Free Electron Lasers (FEL)

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The CERN Accelerator School

Advanced Accelerator Physics
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The CERN Accelerator School

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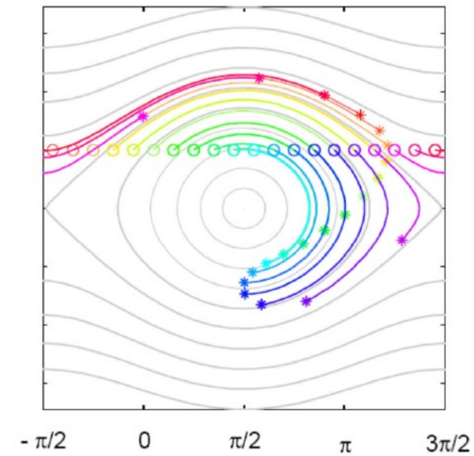
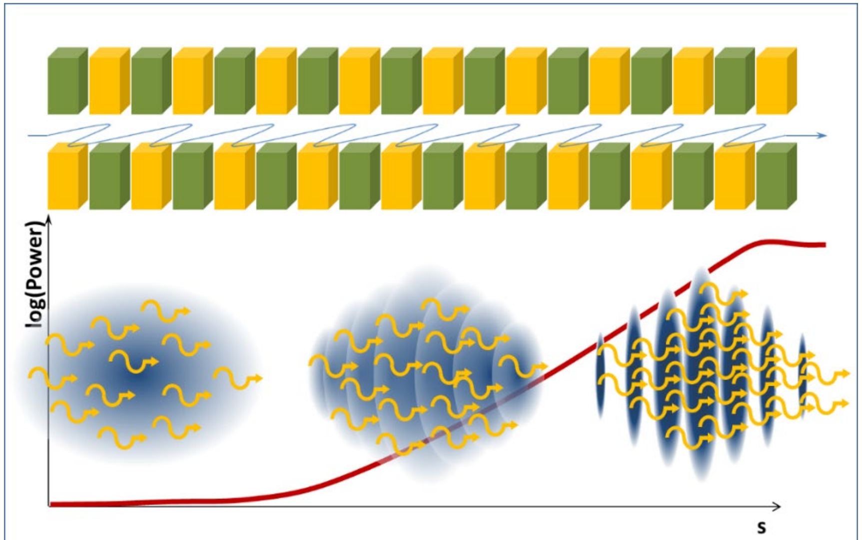
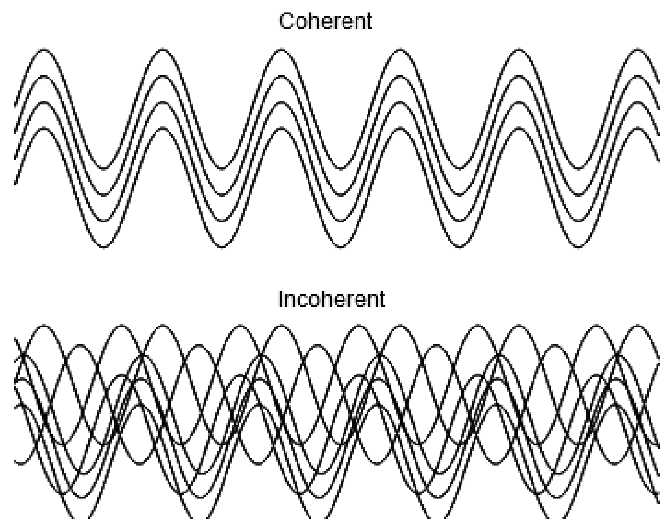




https://www.helmholtz-berlin.de/projects/berlinpro/erl-intro/linac-fel_en.html

Coherence

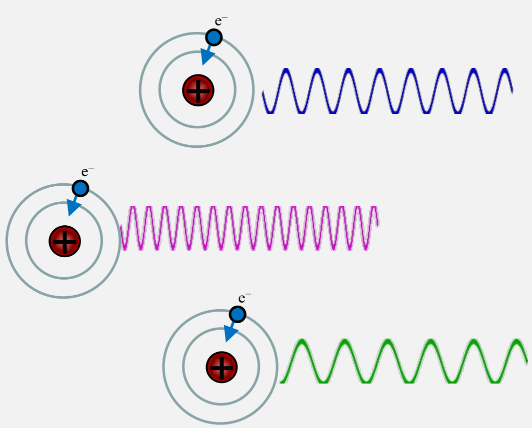
*2 waves are said to be coherent if they have a **constant relative phase!***



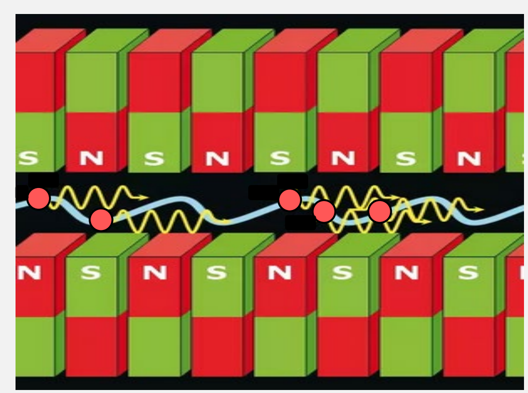
Coherent light can interfere!

Spontaneous emission typically generates incoherent light:

atoms / molecules

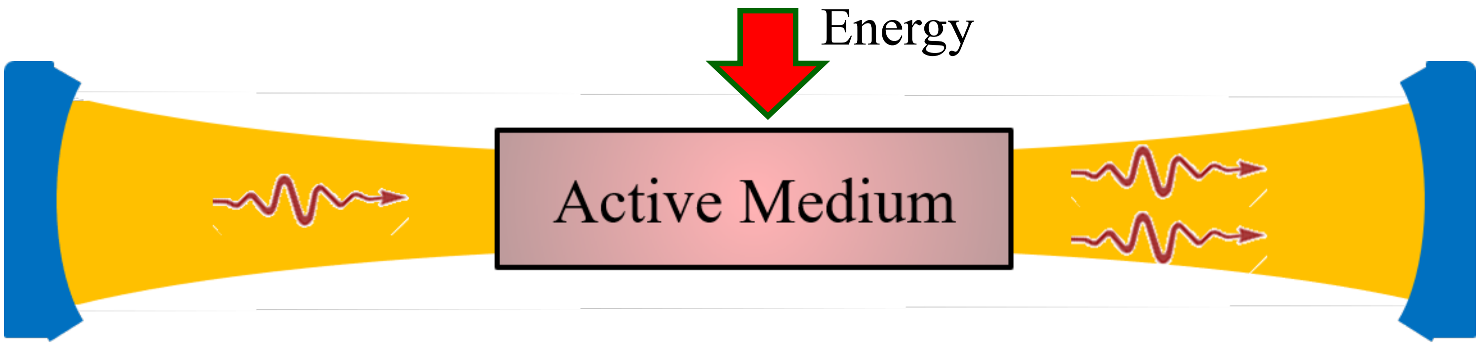


many e⁻ in an undulator



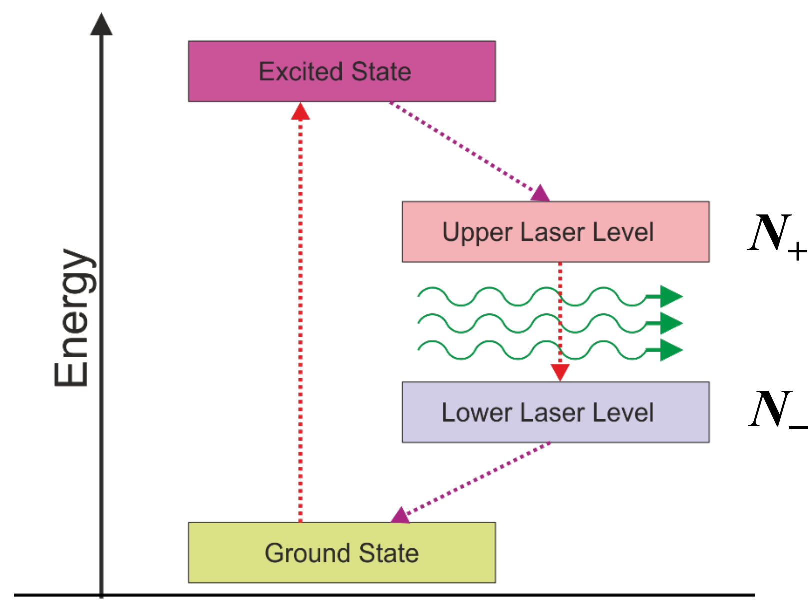
Optical Laser

Laser: **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation



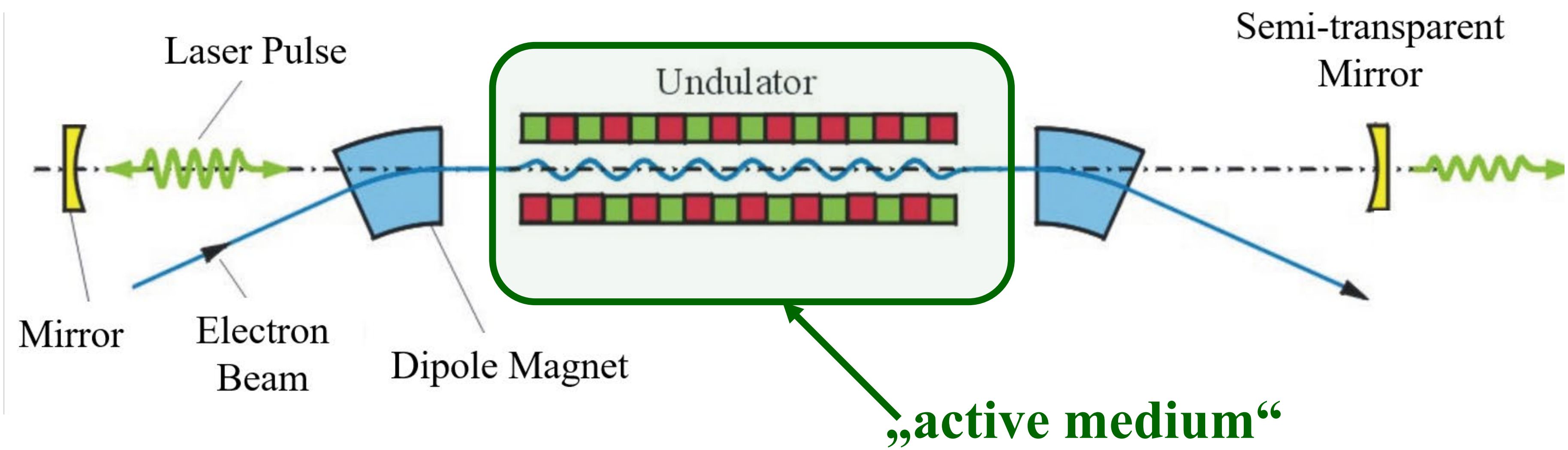
Amplification requires a **population inversion**:

→ 4 Level System



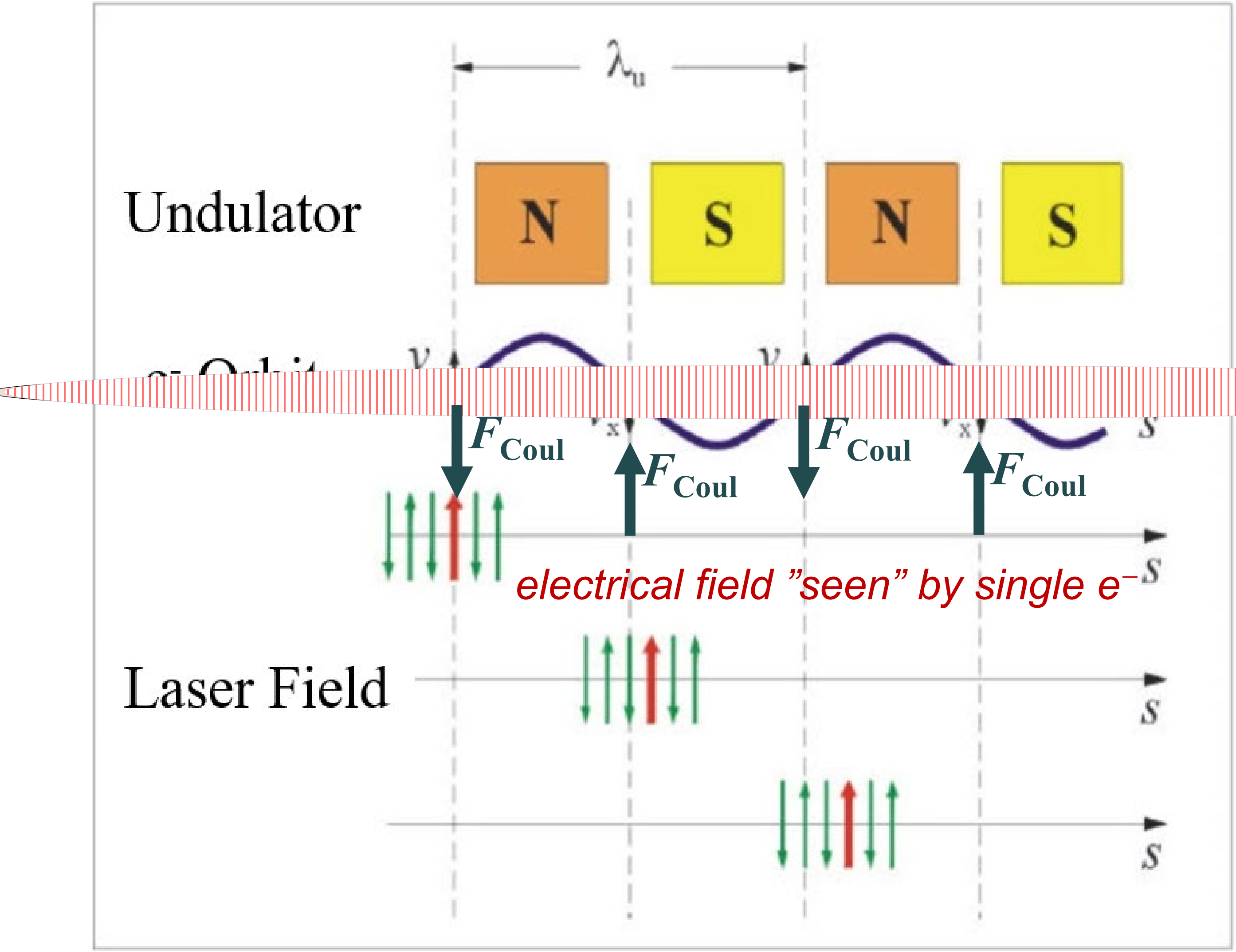
Free Electron Laser

Electron Beam in Undulator serves as active medium!



Population Inversion??

FEL Amplification



Typical Dimensions

$\lambda_U \approx \mathrm{cm}$

$\sigma_e > 0.1\ \mathrm{mm}$

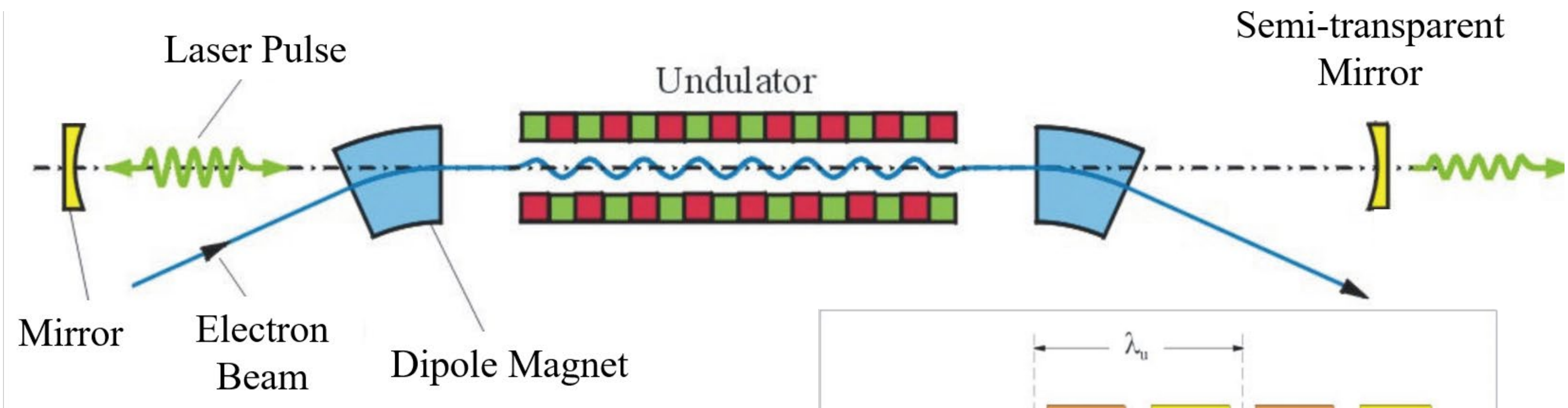
$\lambda_L \approx \mathrm{nm}$

required bunching:

$$\sigma_e < \frac{1}{2}\lambda_L$$

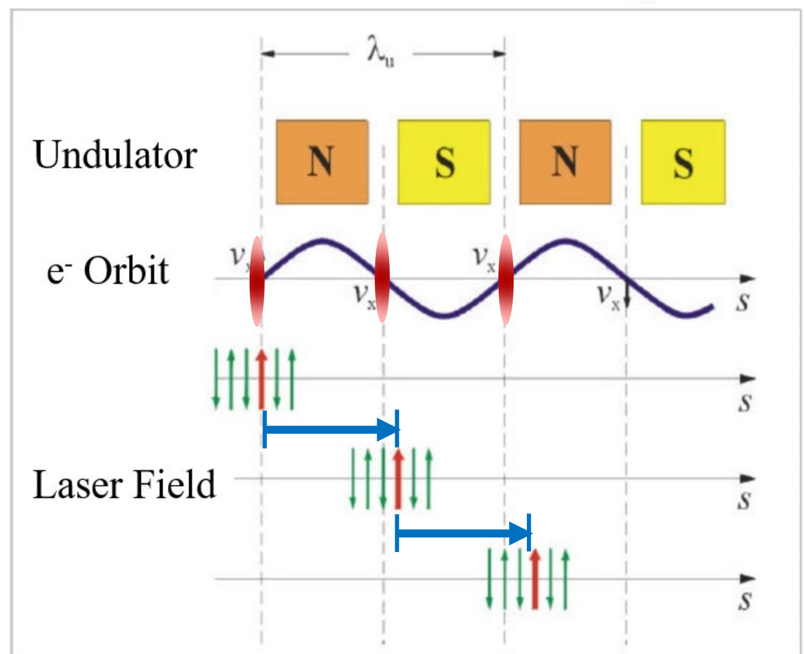
Free Electron Laser

Electron Beam in Undulator serves as active medium!



