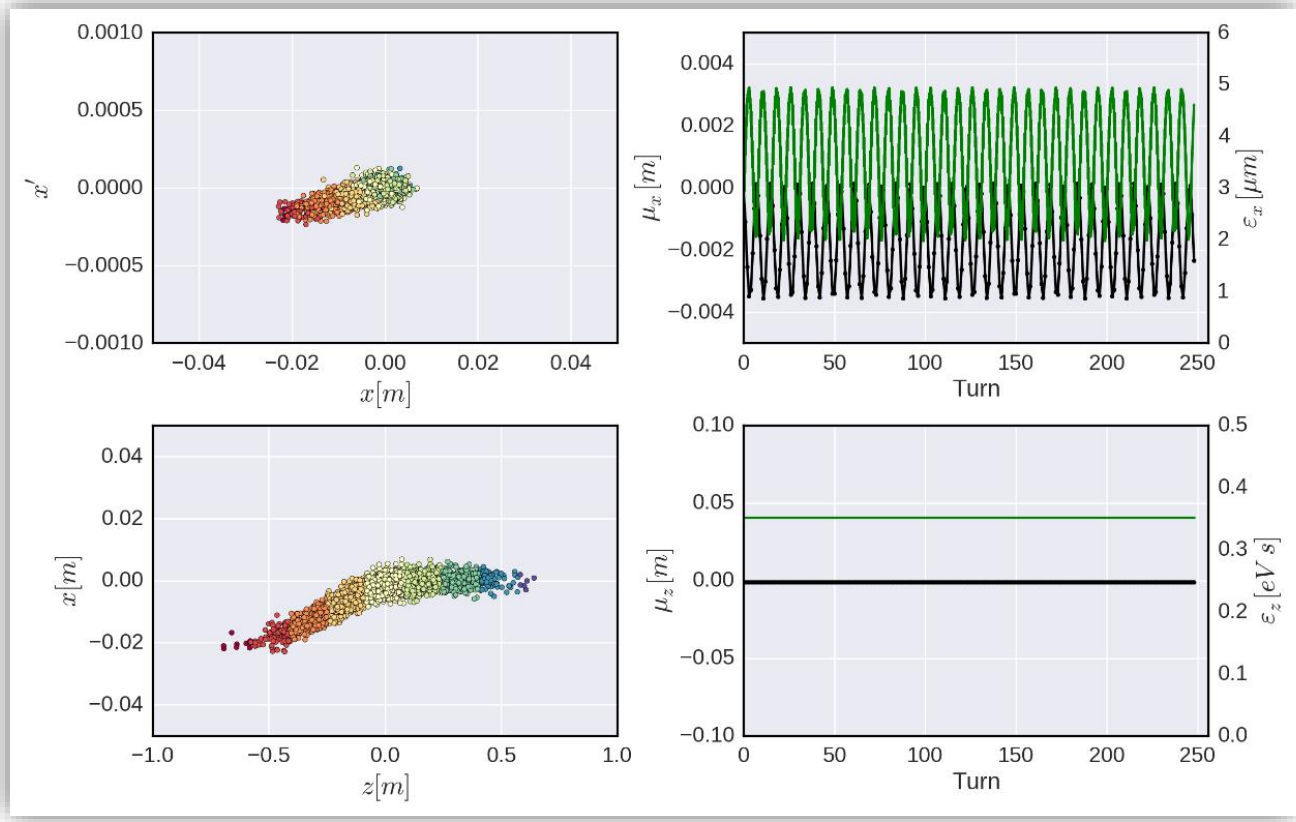
$$\Delta E = e V_{\mathrm{rf}}(z)$$

Examples – constant wakes

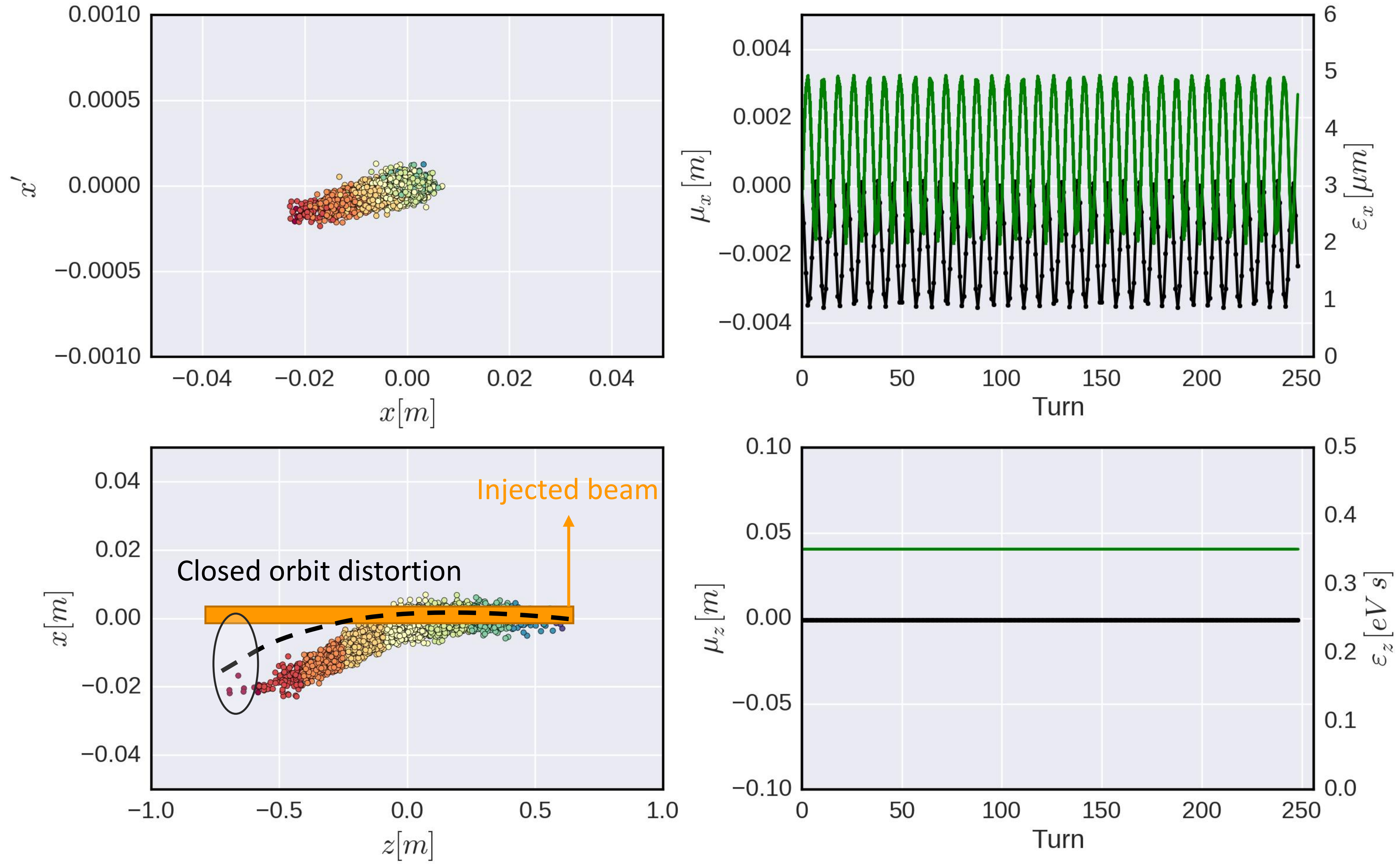
$$\Delta x'[i] = \underbrace{-\frac{e^2}{m\gamma\beta^2c^2}\sum_{j=0}^{i}\boxed{N[j]\cdot W_{Cx}[i-j]}}$$

Dipolar term → orbit kick

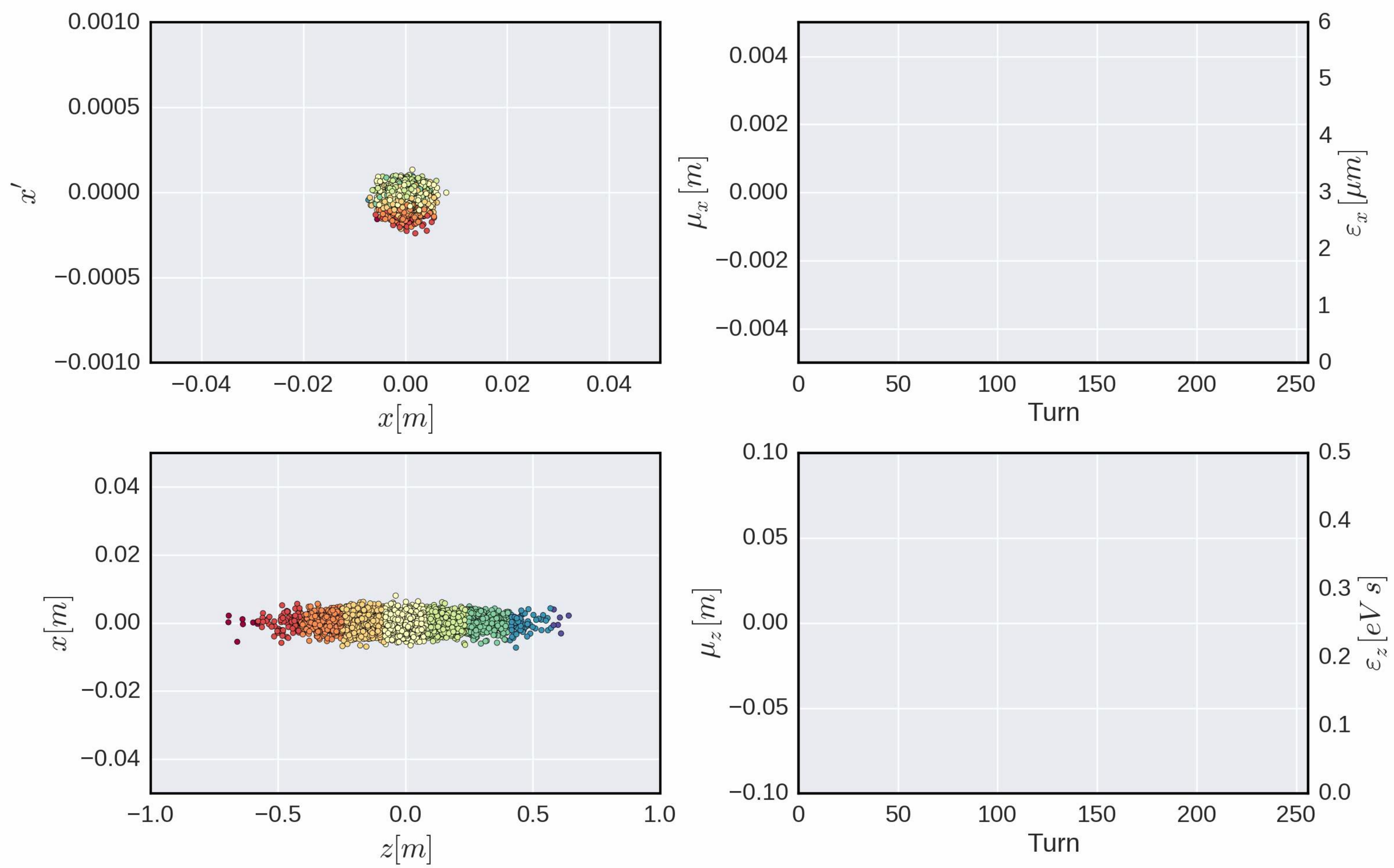
Slice dependent change of closed orbit (if line density does not change)



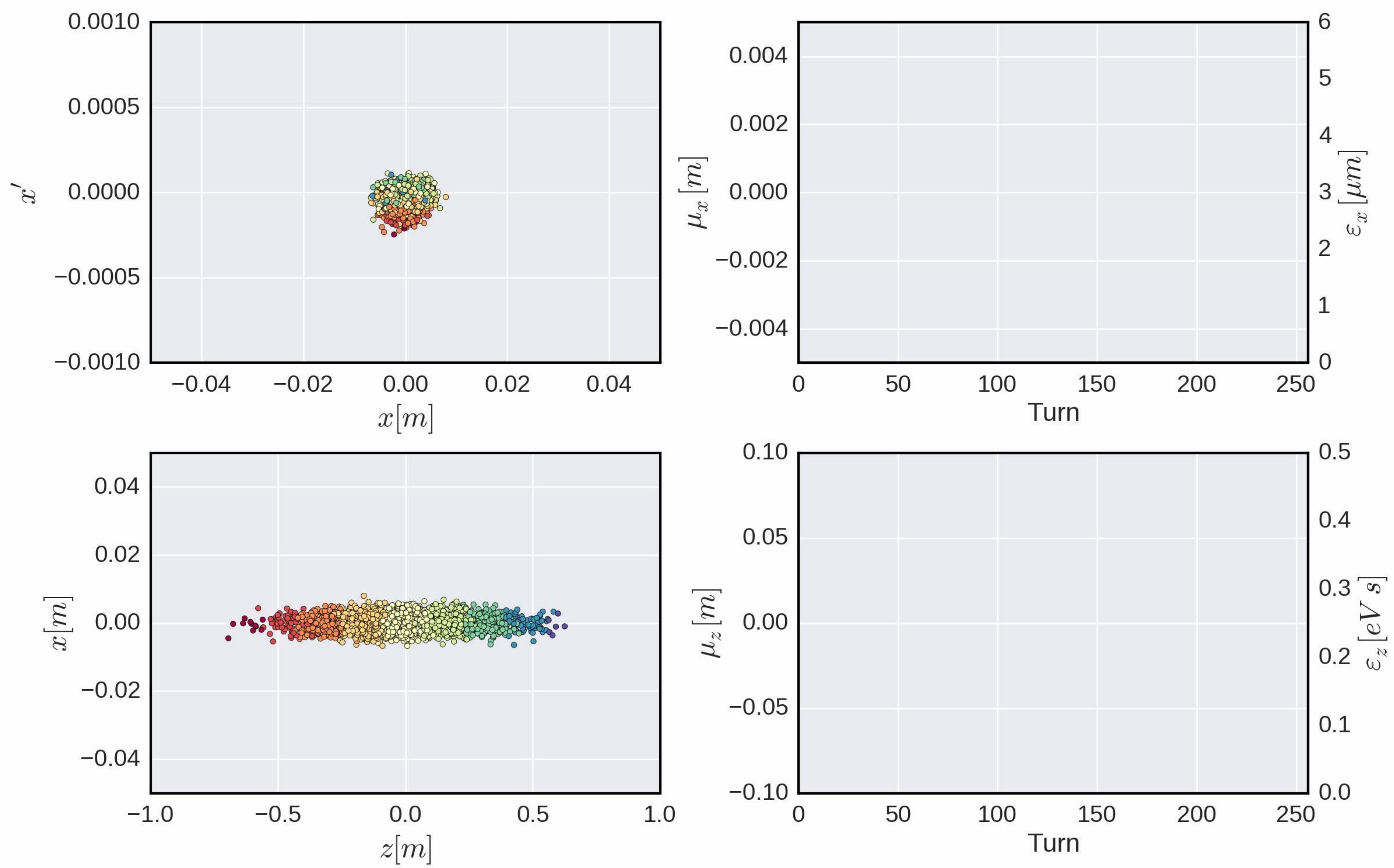
Examples – constant wakes



Examples – constant wakes w/o synchrotron motion



Examples – constant wakes with synchrotron motion



Signpost

We have seen how the impact of **wake fields on charged particle beams** can be **implemented numerically** in an efficient manner via **the longitudinal discretization** of bunches.

We have used the simulation models to show **orbit effects** from transverse wake fields.

