Beam Dynamics with Synchrotron Radiation

Ian Martin
Diamond Light Source

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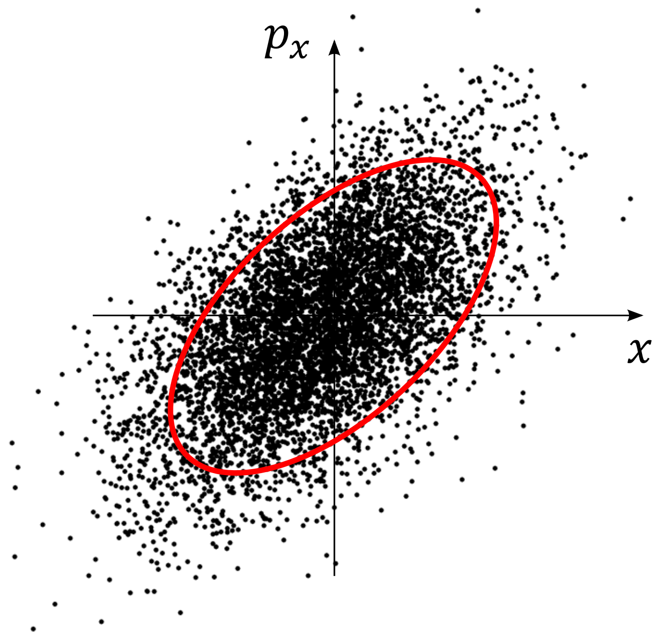
Contents

Emittance

The emittance of an electron beam is a measure of the area occupied by the beam in phase space.



In the absence of coupling and dispersion, the horizontal emittance is given by:

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2} \qquad (1)$$

- x is the horizontal coordinate with respect to a chosen reference trajectory
- $p_x = \gamma m \dot{x} / P_0$ is the horizontal momentum, scaled by a fixed reference momentum, P_0

For $|p_x| \ll 1$, we find $p_x \approx \frac{dx}{ds}$, i.e. p_x is approximately the angle of the particle's trajectory with respect to the reference trajectory

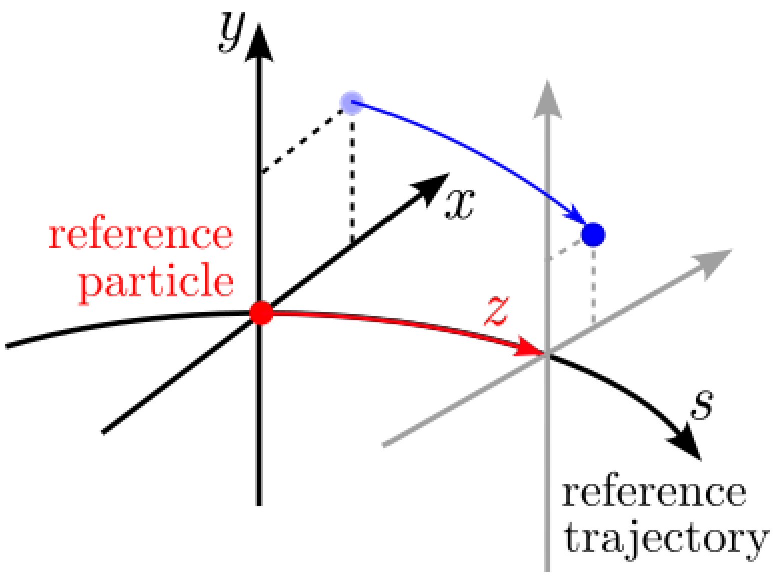
Emittance

Similarly, the vertical emittance is given by:

$$\varepsilon_y = \sqrt{\langle y^2\rangle\langle p_y^2\rangle - \langle y p_y\rangle^2} \qquad (2)$$

and the longitudinal emittance is given by:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle \delta^2\rangle - \langle z\delta\rangle^2} \qquad (3)$$



- z is the longitudinal position of the particle with respect to a reference particle, with $z > 0$ for a particle arriving early
- $\delta \approx \Delta P/P_0$ is the relative deviation from the reference momentum, P_0

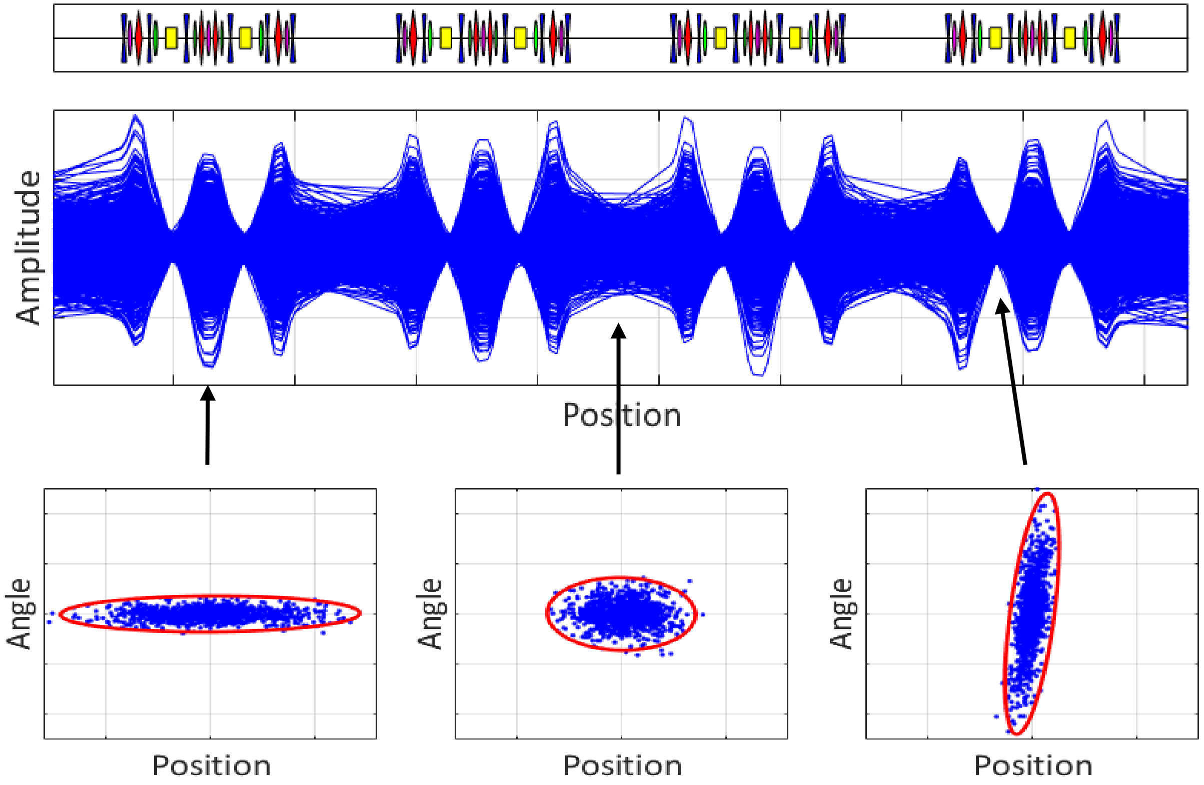
These definitions can be generalised to include betatron coupling and coupling between longitudinal and transverse motion through dispersion.