Amplification requires **microbunching!**

and in addition:
→ correct phase slippage
→ correct relative phasing



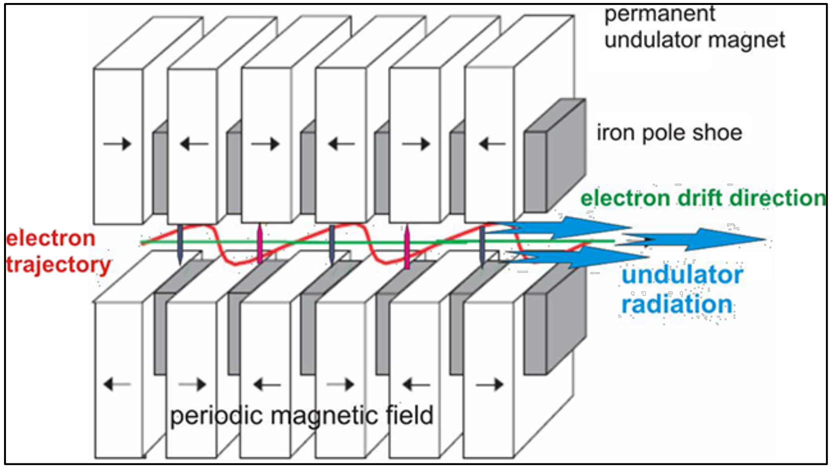
Recap: Undulator Radiation

Particle orbit in an undulator:

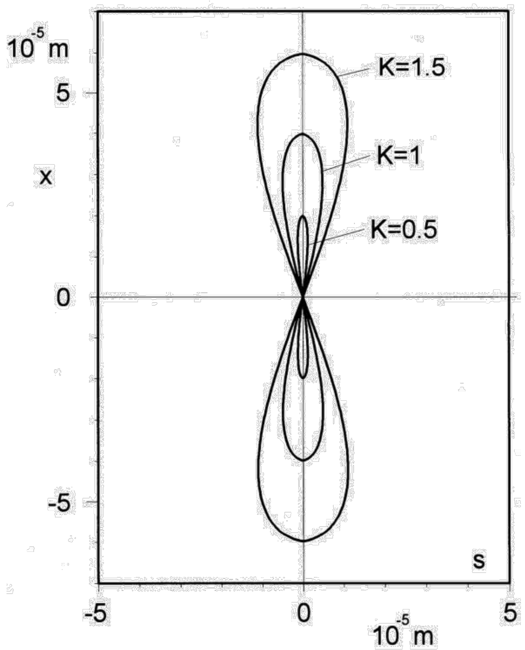
$$x(t) = \frac{K}{\gamma k_u} \cdot \sin(\omega_u t)$$

$$s(t) = \bar{\beta} c t - \frac{K^2}{8\gamma^2 k_u} \cdot \sin(2\omega_u t)$$

$$\bar{\beta} = 1 - \frac{1}{2\gamma^2}\left\{1 + \frac{K^2}{2}\right\}$$



Taken from Schmüser/Dohlus/Rossbach/Behrens



Coherence condition in forward direction:

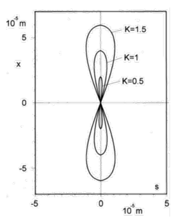
$$\lambda_L = \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \cdot \lambda_u = \left(1 - \bar{\beta}\right) \cdot \lambda_u$$

Radiation power per e⁻ (1st harmonic):

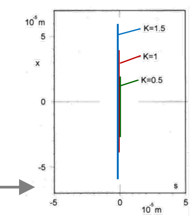
$$P = \frac{e^2 c \gamma^2 K^2 k_u^2}{12\pi\varepsilon_0 \left(1 + K^2/2\right)^2}$$

Single Electron Energy Change with the Laser Field

Remark:
In the following, we want to neglect the longitudinal oscillation completely in order to achieve the aim (understanding!) preferably simply and fast. For a correct treatment, we then would have to modify the K parameter accordingly to (without proof):



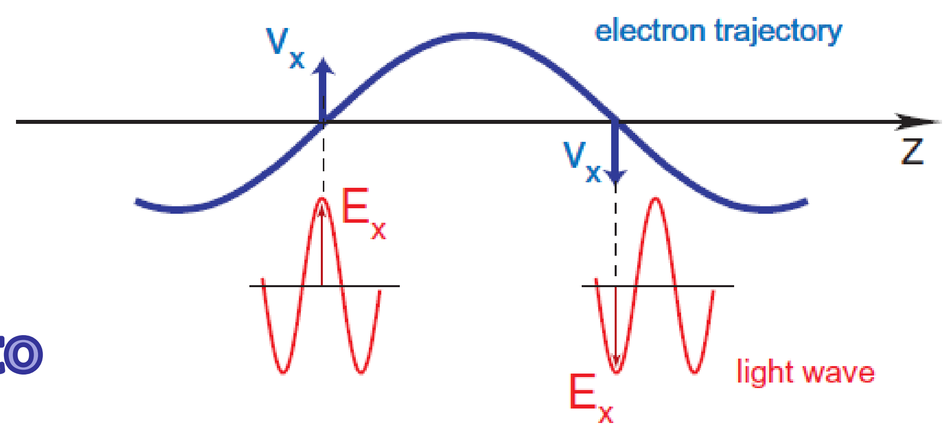
$$K \rightarrow K_{JJ} = K\left\{J_0\left(\frac{K^2}{4+2K}\right) - J_1\left(\frac{K^2}{4+2K}\right)\right\}$$



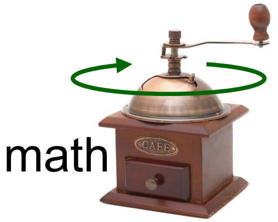
Energy change of a single electron in an externally generated laser field

$$\boxed{\frac{\mathrm{dW}}{\mathrm{d}t} = \vec{F}\cdot\vec{v} = -e\,E_x(t)\,v_x(t)}$$

add. energy gain/loss due to interaction with EM field



Taken from Schmüser/Dohlus/Rossbach/Behrens



Energy Exchange

We derived for the transverse electron orbit

$$v_x = \dot{x} = c \cdot \frac{K}{\gamma} \cos\left(k_u s\right), \qquad k_u = \frac{2\pi}{\lambda_u}$$

and the radiation field

relative phase to electron oscillation

$$E_x(t) = E_0 \cos\left(\omega_L t - k_L s + \phi_L\right), \qquad k_L c = \omega_L$$

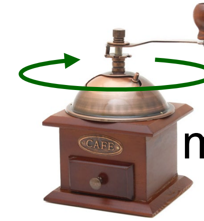
and with $\cos\alpha \cdot \cos\beta = \frac{1}{2}\left\{\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right\}$

$$\frac{\mathrm{dW}}{\mathrm{d}t} = -e E_x(t)\, \dot{x} = -e \frac{K_{JJ} c}{\gamma} \cos\left(k_u s\right) E_0 \cos\left(k_L s - \omega t + \phi_L\right)$$

$$= -e \frac{K_{JJ} c}{2\gamma} E_0 \left\{ \cos\underbrace{\left(\left(k_L + k_u\right) s - \omega t + \phi_L\right)}_{=\psi} + \cos\underbrace{\left(\left(k_L - k_u\right) s - \omega t + \phi_L\right)}_{=\chi} \right\}$$

→ Definition of the two phases ψ and χ !

Energy Exchange



→ Energy variation is depending on 2 phases ψ and χ :

$$\boxed{\frac{\mathrm{dW}}{\mathrm{d}t} = -e\frac{K_{JJ}\,c}{2\gamma}E_0\left(\cos\psi + \cos\chi\right)}$$

The phase Ψ is slowly varying and $\dot{\Psi} = 0$ on resonance!

$$\dot{\psi} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L + k_u\right)s - \omega t + \phi_L\right\} = \left(k_L + k_u\right)\bar{\beta}c - \omega \underset{\uparrow\ \boxed{\omega = k_L c}}{=} \left(k_L + k_u\right)\bar{\beta}c - k_L c$$

$$= \left[k_u\bar{\beta} - \left(1-\bar{\beta}\right)k_L\right]c$$

since for the resonant k_L of the light wave (coherence condition!) we have

Coherence Condition $\xrightarrow{\text{page 9}}$ $k_u = \left(1-\bar{\beta}\right)\cdot k_L \qquad \rightarrow \qquad \dot{\psi} = k_u c\underbrace{\left(\bar{\beta}-1\right)}_{\approx 0} \approx 0$

The other phase χ is rapidly changing (by 4π over one undulator period!):

$$\dot{\chi} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L - k_u\right)s - \omega t + \phi_L\right\} = \left[-k_u\bar{\beta} - \left(1-\bar{\beta}\right)k_L\right]c = -2k_u c$$

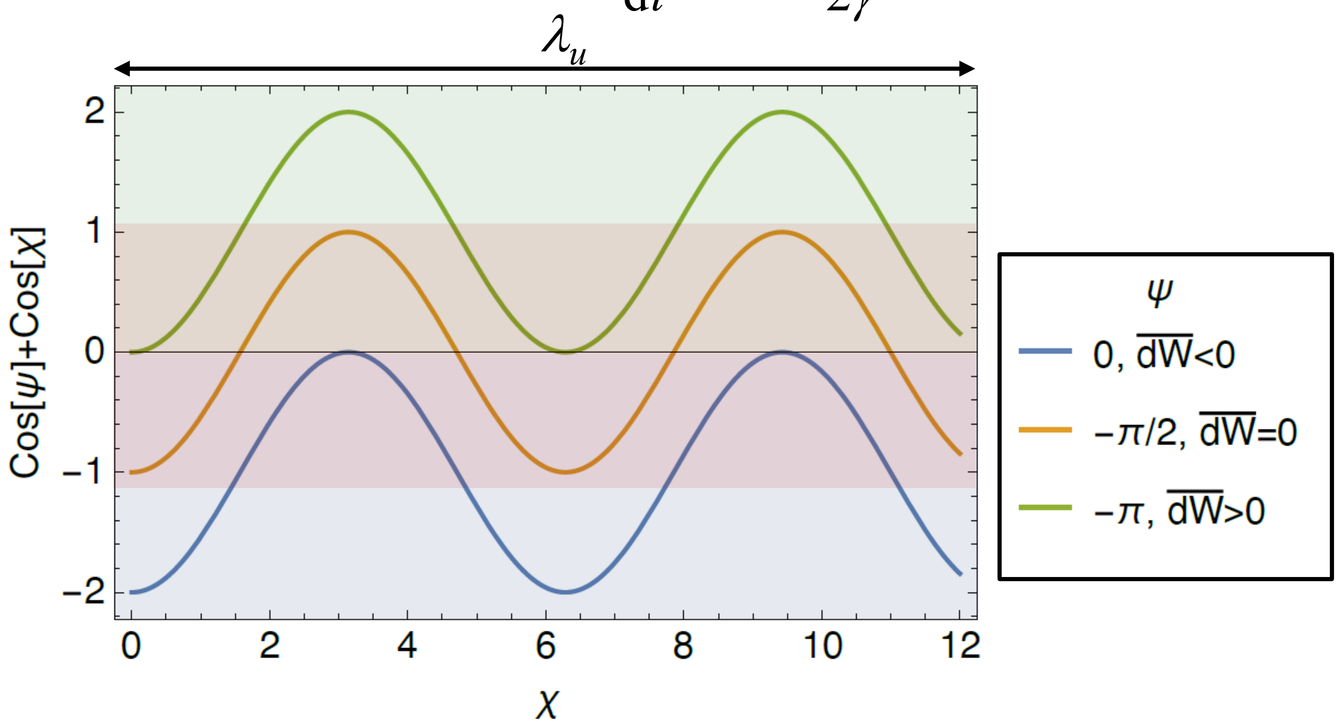
Ponderomotive Phase θ

Phase dependency of the energy exchange: $\frac{\mathrm{dW}}{\mathrm{d}t} = -e\frac{K_{JJ}\,c}{2\gamma}E_0\left(\cos\psi + \cos\chi\right)$

Electron:

Energy Gain →

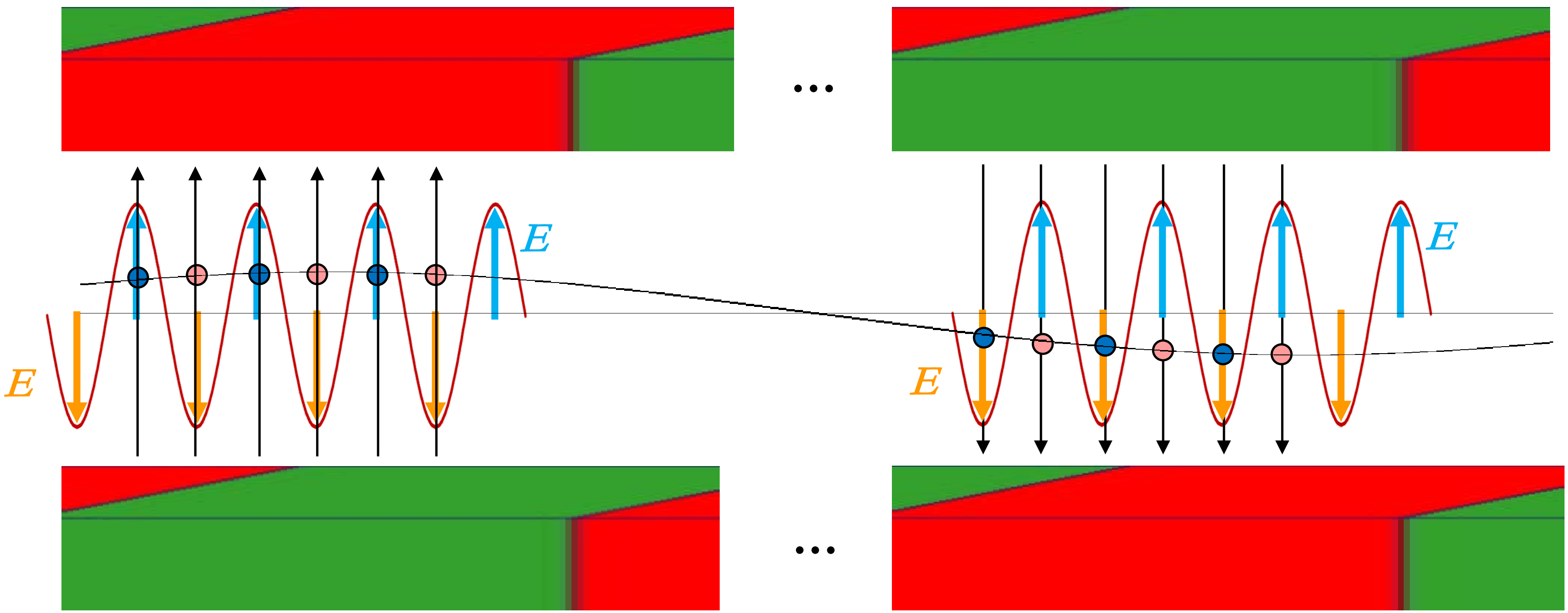
Energy Loss →



Ponderomotive Phase: $\theta = \psi + \pi/2$

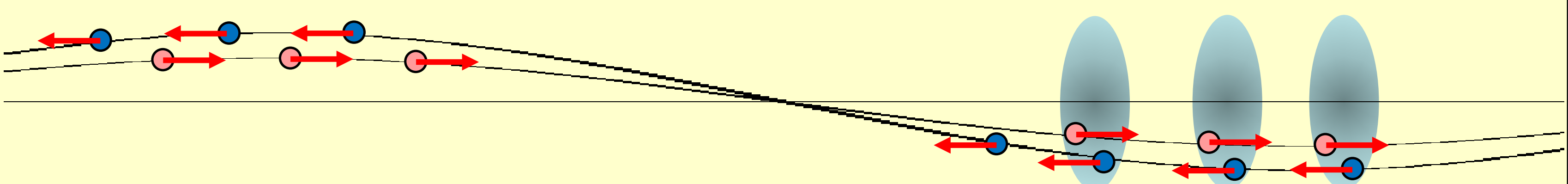
- $-\pi < \theta < 0$: average energy transfer from EM field to electron
- $\theta = 0$: no average energy exchange
- $0 < \theta < +\pi$: average energy transfer from electron to EM field

Micro Bunching



Energy Modulation:
- gain energy
- loose energy

Density Modulation:
- catch up
- fall back



Key Parameters

Findings so far:

- average electron energy loss/gain: $\boxed{\left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle = -e \frac{K_{JJ}\, c}{2\gamma} E_0 \sin\theta}$
- on resonance ($\gamma = \gamma_{res}$), the ponderomotive phase is constant, $\dot{\theta} = 0$!

But:

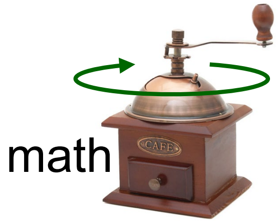
Electron energy loss or gain will cause

- change of electron's kinetic energy and Lorentz γ,
- change of the ponderomotive phase θ.

interrelated!

Key parameters are therefore:

- **ponderomotive phase $\boldsymbol{\theta}$** with: $\theta = \left(k_L + k_u\right)s - \omega t + \phi_L + \pi/2$
- **relative energy deviation $\boldsymbol{\eta}$** with: $\eta = \frac{\gamma - \gamma_{res}}{\gamma_{res}}$
- **normalized field amplitude $\boldsymbol{\varepsilon}$** with: $\varepsilon = \frac{eE_0 K_{JJ}}{2m_0 c^2 \gamma_{res}^2}$



Phase Equation

Change of the ponderomotive phase (cf. page 10):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L + k_u\right)s - \omega t + \phi_L + \pi/2\right\} = \ldots = c\left[k_u\bar{\beta} - \left(1-\bar{\beta}\right)k_L\right]$$

Now:

$$\bar{\beta} = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) = 1 - \frac{X}{\gamma^2} \qquad \text{with} \qquad X = \frac{1}{2}\left(1+\frac{K^2}{2}\right) \overset{K\approx 1}{\approx} 1$$

Coherence Condition: $\xrightarrow{\text{page 9}}$ $k_u = k_L \cdot \dfrac{1}{2\gamma_{res}^2}\left(1+\dfrac{K^2}{2}\right) = k_L \dfrac{X}{\gamma_{res}^2}$

gives:

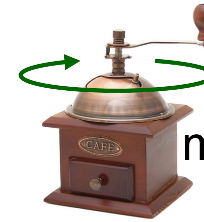
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = c\left[k_u \underbrace{\left(1-\frac{X}{\gamma^2}\right)}_{\bar{\beta}} - \underbrace{\frac{X}{\gamma^2}}_{1-\bar{\beta}} \underbrace{\frac{\gamma_{res}^2}{X}k_u}_{k_L}\right] = ck_u\left[\left(1-\underbrace{\frac{X}{\gamma^2}}_{\approx 0}\right) - \frac{\gamma_{res}^2}{\gamma^2}\right] \approx ck_u\left(1-\frac{\gamma_{res}^2}{\gamma^2}\right)$$

and with:

$$\frac{\gamma_{res}^2}{\gamma^2} = \frac{1}{\left(\eta+1\right)^2} \approx 1 - 2\eta \qquad \text{for} \quad \eta \ll 1$$

Finally: $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 2ck_u\eta \quad \rightarrow \quad \dfrac{\mathrm{d}\theta}{\mathrm{d}s} = 2k_u\eta$

Energy Equation



We rewrite:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{\gamma_{res}} \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{\gamma_{res}} \frac{1}{m_0 c^2} \left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle$$

and with

$$\left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle = -e \frac{K_{JJ}\, c}{2\gamma} E_0 \sin\theta \longleftarrow \boxed{\text{Page 14}}$$

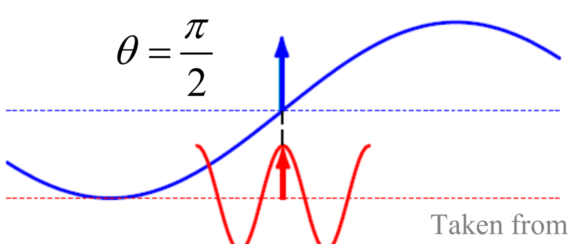
one obtains:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -e \frac{K_{JJ}\, c}{2 m_0 c^2 \gamma_{res}^2} E_0 \sin\theta = -\varepsilon \cdot c \cdot \sin\theta$$

Finally:

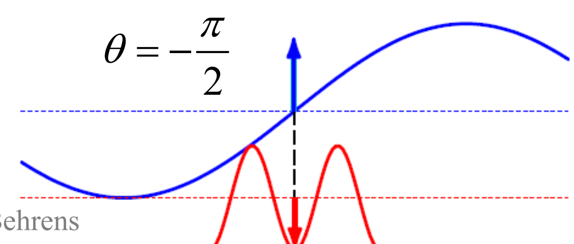
$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\varepsilon c \sin\theta \quad \xrightarrow{s \approx ct} \quad \boxed{\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\varepsilon \sin\theta}$$

energy transfer from electron to light wave



“stimulated emission”

energy transfer from light wave to electron



“absorption”

Taken from Schmüser/Dohlus/Rossbach/Behrens

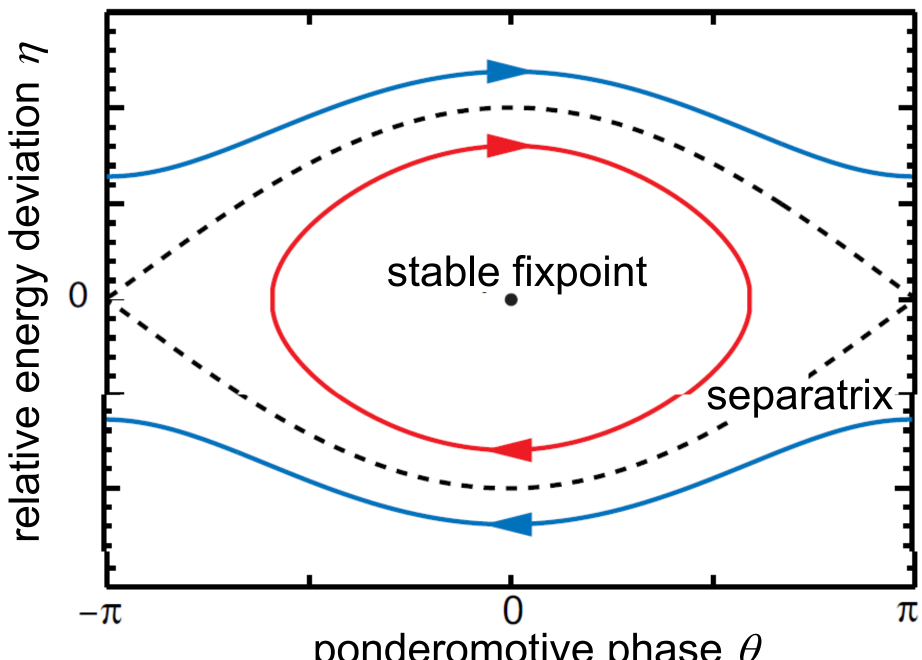
Pendulum Equations

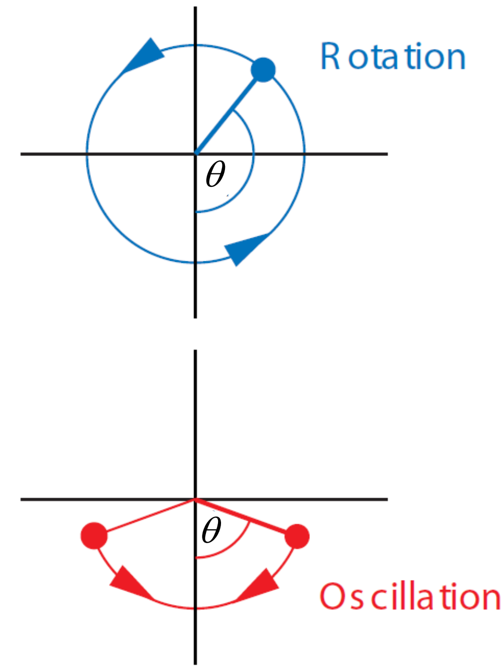
Phase equation: $$\frac{\mathrm{d}\theta}{\mathrm{d}s} = 2k_u\eta$$

Energy equation: $$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\varepsilon \sin\theta$$

combined: $$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2k_u\varepsilon \sin\theta = 0$$

Pendulum!





Important: separatrix depends on intensity of laser field $\eta_{s,\max} = \sqrt{2\varepsilon / k_u}$

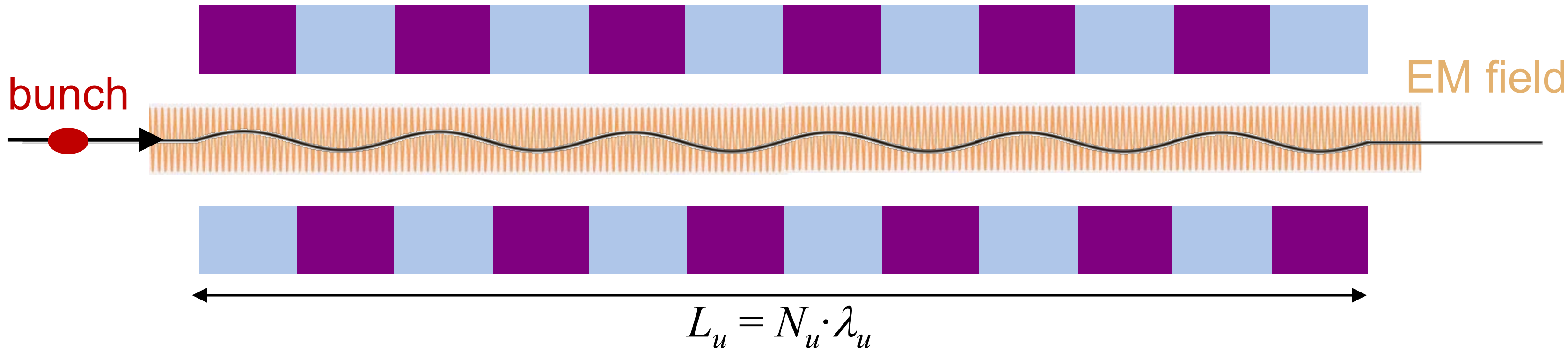
Electron Bunch ↔ Laser Field

So far:
Interaction of a **single electron** with an externally generated laser field when co-propagating through an undulator

Now:
Consider an electron bunch of length $\sigma_b >> \lambda_L$

Simplifying assumptions:

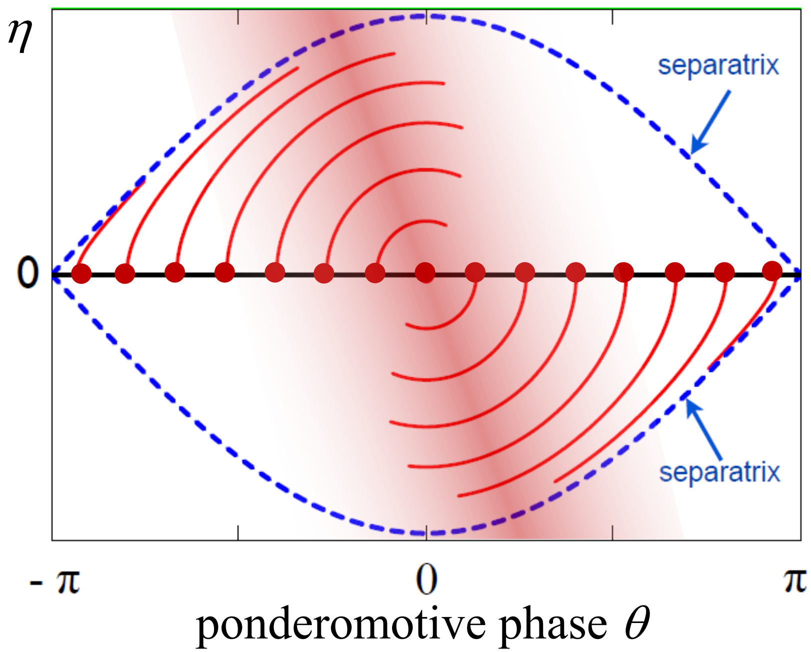
- laser field does not change significantly during bunch passage (**E = const.**)
- "ideal" electron bunch with vanishing energy spread ($\boldsymbol{\sigma_\gamma = 0}$)
- simple quasi 1D treatment of the problem ($\boldsymbol{\sigma_x , \sigma_y \rightarrow 0}$)
- neglect spontaneous emission of undulator radiation



Electron Bunch ↔ Laser Field

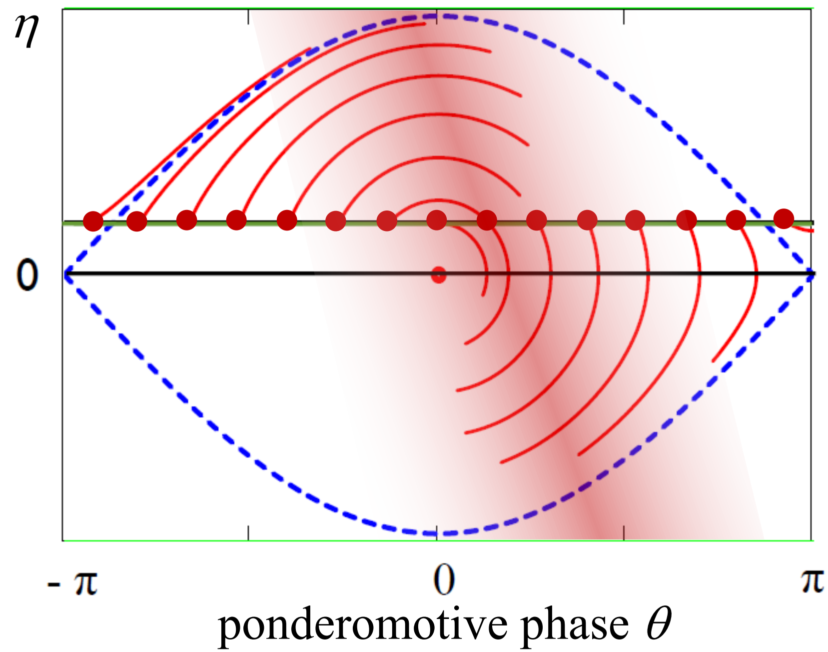
Electron Injection

on resonance: $\gamma = \gamma_{res}$



→ energy modulation
→ density modulation
no net energy transfer!

above resonance: $\gamma > \gamma_{res}$



→ energy modulation
→ density modulation
net energy transfer!

Gain Function

FEL gain function *G* defined as relative growth of laser light intensity I_L:

$$\boxed{G = \frac{\Delta I_L}{I_L}} \qquad \text{with} \qquad I_L = \varepsilon_0 E^2 \cdot V$$

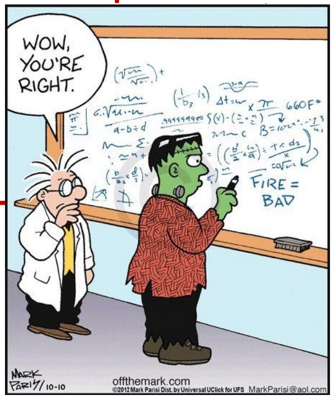
Since amplification = growth of laser light intensity is caused by energy transfer from N_e electrons to the laser field, we have (with $n_e = N_e / V$)

$$G = -\frac{m_0 c^2 N_e \langle \Delta\gamma \rangle}{I_L} = -\frac{\gamma_{res} m_0 c^2 n_e \langle \eta(s = L_u) \rangle}{\varepsilon_0 E_0^2}$$

A somehow lengthy Taylor expansion up to second order for small ε ($\varepsilon << k_u^{-1} L_u^{-2}$) gives

$$G = \frac{\pi e^2 L_u^3 E_0^2 K_{JJ}^2 n_e}{4\gamma_{res}^3 m_0 c^2 \varepsilon_0 \lambda_u} \frac{\mathrm{d}}{\mathrm{d}\xi_0} \left(\frac{\sin \xi_0}{\xi_0} \right)^2$$

where $\qquad \xi_0 = k_u L_u \eta_0 \qquad$ at the undulator entrance ($s = 0$)

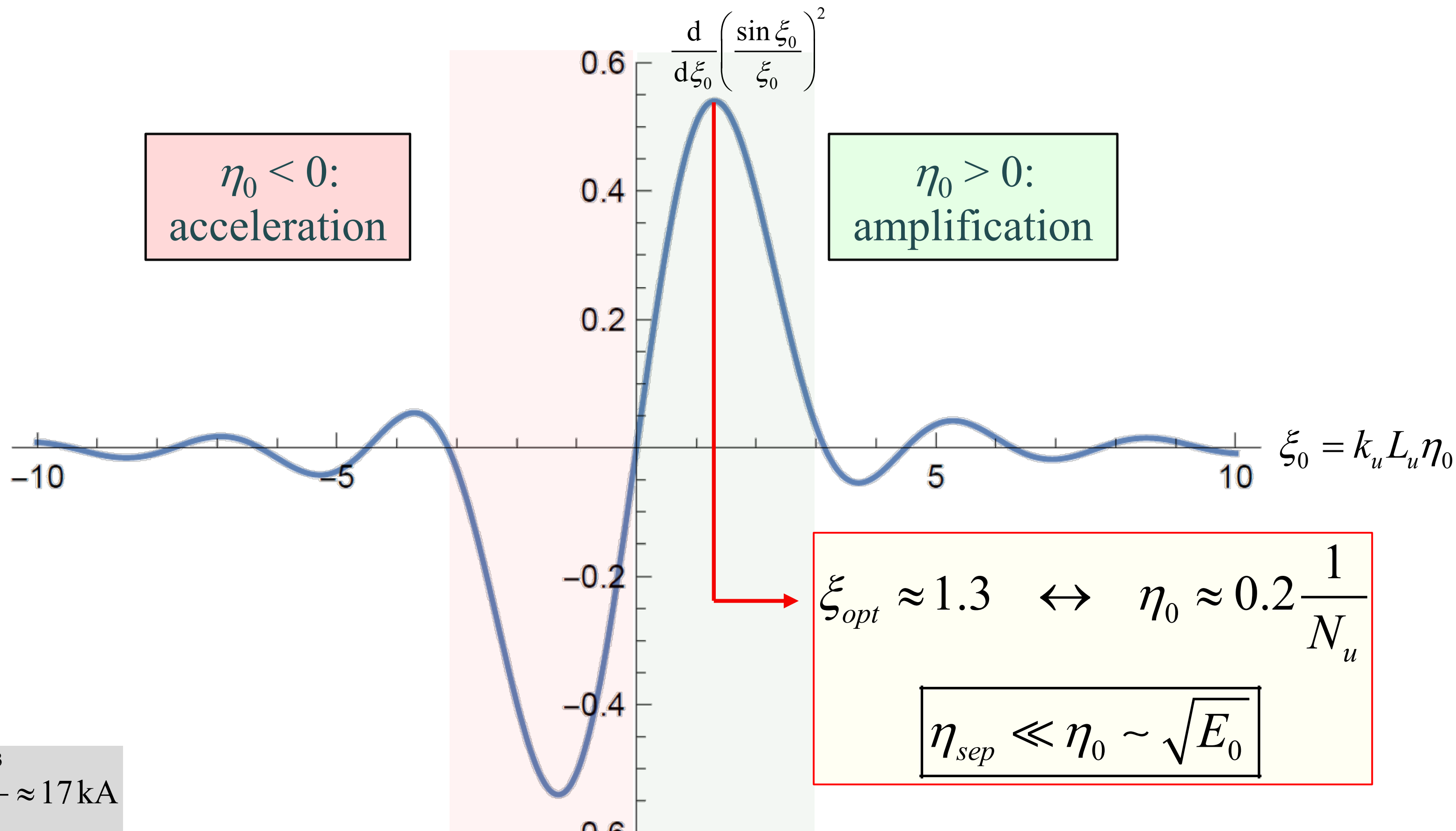


Madey Theorem

$$G=\frac{\pi}{2}\frac{K_{JJ}^{2}N_{u}^{3}}{\gamma_{r}^{3}}\frac{I_{\text{beam}}}{I_{\text{Alfvén}}}\left(\frac{\lambda_{u}}{\sigma_{r}}\right)^{2}\frac{\mathrm{d}}{\mathrm{d}\xi_{0}}\left(\frac{\sin\xi_{0}}{\xi_{0}}\right)^{2}$$

$$\boxed{\rightarrow\quad G\sim\frac{\mathrm{d}}{\mathrm{d}\omega}I_{L}(\omega)}$$

Gain curve ~ derivative of the spontaneous spectrum! (Madey 1979)



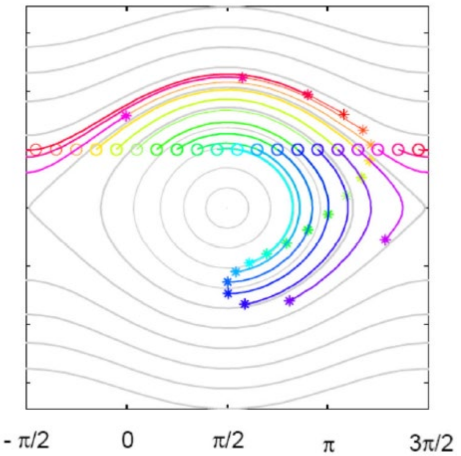
$$I_{\text{Alfvén}}=\frac{4\pi\varepsilon_0 c^3}{e}\approx 17\,\text{kA}$$

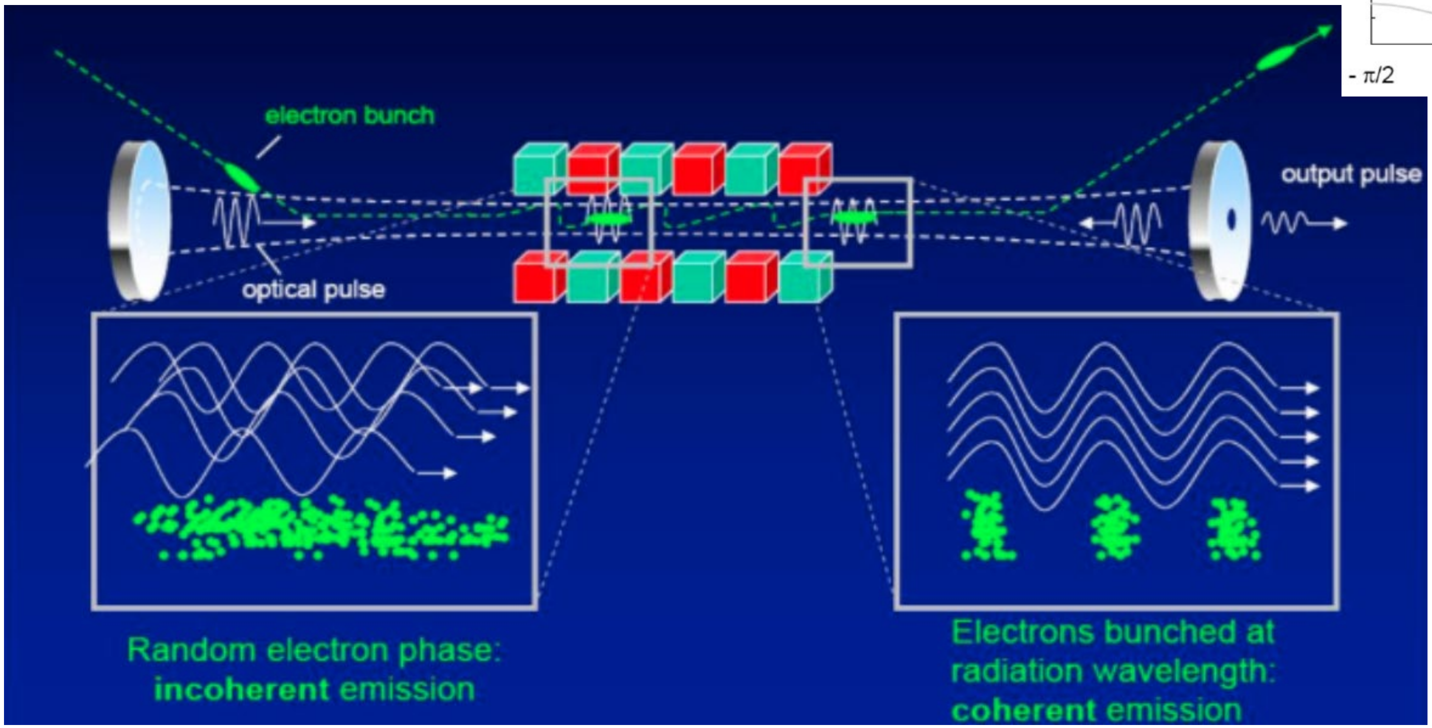
Low Gain FEL

Injection with energy above resonance energy:

- Energy modulation
  - Density modulation
    - Energy transfer

Intensity build-up over many passes!