We will now look at some **transverse instabilities.**

Part 3: Multiparticle dynamics with wake fields – their different types and impact on transverse beam dynamics

- Transverse wake function and impedance
- Effect on a bunch and transverse „potential well distortion“
- Some examples of beam instabilities

Effect of dipole wakes on a particle bunch

$$\Delta x'[i] = \underbrace{-\frac{e^2}{m\gamma\beta^2c^2\,C}\sum_{j=0}^{i}\boxed{N[j]\,\langle x\rangle[j]\cdot W_{Dx}[i-j]}}_{\text{Dipolar term} \rightarrow \text{orbit kick}}$$

Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the **beam is unstable** → beam break-up in linacs, instabilities much faster than synchrotron motion, e.g. close to transition crossing

Effect of dipole wakes on a particle bunch

The CERN Accelerator School

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we can get into a loop

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- With synchrotron motion:
  - Chromaticity = 0
    - Modes related to longitudinal motion appear in transverse motion
    - Existence of an instability threshold
  - Chromaticity ≠ 0
    - **Headtail modes** → beam is unstable (can be very weak and often damped by non-linearities)

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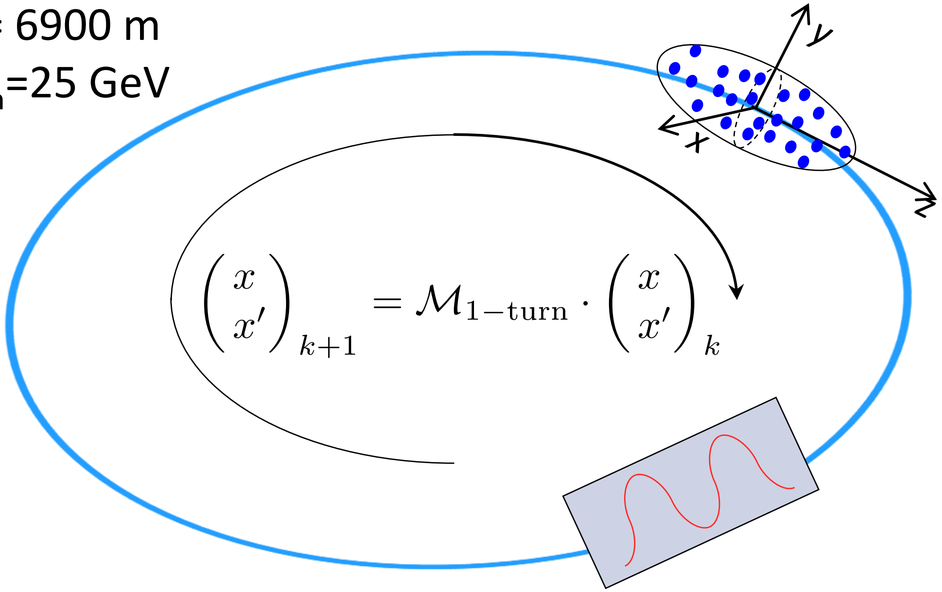

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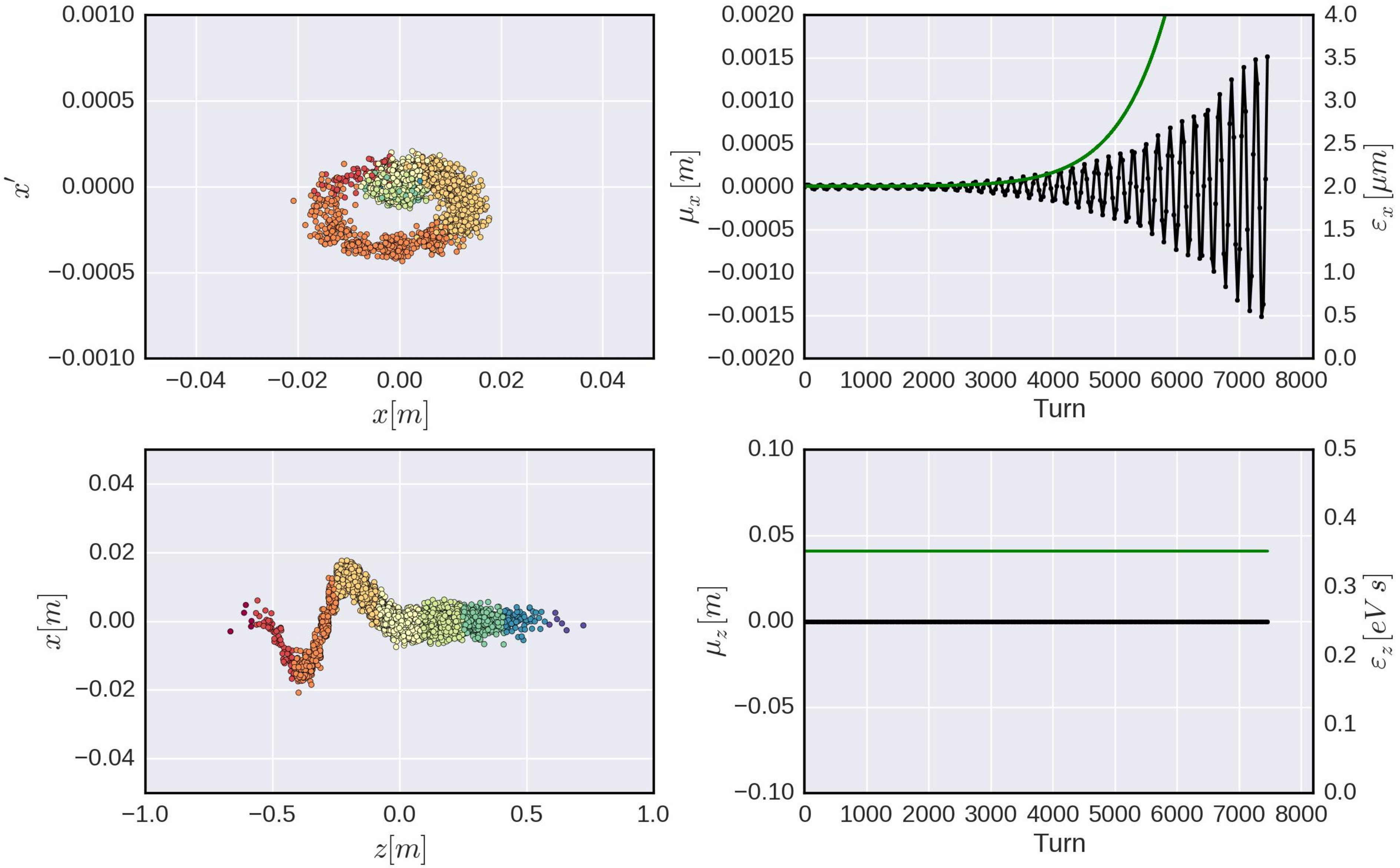
Single Gaussian bunch
σ_z = 0.2 m (0.67 ns)

Dipole horizontal wake in the form of broad-band resonator

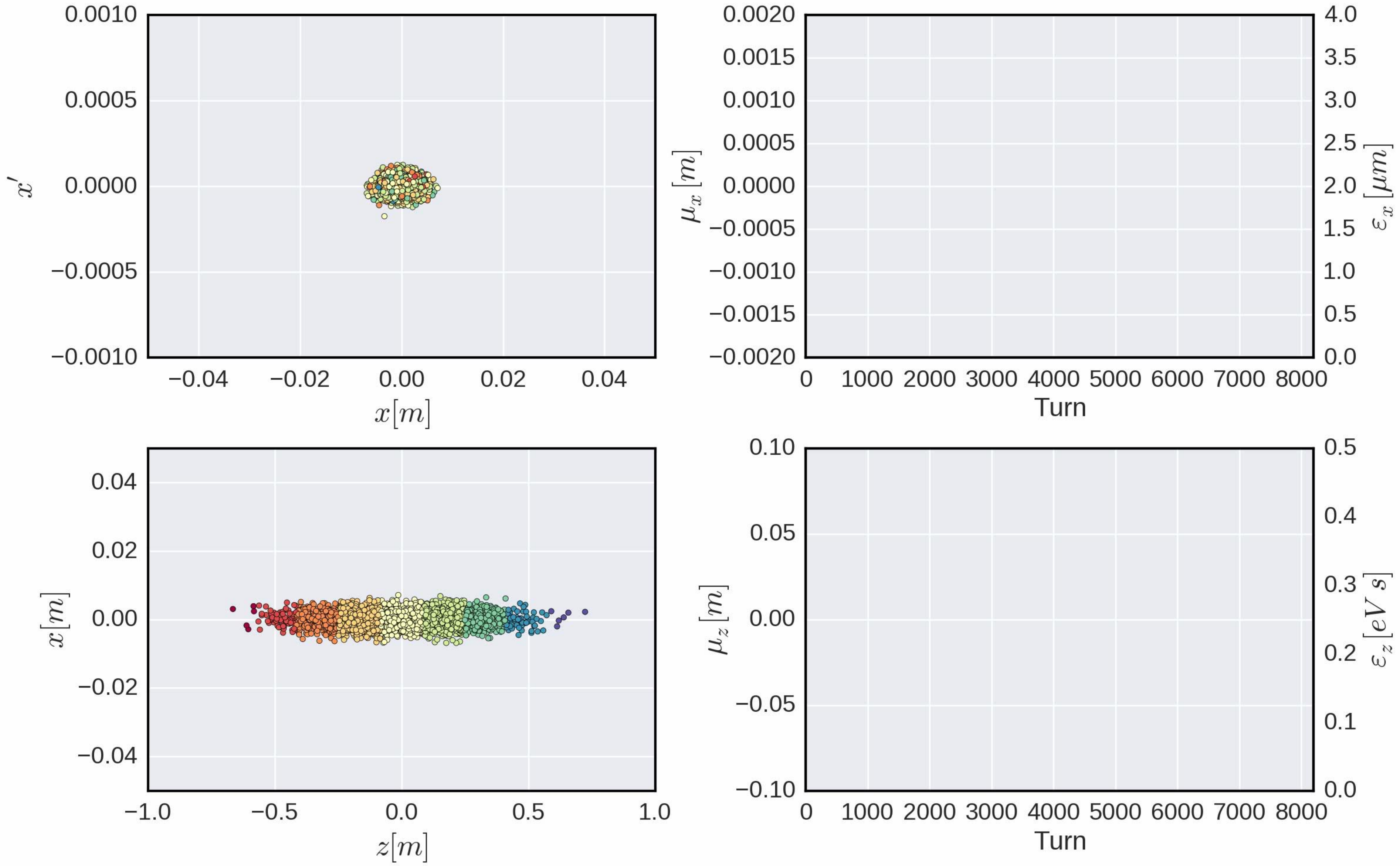
Frozen longitudinal motion or crossing transition ($\eta \approx 0$)

$$\Delta x' = -\frac{e^2}{\beta^2 E_0} \int \lambda(z') \langle x \rangle (z') W_{Dx}^{\mathrm{Ring}}(z - z') dz'$$

Dipole wakes – beam break-up

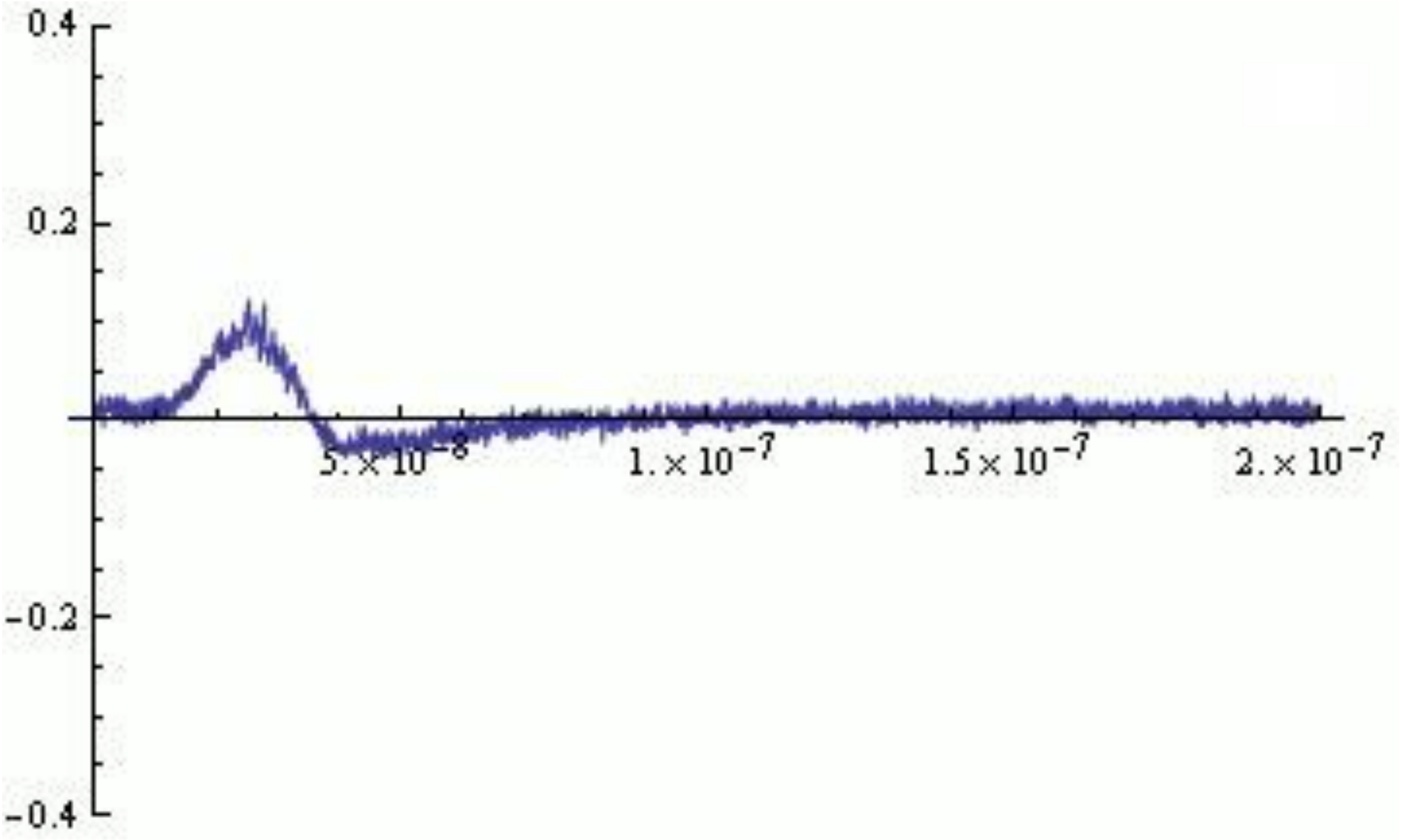


Dipole wakes – beam break-up



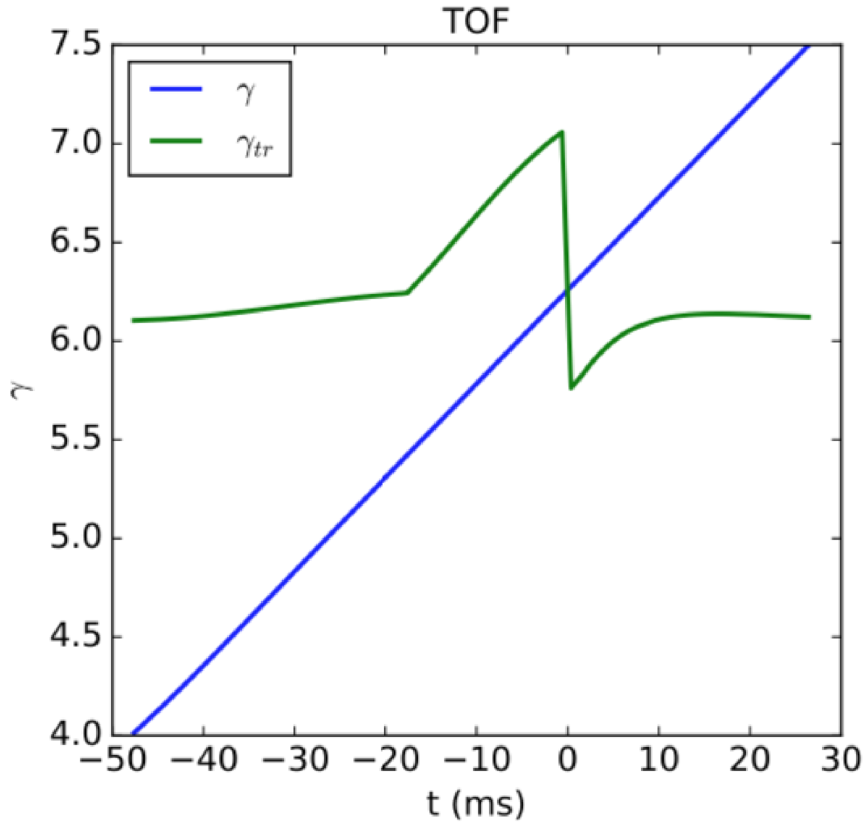
Measurement at CERN PS

- Beam break up type instabilities have been seen in the CERN PS when crossing transition with high intensity beams



Measurement at CERN PS

- Beam break up type instabilities have been seen in the CERN PS when crossing transition with high intensity beams
- To increase the intensity reach, it is necessary to cross transition more quickly, gamma jump scheme implemented



Effect of dipole wakes on a particle bunch

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$$\Delta x'[i] = \underbrace{-\frac{e^2}{m\gamma\beta^2c^2\,C}\sum_{j=0}^{i}\boxed{N[j]\,\langle x\rangle[j]\cdot W_{Dx}[i-j]}}_{}$$

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Offset dependent orbit kick
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Effect of a transverse impedance on a bunch