Why is emittance a useful concept?

The emittance of a beam of particles remains constant as the distribution moves around a storage ring, as long as the following conditions are satisfied:

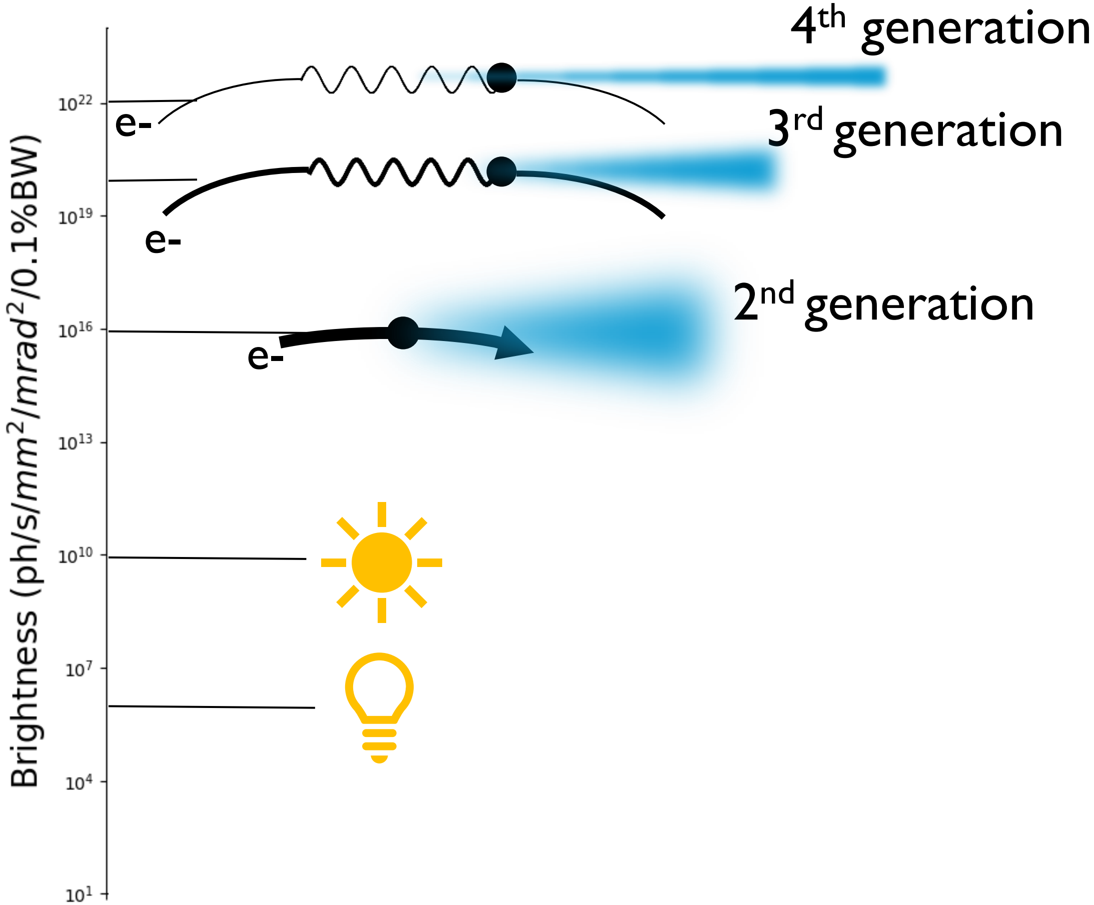
- There is no synchrotron radiation
- There are no collective effects
- There are no external damping or excitation processes, e.g. stochastic cooling

In a lattice with given focussing strength (i.e. fixed optics), a smaller emittance leads to a smaller beam size and divergence.

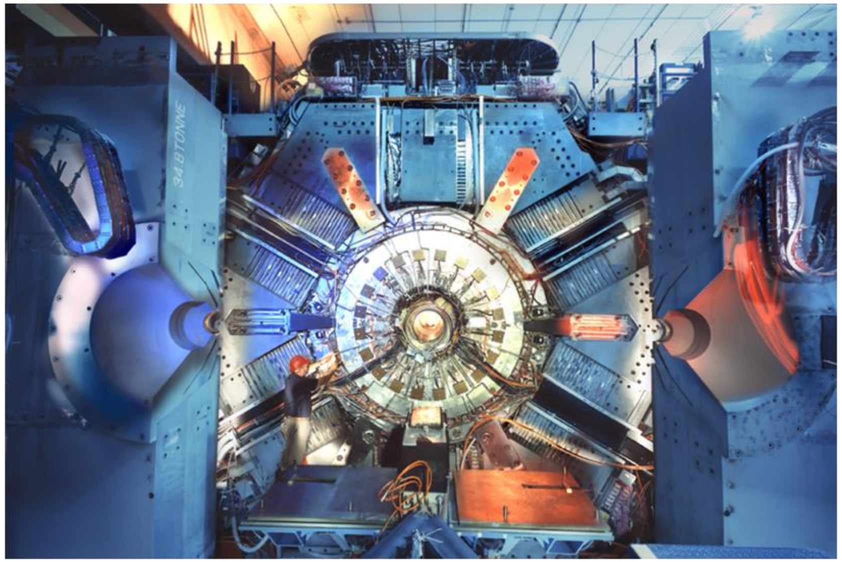


<u>Emittance is a key parameter for both synchrotron light sources and colliders</u>

$$brightness = \frac{radiation\ power\ per\ unit\ bandwidth}{phase\ space\ area\ of\ source} \quad (4)$$



Luminosity depends directly on the horizontal and vertical emittances



Dynamical effects associated with the collisions mean that it is sometimes helpful to increase the horizontal emittance, but generally, reducing the vertical emittance as far as possible helps to increase the luminosity.

Lecture 1 objectives: linear beam dynamics with synchrotron radiation

In this lecture we shall:

- Begin with a reminder of the equations of motion in the absence of synchrotron radiation
- Describe the damping of synchrotron and betatron oscillations by the emission of electromagnetic radiation
- Discuss how quantum excitation leads to equilibrium values for the longitudinal and transverse beam emittances

Many of the parameters that characterise synchrotron radiation damping and quantum excitation depend on integrals of lattice properties such as local beta-functions, dispersion and dipole bend radius.