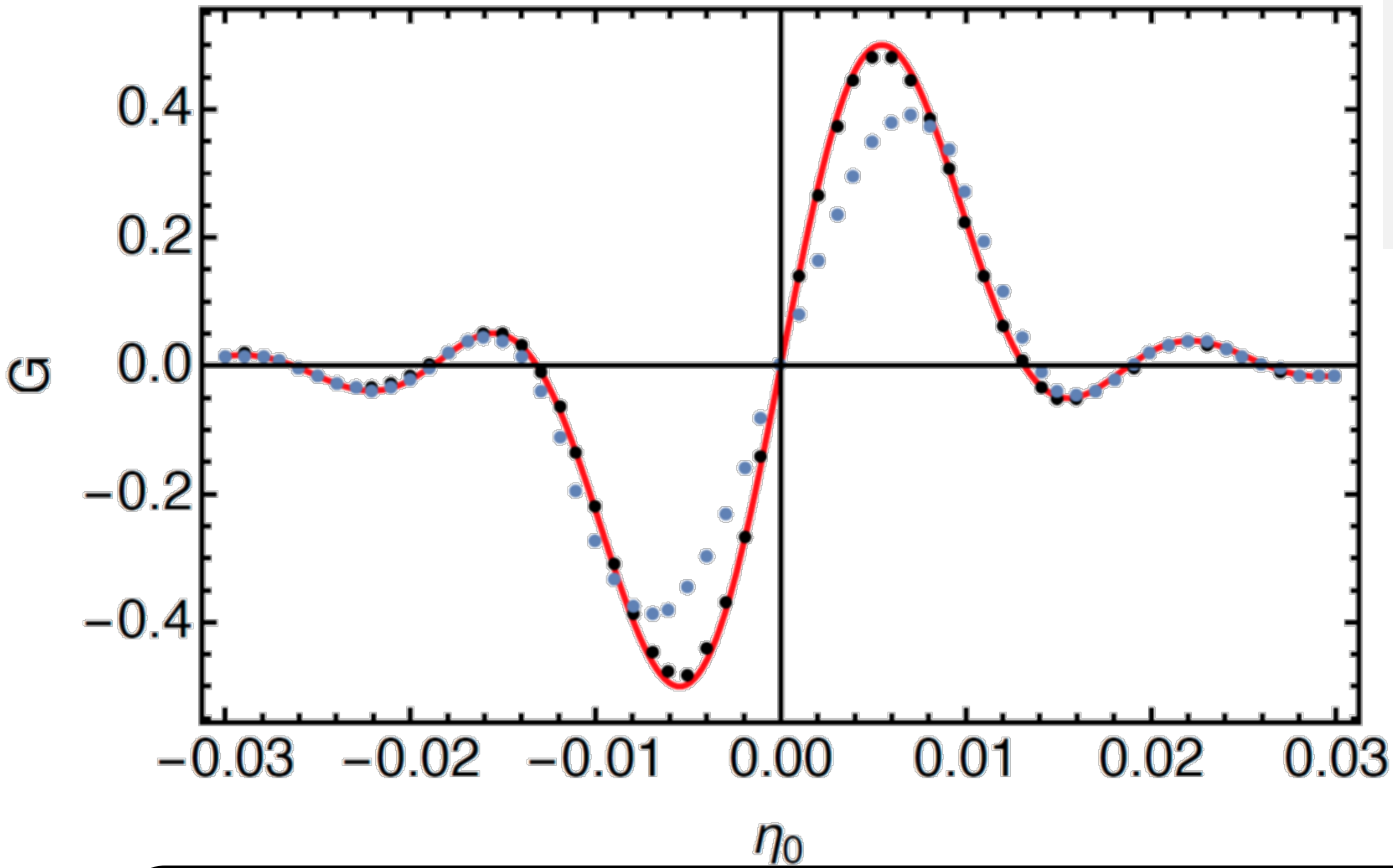


Taken from http://www.stfc.ac.uk/astec/17452.aspx

Gain Curve

Gain curve ↔ Madey Theorem:

Deviation for strong radiation fields



Example:

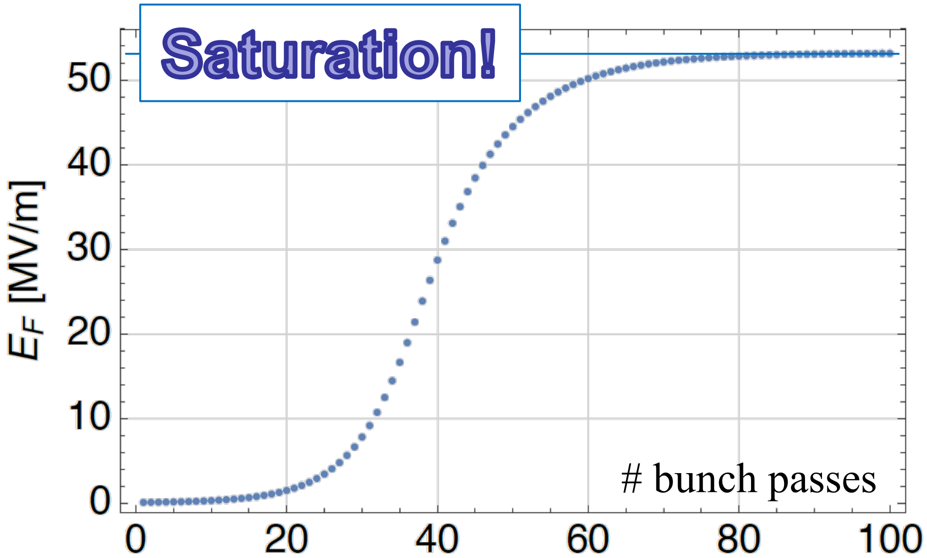
IR-FEL FELIX

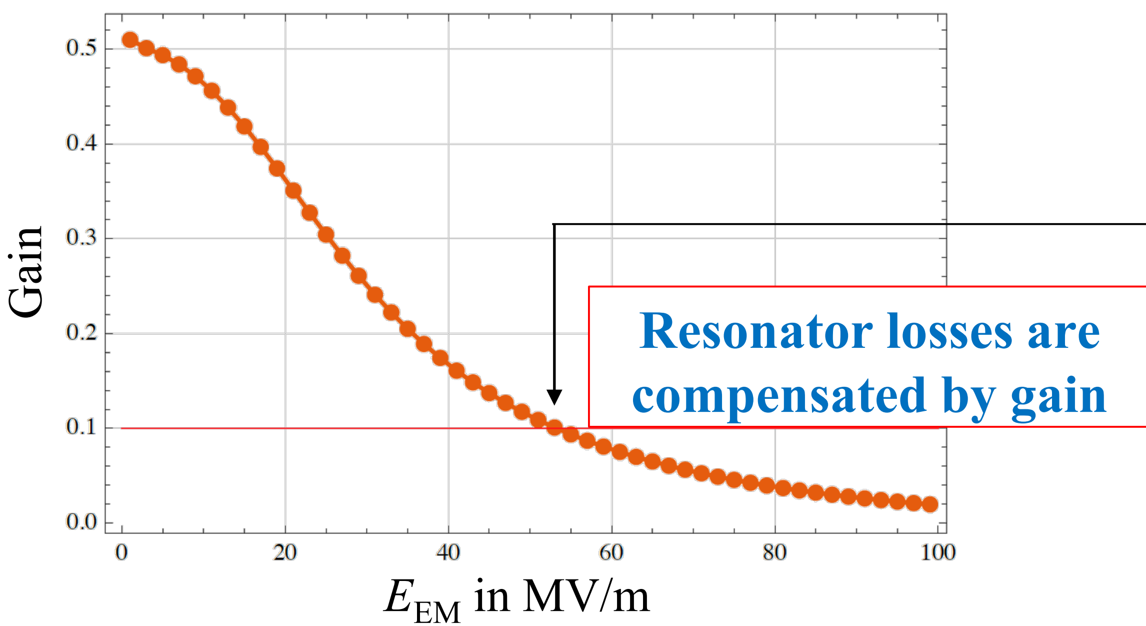
(U Nijmegen)

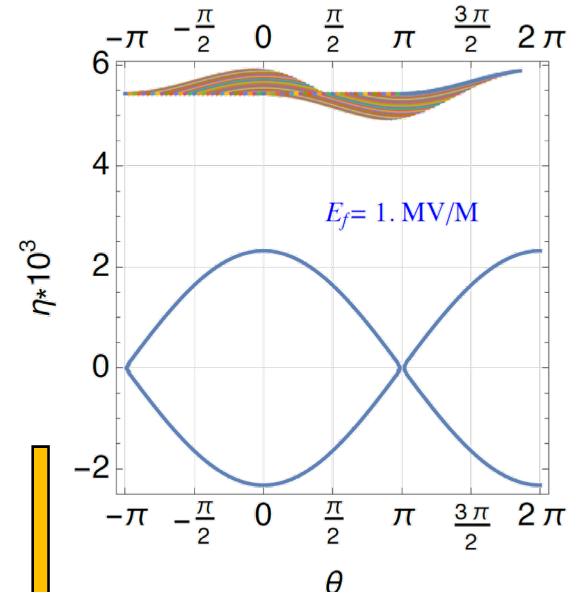
λ_L	$20\,\mu m$	L_u	$2.47\,m$
λ_u	$65\,mm$	N_u	38
K	0.5	γ_r	42.76
K_{JJ}	0.4857	η_{opt}	$5.4 \cdot 10^{-3}$
$\sigma_{x,y}$	$2\,mm$	W_b	$22\,MeV$
σ_z	$0.9\,mm$	I_b	$57.3\,A$
Q_b	$172\,pC$	dummy	0

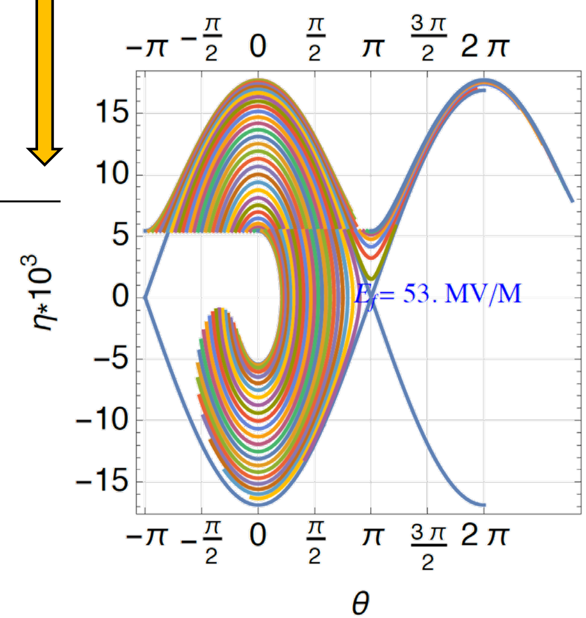
Gain drops with increasing laser intensity!

Saturation


Saturation!
E_F [MV/m]
0
10
20
30
40
50
0
20
40
60
80
100
bunch passes

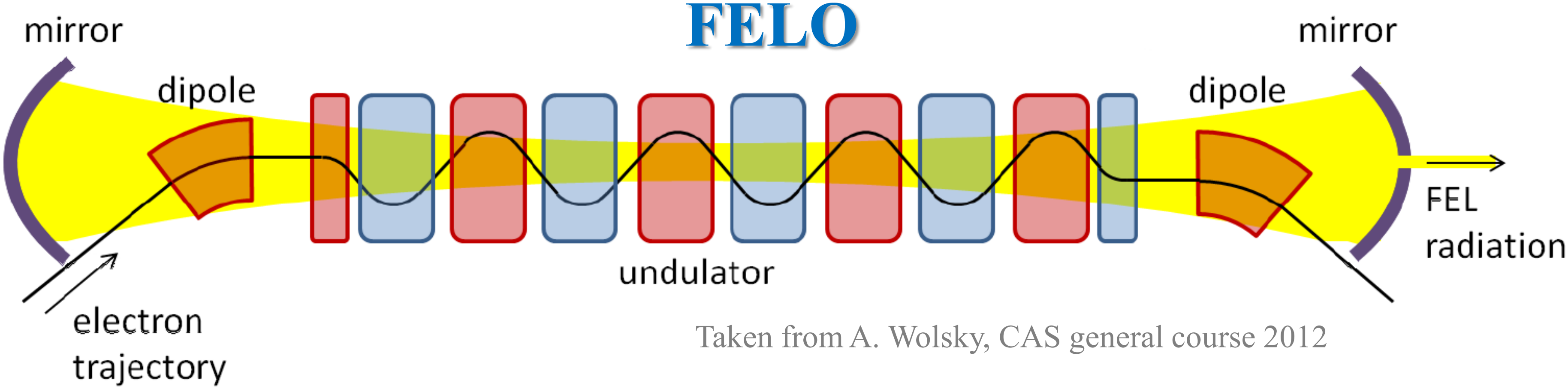

Gain
0.0
0.1
0.2
0.3
0.4
0.5
0
20
40
60
80
100
E_{EM} in MV/m
Resonator losses are compensated by gain


$-\pi$ $-\frac{\pi}{2}$ 0 $\frac{\pi}{2}$ π $\frac{3\pi}{2}$ 2π
$\eta{*}10^3$
6
4
2
0
−2
E_f= 1. MV/M
θ


$-\pi$ $-\frac{\pi}{2}$ 0 $\frac{\pi}{2}$ π $\frac{3\pi}{2}$ 2π
$\eta{*}10^3$
15
10
5
0
−5
−10
−15
E_f= 53. MV/M
θ

Efficiency

FELO



Optimum undulator length for FELO:

- sufficient gain to compensate resonator losses: $G \sim N_u^{\,3}$
- high efficiency of energy transfer: $\Delta\eta_{sat} \approx 3/N_u$

→ ***some ‰ of the beam energy is transferred to the radiation***

How can we produce XUV and hard X-rays where no suitable mirrors are available?

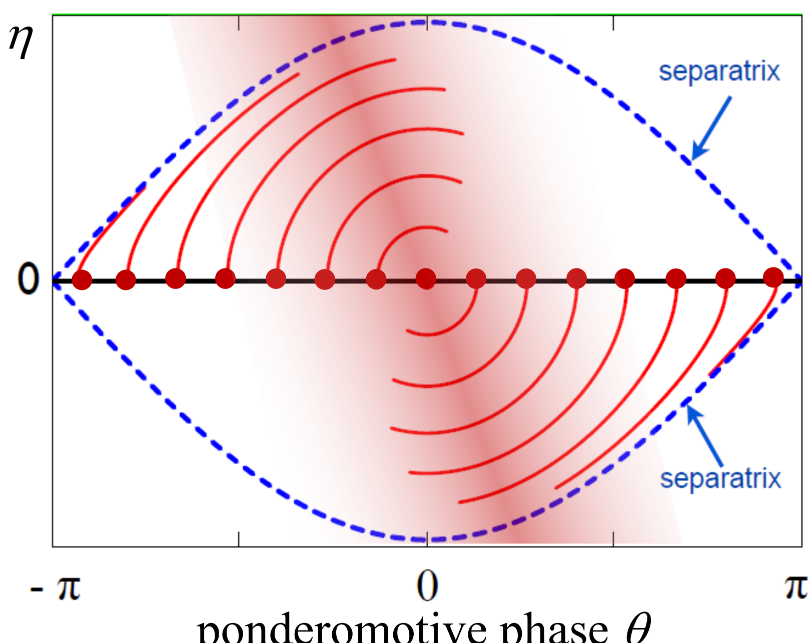
So far…

… we have neglected that the energy exchange between electrons and the laser field will cause a change of the EM field intensity and **set $E = E_0$ = const. for a single passage of the undulator!**

This might be wrong for a "long" undulator!

What happens if we make the undulator "longer" and consider a slowly varying field intensity?

Remember – injection on resonance:

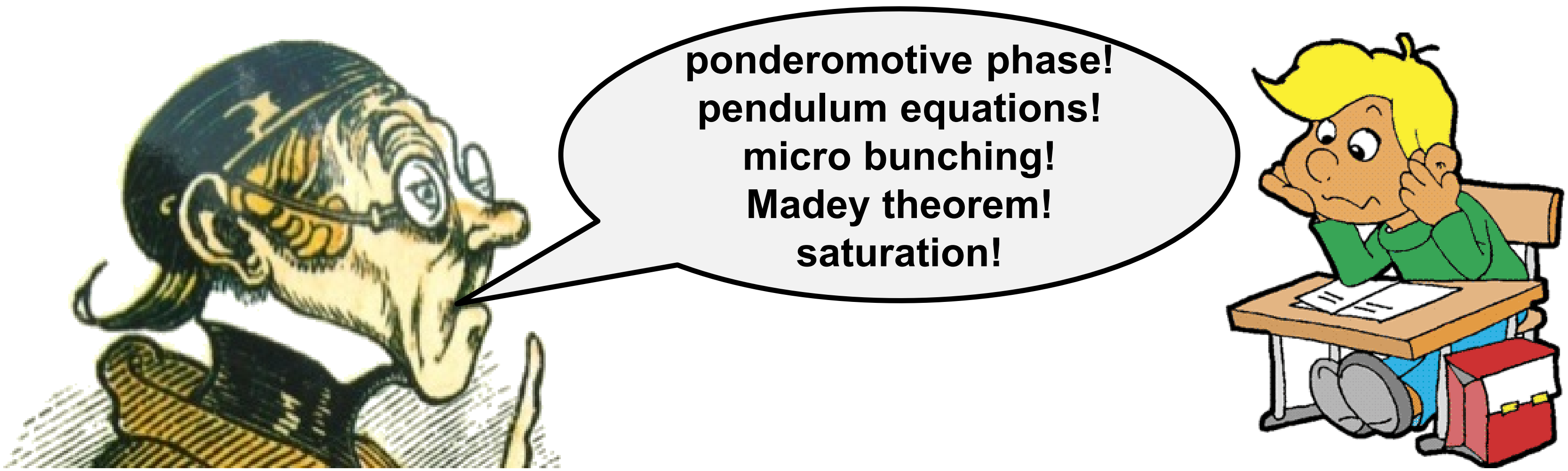


→ energy modulation
→ density modulation
no net energy transfer!

really??

→ ***be careful…***

End of 1st Lecture!

