Non Linear Dynamics - Methods and Tools

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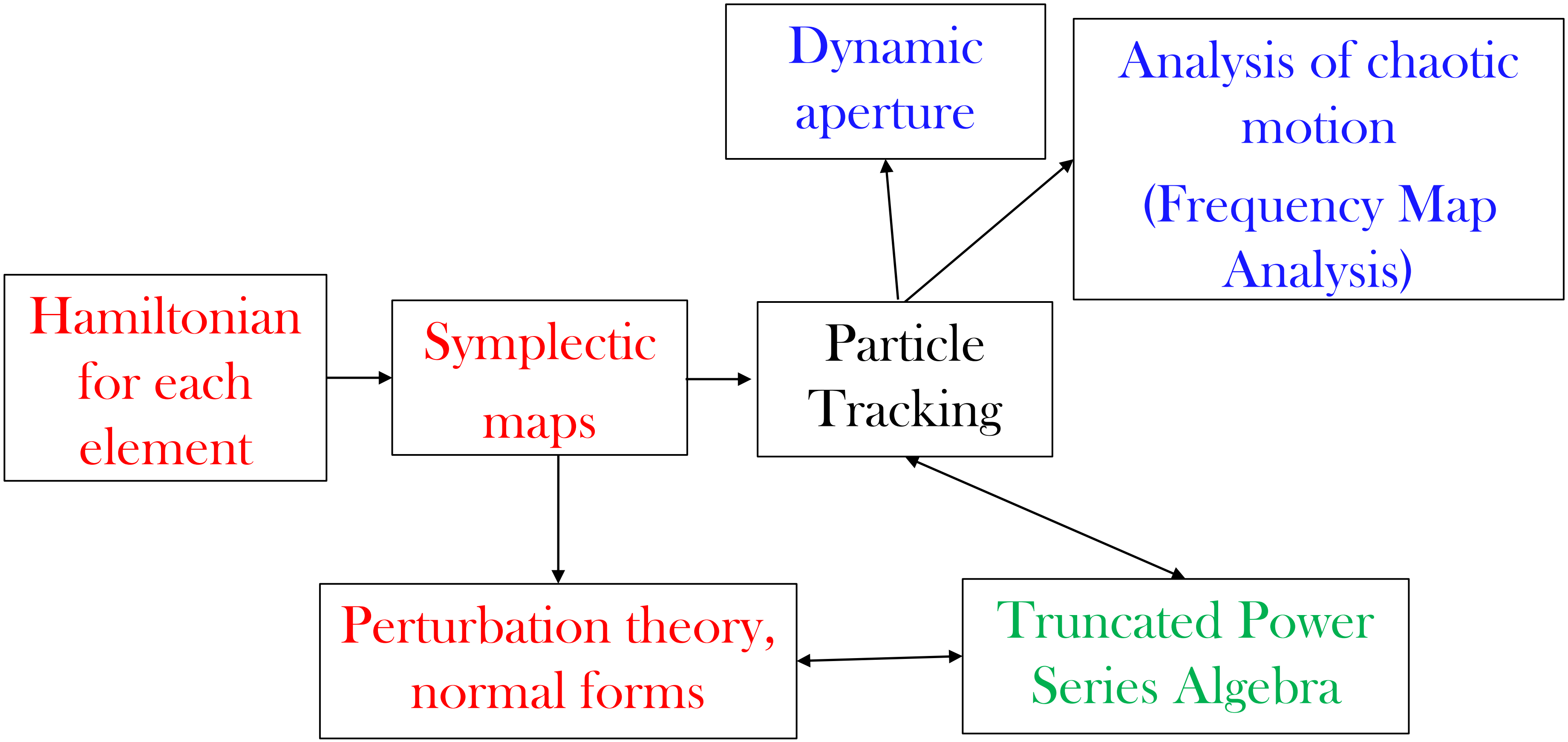
Non-linear dynamics



Content of lecture I

- **Non-linear effects** and their **impact**
- Reminder of **Lagrangian** and **Hamiltonian** formalism, **canonical transformation**, and **symplecticity**
- The **relativistic Hamiltonian** for E/M fields
- **Canonical perturbation theory** and its **limitations**