As such, it is useful to begin by defining some variables that capture this information.

The synchrotron radiation integrals

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds, \tag{5}$$

$$I_2 = \oint \frac{1}{\rho^2} ds, \tag{6}$$

$$I_3 = \oint \frac{1}{|\rho^3|} ds \tag{7}$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1\right) ds \tag{8}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds \tag{9}$$

Where η_x is the horizontal dispersion, ρ is the local bending radius, $k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$ is the quadrupole strength and the chromatic (or dispersion) invariant is defined as:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2 \tag{10}$$

Radiation damping

Our first goal is to understand how the emission of synchrotron radiation leads to the damping of synchrotron and betatron oscillations, starting with synchrotron motion.

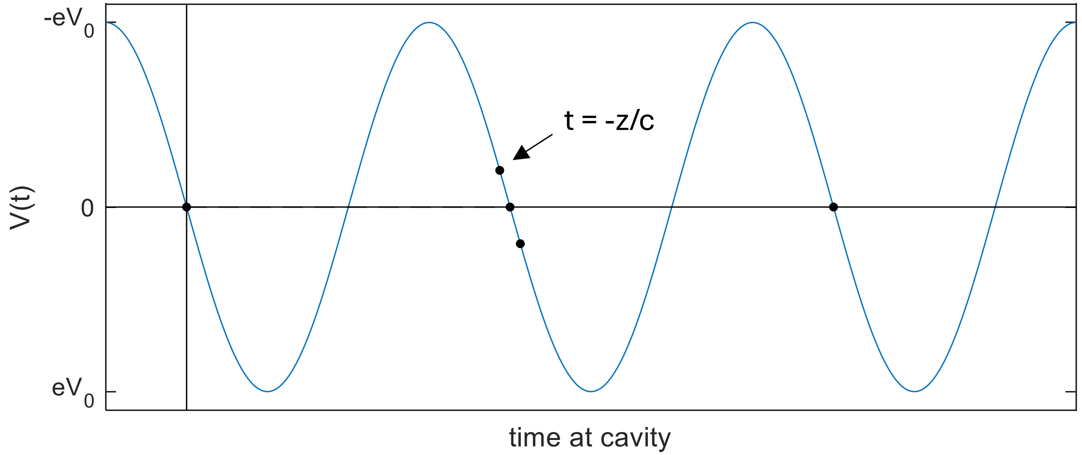
We shall proceed as follows:

- We write down the equations of motion for a particle performing synchrotron motion in the absence of synchrotron radiation
- We include a term in the equation of motion representing the effect of classical radiation as a ‘frictional’ force

Later we will add a term to the equation of motion representing the effect of photon emission (quantum radiation) as stochastic ‘noise’. This adds growth in the amplitude of oscillation, and counters the damping term, eventually leading to equilibrium values.

Synchrotron oscillations without radiation

RF cavities in a storage ring change the energy of particles passing through them.



Consider a cavity with peak voltage V_{RF} and an angular frequency ω_{RF}

A particle passing through the cavity at time $t = -z/c$ with respect to the reference particle sees a voltage:

$$V(t) = V_{RF}\sin(\phi_{RF} + \omega_{RF}t) = V_{RF}\sin\left(\phi_{RF} - \frac{\omega_{RF}z}{c}\right) \tag{11}$$

If z is small compared to the RF wavelength λ_{RF} and taking the convention that $\phi_{RF} = \pi$, the change in δ is:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\frac{\omega_{RF}z}{c}\right) \approx \frac{eV_{RF}}{E_0}\frac{\omega_{RF}z}{c} \tag{12}$$

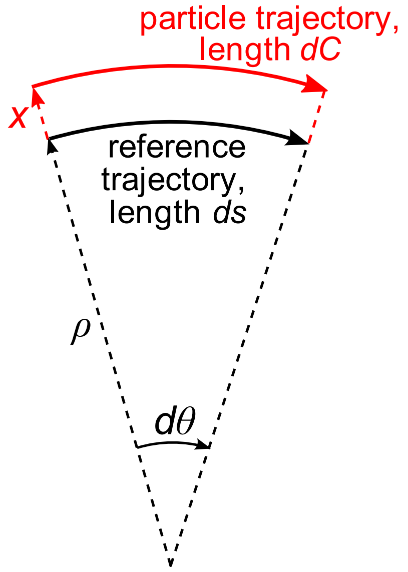
Synchrotron oscillations without radiation

If the storage ring has circumference C_0, then averaged over many turns, the change of δ per unit distance is:

$$\frac{d\delta}{ds} \approx \frac{eV_{RF}}{E_0 C_0} \frac{\omega_{RF} z}{c} \tag{13}$$

We need to take into account of changes in orbit length from changes in energy deviation:

$$\frac{dz}{ds} = -\alpha_c \delta \tag{14}$$



where α_c is the momentum compaction factor:

$$\alpha_c = \frac{1}{C_0} \oint \frac{\eta_x(s)}{\rho(s)} ds = \frac{I_1}{C_0} \tag{15}$$

I_1 is the first synchrotron radiation integral.

Synchrotron oscillations without radiation

Combining equations (13) and (14):

$$\frac{d\delta}{ds} \approx \frac{eV_{RF}}{E_0 C_0}\frac{\omega_{RF} z}{c}, \qquad\qquad \frac{dz}{ds} = -\alpha_c \delta$$

we have the result:

$$\frac{d^2\delta}{ds^2} + \frac{\omega_s^2}{c^2}\delta = 0 \tag{16}$$

Particles perform harmonic oscillations at the synchrotron oscillation frequency, ω_s:

$$\omega_s = \sqrt{\frac{eV_{RF} c \alpha_c \omega_{RF}}{E_0 C_0}} \tag{17}$$

Note that choosing an RF phase $\phi_{RF} = \pi$ ensures stable oscillations ($\omega_s^2 > 0$ for $\alpha_c > 0$)

Synchrotron oscillations with classical radiation

Now let us consider the impact of synchrotron radiation.

In a classical (non-quantum) approximation, synchrotron radiation has two effects:

- First, the energy that a particle loses by radiation must be replaced by the RF cavities (shifting the synchronous phase of the beam with respect to the RF phase)

- Second, the amount of energy radiated by a particle depends on the energy deviation (leading to a damping of its synchrotron motion)

We shall consider each of these effects in turn.

Synchrotron oscillations with classical radiation

First, we consider how much energy is being lost each revolution due to radiation emission.