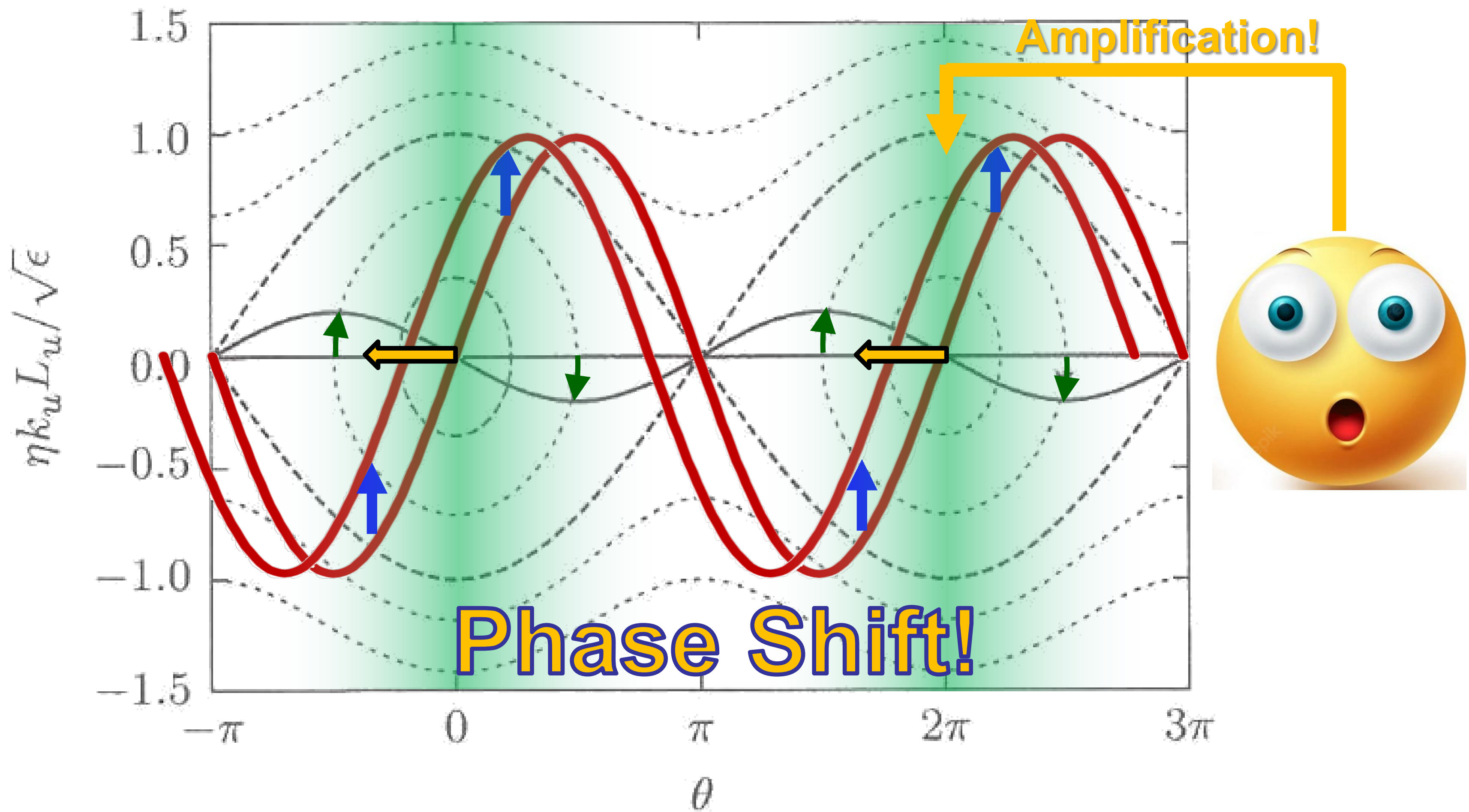


Questions?

Slow Variation of Laser Field

Injection on resonance!

Interaction with external generated laser field



Extended Pendulum Equations

We have to extend the existing pendulum equations

- phase equations $$\frac{\mathrm{d}\theta_j(s)}{\mathrm{d}s} = 2k_u\eta_j(s)$$
- energy equations $$\frac{\mathrm{d}\eta_j(s)}{\mathrm{d}s} = -\varepsilon\sin\theta_j(s)$$

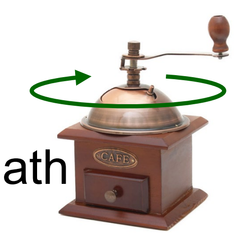
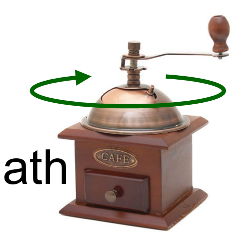
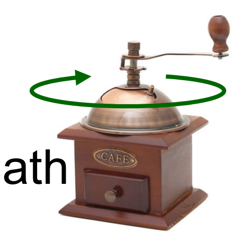
2 N_e equations!

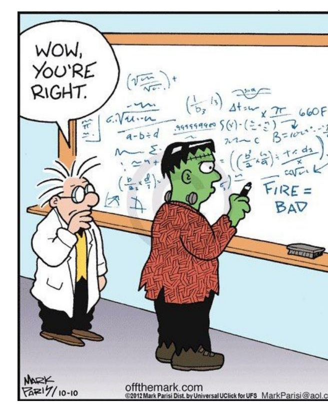
by an additional equation describing the slowly varying EM field

- field equation $$\frac{\mathrm{d}E}{\mathrm{d}s} = ???$$ (remember: $\varepsilon = \dfrac{eE_0K_{JJ}}{2m_0c^2\gamma_{res}^2}$)

and to consider a slowly varying amplitude and phase (→ complex E)!

Warning: What follows is a condensed version of the somehow lengthy math (show how to get there)!

math  math  math  math 



Field Change in 1D Approx.

Slowly varying **amplitude** and **phase** (index "*S*" means *slowly varying*):

$$E_x(s,t) = \hat{E}_S(s,t) \cdot \cos\left(k_L s - \omega t + \phi_S(s,t)\right)$$

Change to **complex field amplitude** defined by:

$$\boxed{\tilde{E}(s,t) = \frac{1}{2}\hat{E}_S(s,t) \cdot e^{i\phi_S(s,t)}}$$

$$\rightarrow \quad E_x(s,t) = \tilde{E}(s,t) \cdot e^{i(k_L s - \omega t)} + \tilde{E}^*(s,t) \cdot e^{-i(k_L s - \omega t)} = 2\,\mathrm{Re}\left\{\tilde{E}(s,t) \cdot e^{i(k_L s - \omega t)}\right\}$$

Wave equation links laser field and electron current in the undulator:

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right\}\vec{E} = -\mu_0 \frac{\partial \vec{j}}{\partial t}$$

1-dim

$$\boxed{\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} - \frac{\partial^2 E_x}{\partial s^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t}}$$

This has to be solved using the approximations and applying some „tricks" ... (→ p. 33-36)

...

Field Change in 1D Approx.

<u>First Trick</u>: decompose the wave operator using

$$\partial_\pm = \frac{1}{c}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \qquad \rightarrow \qquad \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} = \partial_+ \cdot \partial_-$$

Slowly varying complex field amplitude then means:

$$\left|\partial_\pm \tilde{E}\right| \cdot \lambda_L \ll \left|\tilde{E}\right| \qquad \text{or} \qquad \left|\partial_\pm \tilde{E}\right| \ll \left|\tilde{E}\right| \cdot k_L$$

We now have to compute

$$= 0 \quad \leftrightarrow \quad \omega = k_L c$$

$$\partial_+ \cdot \partial_- E_x(s,t) = \partial_+ \cdot \partial_- \left(\tilde{E} \cdot e^{i(k_L s - \omega t)} + \tilde{E}^* \cdot e^{-i(k_L s - \omega t)} \right)$$

$$\approx 0 \quad \leftrightarrow \quad \left|\partial_- \tilde{E}\right| \ll k_L \left|\tilde{E}\right|$$

Using

$$\partial_+ e^{i(k_L s - \omega t)} = -i\left(\frac{\omega}{c} - k_L\right) e^{i(k_L s - \omega t)} = 0 \qquad \partial_- e^{i(k_L s - \omega t)} = -i\left(\frac{\omega}{c} + k_L\right) e^{i(k_L s - \omega t)} = -2i\, k_L e^{i(k_L s - \omega t)}$$

we first get, since $\tilde{E}$ is slowly varying

$$\partial_+ \cdot \partial_- E_x(s,t) = -\partial_+ \left[\left(2ik_L \tilde{E}\right) \cdot e^{i(k_L s - \omega t)} - \left(2ik_L \tilde{E}^*\right) \cdot e^{-i(k_L s - \omega t)} \right]$$

and finally

$$\partial_+ \cdot \partial_- E_x(s,t) = -2ik_L \left[\left(\partial_+ \tilde{E}\right) \cdot e^{i(k_L s - \omega t)} - \left(\partial_+ \tilde{E}^*\right) \cdot e^{-i(k_L s - \omega t)} \right]$$

Field Change in 1D Approx.

We insert the result in the wave equation

$$2ik_L\left[\left(\partial_+\tilde{E}\right)\cdot e^{i\left(k_L s-\omega t\right)}-\left(\partial_+\tilde{E}^*\right)\cdot e^{-i\left(k_L s-\omega t\right)}\right]=\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t} \qquad \Big|\cdot e^{-i\left(k_L s-\omega t\right)}$$

multiply with the phase factor and obtain

$$2ik\cdot\left(\partial_+\tilde{E}\right)-\cancel{2ik\cdot\left(\partial_+\tilde{E}^*\right)\cdot e^{-2i\left(k_L s-\omega t\right)}}=\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t}\cdot e^{-i\left(k_L s-\omega t\right)}$$

<u>*Second Trick*</u>: Since the field amplitude is slowly varying, we average over a small number n of the rapidly oscillating periods T, thus $\Delta t = 2n\pi/\omega$ and use

$$\rightarrow \quad \frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2} 2ik\left(\partial_+\tilde{E}\right)\mathrm{d}t \approx 2ik\left(\partial_+\tilde{E}\right), \qquad \frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2} 2ik\left(\partial_+\tilde{E}^*\right)e^{-2i\left(k_L s-\omega t\right)}\mathrm{d}t \approx 0$$

yielding

$$\boxed{2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{1}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\frac{\partial j_x}{\partial t}\cdot e^{-i\left(k_L s-\omega t\right)}\mathrm{d}t}$$

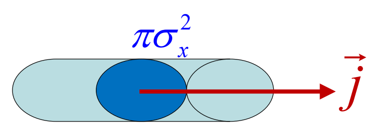
Field Change in 1D Approx.

<u>*Third Trick*:</u> We integrate by parts and assume that j_x is periodic in λ_L

$$2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{1}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\underbrace{\frac{\partial j_x}{\partial t}}_{u'}\cdot\underbrace{e^{-i\left(k_L s-\omega t\right)}}_{v}\mathrm{d}t=\frac{i\omega}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\underbrace{j_x}_{u}\cdot\underbrace{e^{-i\left(k_L s-\omega t\right)}}_{v'/i\omega}\mathrm{d}t$$

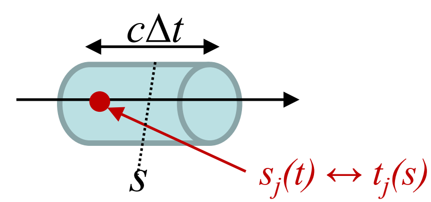
The current density is generated by single electrons (at positions s_j) having a transverse velocity from the undulator motion. Assuming that the bunch "fills" a transverse area $\pi\sigma_x^2$ and $\gamma_j \approx \gamma_{res}$ we obtain

$$j_x=\frac{-e}{\pi\sigma_x^2}\underbrace{\frac{cK}{\gamma_{res}}\cos\left(k_u s\right)}_{=\dot{x}}\sum_{j=1}^{N_e}\delta\left(s-s_j(t)\right)$$



and therewith

$$2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{i\omega}{\varepsilon_0 c^2}\frac{-e}{\pi\sigma_x^2}\frac{cK}{\gamma_{res}}\frac{1}{c\Delta t}\sum_{j=1}^{N_\Delta}\cos\left(k_u s\right)e^{-i\left(k_L s-\omega t_j\right)}$$



which yields with replacing the sum by the average over all $N_\Delta = n_e\left(\pi\sigma_x^2\right)\left(c\Delta t\right)$ electrons in the slice Δt:

$$\partial_+\tilde{E}=-\frac{eKn_e}{2\varepsilon_0\gamma_{res}}\left\langle\cos\left(k_u s\right)e^{-i\left(k_L s-\omega t_j\right)}\right\rangle$$

Field Change in 1D Approx.

We express $\cos(k_u s)$ by its complex representation

$$\partial_+ \tilde{E} = -\frac{eKn_e}{2\varepsilon_0 \gamma_{res}} \left\langle \frac{e^{ik_u s} + e^{-ik_u s}}{2} e^{-i\left(k_L s - \omega t_j\right)} \right\rangle = -\frac{eKn_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{i\left(k_u s - k_L s + \omega t_j\right)} + e^{-i\left(k_u s + k_L s - \omega t_j\right)} \right\rangle$$

<u>Fourth Trick</u>: We now use the definition of the phases ψ and χ (cf. page 9) and neglect again the longitudinal oscillation by replacing $K \to K_{JJ}$:

$$\partial_+ \tilde{E} = -\frac{eK_{JJ} n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\chi_j} + e^{-i\psi_j} \right\rangle \approx -\frac{eK_{JJ} n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\psi_j} \right\rangle = -\frac{eK_{JJ} n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\theta_j} \right\rangle$$

Last step

$$\boxed{\frac{1}{c}\frac{\partial \theta}{\partial t} = 2k_u}$$

$$\partial_+ \tilde{E} = \frac{\partial \tilde{E}(s,t)}{\partial s} + \frac{1}{c}\frac{\partial \tilde{E}(s,t)}{\partial t} = \frac{\partial \tilde{E}(s,\theta)}{\partial s} + \underbrace{2k_u \frac{\partial \tilde{E}(s,\theta)}{\partial \theta}}_{\approx 0 \ \text{(for fundamental)}} \approx \frac{\mathrm{d}\tilde{E}(s,\theta)}{\mathrm{d}s}$$

and finally:

$$\boxed{\frac{\mathrm{d}\tilde{E}}{\mathrm{d}s} = -\frac{eK_{JJ} n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\theta_j} \right\rangle}$$

whew!

Coupled 1D Equations

→ Extension of the pendulum equations to a system of coupled differential equations:

$$\frac{\mathrm{d}\tilde{E}}{\mathrm{d}s} = -\kappa_2 n_e \left\langle e^{-i\theta_j} \right\rangle$$	with bunching factor	$b = \left\langle e^{-i\theta_j} \right\rangle$
$$\frac{\mathrm{d}\theta_j}{\mathrm{d}s} = 2k_u \eta_j$$	with ponderomotive phases	θ_j
$$\frac{\mathrm{d}\eta_j}{\mathrm{d}s} = \kappa_1 \left(\tilde{E} \cdot e^{i\theta_j} + \tilde{E}^* \cdot e^{-i\theta_j} \right)$$	with rel. energy deviations	η_j

Assumptions:

- one-dimensional treatment
- slowly varying field amplitude and phase
- restriction to the fundamental harmonic
- no space charge effects considered (which are small)

Abbreviations:

$$\kappa_1 = \frac{eK_{JJ}}{2\gamma_{res}^2 m_0 c^2}$$

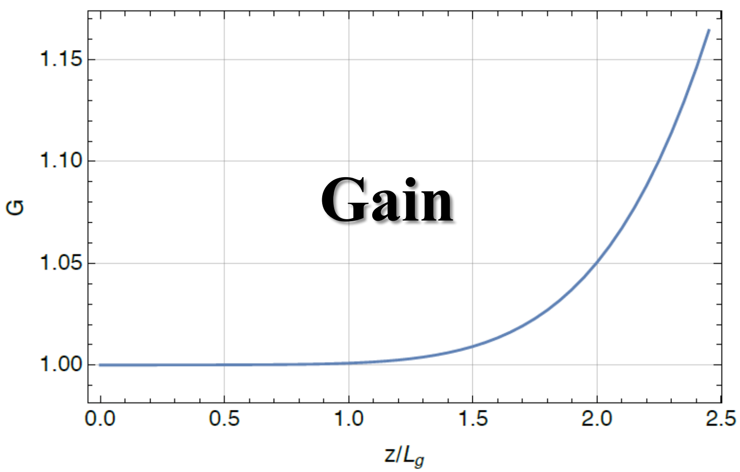
$$\kappa_2 = \frac{eK_{JJ}}{4\varepsilon_0 \gamma_{res}}$$

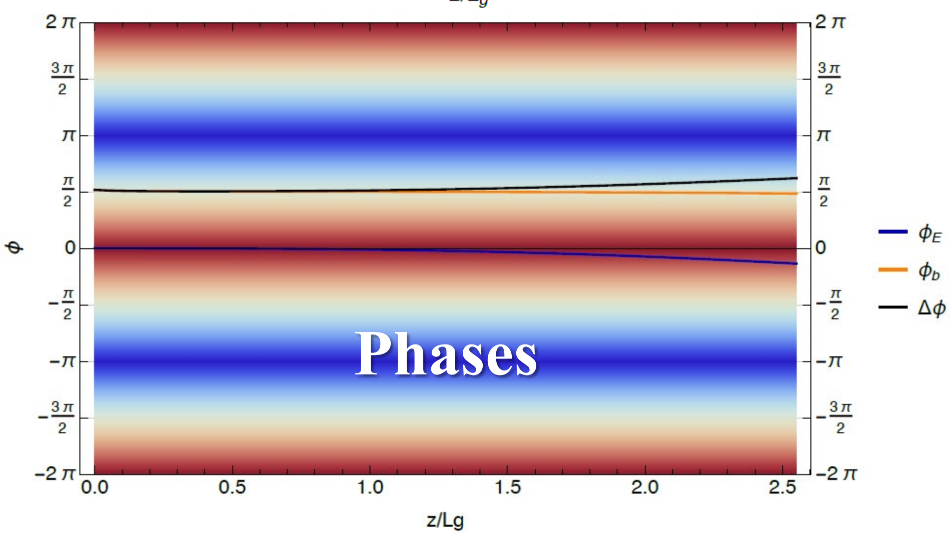
Set of $2N_e+1$ coupled differential equations!!