Non-linear effects

- **Non-linear magnets**, such as chromaticity sextupoles (especially in low emittance rings), octupoles,…
- Magnet **imperfections** and **misalignments**
- **Insertion devices** (wigglers, undulators) for synchrotron radiation storage rings
- Magnet **fringe fields** (especially in high-intensity rings)
- Power supply **ripple**
- **Ground motion** (for e+/e-)
- **Electron (Ion) cloud**
- **Beam-beam** effect (for colliders)
- **Space-charge** effect (especially in high-intensity ring)

Non-linear effects affect performance

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- **Performance** impact
  - Reduced **injection efficiency**
  - **Particle losses** causing
    - Reduced intensity and beam lifetime
    - Radio-activation (equipment maintenance and lifetime)
    - Super-conducting magnet quench
    - Reduced machine availability
  - **Emittance** increase
  - Reduced number of bunches, increased crossing angle, affecting **luminosity** (for colliders)
  - Allow to damp **instabilities** (see lecture on "Landau damping")
  - Can be used for **beam extraction**

…but also cost

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  - Can be used for **beam extraction**
- **Cost** issues
  - Magnet field quality, alignment tolerances
  - Number of magnet corrector, power convertor families and specifications
  - Design of collimation system
  - Operational efficiency (**energy**)

Reminder of Hamiltonian formalism

Hamiltonian formalism

- The **Hamiltonian** of the system is defined as the **Legendre transformation** of the Lagrangian

$$H(\mathbf{q},\mathbf{p},t) = \sum_i \dot{q}_i p_i - L(\mathbf{q},\dot{\mathbf{q}},t)$$

where the **generalised momenta** are $p_i = \frac{\partial L}{\partial \dot{q}_i}$

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- The **generalised velocities** can be expressed as a function of the **generalised momenta** if the previous equation is invertible, and thereby define the Hamiltonian of the system
- **Example**: consider $L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_i m_i \dot{q}_i^2 - V(q_1, \ldots, q_n)$
- From this, the momentum can be determined as $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1, \ldots, q_n)$$

Hamilton's equations

- The **equations of motion** can be derived from the Hamiltonian following the variational principle of **"stationary" action** but also by simply taking the differential of the Hamiltonian (see appendix)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \,, \quad \dot{p}_i = -\frac{\partial H}{\partial q} \,, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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- These are indeed $2n + 2$ equations describing the motion in the **"extended" phase space** $(q_1, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Properties of Hamiltonian flow

- The variables $(q_1, \ldots, q_n, p_1, \ldots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in **phase space**
- These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
- The variables used in the **Lagrangian do not necessarily have** this **property**

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- The variables used in the **Lagrangian do not necessarily have** this **property**
- Hamilton's equations can be written in **vector form** $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\nabla = (\partial q_1, \ldots, \partial q_n, \partial p_1, \ldots, \partial p_n)$
- The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic** matrix

Poisson brackets

- Crucial step in study of Hamiltonian systems is identification of **integrals of motion**
- Consider a **time dependent function** of phase space. Its time evolution is given by

$$\frac{d}{dt}f(\mathbf{p},\mathbf{q},t) = \sum_{i=1}^{n}\left(\frac{dq_i}{dt}\frac{\partial f}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t} = [H,f] + \frac{\partial f}{\partial t}$$

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- If a quantity is explicitly **time-independent** and its Poisson bracket with the Hamiltonian vanishes (i.e. **commutes** with the H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

Poisson brackets' properties

- From the definition, and for any three given functions, the following properties can be shown

$$[af + bg, h] = a[f, h] + b[g, h] \ , a, b \in \mathbb{R}$$ **bilinearity**

$$[f, g] = -[g, f]$$ **anticommutativity**

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$ **Jacobi's identity**

$$[f, gh] = [f, g]h + g[f, h]$$ **Leibniz's rule**

- Poisson brackets operation satisfies a **Lie algebra**

Canonical transformations

Canonical Transformations

- Find a **function** for transforming the Hamiltonian from variable $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$, so system becomes **simpler** to study
- Transformation should be **canonical** (or **symplectic**), so that **Hamiltonian** properties (**phase-space volume**) are preserved

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- These "mixed variable" **generating** functions are derived by

$$F_1(\mathbf{q}, \mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad F_3(\mathbf{Q}, \mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q}, \mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \qquad F_4(\mathbf{p}, \mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general **non-autonomous Hamiltonian** is transformed to

$$H(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F_j}{\partial t}, \quad j = 1, 2, 3, 4$$