SPS ring
C = 6900 m
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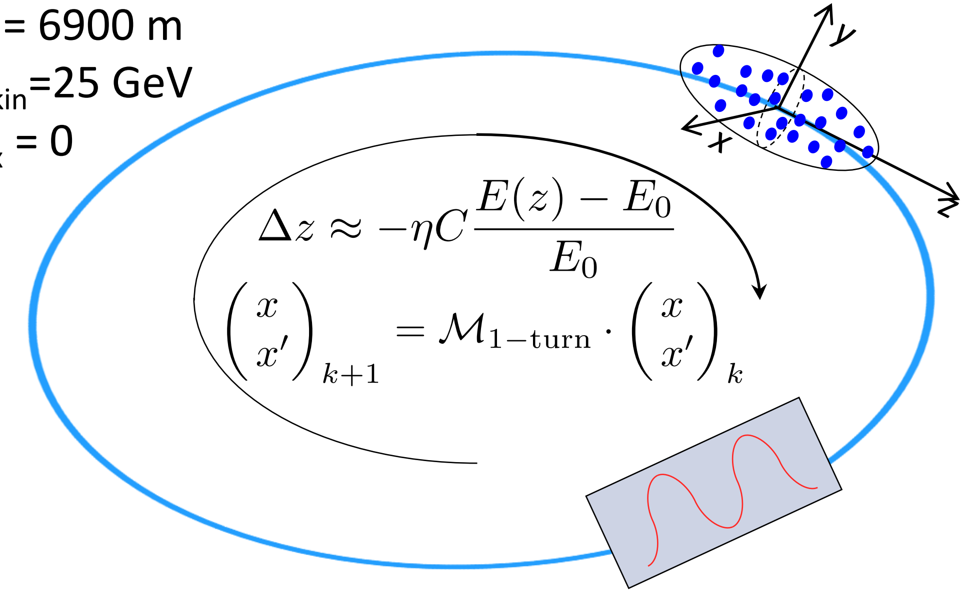
ξ_x = 0



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σ_z = 0.2 m (0.67 ns)

Dipole horizontal wake in the form of broad-band resonator

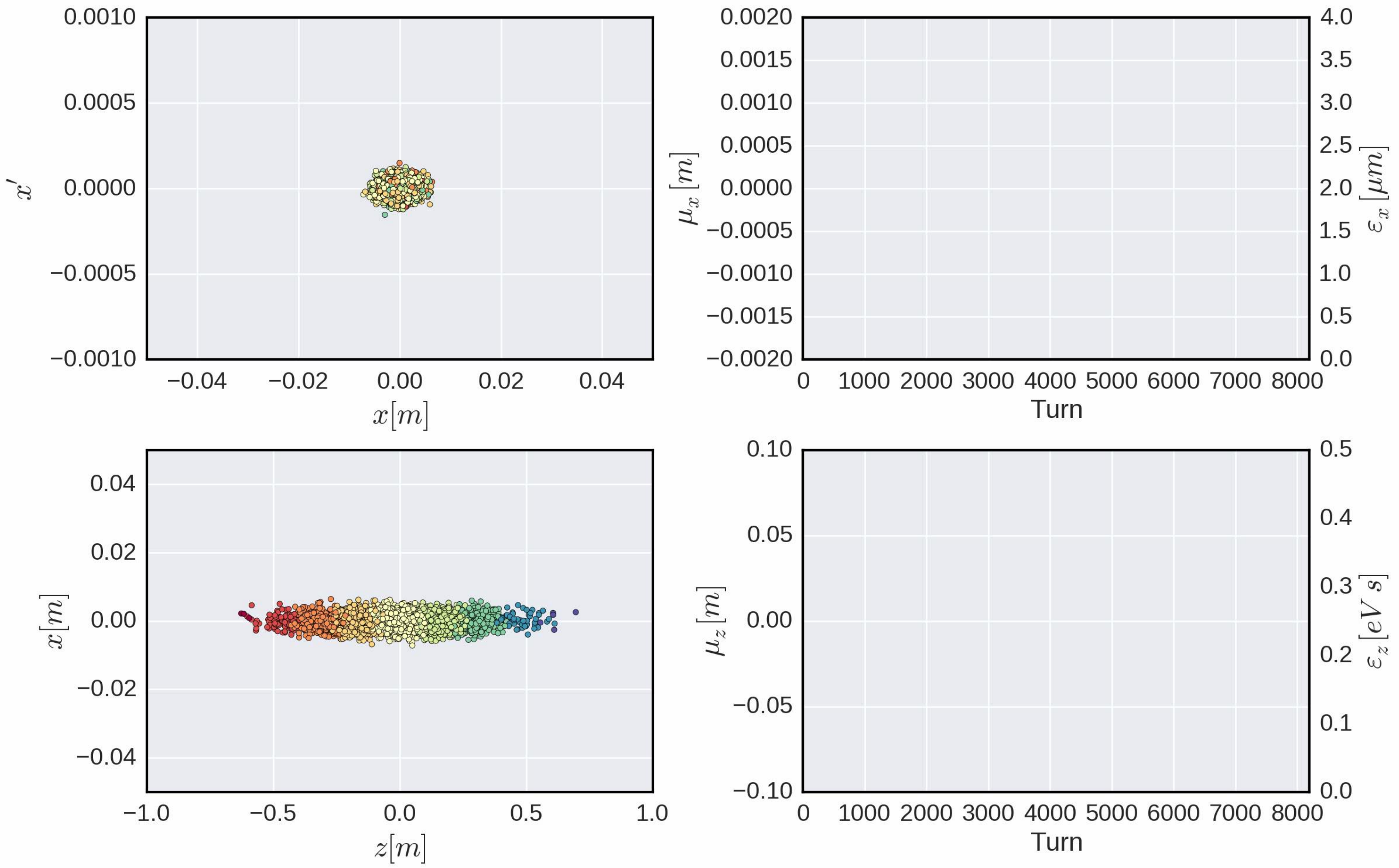
Single RF system
ω_{rf} = 200 MHz
V_{rf}^{max} = 3 MV

$$\Delta x' = -\frac{e^2}{\beta^2 E_0} \int \lambda(z') \langle x \rangle (z') W_{Dx}^{\mathrm{Ring}}(z-z') dz'$$

$$\Delta E = e V_{\mathrm{rf}}(z)$$

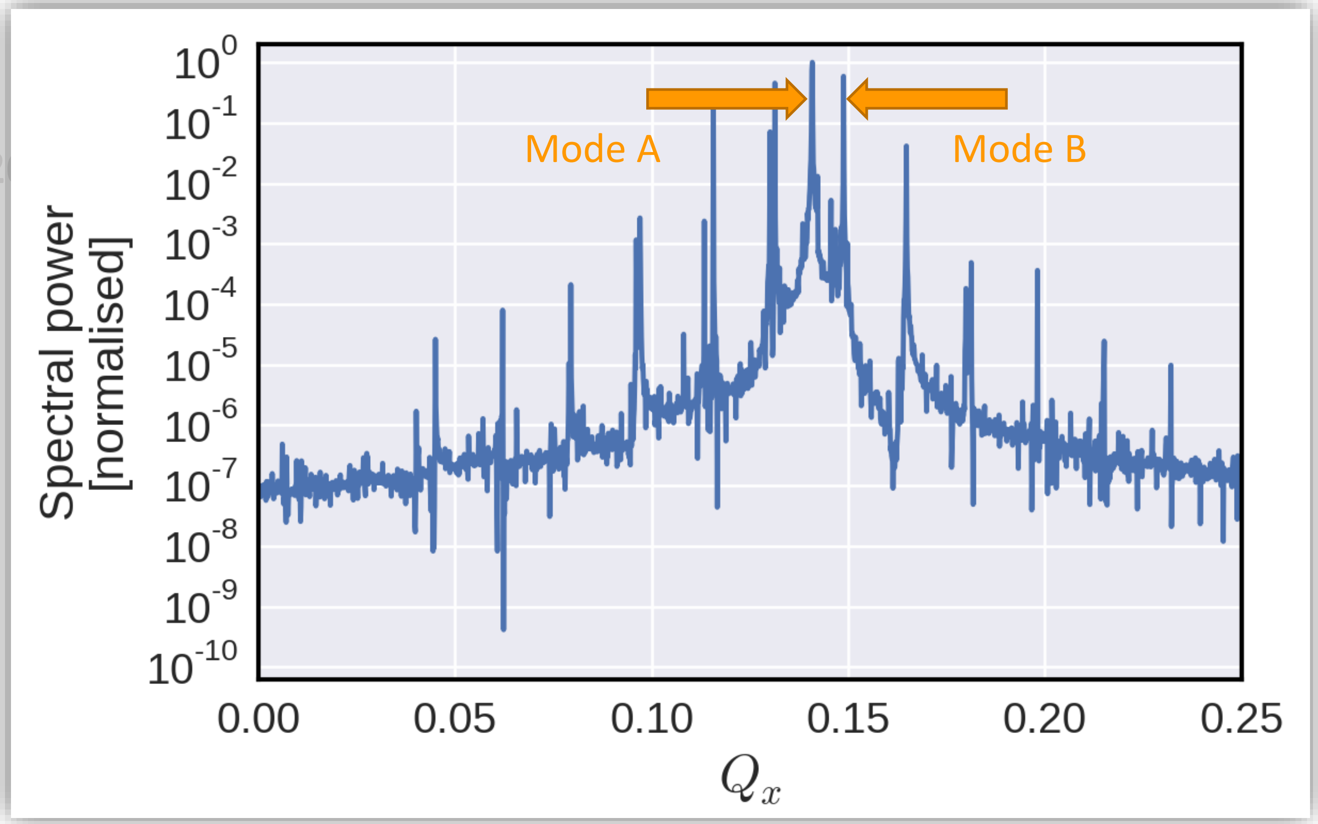
Dipole wakes – below instability threshold

- Bunch is stable up to a certain intensity ($N_b < N_{thr}$)



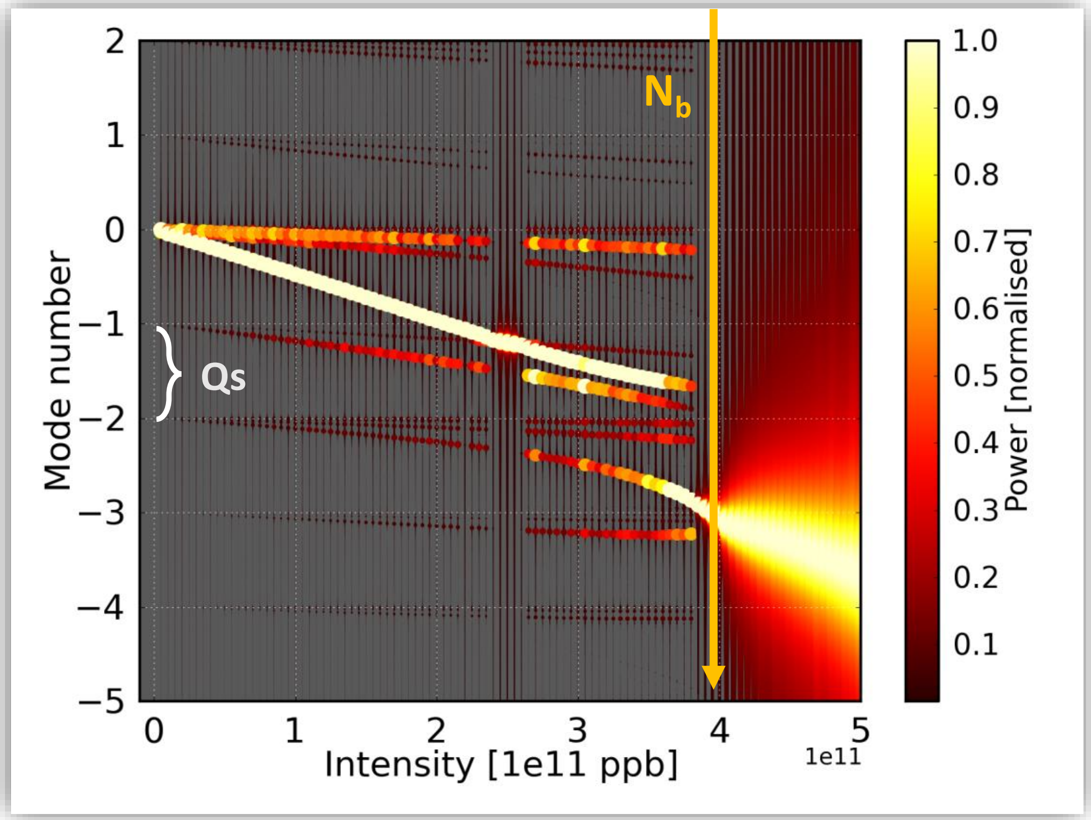
Coherent modes of the bunch

- Bunch is stable up to a certain intensity ($N_b < N_{thr}$)
- Fourier analysis of bunch centroid reveals the existence of many modes



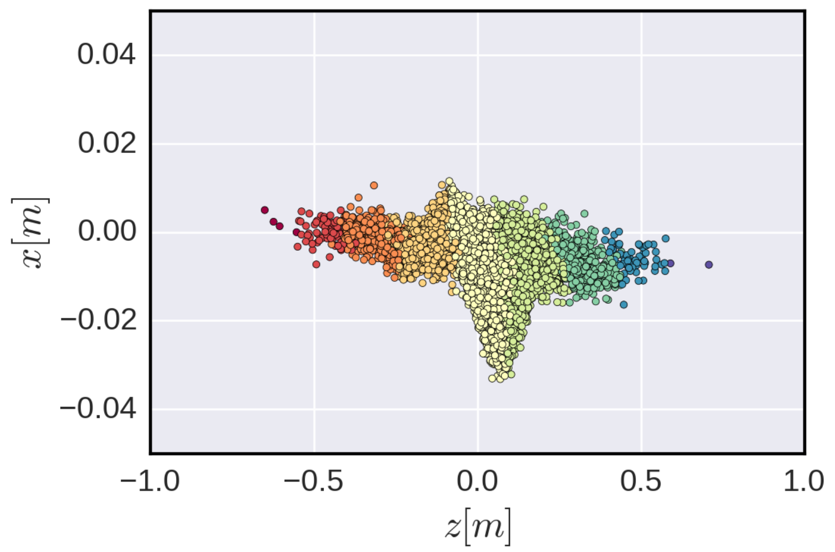
Coherent modes of the bunch

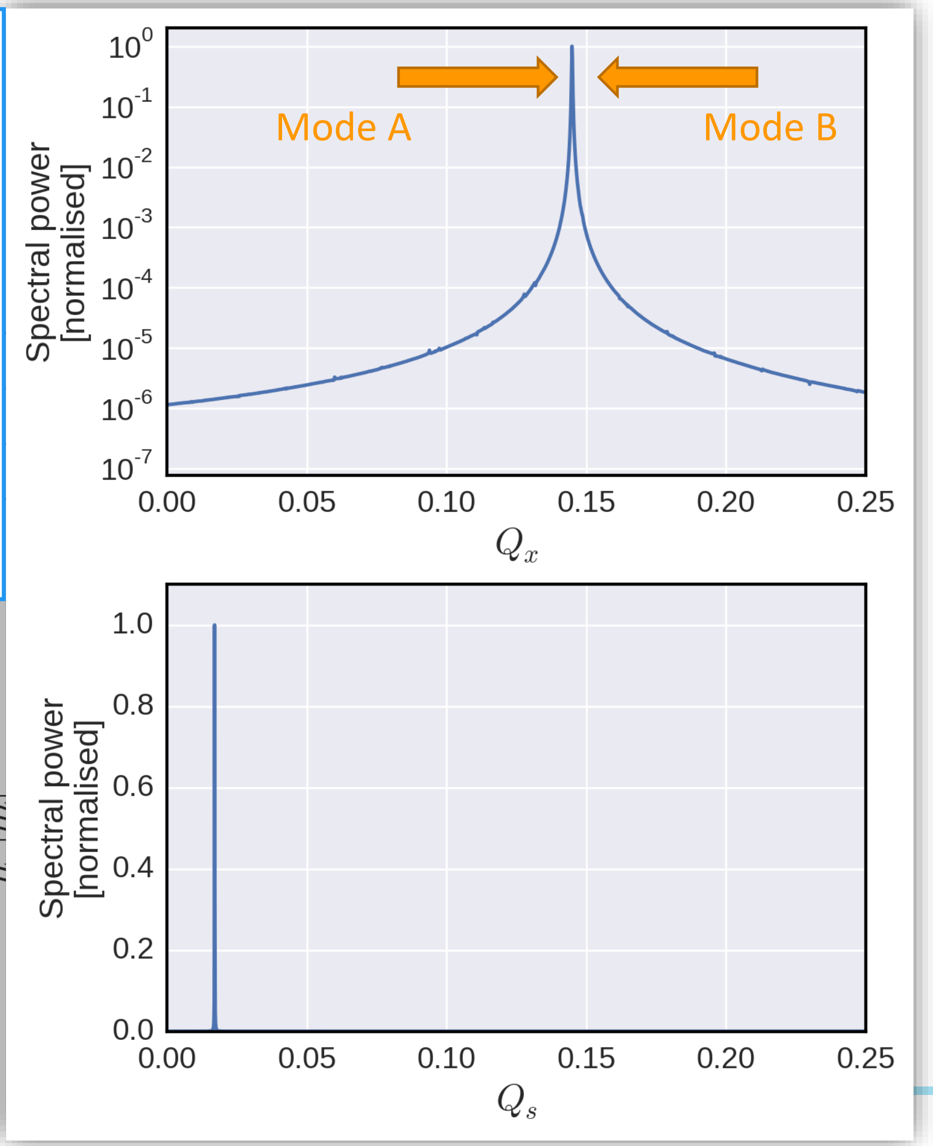
- Bunch is stable up to a certain intensity ($N_b < N_{thr}$)
- Fourier analysis of bunch centroid reveals the existence of many modes
  - Separated by ω_s at very low intensity
  - Shifting closer to each other for increasing intensity and eventually merging