Numerical Solution

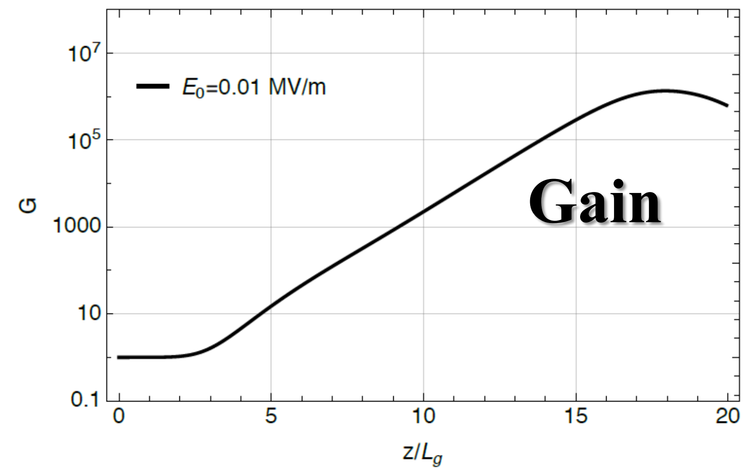
Redefinition of the gain: $G = G_{old} + 1$

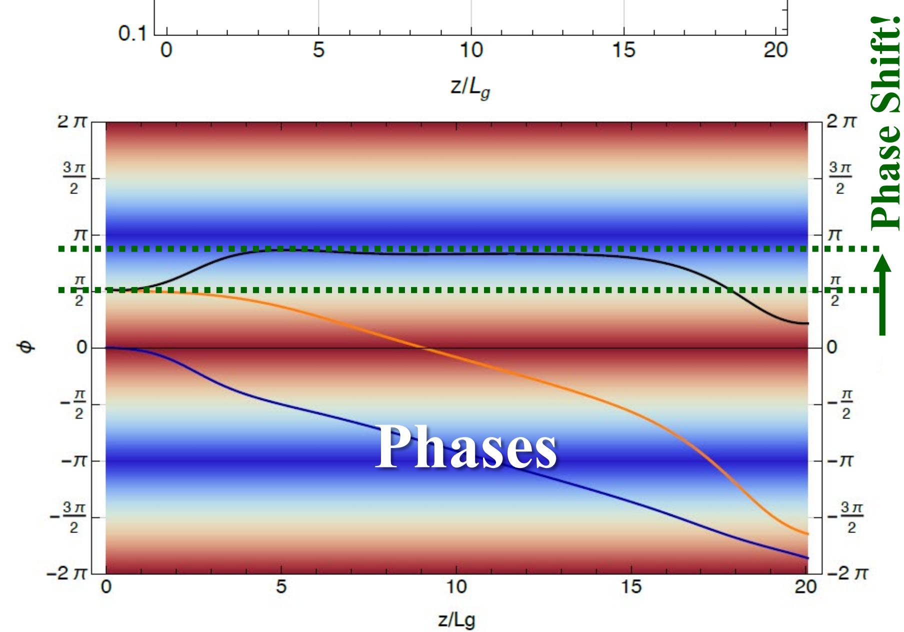
"Short" Undulator





"Long" Undulator





Normalized Parameters

Deeper understanding of the differential equations by defining normalized dimensionless scale parameters:

longitudinal coordinate: $\hat{s} = 2k_u \rho s$ $\qquad \boxed{\dfrac{\mathrm{d}\theta_j}{\mathrm{d}\hat{s}} = \hat{\eta}_j}$

rel. energy deviation: $\hat{\eta} = \dfrac{\eta}{\rho}$ $\qquad \boxed{\dfrac{\mathrm{d}\hat{\eta}_j}{\mathrm{d}\hat{s}} = a e^{i\theta_j} + a^* e^{-i\theta_j}}$

norm. field amplitude: $a = \dfrac{\kappa_1}{2k_u \rho^2} \tilde{E}$ $\qquad \boxed{\dfrac{\mathrm{d}a}{\mathrm{d}\hat{s}} = -\dfrac{\kappa_1 \kappa_2 n_e}{4k_u^2 \rho^3} \left\langle e^{-i\theta_j} \right\rangle = -\left\langle e^{-i\theta_j} \right\rangle}$

Pierce parameter: $\rho = \sqrt[3]{\dfrac{\kappa_1 \kappa_2 n_e}{4k_u^2}} = \sqrt[3]{\dfrac{1}{8\pi} \left(\dfrac{I_{\text{beam}}}{I_{\text{Alfvén}}} \right) \left(\dfrac{K_{JJ}}{1 + K^2/2} \right)^2 \left(\dfrac{\gamma \lambda^2}{2\pi \sigma_x^2} \right)}$

Coupled equations simplify to

neglect higher harmonics

$$\frac{\mathrm{d}a}{\mathrm{d}\hat{s}} = -\left\langle e^{-i\theta_j} \right\rangle = -b, \qquad \frac{\mathrm{d}b}{\mathrm{d}\hat{s}} = -i\left\langle \hat{\eta}_j \cdot e^{-i\theta_j} \right\rangle = -iP, \qquad \frac{\mathrm{d}P}{\mathrm{d}\hat{s}} = a + a^* \left\langle \cancel{e^{-2i\theta_j}} \right\rangle - i\left\langle \hat{\eta}_j^2 \cancel{e^{-i\theta_j}} \right\rangle$$

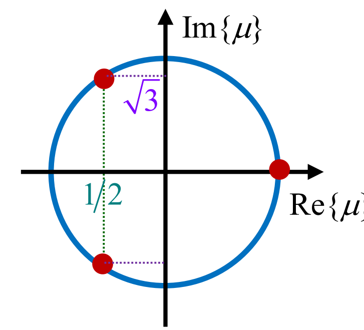
norm. field amplitude $\qquad$ *bunching factor* $\qquad$ *collective momentum*

Cubic Differential Equation

Combination yields a **differential equation of 3rd order**:

$$\frac{\mathrm{d}^3 a}{\mathrm{d}\hat{s}^3} = ia \qquad \text{use Ansatz} \quad a = C \cdot e^{-i\mu\hat{s}} \qquad \rightarrow \qquad \mu^3 = 1$$

which has 3 solutions of the characteristic polynomial:



$$\mu_1 = 1, \qquad \mu_2 = -\frac{1}{2}\left(1 + i\sqrt{3}\right), \qquad \mu_3 = -\frac{1}{2}\left(1 - i\sqrt{3}\right)$$

yielding the **general solution:**

$$a(\hat{s}) = C_1 e^{-i\hat{s}} + C_2 e^{\frac{1}{2}\left(i-\sqrt{3}\right)\hat{s}} + \boxed{C_3 e^{\frac{1}{2}\left(i+\sqrt{3}\right)\hat{s}}}$$

exp. increase

with the initial values ($s = 0$):

- normalized field amplitude $\quad a(0) = \sum C_i$
- bunching factor $\quad b(0) = -\left.\frac{\mathrm{d}a}{\mathrm{d}\hat{s}}\right|_0 = i\sum \mu_i C_i$
- collective momentum $\quad P(0) = i\left.\frac{\mathrm{d}b}{\mathrm{d}\hat{s}}\right|_0 = i\sum \mu_i^2 C_i$

Cubic Differential Equation

Initial values are determined from following system of equations:

$$\begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix} = \mathbf{M}_\mu \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ i\mu_1 & i\mu_2 & i\mu_3 \\ i\mu_1^2 & i\mu_2^2 & i\mu_3^2 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

which yields after matrix inversion:

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \mathbf{M}_\mu^{-1} \cdot \begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{i}{3} & -\frac{i}{3} \\ \frac{1}{3} & \frac{1}{6}(i+\sqrt{3}) & \frac{1}{3}(-1)^{5/6} \\ \frac{1}{3} & \frac{1}{6}(i-\sqrt{3}) & \frac{1}{3}(-1)^{5/6} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix}$$

Considering an initial energy shift $\hat{\eta}_0 = \eta_0 / \rho$:

$$P(\hat{s}) \rightarrow \left\langle \hat{\eta}_j e^{-i\theta_j} \right\rangle + \hat{\eta}_0 \qquad \rightarrow \qquad \mu^3 - 2\hat{\eta}_0 \mu^2 + \hat{\eta}_0^2 \mu - 1 = 0$$

Cubic Differential Equation

Case 1: start from already existing radiation field

Starting conditions:

- no density modulation $\rightarrow b_0 = 0$
- no energy offset and modulation $\rightarrow \eta_0 = 0 \ \rightarrow P_0 = 0$
- Incoming radiation field $\rightarrow \boldsymbol{a_0 > 0}$

$$\boxed{\Rightarrow \quad C_1 = C_2 = C_3 = \frac{1}{3} a_0}$$

Field amplitude:

$$a(\hat{s}) = \frac{a_0}{3} \left\{ e^{-i\hat{s}} + e^{\frac{1}{2}\left(i-\sqrt{3}\right)\hat{s}} + e^{\frac{1}{2}\left(i+\sqrt{3}\right)\hat{s}} \right\}$$

Gain:

$$G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9} \left\{ 3 + e^{-\sqrt{3}\hat{s}} + \boxed{e^{\sqrt{3}\hat{s}}} + 2\cos\left(\frac{3}{2}\hat{s}\right) \cdot \left[e^{-\frac{\sqrt{3}}{2}\hat{s}} + e^{\frac{\sqrt{3}}{2}\hat{s}} \right] \right\}$$

Cubic differential Equation

Case 1: start from existing radiation field

Universal gain curve:

$$\boxed{G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9}\left\{3 + e^{-\sqrt{3}\hat{s}} + \boxed{e^{\sqrt{3}\hat{s}}} + 4\cos\left(\frac{3}{2}\hat{s}\right)\cosh\left(\frac{\sqrt{3}}{2}\hat{s}\right)\right\}}$$

Asymptotical behavior for large $\hat{s}$:

$$G \approx \frac{1}{9}e^{\sqrt{3}\hat{s}} = \frac{1}{9}e^{2\sqrt{3}k_u\rho\cdot s} = \frac{1}{9}e^{s/L_G}$$

Definition of the 1-dim gain length (**power gain length**):

$$\sqrt{3}\hat{s} = 1 \quad \rightarrow \quad \boxed{L_G = \frac{1}{2\sqrt{3}k_u\rho} = \frac{\lambda_u}{4\pi\sqrt{3}\rho}}$$

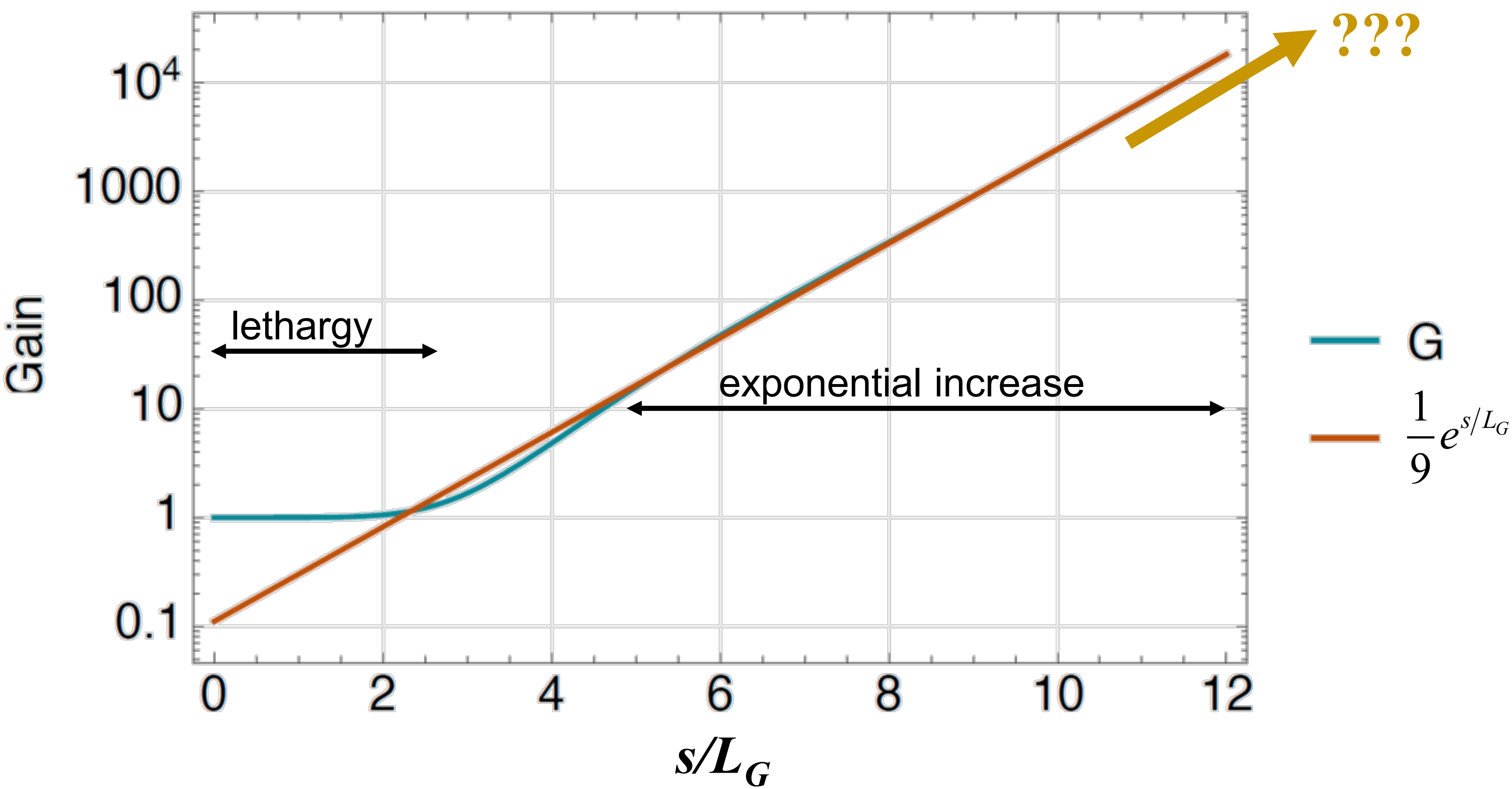
Behavior for small s/L_G (Taylor expansion) $\leftrightarrow$ "**Lethargy**"

$$\boxed{G_{\text{leth}} = 1 + \frac{1}{1080}\left(\frac{s}{L_G}\right)^6 = 1 + \left(\frac{s}{3.2L_G}\right)^6}$$

Universal Gain Curve

$$G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9}\left\{3 + 2\cosh\chi + 4\cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{\chi}{2}\right)\right\}$$

$$\chi = \frac{s}{L_G}$$

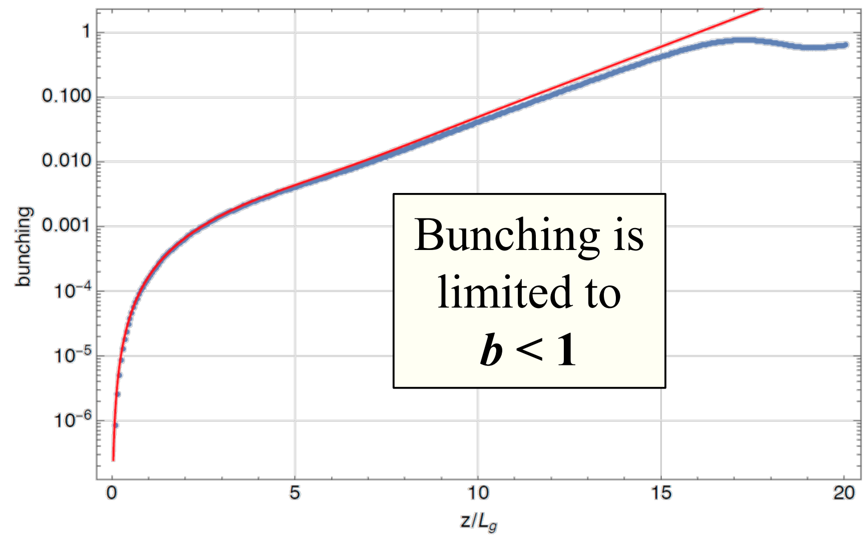


Saturation

Region of exponential increase:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} a = \frac{\sqrt{3}}{2} a = -b \;\rightarrow\; |a|^2 = \frac{4}{3}|b|^2 < \frac{4}{3} \approx 1$$

→ **field amplitude cannot grow larger than $|a| \approx 1$**



$$|a|^2 = \left| \frac{\kappa_1}{2k_u\rho^2} E \right|^2 \le 1 \;\rightarrow\; \boxed{\tilde{E}_{\mathrm{sat}} = \frac{2k_u\rho^2}{\kappa_1}}$$

$$\rightarrow\; \mathcal{W}_{\mathrm{sat}} = \frac{1}{2}\varepsilon_0 E_{sat}^2 = 2\varepsilon_0 \left|\tilde{E}_{sat}\right|^2 = 2\varepsilon_0\rho\left(\frac{2k_u}{\kappa_1}\right)^2\rho^3 = 2\varepsilon_0\rho n_e\left(\frac{\kappa_2}{\kappa_1}\right) = \rho\, n_e \gamma_{res} m_0 c^2 = \rho\,\mathcal{W}_{\mathrm{beam}}$$