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- One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q},\mathbf{P}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{Q}\cdot\mathbf{P}\ , \quad F_3(\mathbf{Q},\mathbf{p}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{q}\cdot\mathbf{p}\ , \quad \ldots$$

with the inner product define as $\mathbf{q}\cdot\mathbf{p} = \sum_i q_i p_i$

Preservation of Phase Volume

- A fundamental property of canonical transformations is the **preservation** of **phase space volume**
- This **volume** preservation in phase space can be represented in the **old** and **new variables** as

$$\int \prod_{i=1}^{n} dp_i dq_i = \int \prod_{i=1}^{n} dP_i dQ_i$$

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- The volume element in old and new variables are related through the **Jacobian**

$$\prod_{i=1}^{n} dp_i dq_i = \frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)} \prod_{i=1}^{n} dP_i dQ_i$$

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- These two relationships imply that the **Jacobian** of a **canonical transformation** should have **determinant** equal to **1**

$$\left|\frac{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}\right| = \left|\frac{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}\right| = 1$$

The Accelerator ring Hamiltonian

Single-particle relativistic Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x},t)$$

- It is generally a **3 degrees of freedom** one plus time (i.e., **4 degrees of freedom**)
- The Hamiltonian represents the **total energy**

$$H \equiv E = \gamma m c^2 + e\Phi$$

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- The **total kinetic momentum** is

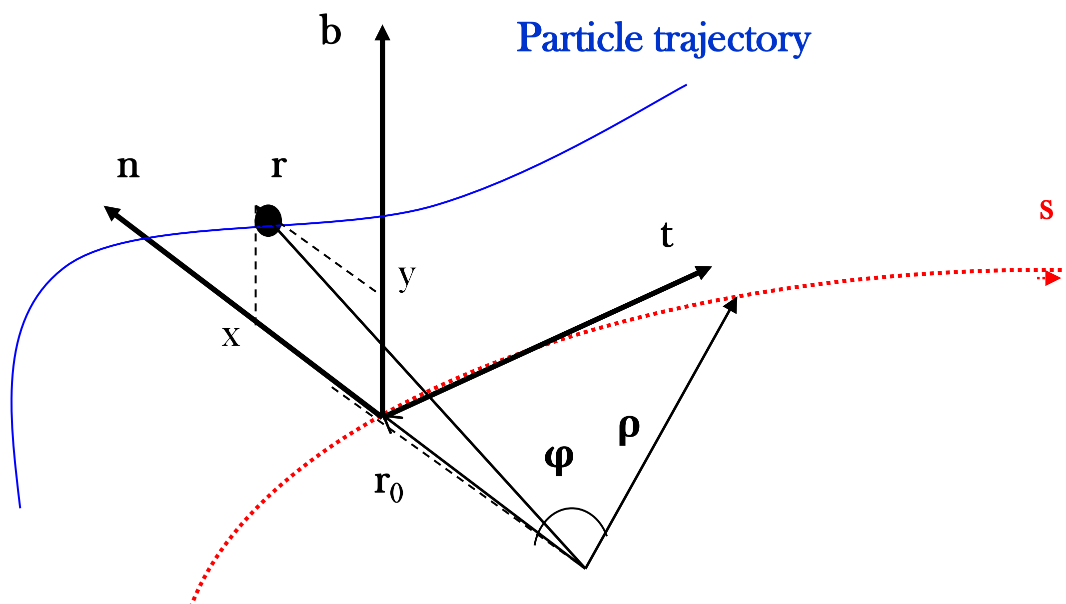
$$P = \left(\frac{H^2}{c^2} - m^2c^2\right)^{1/2}$$

- Using **Hamilton's equations**

$$(\dot{\mathbf{x}},\dot{\mathbf{p}}) = [(\mathbf{x},\mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**



$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P}) \text{ or}$$

$$(x, y, z, p_x, p_y, p_z) \mapsto (X, Y, s, P_x, P_y, P_s)$$

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**
  - Changing the **independent variable** from time t to the **path length** s
  - The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** with conjugate momentum $P_t = -\mathcal{H}$ or $P_s = -\mathcal{H}$

Coordinate tranformations

Canonical transformations and approximations

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**
  - Changing the **independent variable** from time t to the **path length** s
  - **Electric field** set to **zero**, as **longitudinal** (synchrotron) motion is much **slower** than **transverse** (betatron) one
  - Consider **static** and **transverse** magnetic fields

Coordinate tranformations (first two items)

Field approximations (last two items)

Canonical transformations and approximations

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**
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  - Consider **static** and **transverse** magnetic fields
  - **Rescale** the momentum with the reference one and move the **origin** to the **periodic orbit**
  - For the **ultra-relativistic limit** $\beta_0 \to 1\ , \quad \frac{1}{\beta_0^2\gamma^2} \to 0$ the Hamiltonian becomes

Coordinate tranformations (first two items)

Field approximations (items three and four)

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(l)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

with $l = -ct + \frac{s - s_0}{\beta_0}$ and $\frac{P_t - P_0}{P_0} \equiv \delta$

High-energy, large ring approximation

- It is useful for study purposes (especially for finding an "integrable" version of the Hamiltonian) to make an extra **approximation**
- For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller** than **1**, i.e. the square root can be expanded.