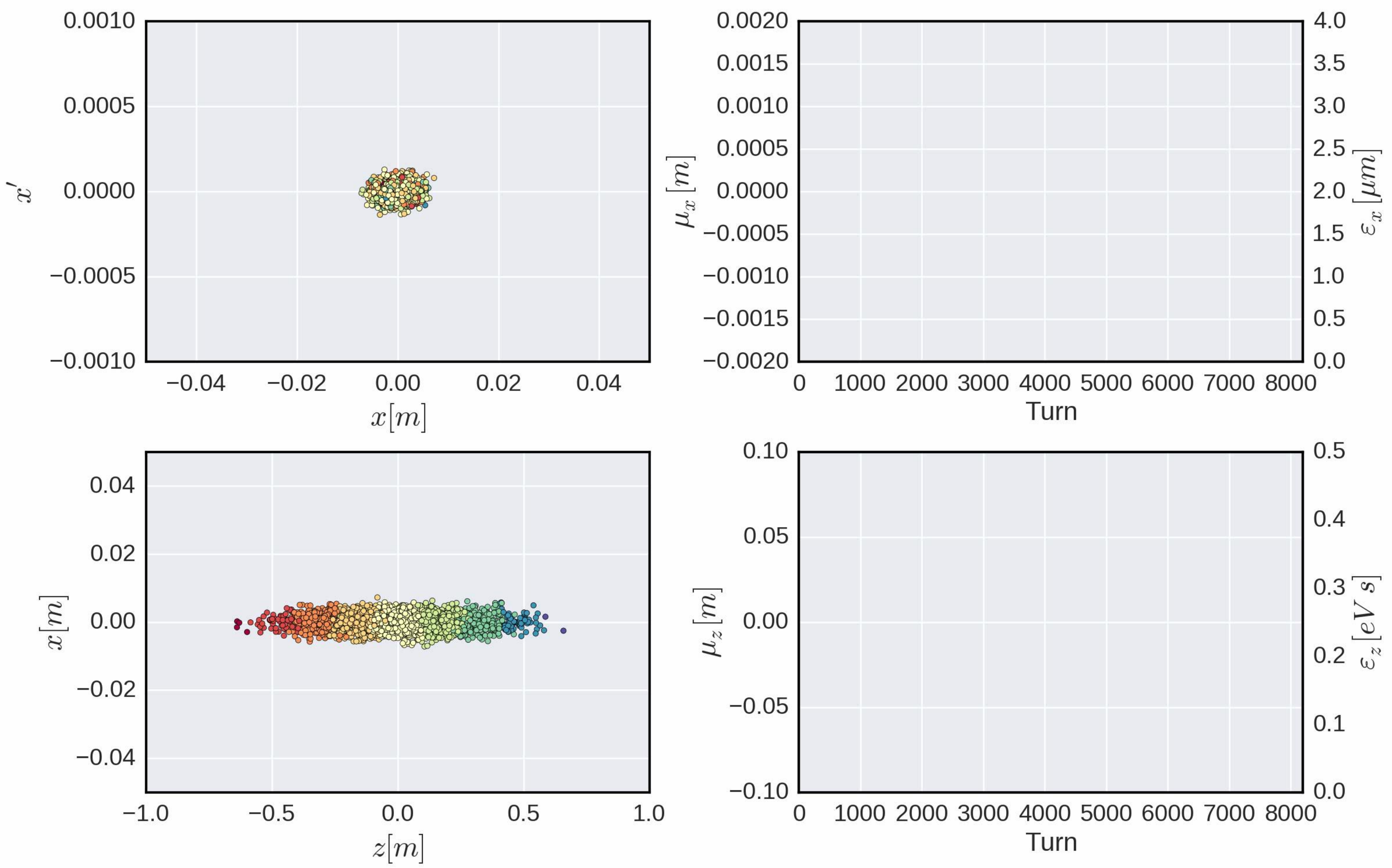
Dipole wakes – above instability threshold

When the two modes merge a fast coherent instability arises – the transverse mode coupling instability (TMCI) which often is a hard intensity limit in many machines





Dipole wakes – above instability threshold



Typical mode shift patterns

- Modes exhibit a complicated shift pattern depending on the bunch parameters
- The shift of the modes can be calculated via Vlasov equation, analytical expression available for low intensity

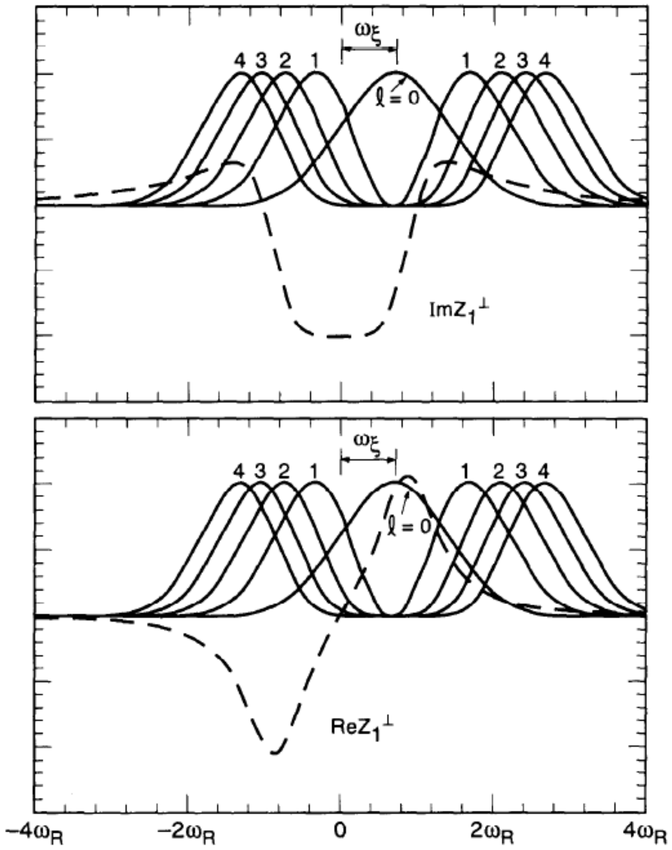
$$\Omega^{(l)} - \omega_\beta - l\omega_s \approx -\frac{i}{4\pi}\frac{\Gamma(l+\frac{1}{2})}{2^l l!}\frac{Ne^2\bar{\beta}_{x,y}}{m_0\gamma C\sigma_z}\frac{\sum_{p=-\infty}^{\infty} Z_1^\perp(\omega')h_l(\omega'-\omega_\xi)}{\sum_{p=-\infty}^{\infty} h_l(\omega'-\omega_\xi)}$$

$$\omega' = p\omega_0 + \omega_{\beta x,y} + l\omega_s$$

$$\omega_\xi = \frac{\xi_{x,y}\omega_{\beta x,y}}{\eta}$$

$$h_l(\omega) = \frac{\left[J_{l+1/2}(\omega\hat{z}/c)\right]^2}{|\omega\hat{z}/c|} \qquad \text{parabolic}$$

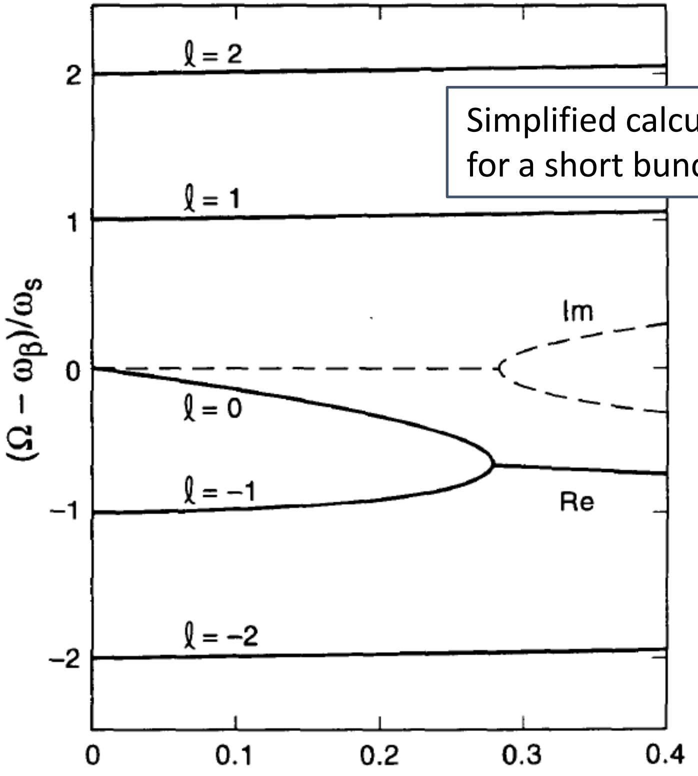
$$h_l(\omega) = \left(\frac{\omega\sigma_z}{c}\right)^{2l}\exp\left(-\frac{\omega^2\sigma_z^2}{c^2}\right) \qquad \text{Gaussian}$$



Spectra of $h_l(\omega - \omega_\xi)$ and real + imaginary part of a broad-band impedance

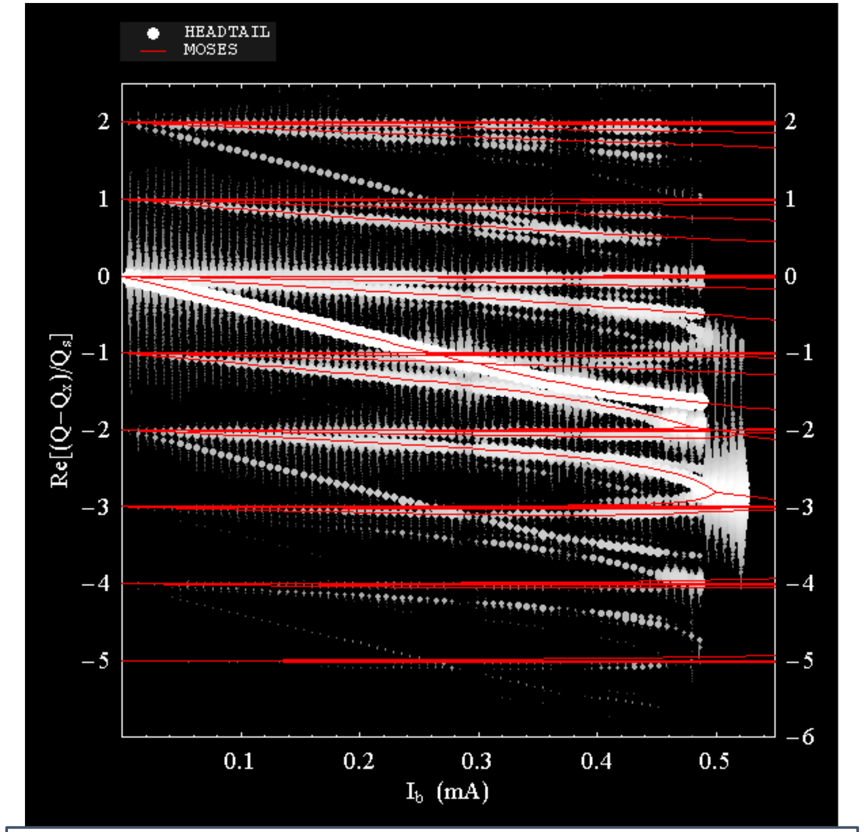
Typical mode shift patterns

- Modes exhibit a complicated shift pattern depending on the bunch parameters
- The shift of the modes can be calculated via Vlasov equation



Simplified calculation for a short bunch

Rough criterion for instability threshold is when $\Delta\omega_\beta|_{l=0} = \omega_s/2$



Full calculation for a long SPS bunch (red lines) + macroparticle simulation (white traces)

Effect of dipole wakes on a particle bunch

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$$\Delta x'[i] = \underbrace{-\frac{e^2}{m\gamma\beta^2c^2\,C}\sum_{j=0}^{i}\boxed{N[j]\,\langle x\rangle[j]\cdot W_{Dx}[i-j]}}$$

Dipolar term → orbit kick

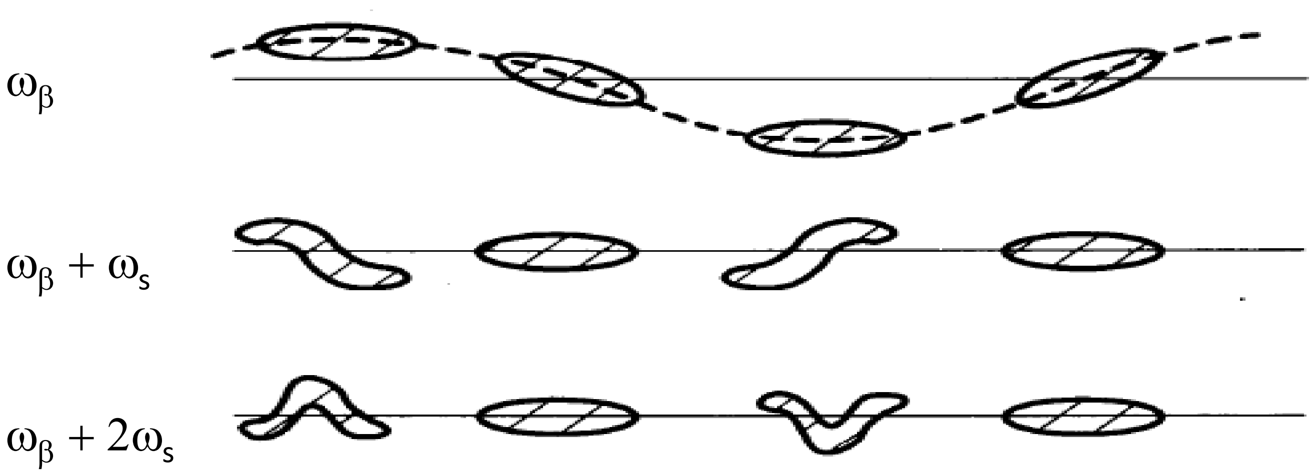
Offset dependent orbit kick
→ kicks can accumulate

With synchrotron motion
we can get into a loop

- Without synchrotron motion:
  kicks accumulate turn after turn – the **beam is unstable** → beam break-up in linacs, instabilities much faster than synchrotron motion, e.g. close to transition crossing
- With synchrotron motion:
  - Chromaticity = 0
    - Modes related to longitudinal motion appear in transverse motion
    - Existence of an instability threshold
  - Chromaticity ≠ 0
    - **Headtail modes** → beam is unstable (can be very weak and often damped by non-linearities)

Dipole wakes – headtail modes

- As soon as **chromaticity is non-zero**, a 'resonant' condition can be met as particles now can 'synchronize' their synchrotron amplitude dependent betatron motion with the action of the wake fields.
- **Headtail modes arise** – the order of the respective mode depends on the chromaticity together with the impedance and bunch spectrum
- Different transverse head-tail modes **correspond to different parts of the bunch** oscillating with relative phase differences, for example:
  - Mode 0 is a rigid bunch mode
  - Mode 1 has head and tail oscillating in counter-phase
  - Mode 2 has head and tail oscillating in phase and the bunch center in opposition



Dipole wakes – headtail modes

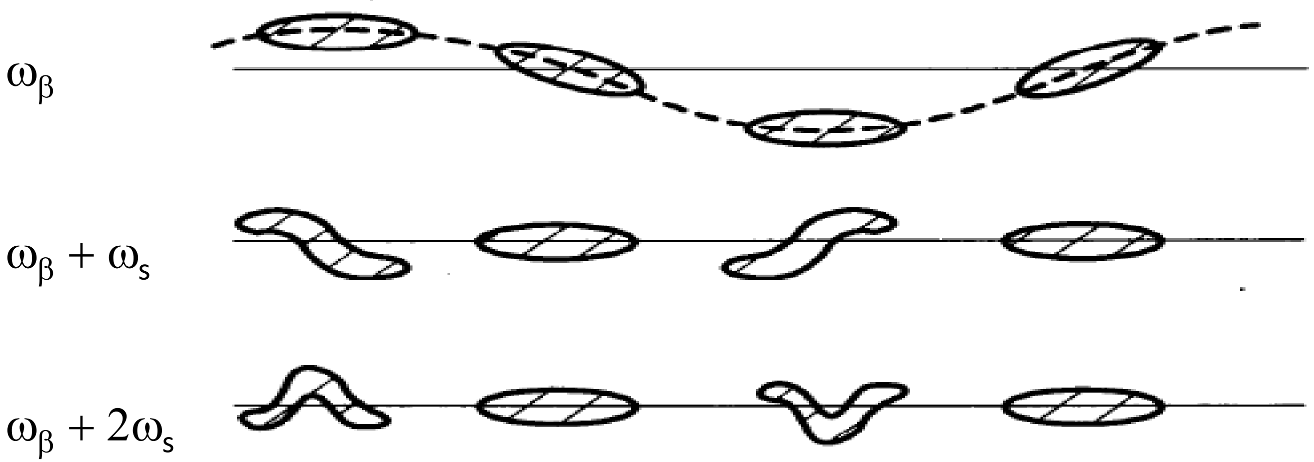
- As soon as **chr** … ‘synchronize’ their synchrot…
- **Headtail mod…** …ther with the impedance an…
- Different trans… …with relative phase differen…
    - Mode 0 is …
    - Mode 1 ha…
    - Mode 2 ha…