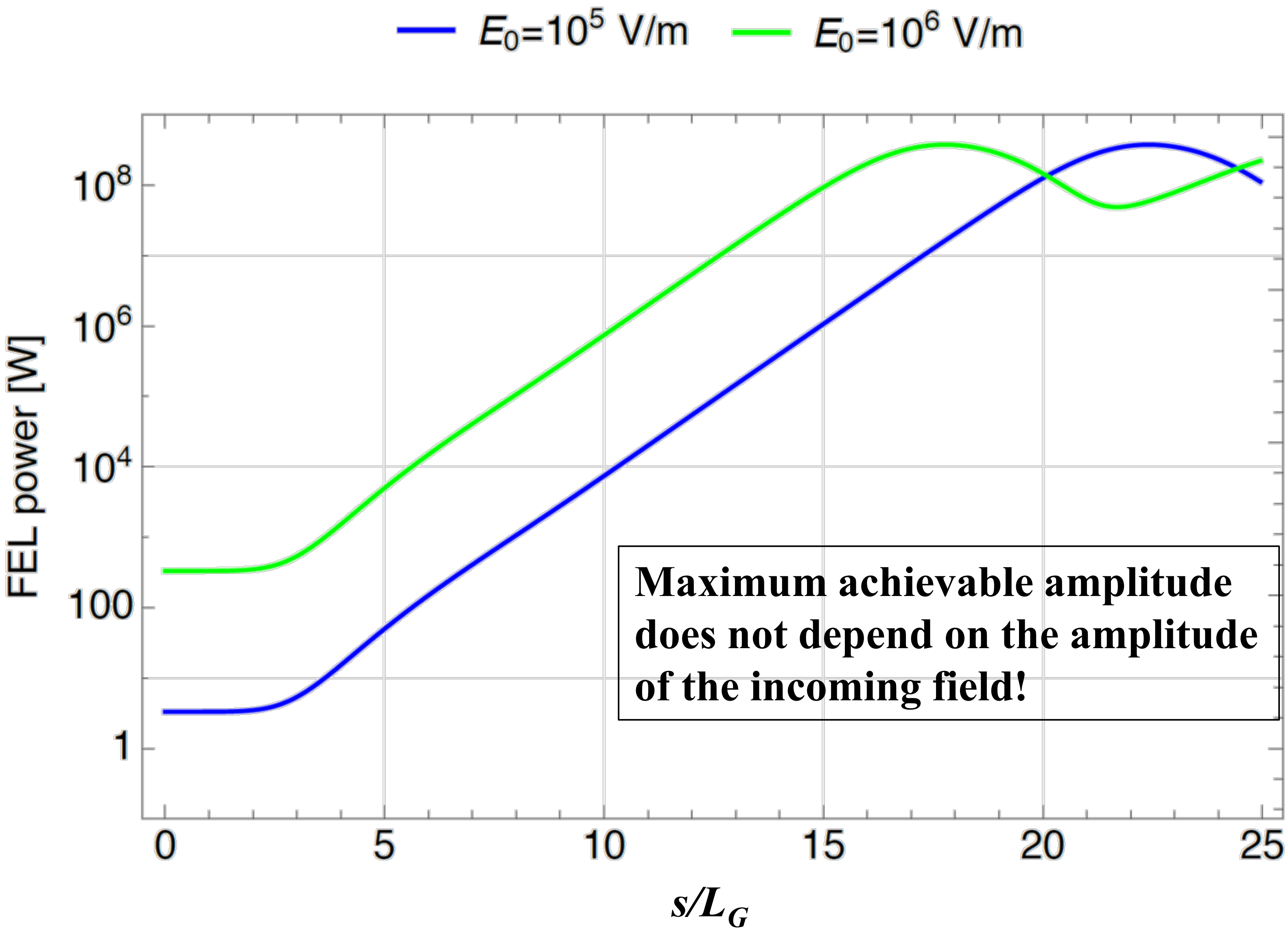
and thus

$$P_{sat} = \rho \cdot I_{\mathrm{beam}} \cdot U_{\mathrm{beam}}$$

Pierce parameter determines efficiency of energy transfer

Typical value: $\rho \approx 10^{-3} = ‰$

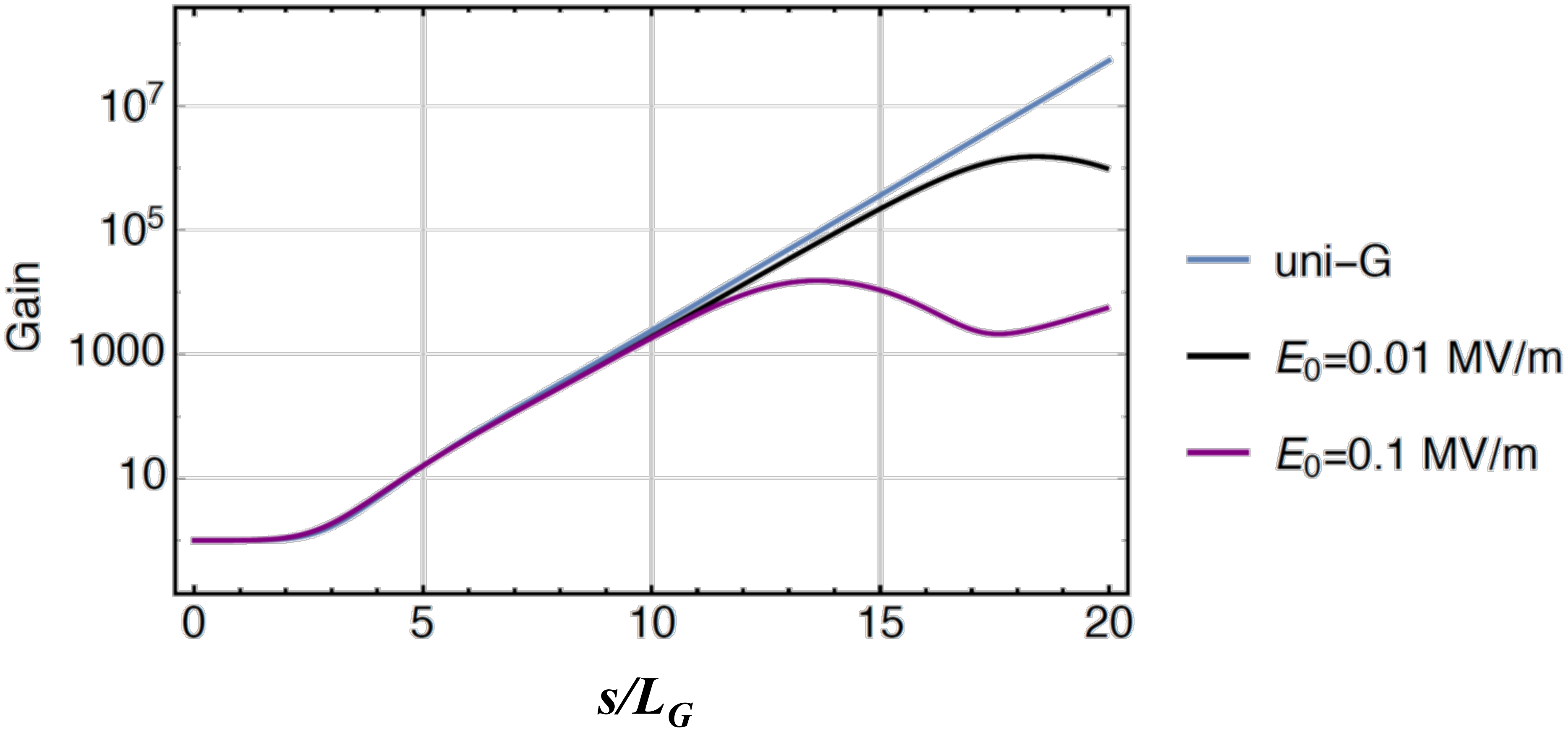
Saturation



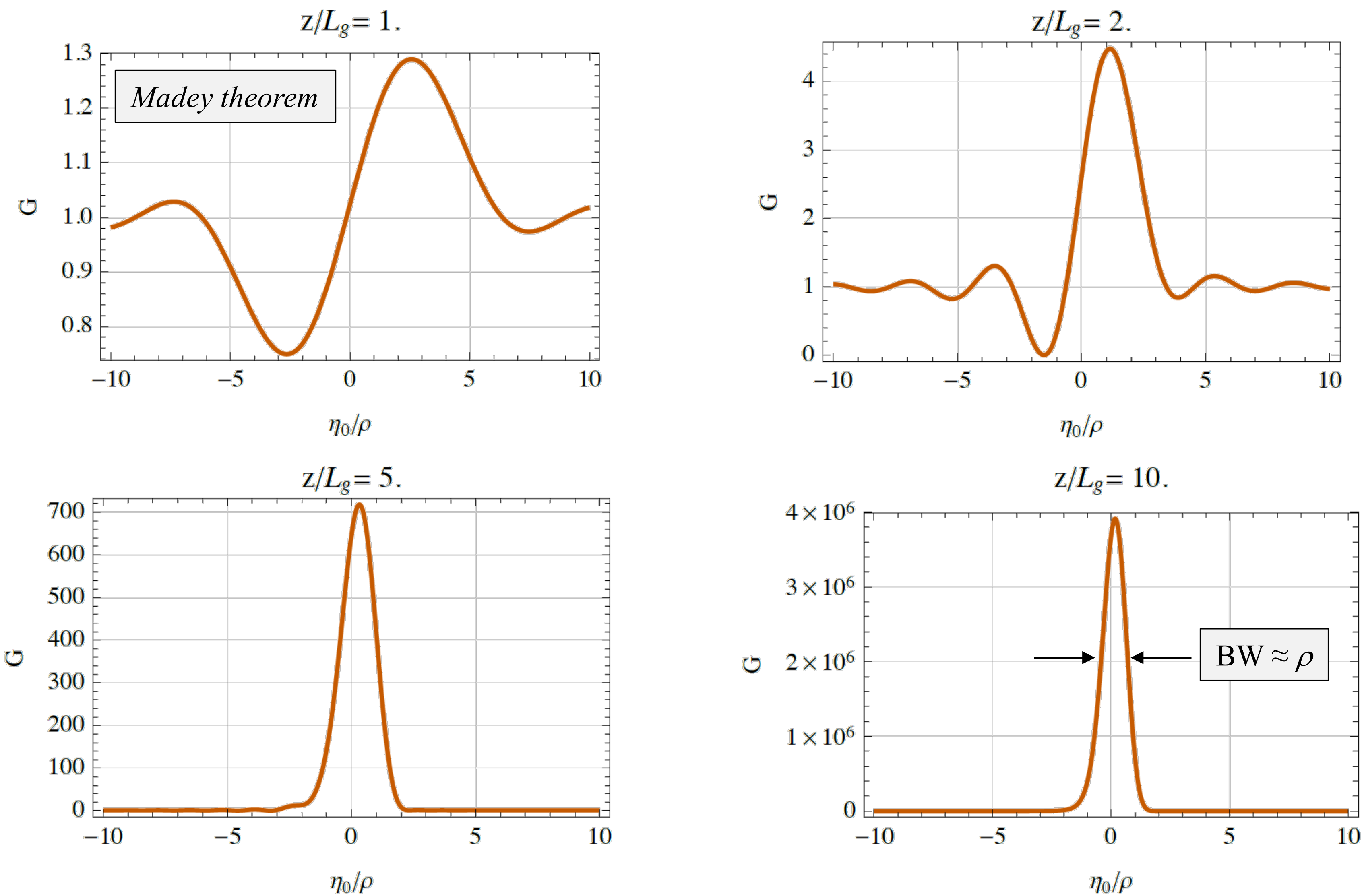
Maximum achievable amplitude does not depend on the amplitude of the incoming field!

Saturation

Maximum achievable gain factor depends on the amplitude of the incoming field



Gain and Bandwidth



Cubic Differential Equation

Case 2: Start from an existing density modulation

Starting conditions:

- Density modulation $\rightarrow b_0 = \left\langle e^{-i\theta_j} \right\rangle\Big|_0$ at $\lambda_m \approx \lambda_r$
- Energy offset → coll. e. modulation! $\rightarrow \eta_i = 0, \ \rightarrow \ P_0 = ib'\big|_0 = \hat{\eta}_0 b_0$
- incoming radiation field $\rightarrow a_0 = 0$

$$\boxed{\Rightarrow \quad C_1 = -i\frac{b_0}{3}, \quad C_2 = (-1)^{5/6}\frac{b_0}{3}, \quad C_3 = (-1)^{1/6}\frac{b_0}{3}}$$

for $\eta_0 = 0$

Field energy:

$$|a|^2 = \frac{2}{9} b_0^2 G(\chi), \qquad \chi = \frac{s}{L_g}$$

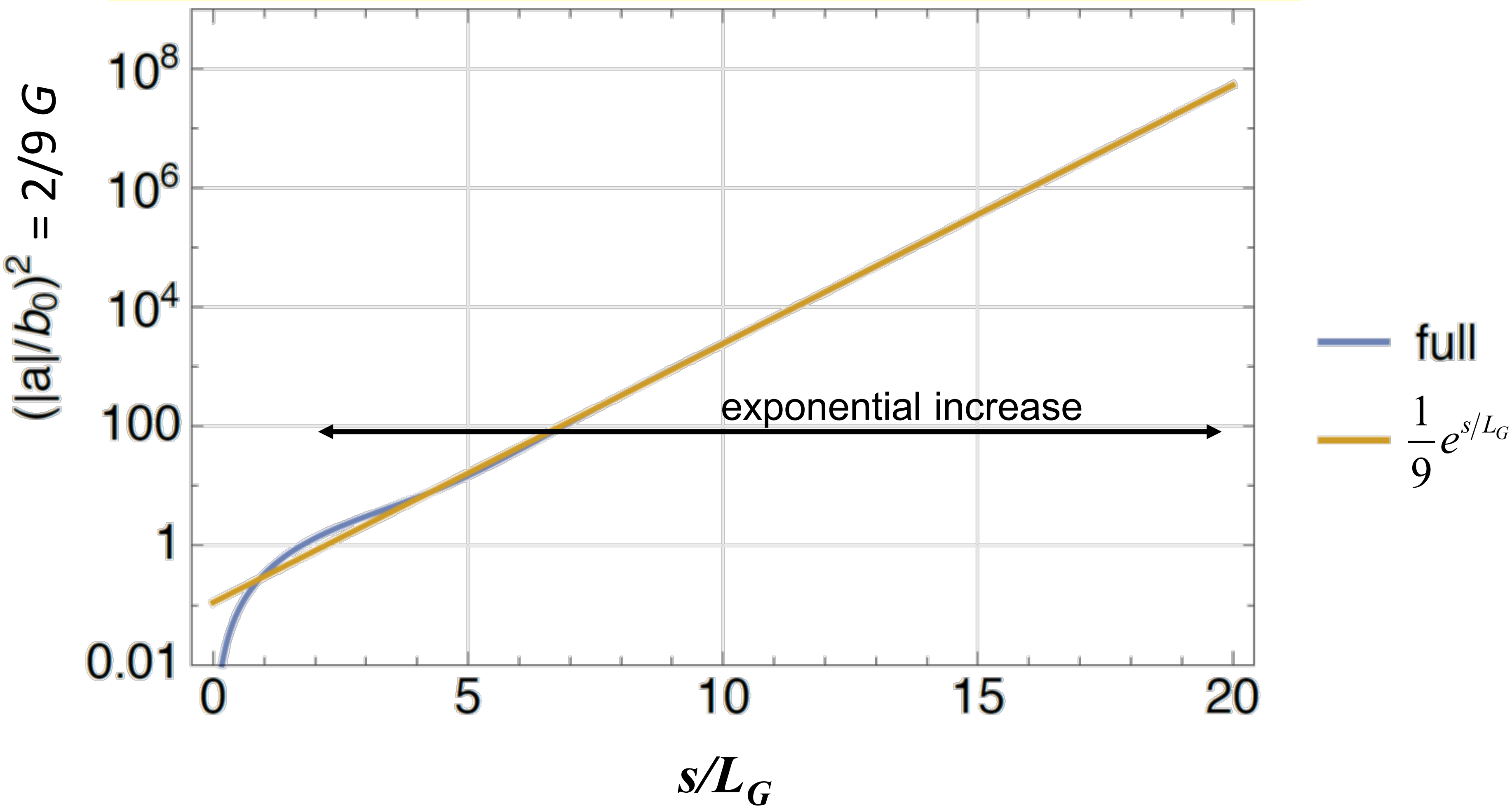
normalization to a_0 is not possible since $a_0 = 0$!

Gain:

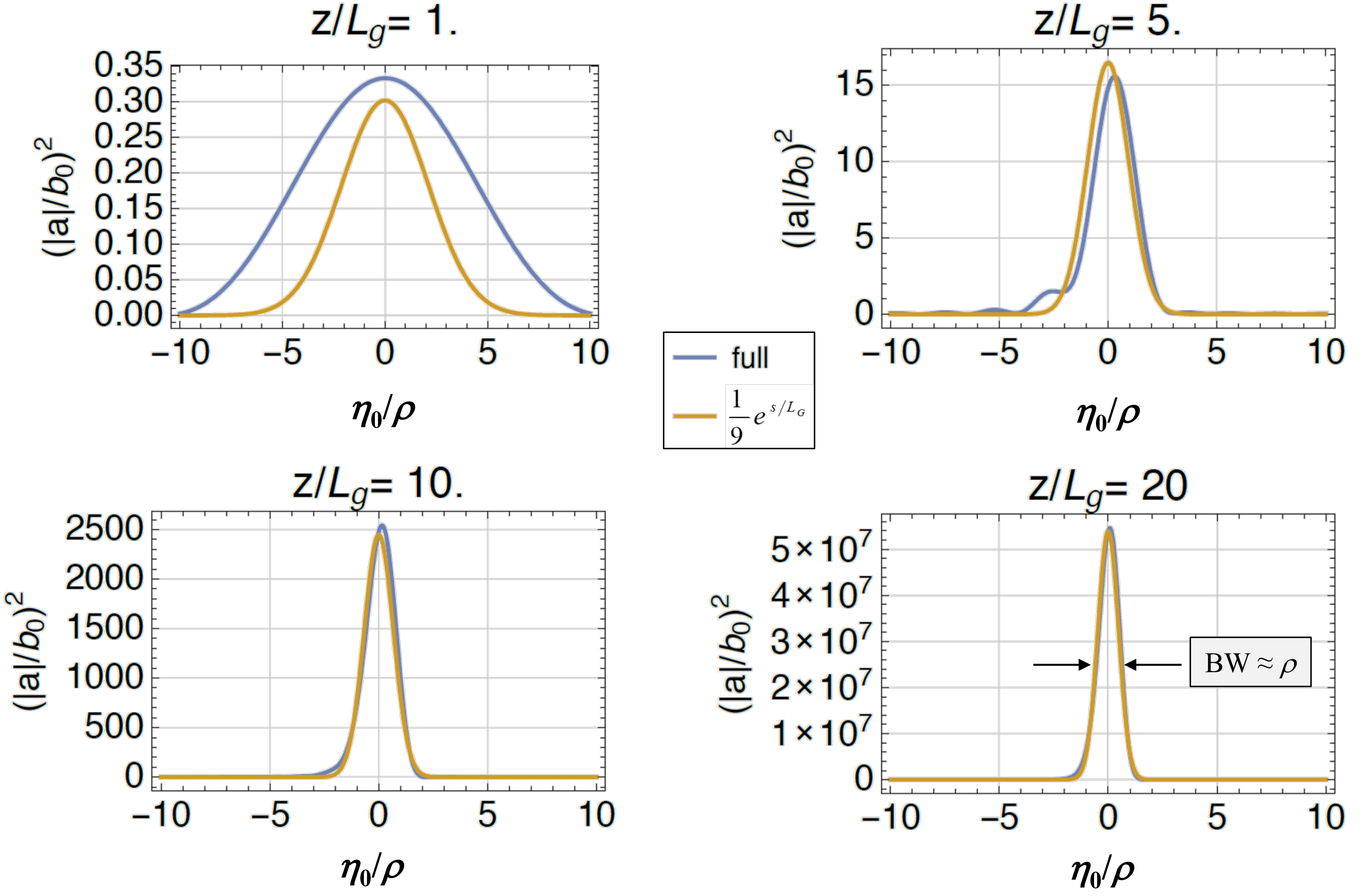
$$G(\hat{s}) = \boxed{\cosh\chi} + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\chi\right)\sinh\left(\frac{1}{2}\chi\right) - \cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{1}{2}\chi\right)$$

Universal Gain Curve

$$G(\hat{s}) = \cosh\chi + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\chi\right)\sinh\left(\frac{1}{2}\chi\right) - \cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{1}{2}\chi\right) \qquad \chi = \frac{s}{L_G}$$

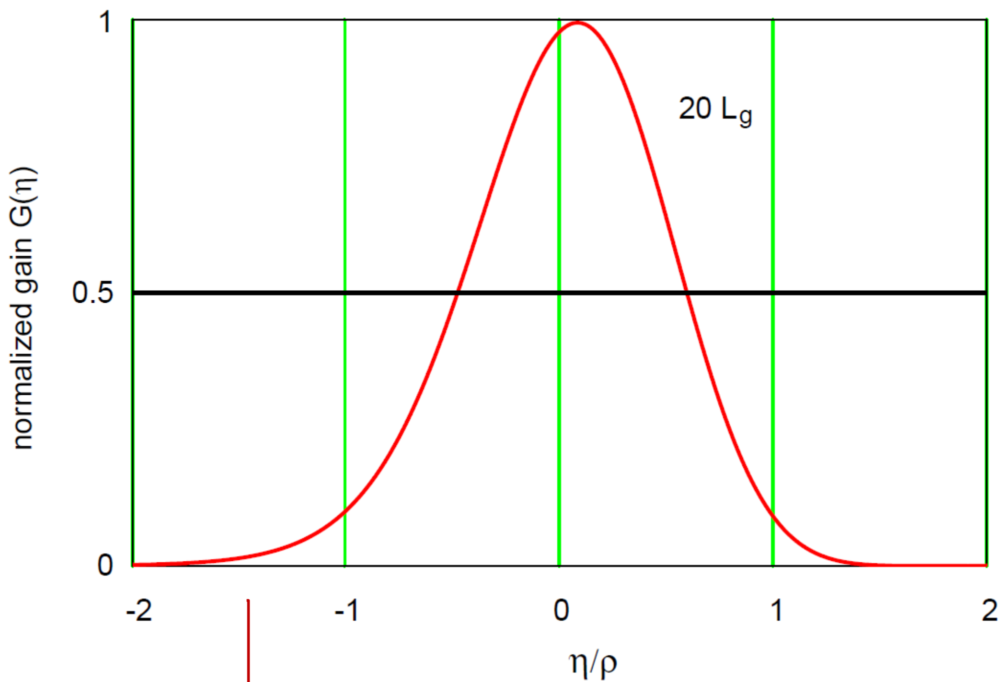


Gain and Bandwidth



Bandwidth

normalized gain @ s = 20 L_g:



Finding:

FEL gain drops significantly, when the relative energy variation η exceeds the Pierce parameter ρ!

s-dependent energy bandwidth

$$\boxed{\Delta\eta(s) = 3\sqrt{\pi}\rho\sqrt{\frac{L_g}{s}}}$$

$$\rightarrow \quad \Delta\eta(20L_g) \approx \rho$$

Gain curve has a FWHM $\approx \rho$

$\rightarrow \rho$ determines spectral width of the generated radiation!

SASE

Self Amplified Spontaneous Emission (SASE)

Was proposed in the beginning of the 1980s to produce high power short wavelength FEL radiation. 2 ways of considering the start of the FEL process:

- **spontaneous emission** at the beginning of the undulator is amplified,
- **random longitudinal distribution** of electrons leads to bunching non-vanishing factor at resonant frequency starting the FEL process.

Both pictures are fully equivalent!

<u>Time structure:</u>
Not the full bunch is contributing to the SASE start-up! Number of contributing electrons are determined by the undulator amplification bandwidth $\sigma_\omega \approx \rho\omega$!