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- For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller** than **1**, i.e. the square root can be expanded.
- Considering also the large machine approximation $x << \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x(1+\delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may **not** be **a good idea**, especially for **low energy**, **small** size **rings**

General non-linear Accelerator Hamiltonian

- Considering the **general expression** of the the **longitudinal component** of the **vector potential** is (see appendix)
  - In curvilinear coordinates (curved elements)

$$A_s = (1 + \frac{x}{\rho(s)}) B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1} (x + iy)^{n+1}$$

  - In Cartesian coordinates $A_s = B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1} (x + iy)^{n+1}$

with the **multipole coefficients** being written as

$$a_n = \frac{1}{B_0\, n!} \frac{\partial^n B_x}{\partial x^n}\Big|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0\, n!} \frac{\partial^n B_y}{\partial x^n}\Big|_{x=y=0}$$

- The **general non-linear Hamiltonian** can be written as

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x,k_y}(s) x^{k_x} y^{k_y}$$

with the **periodic functions** $h_{k_x,k_y}(s) = h_{k_x,k_y}(s + C)$

Magnetic element Hamiltonians

- Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Linear magnetic fields

Linear magnetic fields

- Assume a simple case of **linear transverse magnetic fields,**

$$\begin{aligned} B_x &= \quad b_1(s)y \\ B_y &= - b_0(s) + b_1(s)x \end{aligned},$$

  - main bending field $-B_0 \equiv b_0(s) = \frac{P_0 c}{e\rho(s)}$ [T]
  - normalized quadrupole gradient $K(s) = b_1(s)\frac{e}{cP_0} = \frac{b_1(s)}{B\rho}$ [1/m^2]
  - magnetic rigidity $B\rho = \frac{P_0 c}{e}$ [T · m]

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- The vector potential has only a **longitudinal component** which in curvilinear coordinates is

$$B_x = -\frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial y} \ , \quad B_y = \frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial x}$$

- The previous expressions can be integrated to give

$$A_s(x,y,s) = \frac{P_0 c}{e}\left[-\frac{x}{\rho(s)} - \left(\frac{1}{\rho(s)^2} + K(s)\right)\frac{x^2}{2} + K(s)\frac{y^2}{2}\right] = P_0 c\ \hat{A}_s(x,y,s)$$

The integrable Hamiltonian

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2}(x^2 - y^2)$$

- Hamilton's equation are

$$\frac{dx}{ds} = \frac{p_x}{1+\delta}\,, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left(\frac{1}{\rho^2(s)} + K(s)\right)x$$

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$$\frac{dy}{ds} = \frac{p_y}{1+\delta}, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. **Hill's equations** (see **Transverse Dynamics lecture**)

$$x'' + \frac{1}{1+\delta}\overbrace{\left(\frac{1}{\rho(s)^2} + K(s)\right)}^{K_x} x = \frac{\delta}{\rho(s)}$$

$$y'' - \frac{1}{1+\delta}\underbrace{K(s)}_{K_y}y = 0$$

with the usual solution for $\delta = 0$ and $u = x, y$

$$u(s) = \sqrt{\epsilon_u \beta_u(s)}\cos(\psi_u(s) + \psi_{u0})$$

$$u'(s) = \frac{du}{ds} = \sqrt{\frac{\epsilon_u}{\beta_u(s)}}(\sin(\psi_u(s) + \psi_{u0}) + \alpha_u \cos(\psi_u(s) + \psi_{u0}))$$

The CERN Accelerator School

Action-Angle Variables

Action-angle variables

- There is a canonical transformation to some **optimal set** of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$ over closed paths in phase space.