From classical electromagnetism (e.g. ref. [1]), the power P_γ radiated by a charged particle moving with energy E through a constant magnetic field B is given by:

$$P_\gamma \approx \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}, \quad \text{where} \quad C_\gamma = \frac{e^2}{3\epsilon_0 (mc^2)^4} \tag{18}$$

For electrons, $C_\gamma \approx 8.846 \times 10^{-5}$ m/GeV^3, for protons $C_\gamma \approx 7.783 \times 10^{-18}$ m/GeV^3.

By multiplying the radiation power by the time spent in the dipoles ($\mathrm{T_B} = 2\pi\rho/c$), we find that the reference particle with energy E_0 has an energy loss per turn:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \quad \text{where} \quad I_2 = \oint \frac{1}{\rho(s)^2} ds \tag{19}$$

I_2 is the second synchrotron radiation integral.

Synchrotron oscillations with classical radiation

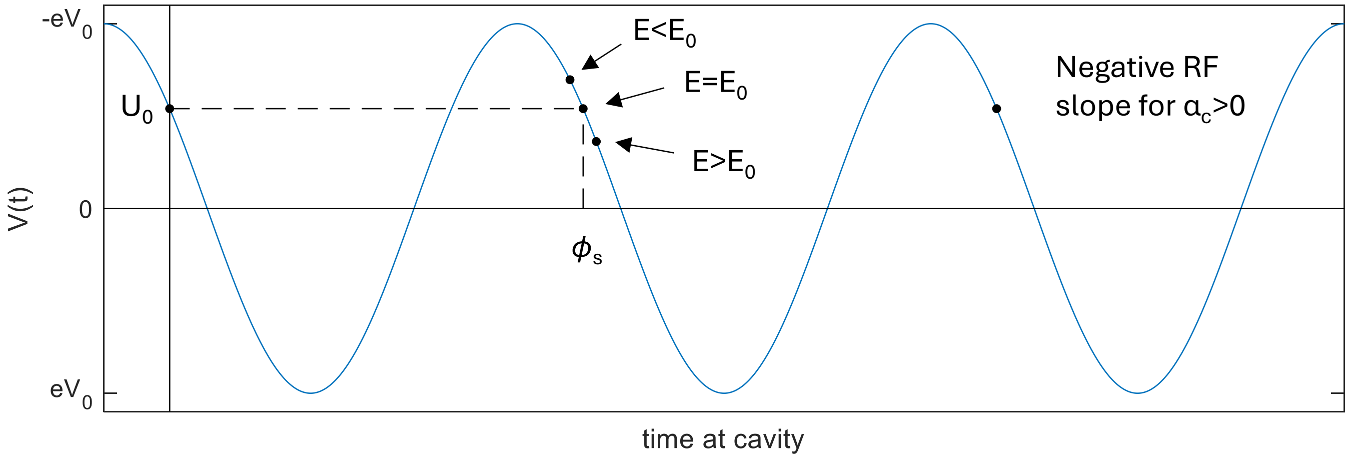
The energy lost by particles through synchrotron radiation must be replaced by the RF cavities. This shifts the phase at which particles cross the cavities.

The reference particle (losing energy U_0) crosses the RF cavities at the synchronous phase ϕ_s:

$$\sin(\phi_s) = \frac{U_0}{eV_{RF}} \tag{20}$$

Then, the energy change of a particle (per turn) is in general:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\phi_s - \frac{\omega_{RF}z}{c}\right) - \frac{U}{E_0} \tag{21}$$



Synchrotron oscillations with classical radiation

Equation (18) shows that the radiated power depends on the square of the particle energy ($E^4/\rho^2 \propto E^2 B^2$). We can write (for $|\delta| \ll 1$)

$$E^2 = (1+\delta)^2 E_0^2 \approx (1+2\delta)E_0^2, \quad \text{so} \quad U \approx (1+2\delta)U_0 \tag{22}$$

Then, including the energy gain from the RF cavities, the rate of change of the energy deviation (13) becomes

$$\frac{d\delta}{ds} = \frac{\omega_s^2}{\alpha_c c^2} z - (1+2\delta)\frac{U_0}{E_0 C_0} \tag{23}$$

Taking the derivative with respect to s and substituting dz/ds from (14) gives the equation of motion:

$$\frac{d^2\delta}{ds^2} + \frac{2U_0}{E_0 C_0}\frac{d\delta}{ds} + \frac{\omega_s^2}{c^2}\delta = 0 \tag{24}$$

Synchrotron oscillations with classical radiation

The term in $\mathrm{d}\delta/ds$ means that synchrotron radiation has a damping effect similar to friction in a mechanical oscillator.

The equation of motion can be written:

$$\frac{d^2\delta}{ds^2} + \frac{2}{c\tau_z}\frac{d\delta}{ds} + \frac{\omega_s^2}{c^2}\delta = 0 \tag{25}$$

Where the synchrotron oscillation frequency ω_s is now

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\cos(\phi_s)\frac{\omega_{RF}}{T_0}\alpha_c \tag{26}$$

$T_0 \approx C_0/c$ is the revolution period, and the damping time τ_z is

$$\tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0 \tag{27}$$

τ_z is roughly the time it would take the reference particle to lose all its energy, if it lost energy at a constant rate P_γ

Synchrotron oscillations with classical radiation

The longitudinal damping partition number j_z (typically ~2) takes into account the variation in magnetic field associated with any quadrupole component in the dipoles:

$$j_z = 2 + \frac{I_4}{I_2} \tag{28}$$

I_2 is defined in (6), and the fourth synchrotron radiation integral I_4 (8) is:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x} \tag{29}$$

η_x is the horizontal dispersion, ρ is the local radius of curvature of the trajectory, and k_1 is the local quadrupole strength.

A negative gradient in the bending magnets will reduce I_4, decreasing j_z and causing the longitudinal damping time to increase (i.e. it reduces the amount of damping in the longitudinal plane).

Synchrotron oscillations with classical radiation

The longitudinal coordinate z obeys an identical equation of motion to the energy deviation δ:

$$\frac{d^2 z}{ds^2} + \frac{2}{c\tau_z}\frac{dz}{ds} + \frac{\omega_s^2}{c^2} z = 0 \tag{30}$$