Remark:

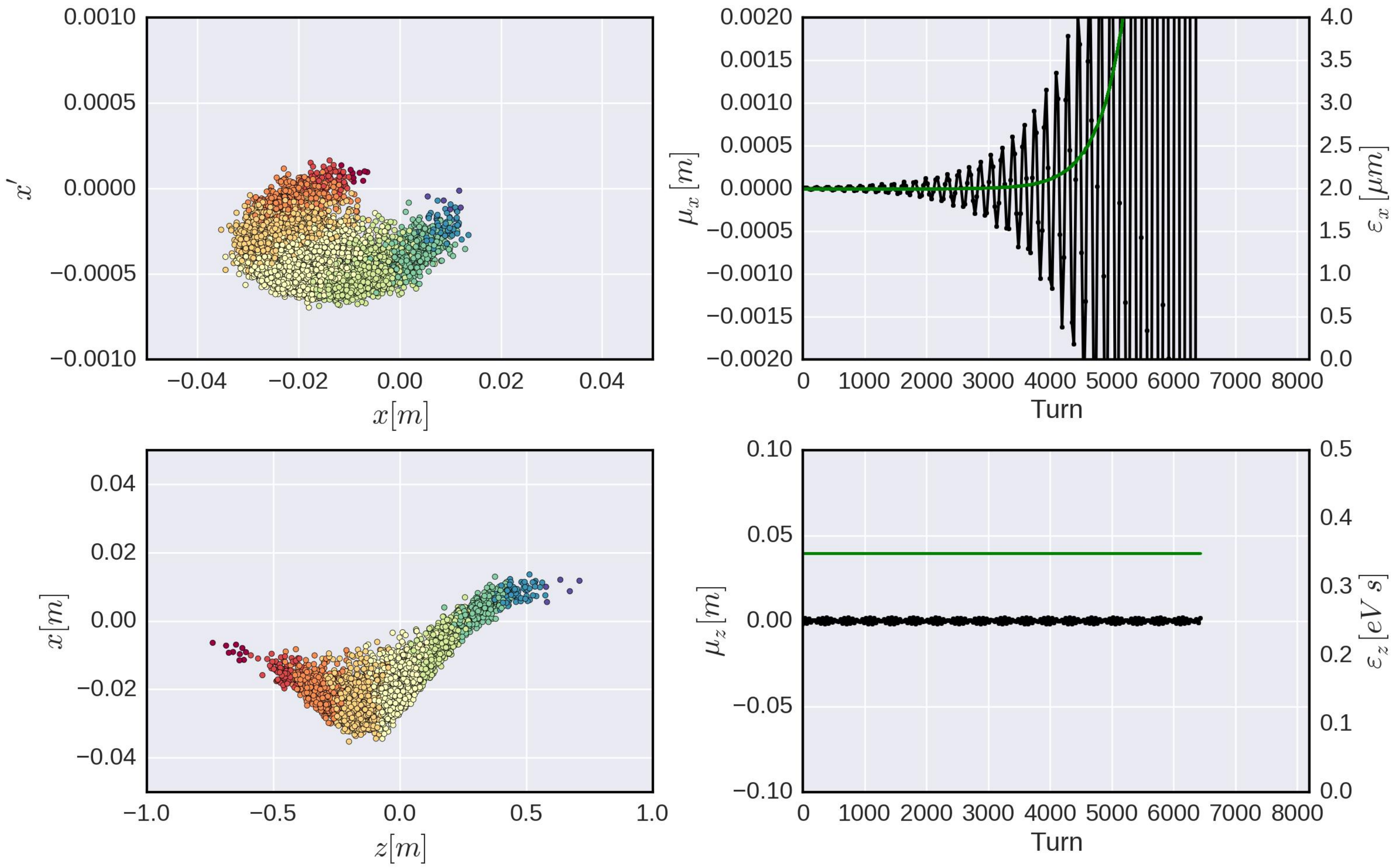
Due to this ‚synchronicity‘, **below transition** (η<0):

- Mode 0 is unstable if Q‘>0.
- Higher order modes tend to be unstable if Q‘<0 (though at lower growth rates).

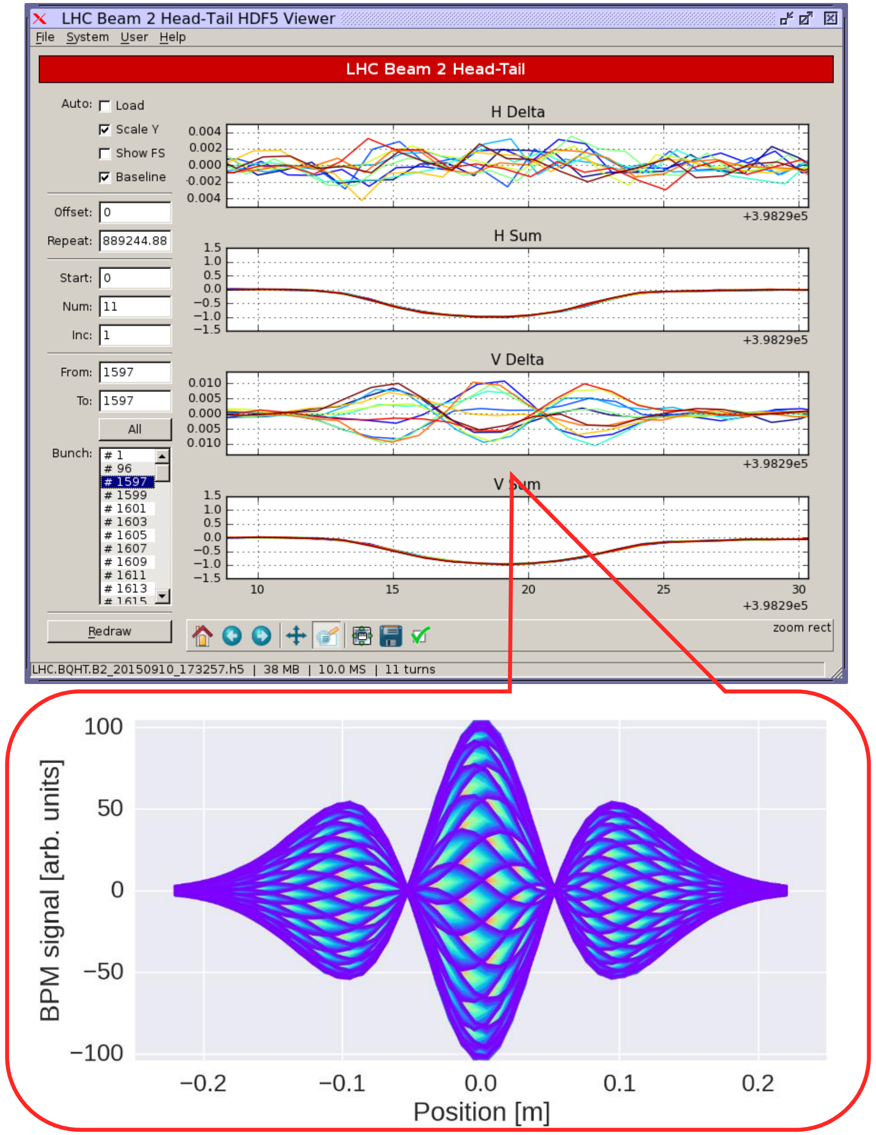
The **situation is reversed** when a machine is operated above transition.

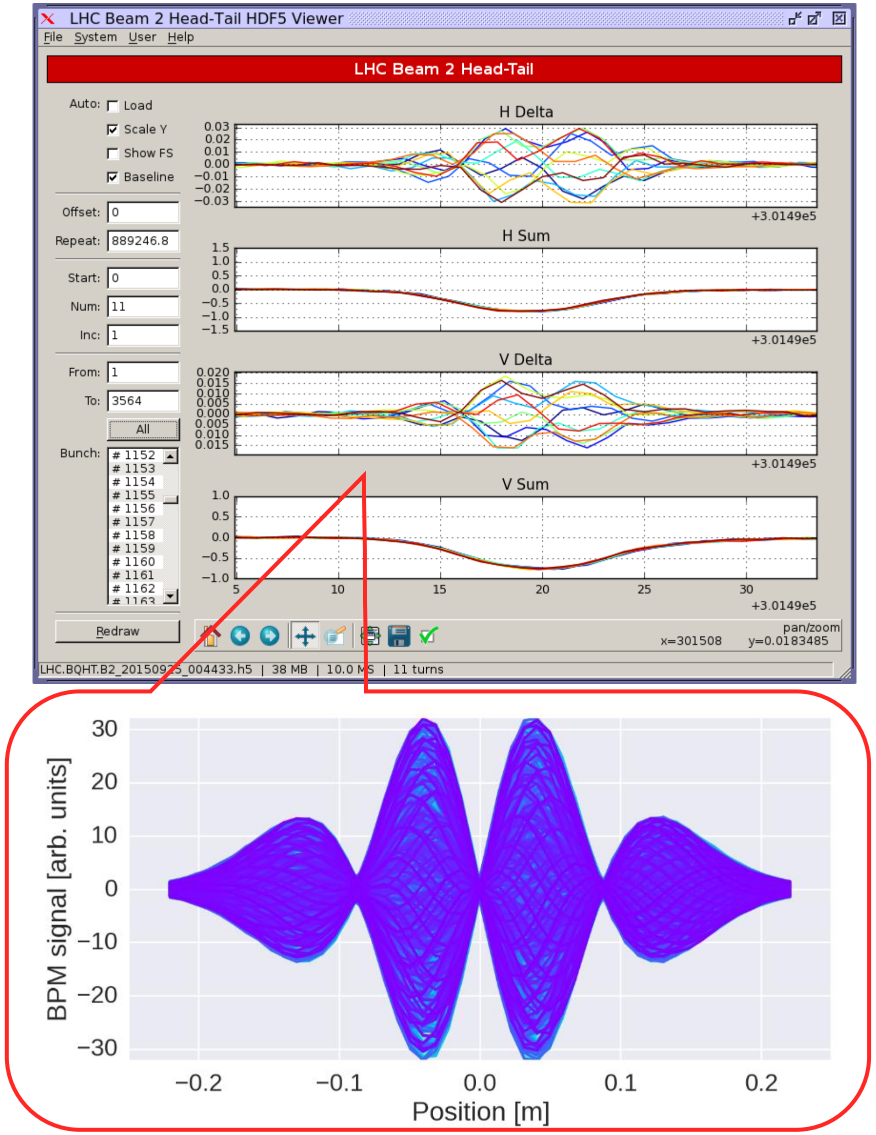


Dipole wakes – headtail modes



Example: Headtail modes in the LHC





Signpost

We have **discussed transverse wake fields** and impedances, their classification into different types along with their impact on the beam dynamics.

We have modeled the **wake field interaction** with a charged particle beam.

We have seen some examples of the effects of transverse wake fields on the beam such as

- **Closed orbit distortion**
- Some types of **transverse beam instability**

Tomorrow Part 4

→ Electron cloud build up and effects on beam dynamics

End part 3

CERN

The Strong Head Tail Instability

- Aka the Transverse Mode Coupling Instability:
  - To illustrate TMCI we will need to make use of **some simplifications**:
    - The bunch **is represented through two particles** carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
    - They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
    - Zero chromaticity ($Q'_{x,y}=0$)
    - Constant transverse wake left behind by the leading particle
    - Smooth approximation → constant focusing + distributed wake

We will:

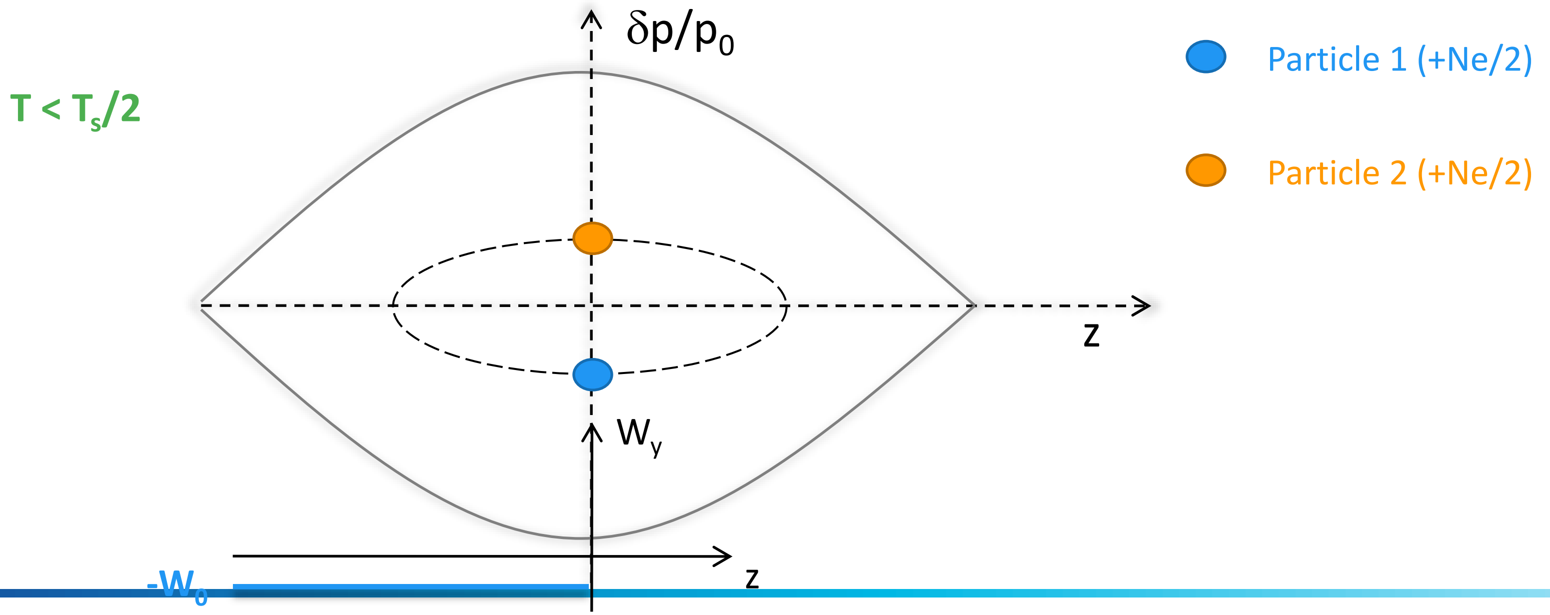
- Calculate a stability condition (threshold) for the transverse motion
- Have a look at the excited oscillation modes of the centroid