n

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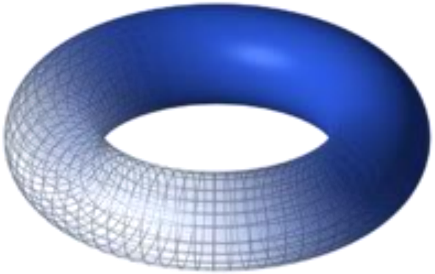
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$ over closed paths in phase space.
- An **integrable Hamiltonian** is written as a function of only the actions, i.e. $H_0 = H_0(\mathbf{J})$. Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$

$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$



i.e. the **actions are integrals of motion** and the **angles** are **evolving linearly with time**, with **constant frequencies** which depend on the actions

- The actions define the surface of an **invariant torus**, topologically equivalent to the product of n circles

Harmonic oscillator revisited

- The Hamiltonian for the harmonic oscillator can be written as

$$H(u, p_u) = \frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right)$$

with the **canonical position** and **momentum** (u, p_u)

- From definition of the action

$$J_u = \frac{1}{2\pi}\oint p_u du = \frac{1}{2\pi}\oint \sqrt{2H - \omega_0^2 u^2 du} = \frac{1}{\pi}\int_{-u_{\text{ext}}}^{u_{\text{ext}}} \sqrt{2H - \omega_0^2 u^2 du} = \frac{H}{\omega_0}$$

with $u_{\text{ext}} = \frac{\sqrt{2H}}{\omega_0}$ the position extrema, obtained for $p_u = 0$.

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with $u_{\rm ext} = \frac{\sqrt{2H}}{\omega_0}$ the position extrema, obtained for $p_u = 0$.

- The Hamiltonian in these new variables $H(\phi_u, J_u) = \omega_0 J_u$
- The **phase** is found by Hamilton's equations as

$$\dot{\phi}_u = \frac{\partial H(\phi_u, J_u)}{\partial J_u} = \omega_0 \quad \text{and hence} \quad \phi_u = \omega_0 t + \phi_{u,0}$$

- The **action** is $\dot{J}_u = -\frac{\partial H(\phi_u, J_u)}{\partial \phi_u} = 0$, i.e. $J_u = \text{const.}$ an integral of motion.

Harmonic oscillator revisited

- Another way to calculate the action is through canonical transformation using a **generating function**
- First, observe from **solution** of harmonic oscillator that

$$p_u = -\omega_0 u \tan(\omega_0 t + \phi_{u,0}) = -\omega_0 u \tan(\phi_u)$$

relationship already connecting **phase** with **old variables**

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$$p_u = -\omega_0 u \tan\left(\omega_0 t + \phi_{u,0}\right) = -\omega_0 u \tan\left(\phi_u\right)$$
relationship already connecting **phase** with **old variables**
- Using **first generating function** $F_1(u, \phi_u)$
$$p_u = \frac{\partial F_1}{\partial u} = -\omega_0 u \tan\left(\phi_u\right)$$
- By integrating, we obtain $F_1 = \int p_u du = -\frac{\omega_0 u^2}{2}\tan(\phi_u)$
- **New momentum** conjugate to the phase is given by
$$J_u = -\frac{\partial F_1}{\partial \phi_u} = \frac{\omega_0 u^2}{2}(1 + \tan^2(\phi_u)) = \frac{1}{2\omega_0}(\omega_0^2 u^2 + p^2) = \frac{H}{\omega_0}$$
i.e. exactly the **same relationship** as with the previous method.

Accelerator Hamiltonian in action-angle variables

- Considering **on-momentum** motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2} + \frac{K_x(s)x^2 - K_y(s)y^2}{2}$$

- As for harmonic oscillator, use Courant-Snyder solutions to build **generating function** from original to action-angles

$$F_1(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x(s)}\left[\tan\phi_x(s) + a_x(s)\right] - \frac{y^2}{2\beta_y(s)}\left[\tan\phi_y(s) + a_y(s)\right]$$

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- The **old variables** with respect to **actions** and **angles** are

$$u(s) = \sqrt{2\beta_u(s)J_u}\cos\phi_u(s)\ , \quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}}\left(\sin\phi_u(s)+\alpha_u(s)\cos\phi_u(s)\right)$$

and the Hamiltonian takes the form

$$\mathcal{H}_0(J_x,J_y,s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

Normalised coordinates

- The transformation to **normalized coordinates**

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos(\phi) \\ -\sin(\phi) \end{pmatrix}$$

transforms motion to simple rotations.