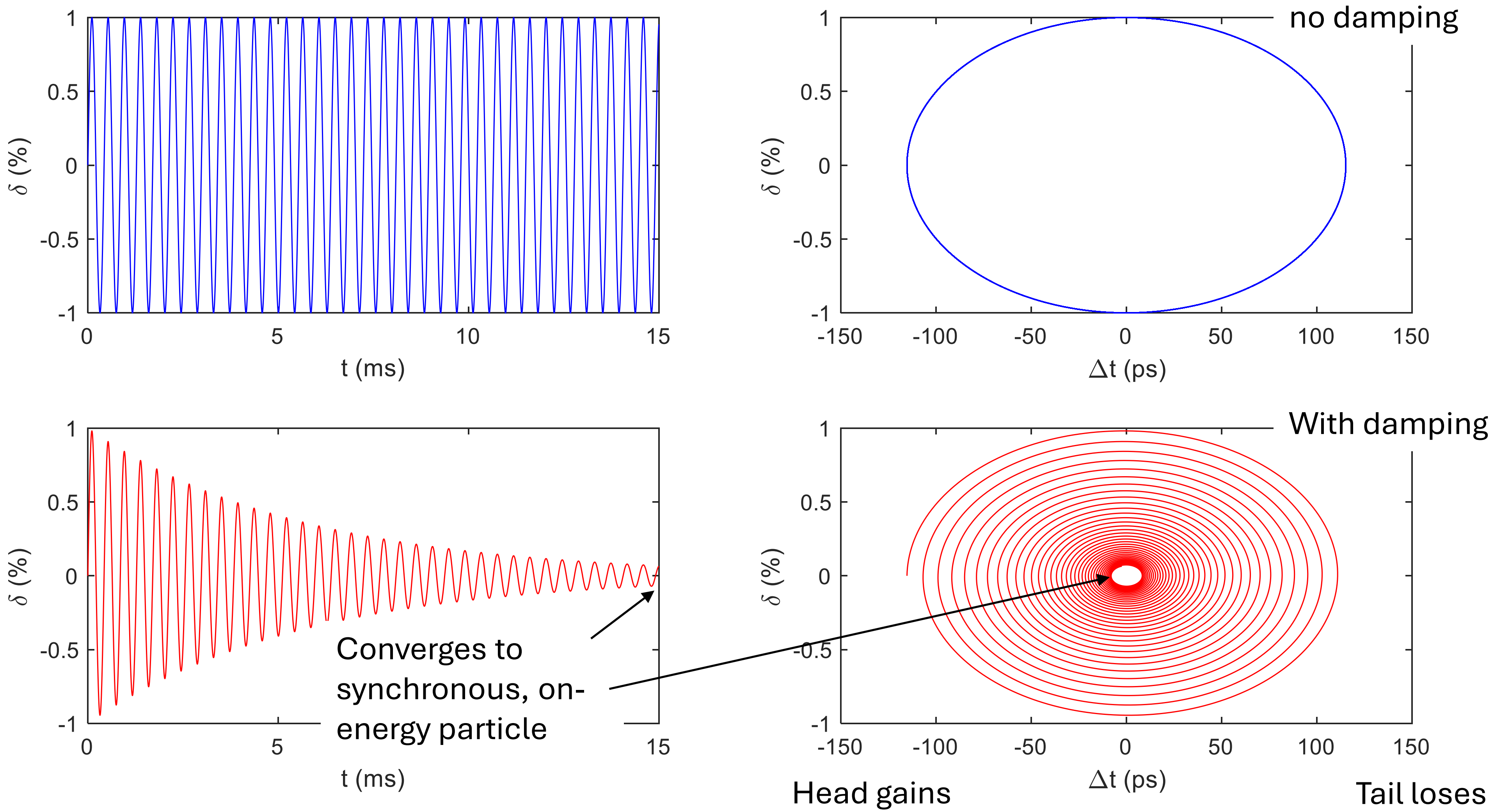
If the synchrotron oscillation period is short compared to the damping time (so that $\frac{1}{\omega_s} \ll \tau_z$), then the solution to the equations of motion can be written:

$$\delta(t) = A_\delta e^{-t/\tau_z} \sin(\omega_s t - \theta_0) \tag{31}$$

$$z(t) = \frac{\alpha_c c}{\omega_s} A_\delta e^{-t/\tau_z} \cos(\omega_s t - \theta_0) \tag{32}$$

Where A_δ and θ_0 are constants (respectively, the amplitude and phase of the oscillation at time $t = 0$)

Synchrotron oscillations with classical radiation



Head gains energy for $\alpha_c > 0$

Tail loses energy for $\alpha_c > 0$

Radiation damping of longitudinal emittance

From the definition of the longitudinal emittance:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle\delta^2\rangle - \langle z\delta\rangle^2} \tag{33}$$

We find using (31) and (32) that synchrotron radiation leads to an exponential damping of the longitudinal emittance with a damping time that is half the damping time of the synchrotron oscillation amplitude.

$$\varepsilon_z(t) = \varepsilon_z(0)\exp\left(-2\frac{t}{\tau_z}\right) \tag{34}$$

It appears that in an electron storage ring, over many damping times the longitudinal emittance will approach zero.

But, as we will see later, the longitudinal emittance approaches a non-zero equilibrium value due to the quantum nature of synchrotron radiation …

Damping of betatron oscillations

Let us now consider the effect of synchrotron radiation on betatron oscillations.

In the case of synchrotron oscillations, we assumed that the synchrotron frequency was small compared to the revolution frequency.

But for betatron motion, the oscillation frequency in a storage ring is usually much larger than the revolution frequency: we shall have to take a different approach to the analysis.

We shall first consider vertical betatron oscillations: this turns out to be a simpler case than horizontal betatron oscillations.

Betatron motion without synchrotron radiation

Betatron oscillations can be expressed as harmonic motion:

$$y = \sqrt{2\beta_y J_y}\cos(\phi_y), \qquad p_y = -\sqrt{\frac{2J_y}{\beta_y}}\left(\sin(\phi_y) + \alpha_y \cos(\phi_y)\right) \tag{35}$$

where β_y is the vertical beta function, ϕ_y is the betatron phase, and J_y is the vertical action:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2 \tag{36}$$

The beam emittance is the average action over all particles:

$$\varepsilon_y = \langle J_y \rangle \tag{37}$$

Using $\beta_y\gamma_y - \alpha_y^2 = 1$, (35), (36) and (37) give (2):

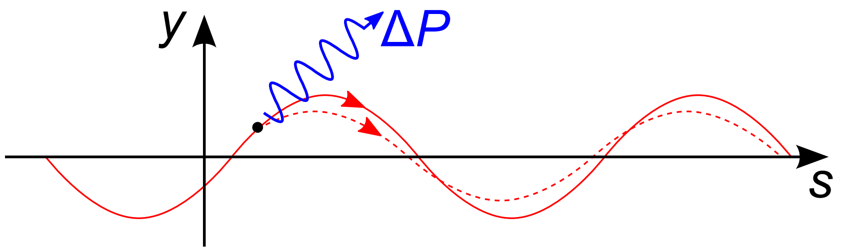
$$\varepsilon_y = \sqrt{\langle y^2\rangle\langle p_y^2\rangle - \langle y p_y\rangle^2} \tag{38}$$

Damping of vertical emittance

In the absence of radiation, the action J_y of each particle remains constant as the particles move around a storage ring.

Radiation damping is a result of particles losing momentum by emitting synchrotron radiation.