The Strong Head Tail Instability

- During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

$$\begin{cases} \dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = 0 \\ \dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N W_0}{2\gamma C} y_1(s) \end{cases}$$

$$0 < s < \frac{\pi c}{\omega_s}$$

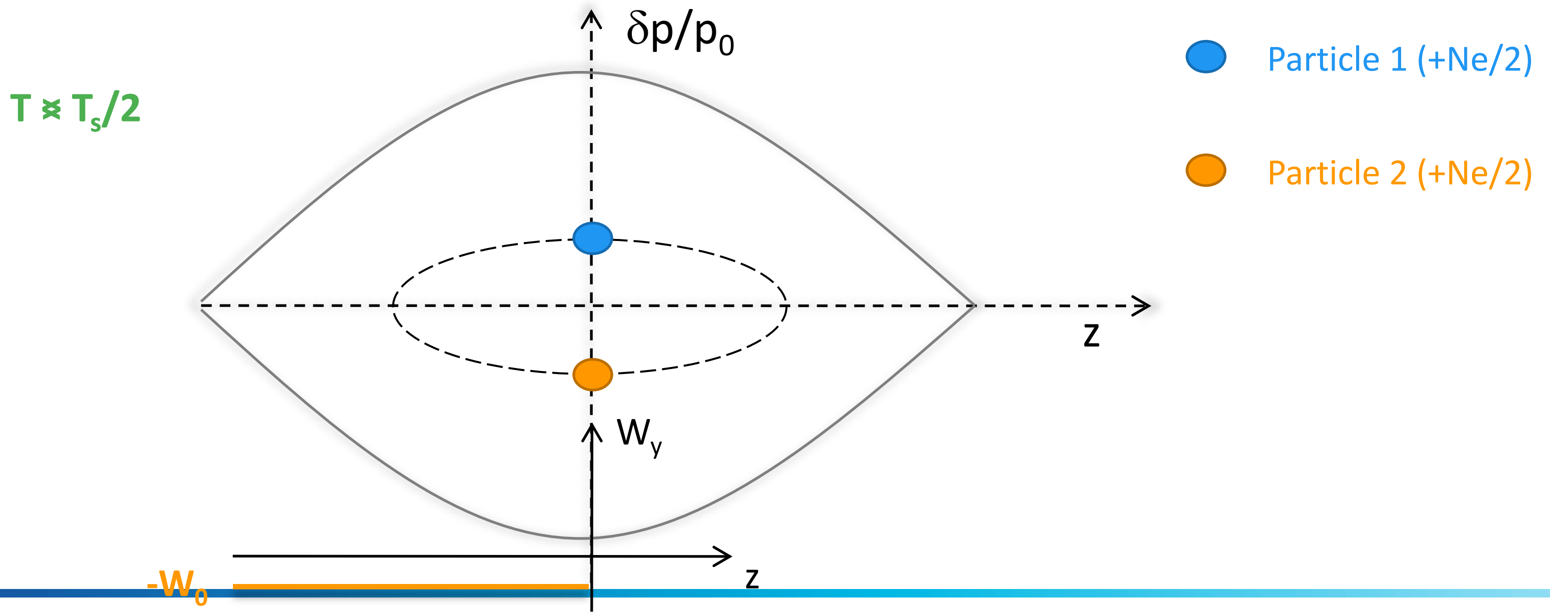


The Strong Head Tail Instability

- During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1
- During the second half of the synchrotron period, the situation is reversed:

$$\begin{cases} \dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N\,W_0}{2\gamma C}\, y_2(s) \\ \dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = 0 \end{cases}$$

$$\frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}$$



The Strong Head Tail Instability

- We solve with respect to the complex variables defined below during the first half of synchrotron period
- y1(s) is a free betatron oscillation
- y2(s) is the sum of a free betatron oscillation plus a driven oscillation with y1(s) being its driving term

$$\tilde{y}_{1,2}(s) = y_{1,2}(s) + i\frac{c}{\omega_\beta}\, y'_{1,2}(s)$$

$$\tilde{y}_1(s) = \tilde{y}_1(0)\,\exp\left(-\frac{i\omega_\beta s}{c}\right)$$

$$\tilde{y}_2(s) = \underbrace{\tilde{y}_2(0)\,\exp\left(-\frac{i\omega_\beta s}{c}\right)}_{\text{Free oscillation term}}$$

since we consider $s = \dfrac{\pi c}{\omega_s}$

$$\underbrace{+\,i\frac{Ne^2W_0}{4\,m_0\gamma c C\omega_\beta}\left(\cancel{\frac{c}{\omega_\beta}\,\tilde{y}_1^*(0)\,\sin\left(\frac{\omega_\beta s}{c}\right)} + \tilde{y}_1(0)\,s\,\exp\left(-\frac{i\omega_\beta s}{c}\right)\right)}_{\text{Driven oscillation term}}$$

- Second term in RHS equation for y2(s) negligible if $\omega_s << \omega_\beta$
- We can now transform these equations into linear mapping across half synchrotron period

The Strong Head Tail Instability

- We can now transform these equations into **linear mapping** across half synchrotron period

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \left[\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}, \qquad \Upsilon = \frac{\pi N e^2 W_0}{4\, m_0\gamma\, C\omega_\beta\omega_s}$$

- In the second half of synchrotron period, **particles 1 and 2 exchange their roles** – we can therefore find the transfer matrix over the full synchrotron period for both particles. We can **analyze the eigenvalues** of the two particle system

$$\begin{aligned}\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=2\pi c/\omega_s} &= \left[\exp\left(-\frac{i\,2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & i\Upsilon\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}\\ &= \left[\exp\left(-\frac{i\,2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1-\Upsilon^2 & i\Upsilon\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}\end{aligned}$$

Strong Head Tail Instability – stability condition

$$\lambda_1 \cdot \lambda_2 = 1 \Rightarrow \lambda_{1,2} = \exp\left(\pm i\varphi\right)$$

$$\lambda_1 + \lambda_2 = 2 - \Upsilon^2 \Rightarrow \sin\left(\frac{\varphi}{2}\right) = \frac{\Upsilon}{2}$$

$$\boxed{\Rightarrow \Upsilon = \frac{\pi N e^2 W_0}{4\, m_0 \gamma\, C \omega_\beta \omega_s} \leq 2}$$

- Since the product of the eigenvalues is 1, the only condition for stability is that they both be purely imaginary exponentials
- From the second equation for the eigenvalues, it is clear that this is true only when sin(φ/2)<1
- This translates into a **stability condition** on the beam/wake parameters

Strong Head Tail Instability – stability condition

$$N \leq N_{\text{threshold}} = \frac{8}{\pi e^2} \boxed{\frac{p_0 \omega_s}{\beta_y}} \boxed{\left(\frac{C}{W_0}\right)}$$

- Proportional to p_0 → bunches with higher energy tend to be more stable
- Proportional to ω_s → the quicker is the longitudinal motion within the bunch, the more stable is the bunch
- Inversely proportional to β_y → the effect of the impedance is enhanced if the kick is given at a location with large beta function

- Inversely proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) lowers the instability threshold

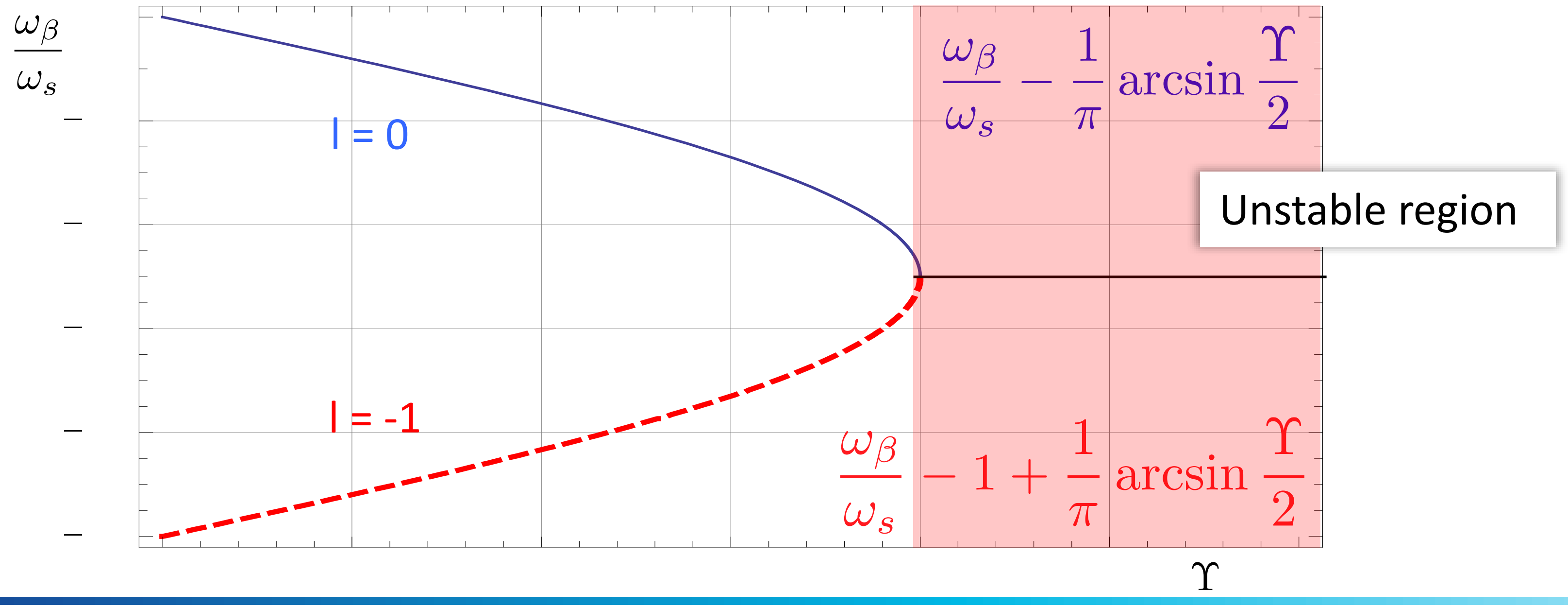
Strong Head Tail Instability – mode frequencies

- The evolution of the eigenstates follows:

$$\begin{pmatrix} \tilde{V}_{+n} \\ \tilde{V}_{-n} \end{pmatrix} = \exp\left(-i\frac{2\pi\omega_\beta}{\omega_s}n\right)\cdot\begin{pmatrix} \exp\left[-2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] & 0 \\ 0 & \exp\left[2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] \end{pmatrix}\begin{pmatrix} \tilde{V}_{+0} \\ \tilde{V}_{-0} \end{pmatrix}$$

Eigenfrequencies: $\omega_\beta + l\omega_s \pm \frac{\omega_s}{\pi}\arcsin\frac{\Upsilon}{2}$

They shift with increasing intensity



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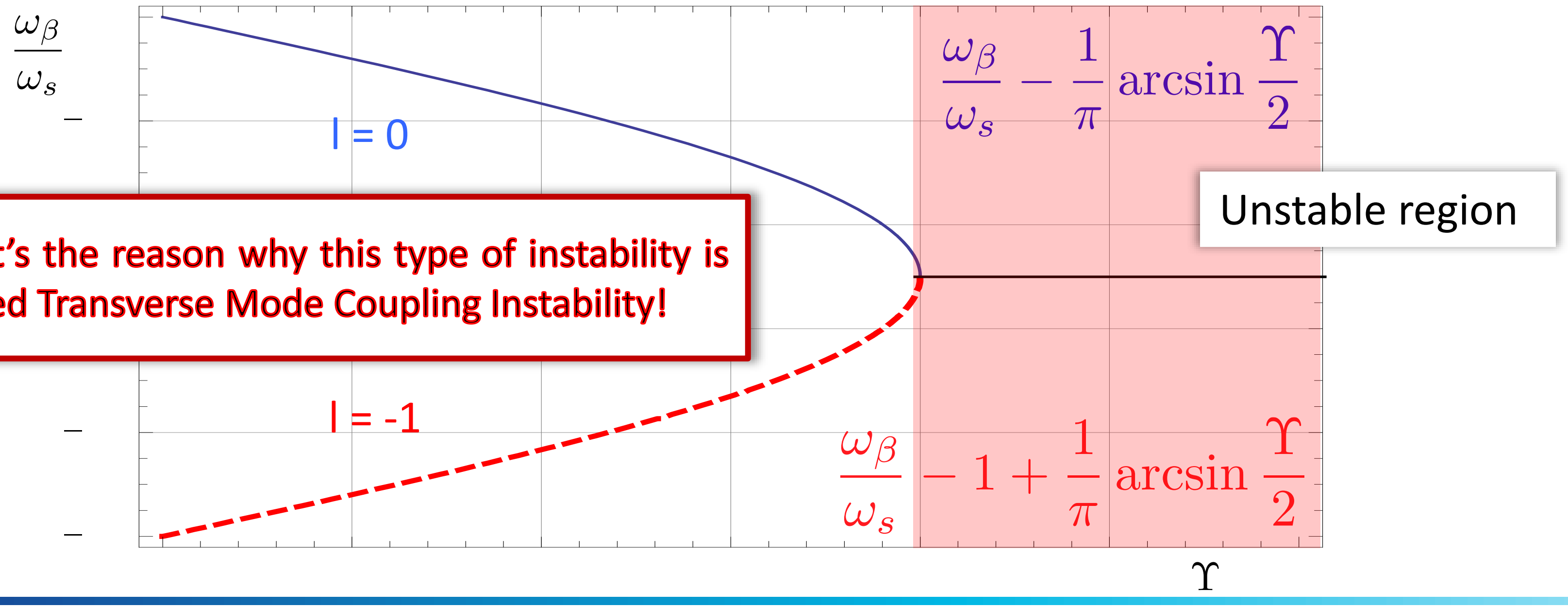
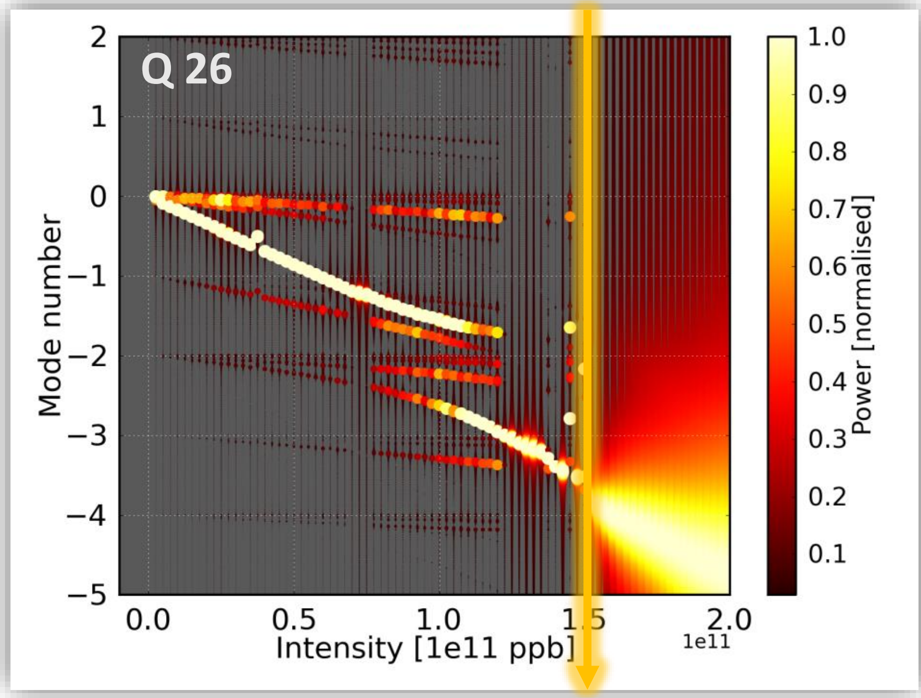


That's the reason why this type of instability is called Transverse Mode Coupling Instability!

Raising the TMCI threshold – SPS Q20 optics

- In **simulations** we have the possibility to perform **scans of variables**, e.g. we can run **100 simulations in parallel** changing the beam intensity
- We can then perform a **spectral analysis** of **each simulation**…
- … and stack all obtained plot behind one another to obtain…
- … the typical **visualization plots of TMCI**



TMCI threshold

The mode number is given as

$$m = \frac{Q_x - Q_{x0}}{Q_s}$$

The modes are separated by the synchrotron tune.

Backup

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s) \sum_{mn} x^n x_s^m\, W_{mn}(z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \sum_{mn} \int x^n \int \lambda_m(z_s)\, W_{mn}(z - z_s - kC)\, dz_s\, dx$$

$$\lambda_m(z_s) = \int \rho(x_s, z_s) x_s^m\, dx_s$$

- Expansion

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2 c^2\, C} \sum_{mn} \int x^n \int \lambda_m(z_s)\, W_{mn}(z - z_s - kC)\, dz_s\, dx$$

$$\lambda_m(z_s) = \int \rho(x_s, z_s) x_s^m \, dx_s$$

$$H = \frac{1}{2}p_x^2 + C + \boxed{A\,x} + \boxed{\frac{1}{2}B\,x^2} + \dots, \qquad \text{with } \frac{dq}{ds} = \frac{\partial H(p,q)}{\partial p}, \quad \frac{dp}{ds} = -\frac{\partial H(p,q)}{\partial q}$$

Dipole term (n=1) → change of orbit

Quadrupole term (n=2) → change of tune

- Expansion – up to second order:

n	m	type
0	0, 1	
1	0	

Constant transverse wake (n=0, m=0)

Dipole transverse wake (n=0, m=1)

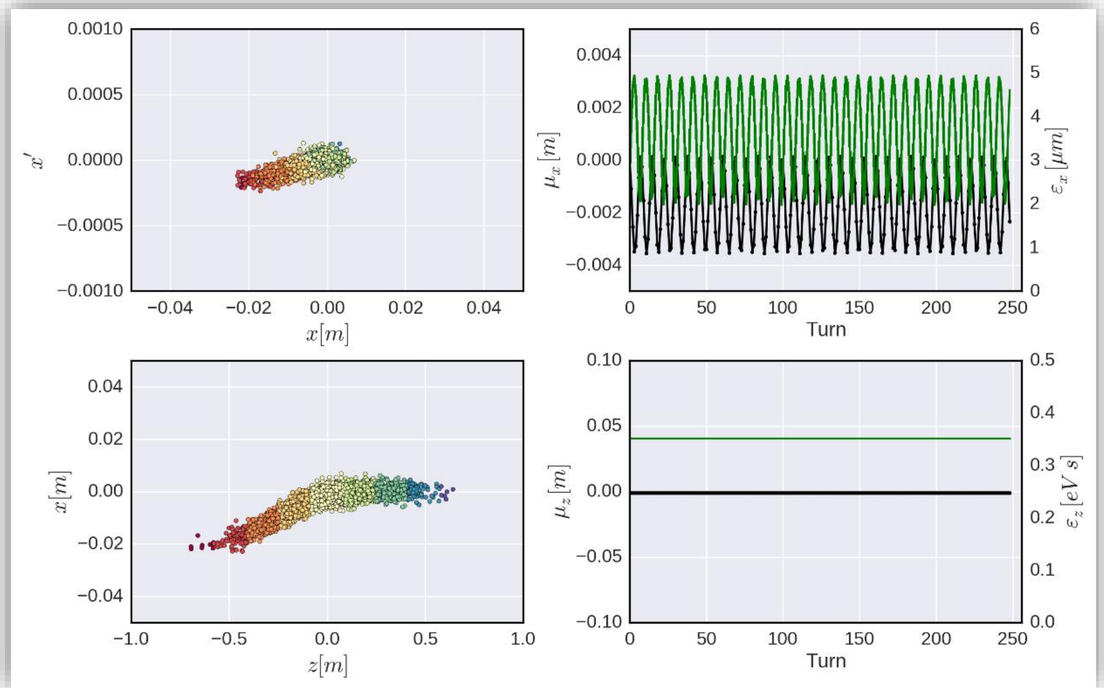
Quadrupole transverse wake (n=1, m=0)