Coherence or cooperation length L_C

can be roughly determined from time-bandwidth product $\tau \cdot \sigma_\omega$:

$$\tau_c = \frac{\sqrt{\pi}}{\sigma_\omega} \approx \frac{\sqrt{\pi}}{\rho\omega} = \frac{\lambda_L}{2\sqrt{\pi}\,\rho c} \quad \rightarrow \quad \boxed{L_C = c\tau_c = \frac{\lambda_L}{2\sqrt{\pi}\,\rho} \approx 300\,\lambda_L}$$

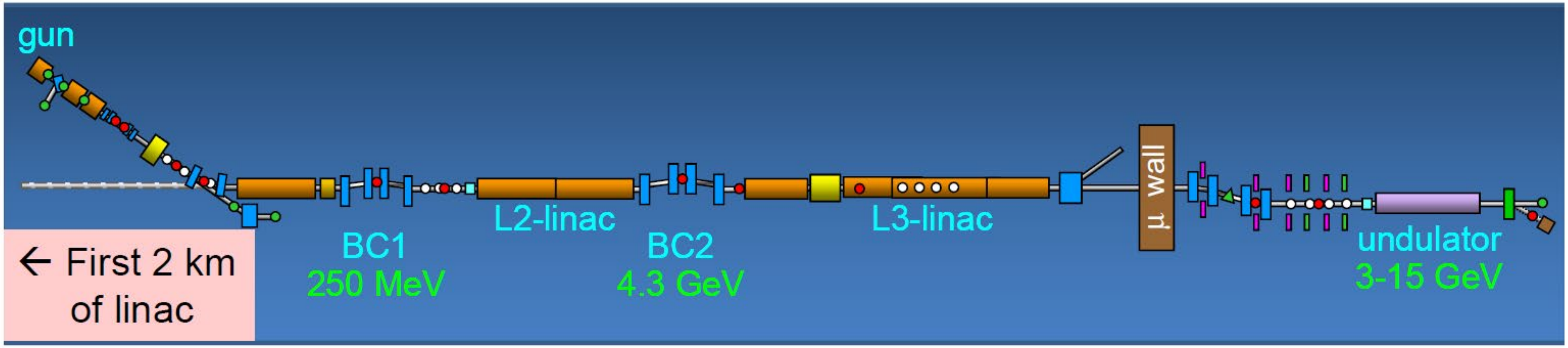
Within the bunch, several areas can start a SASE process individually!

High Gain SASE FEL

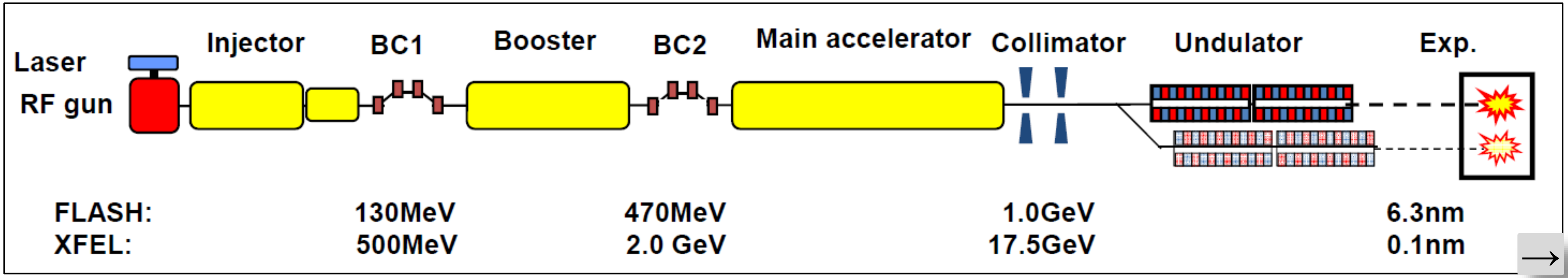
e beam requirements:

- transv. emittance $\varepsilon_{x,y} \leq \lambda_L / 4\pi$
- energy spread $\sigma_\gamma / \gamma < \rho$
- energy, current $E_{beam} \approx \text{GeV}, \quad I_{peak} \approx \text{kA}$

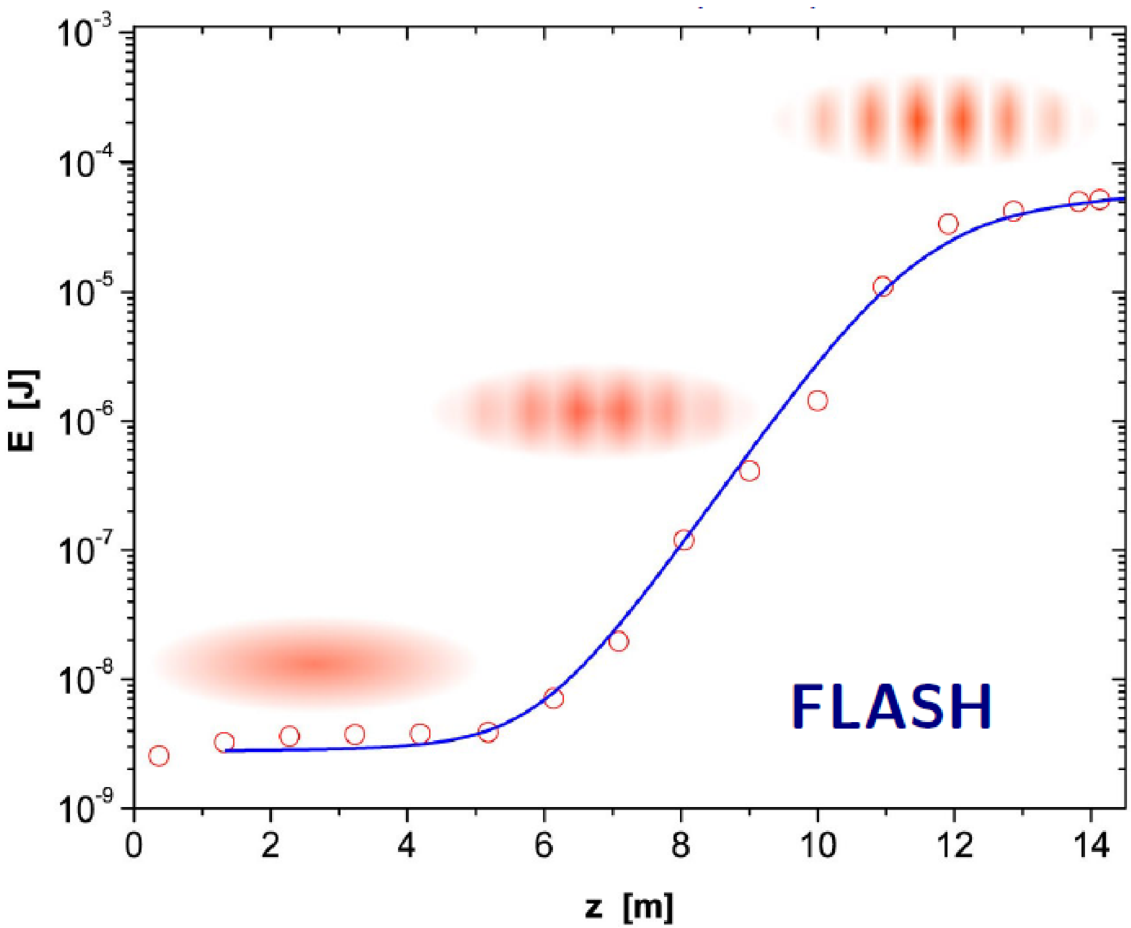
Linac Coherent Light Source LCLS: the blue pint of all SASE FELs

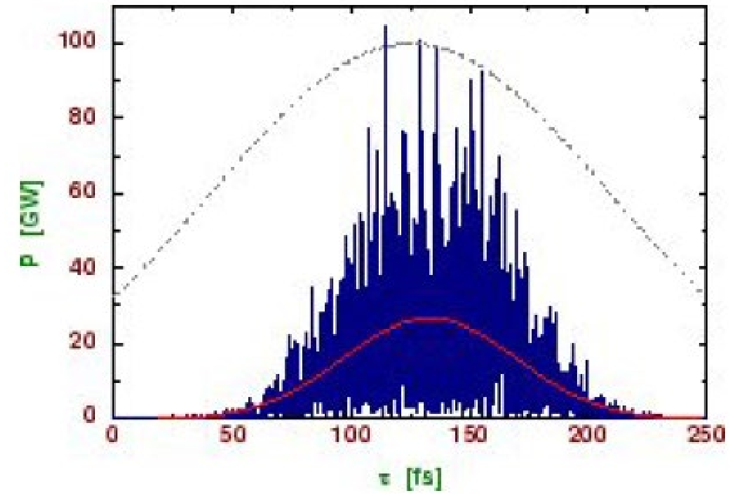


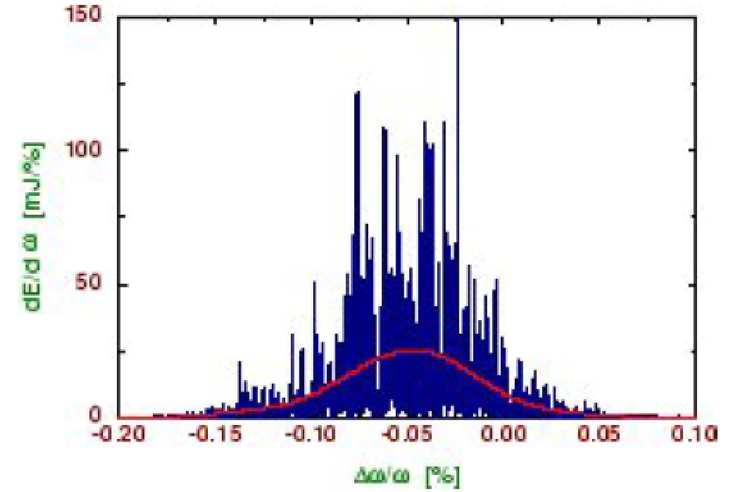
FLASH and European XFEL: long pulse trains from s.c. Linacs

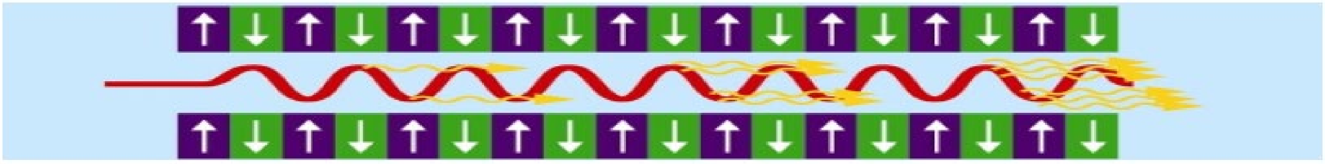


High Gain SASE FEL









Low temporal coherence!

Peak Brilliance

Figure of merit: peak brillance

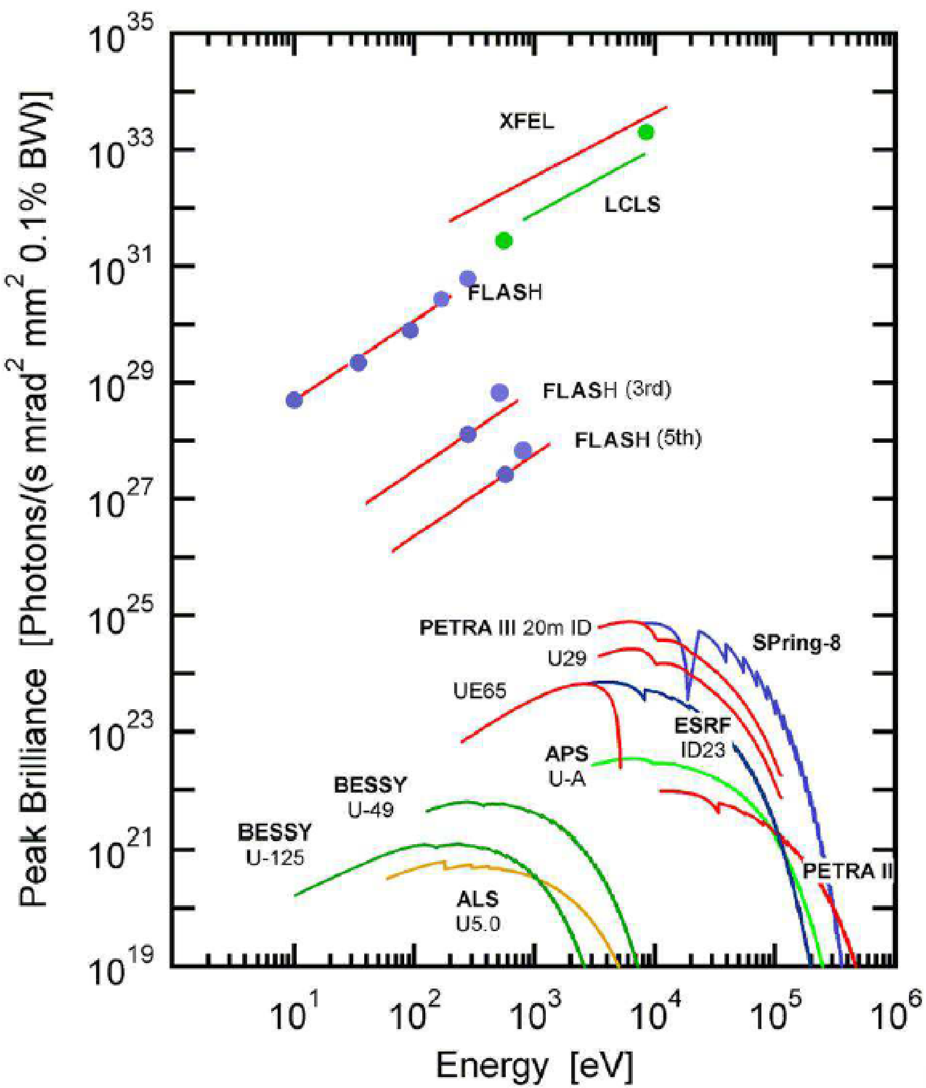
number of photons

$$B = \frac{\frac{d}{dt} n}{4\pi^2 \Sigma_x \Sigma_y \Sigma_{x'} \Sigma_{y'} \, d\omega / \omega}$$

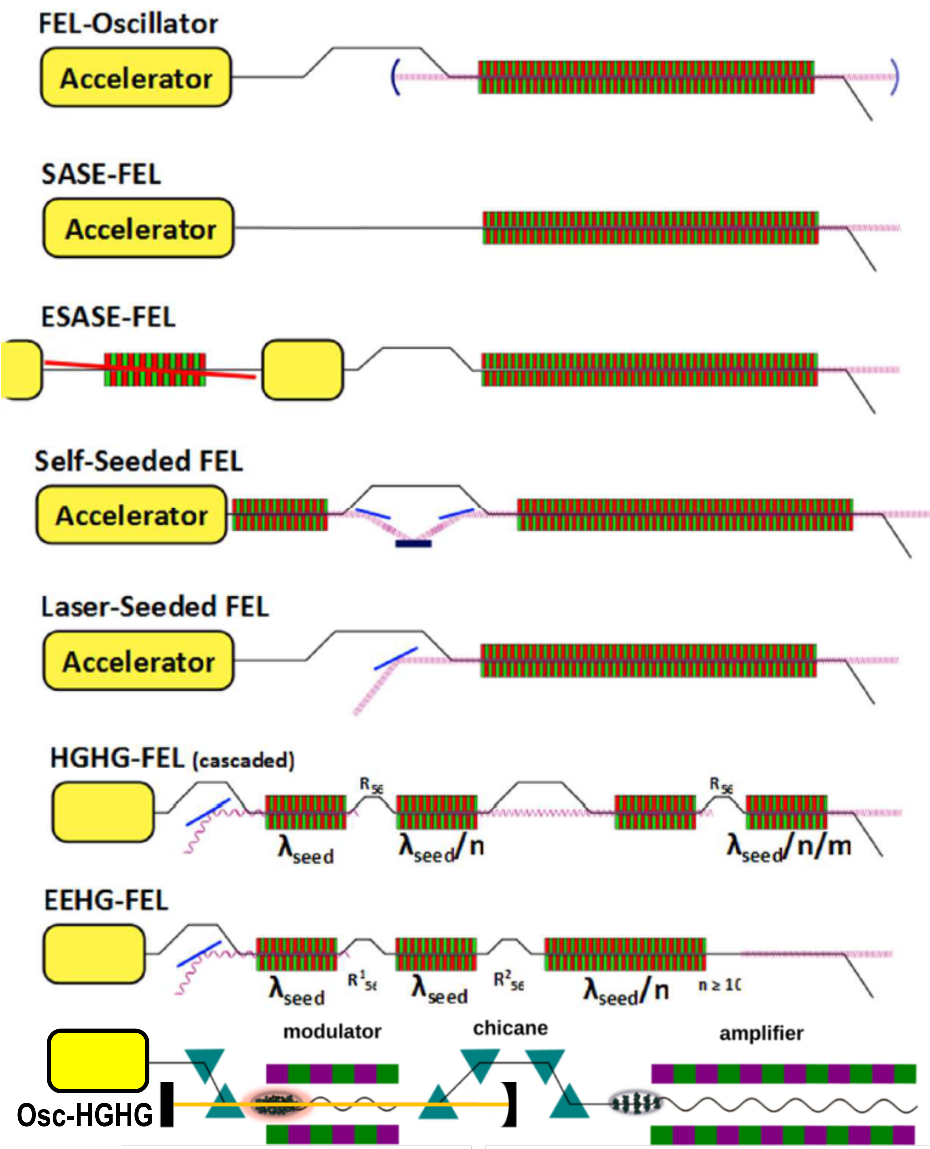
$$\Sigma^2 \approx \sigma_\gamma^2 + \sigma_e^2$$

Undulator beam lines:

$$B_{peak} \approx 10^{25} \mathrm{mm}^{-2} \mathrm{mrad}^{-2} \mathrm{s}^{-1} (0.1\%)^{-1}$$



Outlook: Seeding



Requires MHz electron bunch repetition (storage ring or cw linac)
Bandwidth determined by mirror system

e.g. XFELO planned @ Eur. XFEL

„Seeding“ by spontaneous synchrotron radiation, i.e. by shot noise

e.g. Eur. XFEL $\lambda_L < 1$ Å

Increase peak current within mirco-bunches generated through laser modulation and subsequent compression

Cut out monochromatic portion from initial SASE FEL for seeding

e.g. LCLS Eur. XFEL

Generate coherent seeding pulse by external laser (synchronized to e-beam!)

FLASH: $\lambda_L = 38$ nm

Dto., but also produce higher FEL harmonics for further seeding stages.

Like HGHG, but generate very high harmonics by multiple compression and multiple seeding.

e.g. FERMI @ ELETTRA $\lambda_L \approx 4$ nm

HGHG seeding with an oscillator starting from shot noise

planned @ FLASH

Literature

Recommended Textbooks:

- J.A. Clark, ***The Science and Technology of Undulators and Wigglers***, Oxford Science Publications, ISBN 019850855: *Synchrotron Radiation, Undulators and Wigglers, includes technical aspects and many details*
- P. Schmüser, M. Dohlus, J. Rossbach, C. Behrens, ***Free-Electron Lasers in the Ultraviolet and X-Ray Regime***, Second Edition (2014), Springer, ISBN 9783319040806: *The Hamburg Blue-Book on Free Electron Lasers*
- K.-J. Kim, Z. Huang, R. Lindberg, ***Synchrotron Radiation and Free-Electron Lasers***, Cambridge University Press (2017), ISB 9781107162617*: Excellent Book going deep into the theory of FEL way beyond the scope of this lecture*

- K. Wille, ***The Physics of Particle Accelerators. An Introduction***. Oxford University Press, Oxford (2001): *A compact book with some insights in LG FELs*
- …