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- The 1-turn Hamiltonian is

$$\bar{\mathcal{H}}_0(J_x, J_y) = J_x \oint \frac{ds}{\beta_x(s)} + J_y \oint \frac{ds}{\beta_y(s)} = 2\pi \left(Q_x J_x + Q_y J_y\right)$$

- The motion is the one of two linearly independent harmonic oscillators with frequencies the **tunes**

Canonical perturbation theory

Canonical perturbation theory

- Consider a general Hamiltonian with n degrees of freedom
$$H(\boldsymbol{J},\boldsymbol{\varphi},\theta) = H_0(\mathbf{J}) + \epsilon H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$$
where the **non-integrable** part $H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta)$ is 2π -periodic on the angles $\boldsymbol{\varphi}$ and the "time" θ
- Provided that ϵ is sufficiently small, **tori** should still exist but they are **distorted**
- We seek a canonical transformation that **could "straighten up" the tori**, i.e. it could transform the non-integrable part of the Hamiltonian (at first order in ϵ) to a **function only of some new actions** $\bar{H}(\bar{\boldsymbol{J}})$ plus higher orders in ϵ

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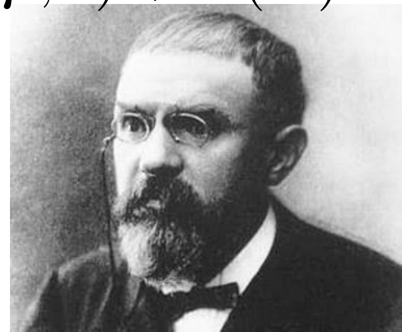
- This can be performed by a **mixed variable** close to identity **generating function** $S(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) = \bar{\boldsymbol{J}}\cdot\boldsymbol{\varphi} + \epsilon S_1(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$ for transforming old variables to new ones $(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$
- In principle, this procedure can be carried to **arbitrary powers** of the perturbation



Canonical perturbation theory

- By the canonical transformation equations (slide 19), the the **old action** and **new angle** can be also represented by a power series in ϵ

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \boldsymbol{\varphi}} + \mathcal{O}(\epsilon^2)$$

$$\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

or

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} - \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

Canonical perturbation theory

- By the canonical transformation equations (slide 23), the the **old action** and **new angle** can be also represented by a power series in ϵ

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \boldsymbol{\varphi}} + \mathcal{O}(\epsilon^2) \qquad \boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2) \quad \text{or} \quad \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} - \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

- The previous equations expressing the old as a function of the new variables assume that there is possibility to **invert** the equation on the left, so that $S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)$ becomes a function of the new variables
- The **new Hamiltonian** is then

$$\bar{H}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta) = H(\boldsymbol{J}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \boldsymbol{\varphi}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \theta) + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \mathcal{O}(\epsilon^2)$$

- The second term is appearing because of the "**time**" dependence through θ

Form of the generating function

- The question is what is the form of the **generating function** that eliminates the angle dependence
- The procedure is cumbersome (see appendix for details), but here is the final result,

$$S(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}} \cdot \bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k} \neq \mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k} \cdot \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + p} e^{i(\boldsymbol{k} \cdot \bar{\boldsymbol{\varphi}} + p\theta)} + \mathcal{O}(\epsilon^2)$$

with the frequency vector $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$
and the integers $\boldsymbol{k}, p \neq \mathbf{0}$

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  with the frequency vector $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$
  and the integers $\boldsymbol{k}, p \neq \mathbf{0}$
- If the denominator vanishes, i.e. for the **resonance condition** $\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p=0$, the Fourier series coefficients (**driving terms**) become **infinite**
- It actually implies that even at **first order** in the perturbation parameter and in the vicinity of a resonance, it is **impossible** to construct a **generating function** for seeking some **approximate integrals of motion**

Problem of small denominators

- In principle, the **technique works** for **arbitrary order**, but the **disentangling** of **variables** becomes difficult even to 2nd order!!!
- The solution was given in the late 60s by introducing the **Lie transforms** (e.g. see Deprit 1969), which are **algorithmic** for **constructing generating functions** and were adapted to beam dynamics by Dragt and Finn (1976)

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- The solution was given in the late 60s by introducing the **Lie transforms** (e.g. see Deprit 1969), which are **algorithmic** for **constructing generating functions** and were adapted to beam dynamics by Dragt and Finn (1976)
- On the other hand, the problem of **small denominators** due to **resonances** is not just a mathematical one. The inability to construct solutions close to a **resonance** has to do with the **un-predictable nature of motion** and the **onset** of **chaos**
- **KAM theory** (see appendix) developed the mathematical framework into which local solutions could be constructed, provided some general conditions on the size of the perturbation and the distance of the system from resonances are satisfied
- Very difficult though to apply **directly** this theorem to realistic physical systems, such as a particle accelerator