Examples – constant wakes

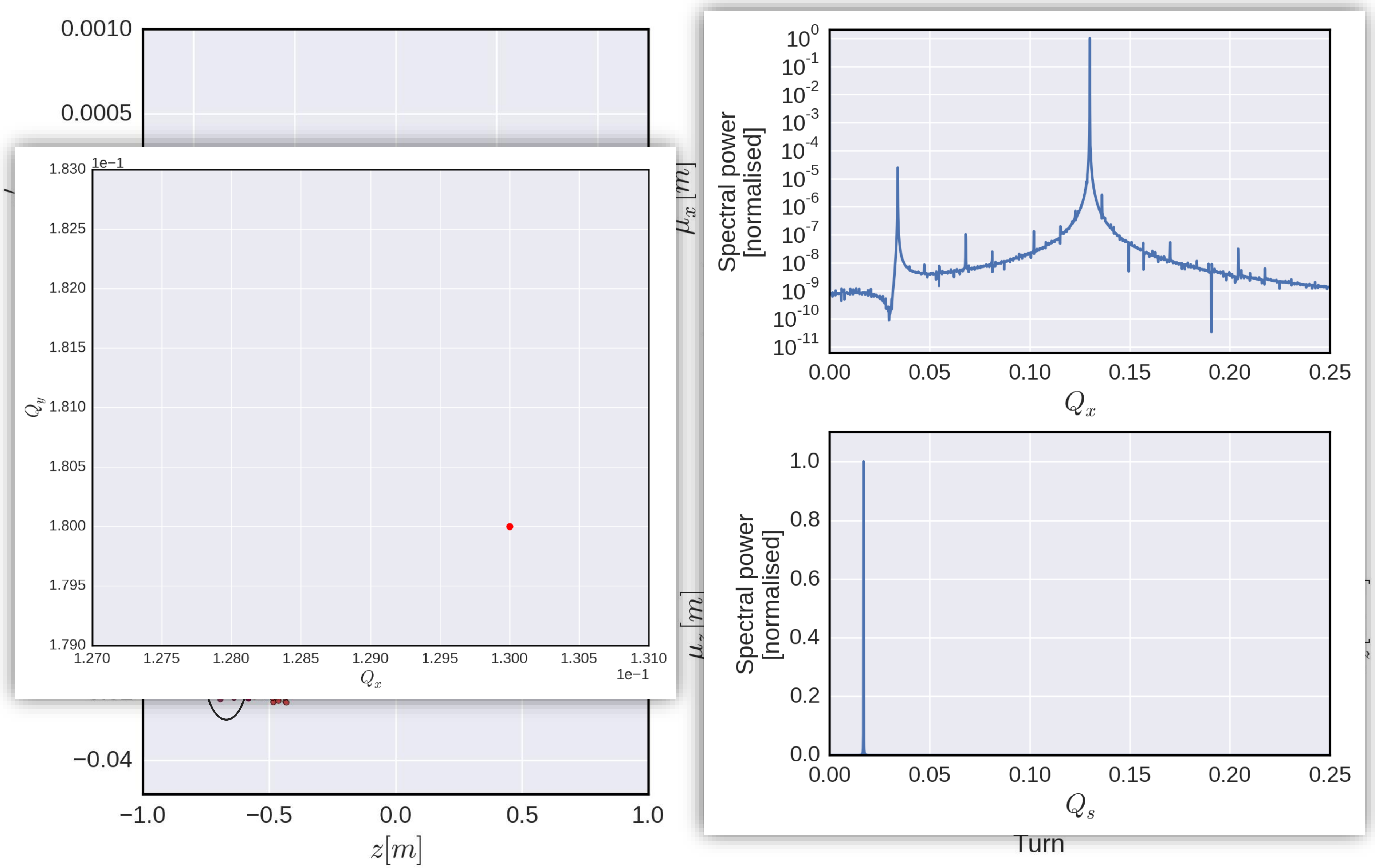
$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\boxed{x}\sum_{j=0}^{\text{n\_slices-1}}\boxed{\lambda(z_j)\,W_{01}(z-z_j)\,\Delta z_j}$$

Dipolar term → orbit kick

Slice dependent change of closed orbit (if line density does not change)



Examples – constant wakes

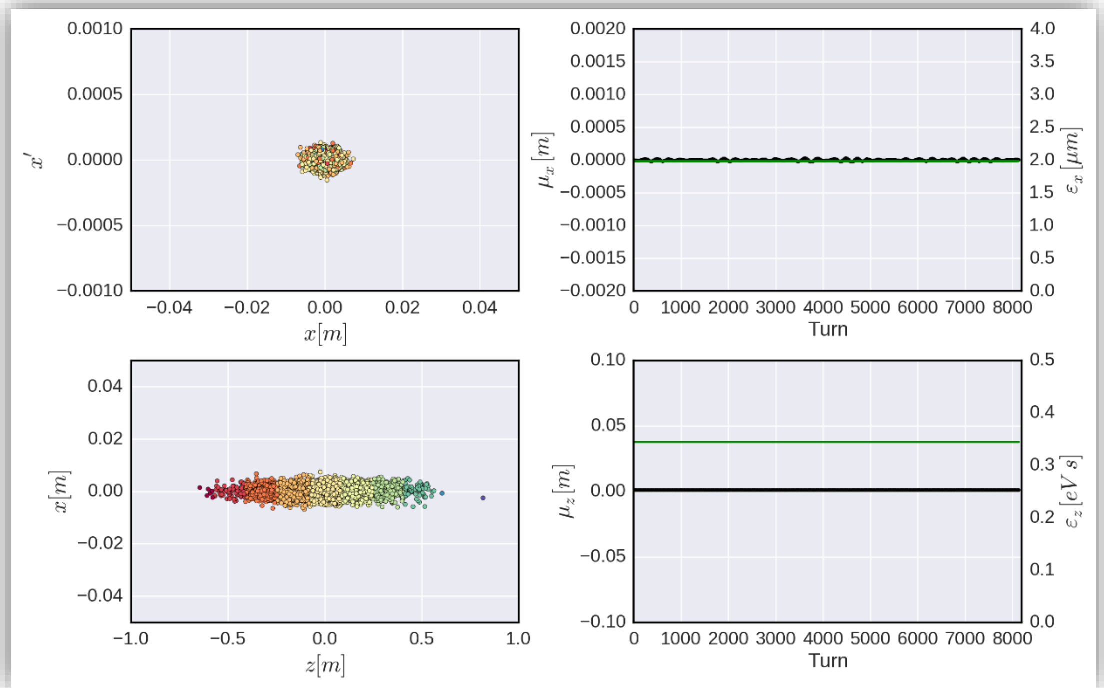


Examples – quadrupole wakes

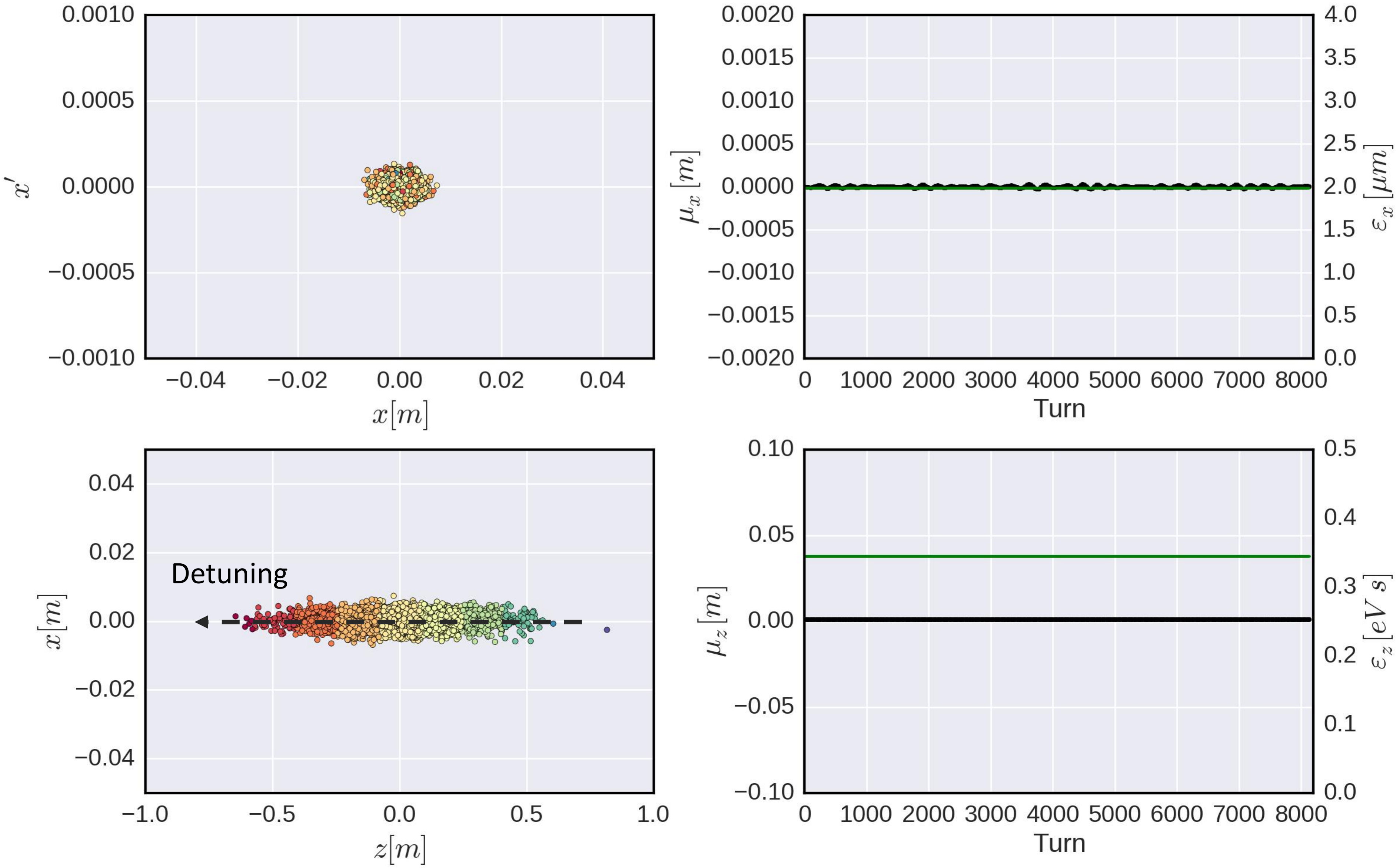
$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\boxed{x^2}\sum_{j=0}^{\text{n\_slices-1}}\boxed{\lambda(z_j)\,W_{02}(z-z_j)\,\Delta z_j}$$

Quadrupole term → tune kick

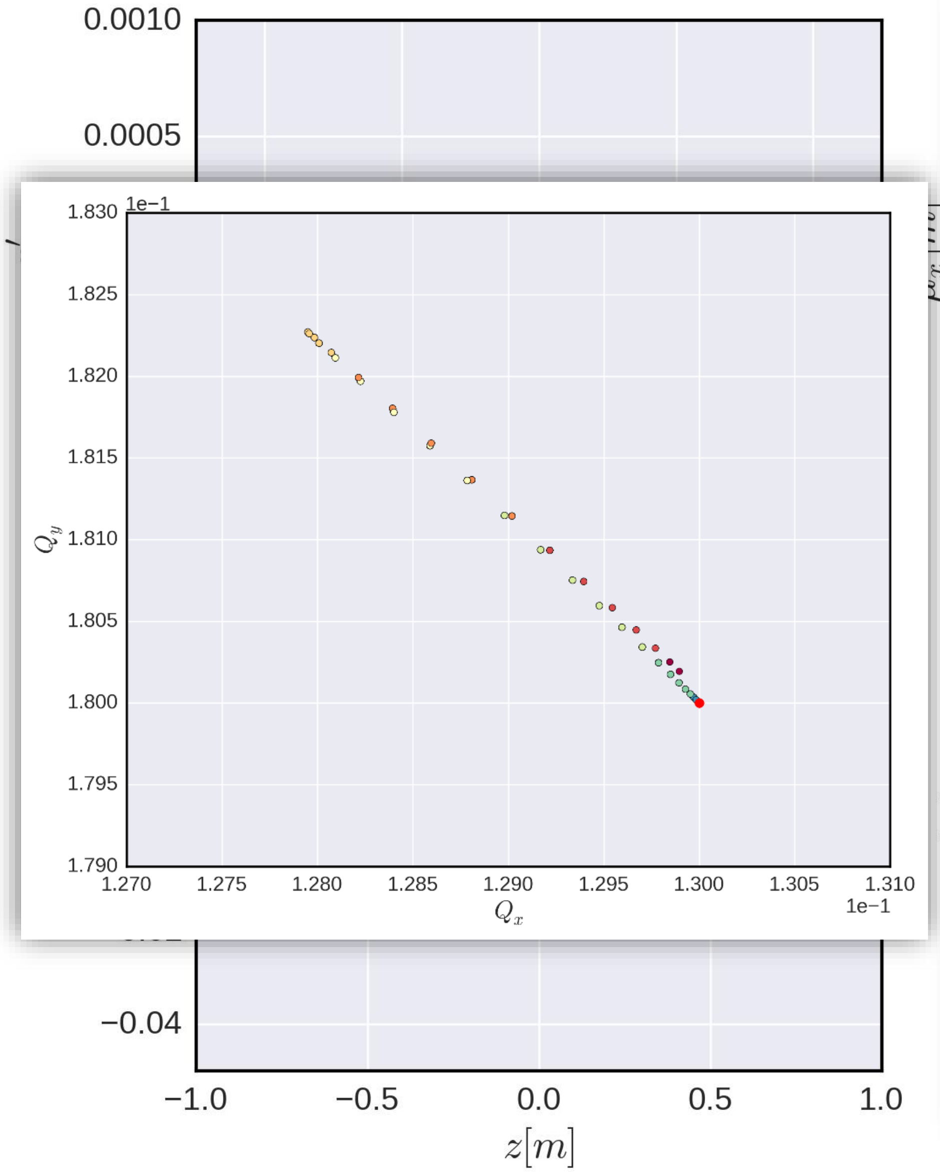
Slice dependent change of tune
(if line density does not change)

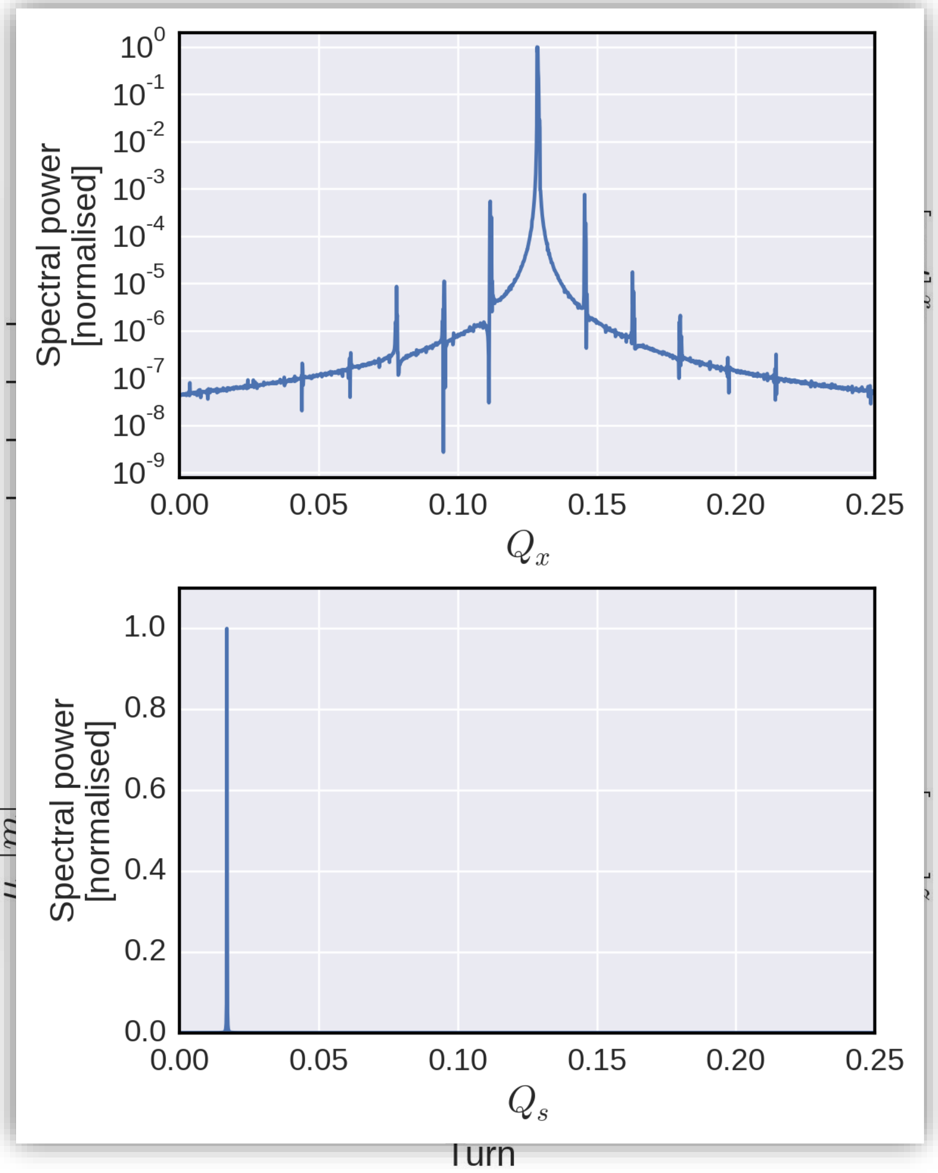


Examples – quadrupole wakes



Examples – quadrupole wakes





Examples – dipole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\boxed{x}\sum_{j=0}^{n\_\text{slices-1}}\boxed{\lambda(z_j)\,\langle x\rangle|_j\;W_{11}(z-z_j)\,\Delta z_j}$$