Synchrotron radiation is emitted in a narrow cone (opening angle $1/\gamma$) around the direction of motion of the particle.



This means that each component of the particle momentum changes in the same way. If the radiation carries momentum ΔP, then

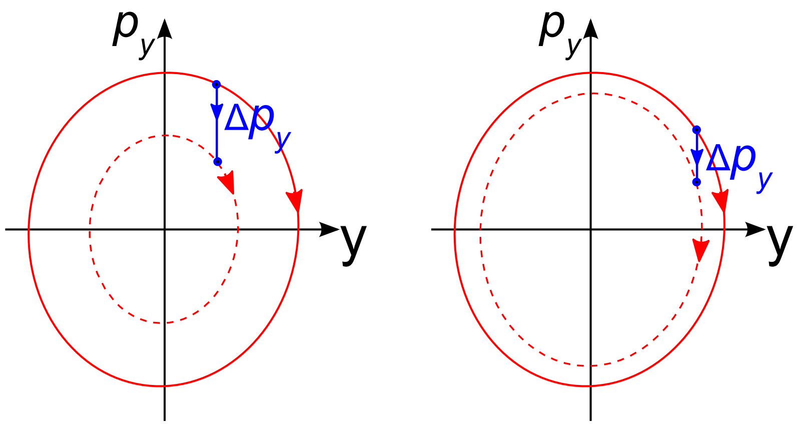
$$\frac{\Delta p_x}{p_x} = \frac{\Delta p_y}{p_y} = \frac{\Delta p_z}{p_z} = -\frac{\Delta P}{P_0} \tag{39}$$

Damping of vertical emittance

In the vertical plane, we have:

$$\Delta p_y = -\frac{\Delta P}{P_0} p_y \tag{40}$$

This change in vertical momentum will result in a change in the vertical action J_y, but the amount by which it changes is dependent upon the phase space coordinates of the particle at the time of emission.



If all particles (at random betatron phases) lose equal momentum ΔP, after substituting (40) into equations (35) and (36) and averaging over all phases, we find the expectation value for the change in vertical emittance to be:

$$\Delta \varepsilon_y = \langle \Delta J_y \rangle = -\varepsilon_y \frac{\Delta P}{P_0} \tag{41}$$

Damping of vertical emittance

If the momentum lost through radiation on each turn is small (compared to the total momentum of each particle), then the rate of change of the emittance can be found by averaging the momentum loss around the ring:

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint \frac{dP}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y = -\frac{2}{\tau_y}\varepsilon_y \qquad (42)$$

Here, T_0 is the revolution period, E_0 is the reference energy, U_0 is the energy loss per turn and we have introduced the vertical damping time τ_y.

The approximation in the above formula is valid for an ultra-relativistic particle, for which $E \approx Pc$

Damping of vertical emittance

Similar to the longitudinal emittance (34), the evolution of the vertical emittance is given by

$$\varepsilon_y(t) = \varepsilon_y(0) \exp\left(-2\frac{t}{\tau_y}\right) \tag{43}$$

and the vertical damping time τ_y takes a similar form to the longitudinal damping time (27)

$$\tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 \tag{44}$$

The vertical damping partition number in this case is $j_y = 1$, whereas for the longitudinal damping partition number (28) it is typically $j_z \approx 2$.

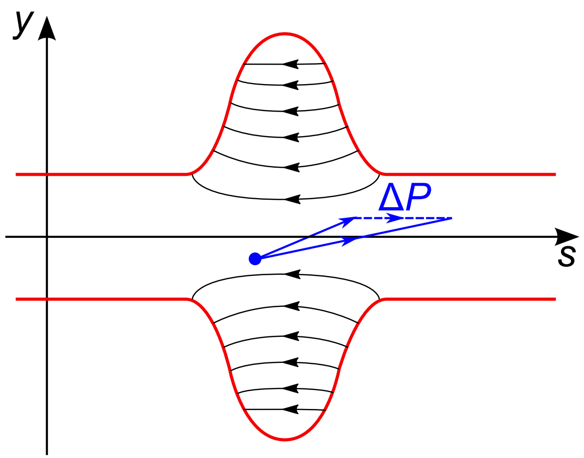
As such, the vertical damping is usually about twice the longitudinal damping time.

Damping of vertical emittance

Typically, in an electron storage ring, the damping time is of the order of tens of milliseconds, while the revolution time is of the order of a microsecond. Therefore, radiation effects are indeed ‘slow’ compared to the revolution frequency.

Note that we made the assumption that the momentum of the particle was close to the reference momentum, (i.e. $P \approx P_0$). If the particle continues to radiate away without any restoration of energy, eventually this assumption will no longer be valid …

However, the lost energy will be restored by the RF cavities. These are usually designed to provide a longitudinal electric field.



There is then no change in the transverse momentum when a particle passes through the cavity.

Therefore, we do not have to consider explicitly the effects of RF cavities on the emittance of the beam.

Damping of horizontal emittance

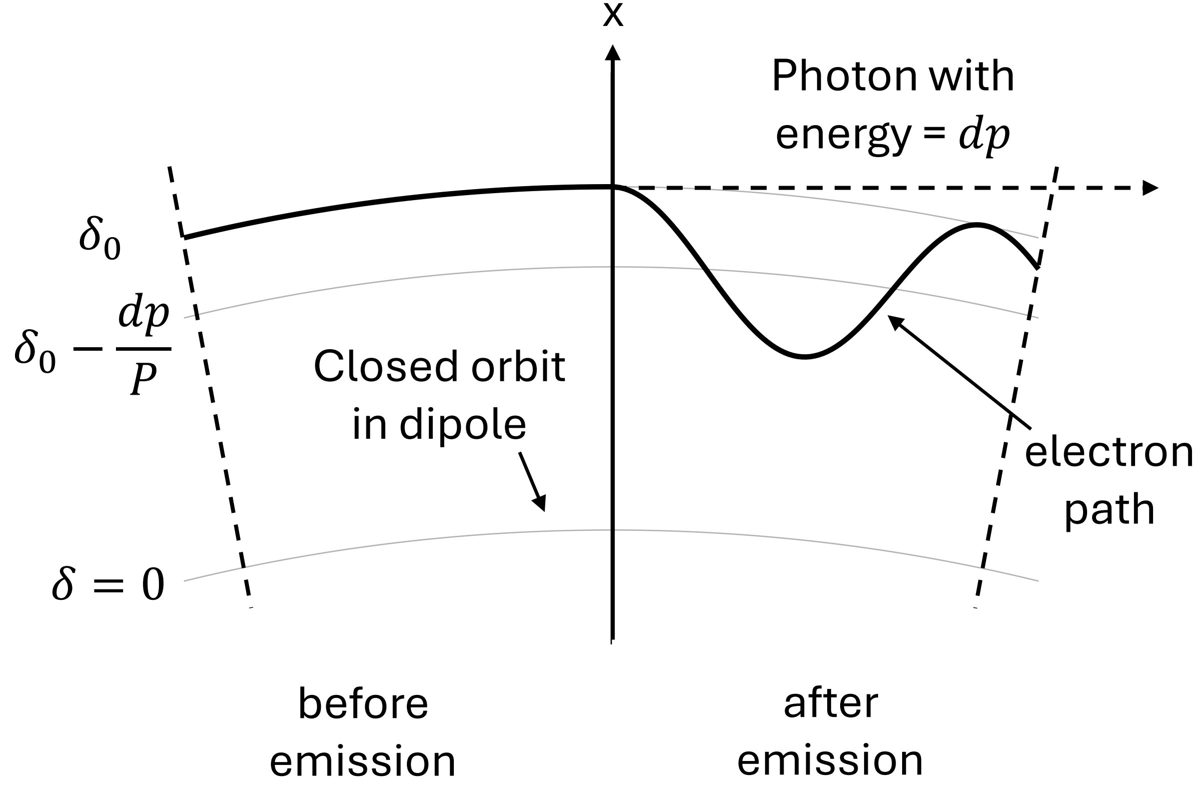
Analysis of radiation effects on the vertical emittance was relatively straightforward.