Example: Perturbation treatment of a sextupole

Perturbation treatment for a sextupole

- Consider the simple case of a **periodic sextupole perturbation** and restrict the study only to one plane. The **Hamiltonian** is written as,

$$H(x, p_x, s) = \frac{p_x^2 + K(s)x^2}{2} + \frac{K_s(s)x^3}{3}$$

where $K(s)$ and $K_s(s)$ are periodic functions of time.

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- We proceed to the **transformation** in **action angle variables** to write the Hamiltonian in the form

$$H = H_0(J) + H_1(\phi, J) = \frac{J}{\beta(s)} + \frac{2\sqrt{2}K_s(s)}{3}(J\beta(s))^{3/2}\cos^3\phi$$

$$= \frac{J}{\beta(s)} + \frac{K_s(s)}{3\sqrt{2}}(J\beta(s))^{3/2}(\cos 3\phi + 3\cos\phi)$$

- The perturbation procedure implies to split the perturbation in an average part over the angles and an oscillating part

$$H_1 = \langle H_1 \rangle_{\phi} + \{H_1\} = \frac{\sqrt{2}k_2(s)}{12} (J\beta(s))^{3/2} (\cos 3\phi + 3\cos\phi)$$

where $\langle H_1 \rangle_{\varphi} = \left(\frac{1}{2\pi}\right) \oint H_1(J, \varphi) d\varphi$

and $\{H_1\} = H_1 - \langle H_1 \rangle_{\boldsymbol{\varphi}}$

$$= \sum_{k,p} H_{1k}(J) e^{i(k\cdot\varphi + p\theta)}$$

Tuneshift from a sextupole

- The **average part** should be only a **function of the action**
- Its derivative with respect to the action should provide the **frequency shift (tune-shift)** due to the non-linearity
- It can be shown that this quantity vanishes for a sextupole perturbation

$$\langle\frac{\partial H_1(\phi,J)}{\partial J}\rangle_\phi = \frac{k_2(s)\beta(s)}{8\sqrt{2}\pi}\,(J\beta(s))^{1/2}\int_0^{2\pi}(\cos 3\phi + 3\cos\phi)d\phi = 0$$

- Sextupoles do not provide any tune-shift at **first order**
- But we know by experience that this is not true, i.e. first order perturbation theory **fails** to give the correct answer
- One has to go to **higher order** (see appendix)

Perturbation treatment for a sextupole

- The **oscillating part** is then the same as the original Hamiltonian

$$\{H_1\} = H_1 - \langle H_1\rangle_{\bar\phi} = H_1 = \frac{K_s(s)}{3\sqrt{2}}\left(\bar J\beta(s)\right)^{3/2}(\cos 3\phi + 3\cos\phi)$$

- Following the canonical perturbation procedure the **generating function** is

$$S(\bar J,\bar\phi) = \bar J\cdot\bar\phi + i\sum_{k,p\neq 0}\frac{H_{1k}(\bar J)}{k\cdot\nu(\bar J)+p}e^{i(k\cdot\bar\phi+p\theta)} + \ldots$$

- The **only non-zero Fourier terms** are for $k=1,3$ and

$$S(\bar J,\bar\phi) = \bar J\cdot\bar\phi + i\frac{K_s(s)}{6\sqrt{2}}\left(\bar J\beta(s)\right)^{3/2}\sum_{p=-\infty}^{\infty}\left(\frac{e^{i(3\bar\phi+p\theta)}}{3\nu+p} + \frac{3e^{i(\bar\phi+p\theta)}}{\nu+p}\right)$$