Dipolar term → orbit kick

Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs

Examples – dipole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\boxed{x}\sum_{j=0}^{n\_\text{slices}-1}\boxed{\lambda(z_j)\,\langle x\rangle|_j\;W_{11}(z-z_j)\,\Delta z_j}$$

Dipolar term → orbit kick

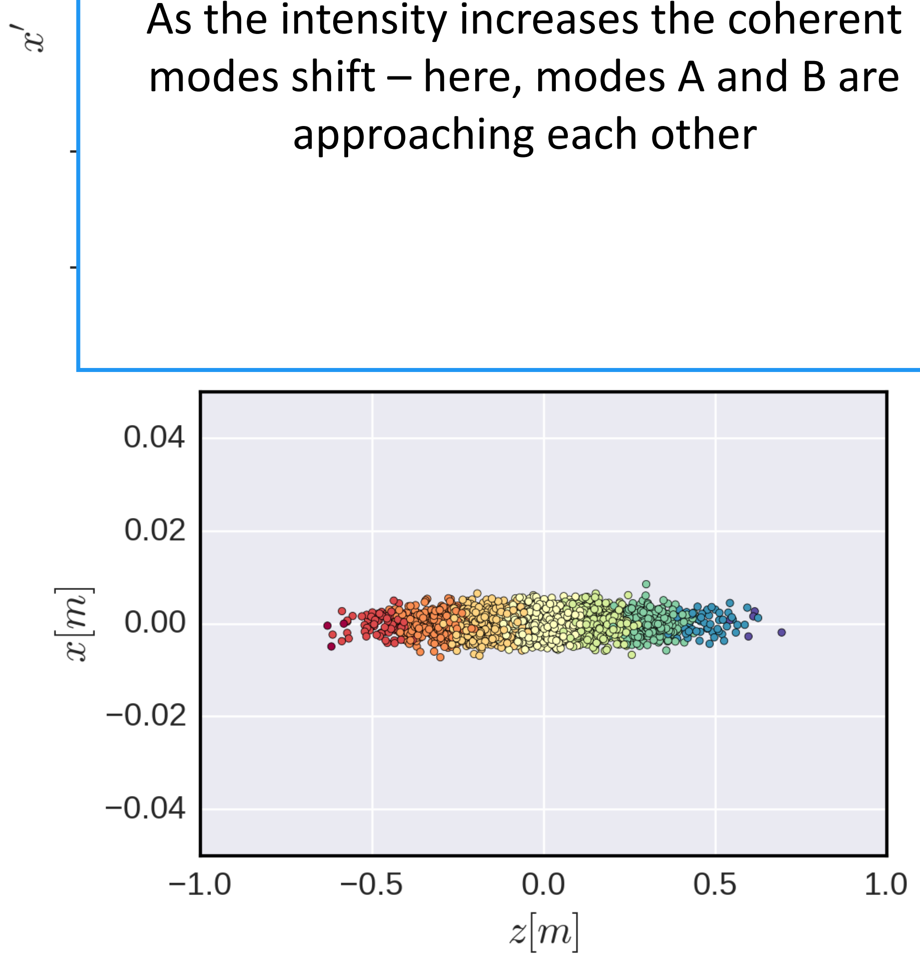
With synchrotron motion we can get into a feedback loop

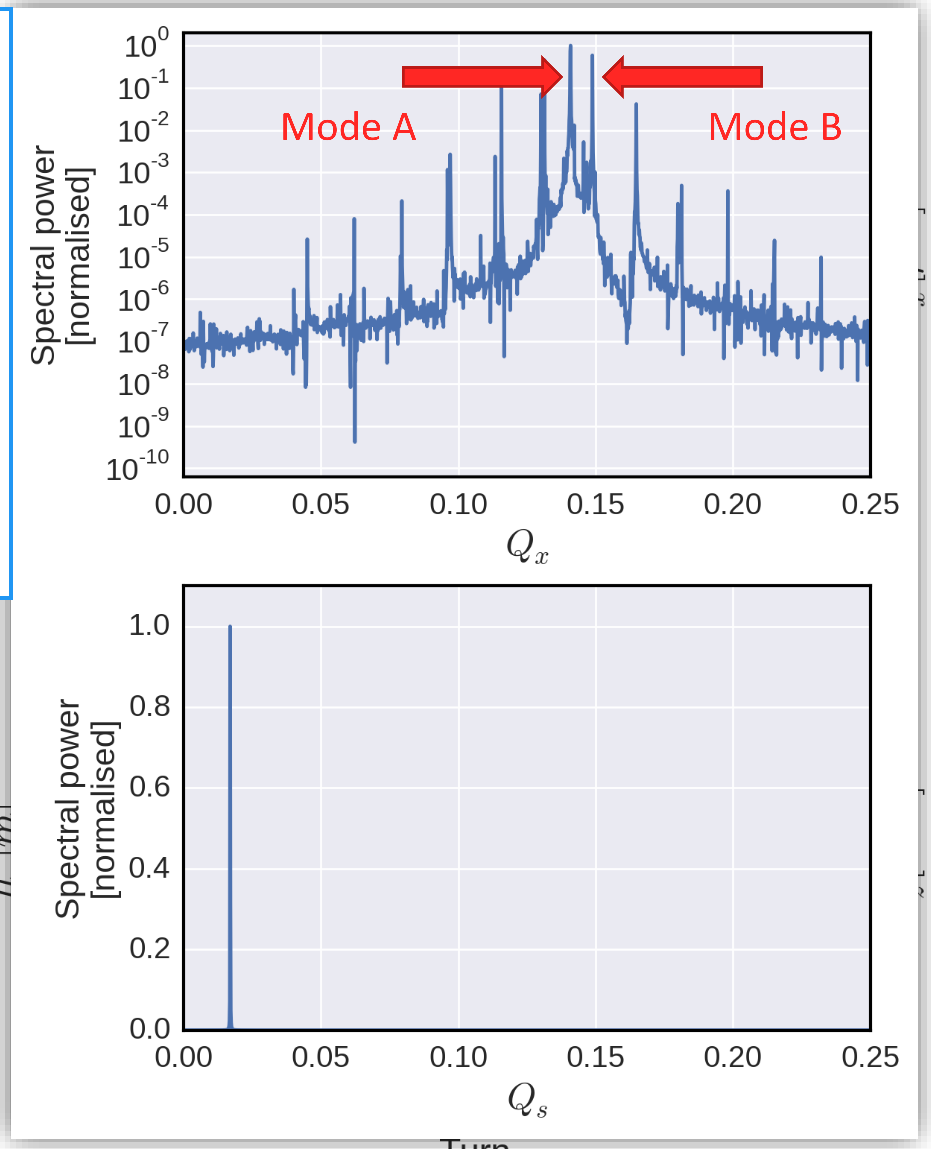
Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs
- With synchrotron motion:
  - Chromaticity = 0
    - Synchrotron sidebands are well separated → beam is stable
    - Synchrotron sidebands couple → (transverse) mode coupling instability
  - Chromaticity ≠ 0
    - Headtail modes → beam is unstable (can be very weak and often damped by non-linearities)

Dipole wakes – TMCI below threshold

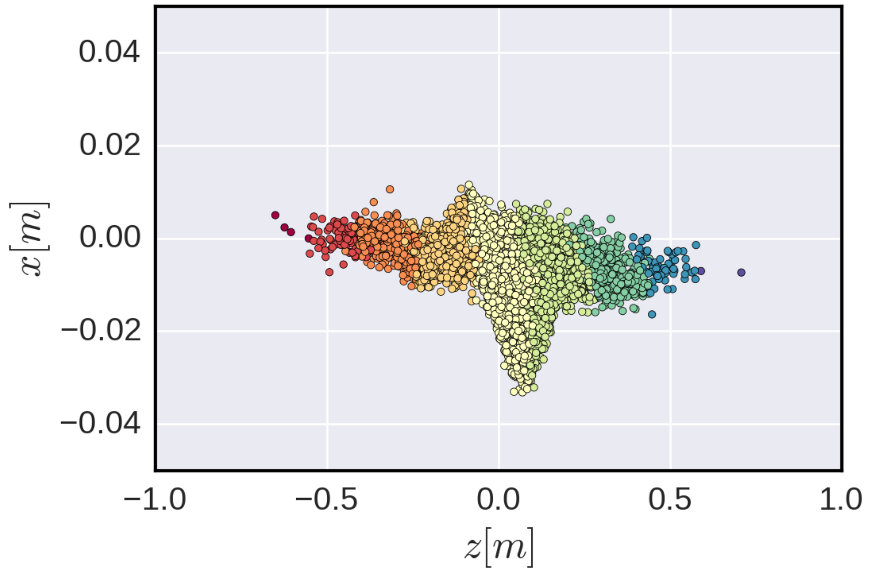
As the intensity increases the coherent modes shift – here, modes A and B are approaching each other

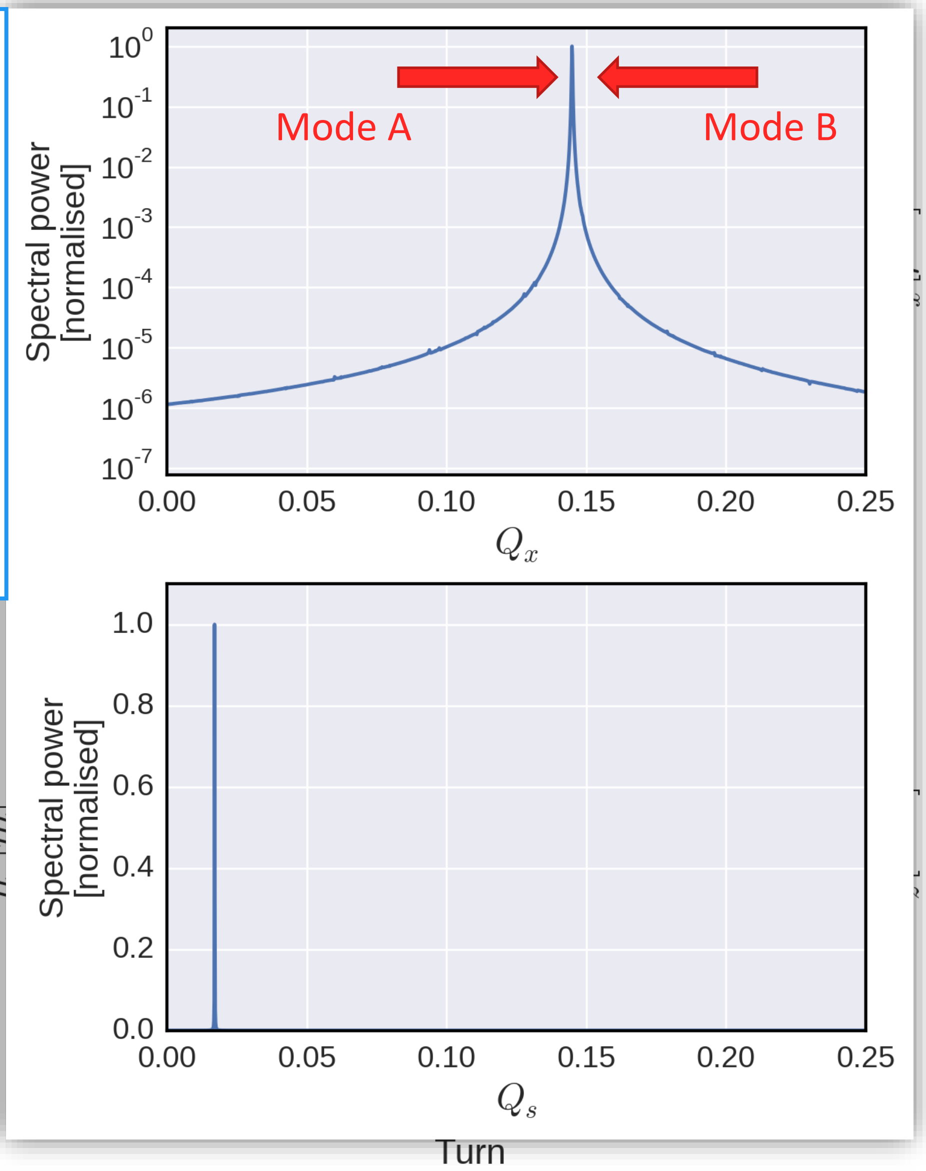




Dipole wakes – TMCI above threshold

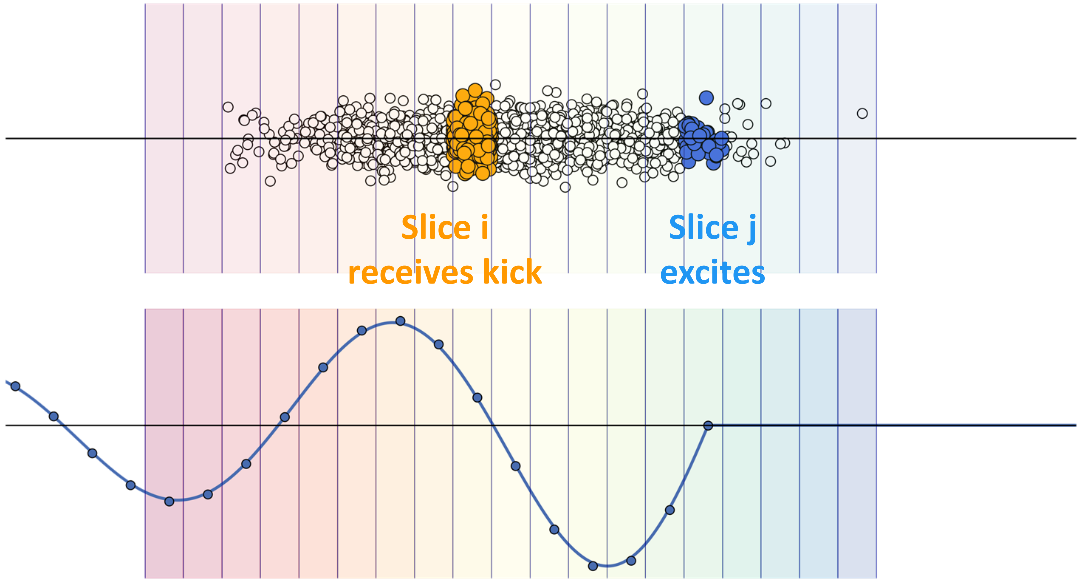
When the two modes merge a fast coherent instability arises – the transverse mode coupling instability (TMCI) which often is a hard intensity limit in many machines





Transverse impedance kick

- Single traversal of a bunch through an impedance source
  - Including the quadrupolar wake, kicks are different on different particles in slice i



$N[i] \to \lambda(z)dz \quad N[j] \to \lambda(z')dz'$

$$\Delta x'_{ik} = -\frac{e^2}{\beta^2 E_0} \sum_{j=0}^{i} N[j] \left[W_{Cx}[(i-j)\Delta z] + \langle x \rangle [j] W_{Dx}[(i-j)\Delta z] + x_{ik} W_{Qx}[(i-j)\Delta z] \right]$$

With k = 1 … # of particles in slice i

$$\Delta x' = -\frac{e^2}{\beta^2 E_0} \int \lambda(z') \left[W_{cx}(z-z') + \langle x \rangle (z') W_{Dx}(z-z') + x W_{Qx}(z-z') \right] dz'$$