When we consider the horizontal emittance, there are three complications that we need to address:

1) The horizontal motion of a particle is often strongly coupled to the longitudinal motion (by dispersion): when a particle emits radiation, its horizontal coordinate with respect to the closed orbit will change.

2) Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal coordinate with respect to the reference trajectory

3) Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal coordinate of the particle.



Damping of horizontal emittance

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance:

1) Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion)
2) Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in the action resulting from the radiation emission
3) Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle
4) Integrate around the ring (taking into account changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome and is not especially enlightening. See for example ref. [2] for more details. Here, we just quote the result …

Damping of horizontal emittance

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x \tag{45}$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \tag{46}$$

The horizontal damping partition number j_x is:

$$j_x = 1 - \frac{I_4}{I_2} \tag{47}$$

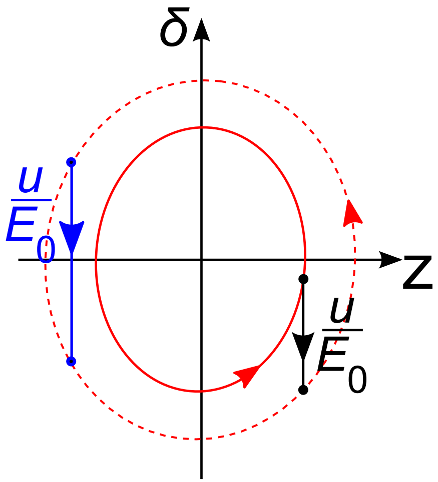
where the fourth synchrotron radiation integral is given by (8):

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds\,, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x} \tag{48}$$

Quantum excitation of longitudinal emittance

The longitudinal emittance is a measure of the average amplitude of synchrotron oscillations of particles in a bunch.

The emission of a photon leads to a 'jump' in the amplitude of synchrotron oscillations performed by a particle.



The change in the synchrotron oscillation amplitude from the emission of a photon depends on the energy of the photon and on the synchrotron phase of the particle at the point of emission.

The effect is analogous to a pendulum receiving random kicks at random times during its oscillation.

Quantum excitation of longitudinal emittance

If a particle emits a photon with energy u_γ, the change in the energy deviation is

$$\Delta\delta = -\frac{u_\gamma}{E_0} \tag{49}$$

Since the rms energy spread in a bunch of particles is given by:

$$\sigma_\delta^2 = \langle \delta^2 \rangle \tag{50}$$

and $\langle \delta \rangle = 0$, the change in the rms energy spread when the particles emit a large number of photons is:

$$\Delta\sigma_\delta^2 = \langle (\delta + \Delta\delta)^2 \rangle - \langle \delta \rangle^2 = \langle \Delta\delta^2 \rangle = \frac{\langle u_\gamma^2 \rangle}{E_0^2} \tag{51}$$

Photon emission leads to an increase in the rms energy spread, and hence an increase in the longitudinal emittance that depends on the rms energy of the emitted photons.

Quantum excitation of longitudinal emittance

Using knowledge of the photon energy distribution from bending magnet radiation and including the effects of radiation damping (see for example refs. [3-5]), it can be shown that the rate of change of the mean square energy deviation is:

$$\frac{d\sigma_\delta^2}{dt} = C_q \gamma^2 \frac{2}{j_z \tau_z} \frac{I_3}{I_2} - \frac{2}{\tau_z} \sigma_\delta^2 \qquad (52)$$

Where γ is the relativistic factor for the electrons and the 'quantum radiation constant' C_q is given by:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \qquad (\approx 3.832 \times 10^{-13} \text{ for electrons}) \qquad (53)$$

Natural energy spread

Quantum excitation gives a steady increase in the mean square energy spread, while damping gives an exponential decay.

It follows that there is an equilibrium energy spread for which the quantum excitation is exactly balanced by the damping. The equilibrium can be found by setting $\frac{d\sigma_\delta^2}{dt} = 0$:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2} \tag{54}$$

This is often referred to as the 'natural' energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined by the beam energy and by the bending radii and gradient of the dipoles: note that it does not depend on the RF parameters (voltage or frequency).

Natural bunch length

The equilibrium bunch length σ_z in a distribution with energy spread σ_δ is then:

$$\sigma_z = \frac{\alpha_c c}{\omega_s}\sigma_\delta \tag{55}$$

For a given energy spread, we can reduce the bunch length, either

- By increasing the RF voltage, or

- By increasing the RF frequency

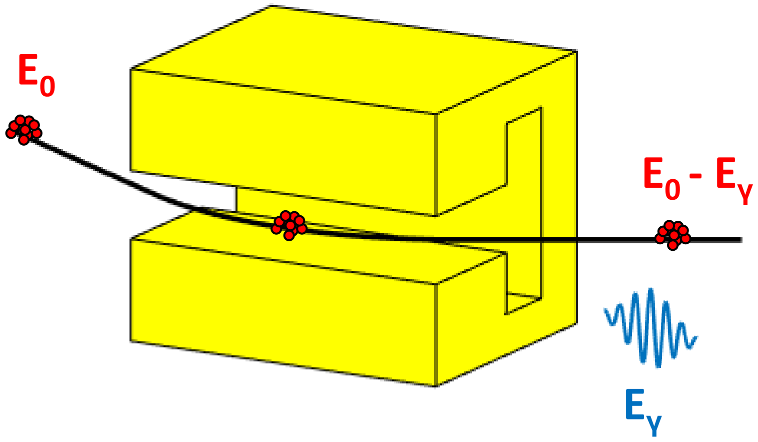
An increase in RF voltage or frequency increases the synchrotron frequency ω_s but does not change the energy spread.

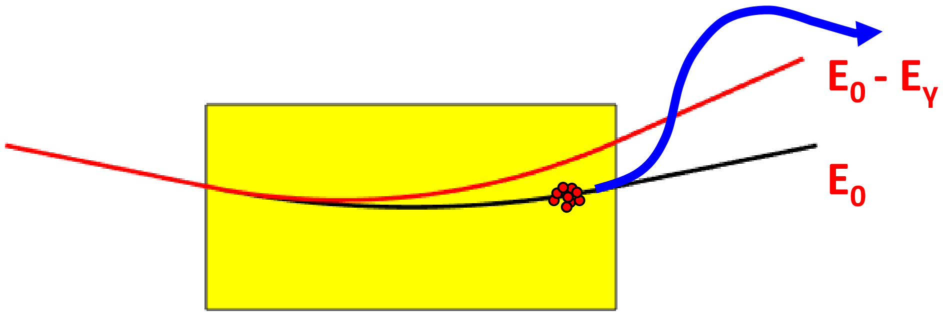
Quantum excitation of horizontal emittance

As in the case of longitudinal motion, if radiation were a purely classical process, the analysis of radiation damping effects suggests the horizontal emittance should eventually damp to zero.

However, radiation is emitted in discrete quanta (photons).

Because of dispersion, the horizontal betatron amplitude of a particle can change when it emits a photon.





Quantum excitation of horizontal emittance

Photon emission gives random 'kicks' to the horizontal motion, just as it does to the synchrotron motion.

The difference is that the horizontal kicks depend on the dispersion in the lattice.

It can be shown that (see for example refs. [3-5]), with radiation damping and quantum excitation, the evolution of the emittance is given by:

$$\frac{d\varepsilon_x}{dt} = C_q \gamma^2 \frac{2}{j_x \tau_x} \frac{I_5}{I_2} - \frac{2}{\tau_x} \varepsilon_x \qquad (56)$$

Where C_q is the quantum radiation constant that we saw earlier (53).

The fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|} ds, \qquad \text{where} \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2 \qquad (57)$$

Equilibrium horizontal emittance

Using (56) we see that there is an equilibrium horizontal emittance ε_0, for which that damping an excitation rates are equal:

$$\frac{d\varepsilon_x}{dt} = 0 \quad \text{when} \quad \varepsilon_x = \varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \tag{58}$$

Note that ε_0 is determined by the square of the beam energy, the lattice functions (beta and dispersion) in the dipoles, and the bend radius and gradient of the dipoles.

ε_0 is sometimes called the 'natural emittance' of the lattice, since it includes only the most fundamental effects that contribute to the emittance: radiation damping and quantum excitation (collective effects are neglected).

Typically, third generation synchrotron light sources have natural emittances of the order of a few nanometres and the latest fourth generation of rings are reaching values below 100 picometres. With beta functions of a few metres, this implies horizontal beam sizes in the range of tens of microns.

Quantum excitation of vertical emittance

Finally, let us consider the quantum excitation of vertical emittance.

In principle, we can apply the formulae that we derived for the quantum excitation of the horizontal emittance, making the appropriate substitutions of vertical quantities for the horizontal ones.

In many storage rings however, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$.

However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

Quantum excitation of vertical emittance

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by (see ref. [6]):

$$\mathcal{E}_y = \frac{13}{55} C_q \frac{1}{j_y I_2} \oint \frac{\beta_y}{|\rho^3|} ds \tag{59}$$

Note that, unlikely the horizontal emittance, the natural vertical emittance does not scale with beam energy.

In most storage rings the natural vertical emittance is an extremely small value, typically 3-4 orders of magnitude smaller than the horizontal emittance.

In practice, the actual vertical emittance is dominated by magnet alignment errors or deliberately-introduced betatron coupling and vertical dispersion. Storage rings typically operate with a vertical emittance that is of the order 0.1-1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

Summary (1): synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma \approx 8.846 \times 10^{-5} \text{m/GeV}^3 \tag{60}$$

The emittances (ε_x, ε_y and ε_z) damp exponentially:

$$\varepsilon(t) = \varepsilon(0) \exp\left(-2\frac{t}{\tau}\right) \tag{61}$$

The radiation damping times are:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0} T_0, \qquad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0} T_0, \qquad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0} T_0 \tag{62}$$

The damping partition numbers are (Robinson theorem: $j_x + j_y + j_z = 4$):

$$j_x = 1 - \frac{I_4}{I_2}, \qquad j_y = 1, \qquad j_z = 2 + \frac{I_4}{I_2} \tag{63}$$

Summary (2): beam dynamics with synchrotron radiation

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-2\frac{t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-2\frac{t}{\tau}\right)\right] \qquad (64)$$

The natural emittance is:

$$\varepsilon_0 = C_q\gamma^2\frac{I_5}{j_x I_2}, \qquad C_q = 3.832\times 10^{-13}\mathrm{m} \qquad (65)$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q\gamma^2\frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_c c}{\omega_s}\sigma_\delta \qquad (66)$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\cos(\phi_s)\frac{\omega_{RF}}{T_0}\alpha_c, \qquad \sin(\phi_s) = \frac{U_0}{eV_{RF}} \qquad (67)$$

Summary (3): synchrotron radiation integrals

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds, \tag{68}$$

$$I_2 = \oint \frac{1}{\rho^2} ds, \tag{69}$$

$$I_3 = \oint \frac{1}{|\rho^3|} ds \tag{70}$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \tag{71}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{\rho} ds \tag{72}$$

The chromatic (or dispersion) invariant is:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2 \tag{73}$$

References

These lectures are based on ones prepared by A. Wolski and given at the CERN Advanced Accelerator School in November 2022, Sevrier, France:

https://indico.cern.ch/event/1126689/

Other useful references on this topic include:

[1] JD Jackson, “Classical Electrodynamics”, 3rd Ed. Wiley, (1999)

[2] S.Y. Lee, “Accelerator Physics”, World Scientific Publishing, (1999)

[3] M. Sands, “The Physics of Electron Storage Rings: An Introduction”, SLAC-121, (1970)

[4] H. Weidemann, “Particle Accelerator Physics I”, Springer, (2003)

[5] H. Weidemann, “Particle Accelerator Physics II”, Springer, (2003)

[6] T. Raubenheimer, “The Generation and acceleration of low emittance flat beams for future linear colliders”, SLAC Report 387, p.19, (1991)

Of course, this list is far from complete!