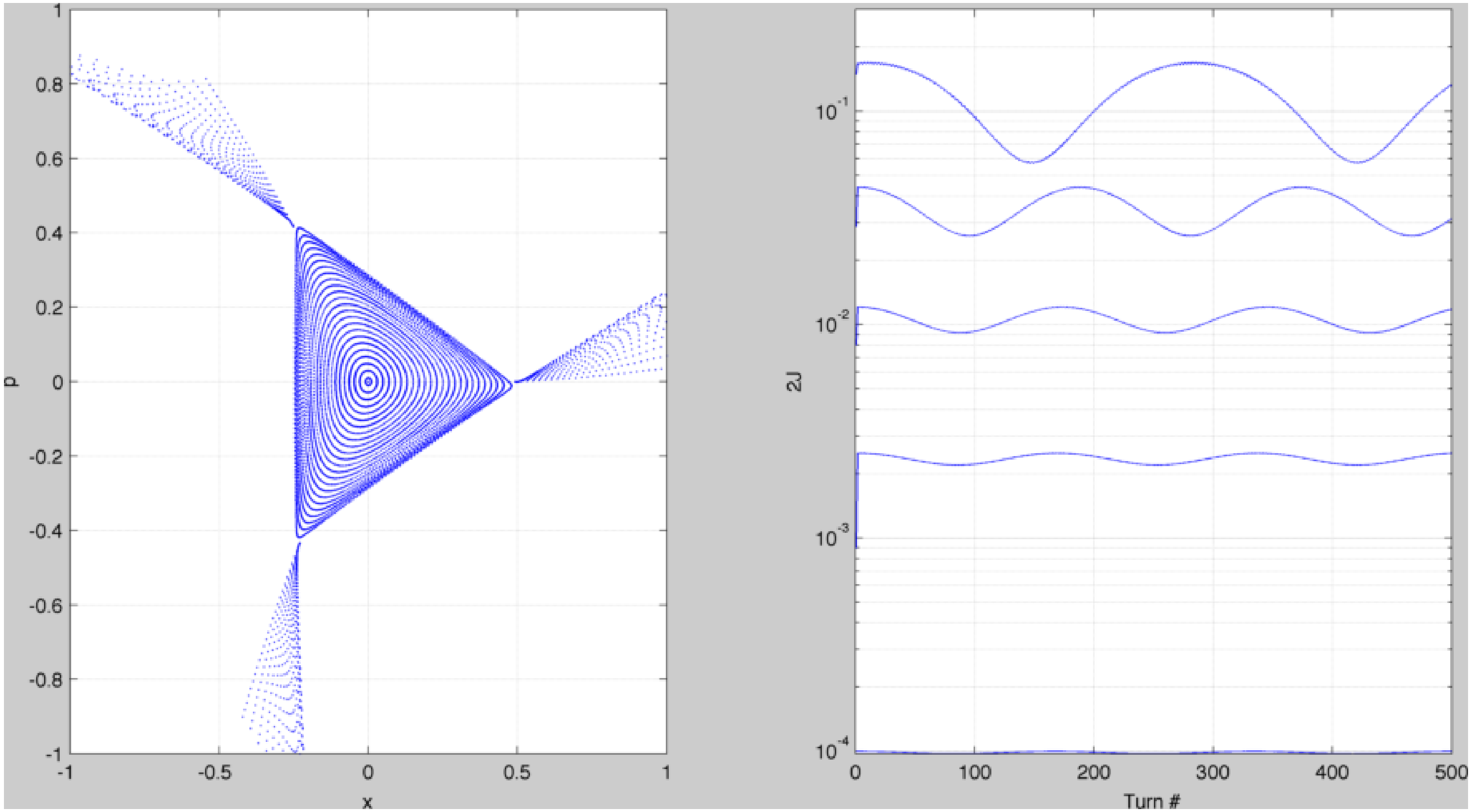
Perturbation treatment for a sextupole

- We derived (with a lot of effort) the common result that sextupoles **at first order** excite **integer** and **third integer** resonances
- Again, this is **not the full story**! It is known that sextupoles can drive **any resonance,** either because their **strength** is **large**, or because the **particle** is **far** away from the **closed orbit**
- This can be shown again by pursuing the perturbation approach to **second order** (as for the tune-shift)
- A useful application is to use the **generating function** for computing the correction to the **original invariant**, as the new one should be an integral of motion (at first order)

$$J \approx \bar{J} + \frac{\partial S_1(\bar{J}, \varphi, \theta)}{\partial \varphi}$$

Phase space for sextupole perturbation

- For small perturbations, the **new action variable** is almost an **invariant** but for larger ones phase space gets deformed
- Close to the integer or third integer resonance, canonical perturbation theory cannot be applied
- The solution is provided by **secular perturbation theory** (see appendix)



Content of lecture II

- From **linear** to **non-linear** or from **matrices** to **maps**
- **Lie** formalism for building **maps**
- **Symplectic integration**
- **Normal forms** for non-linear systems
- **Truncated Power Series** through **differential Algebra**

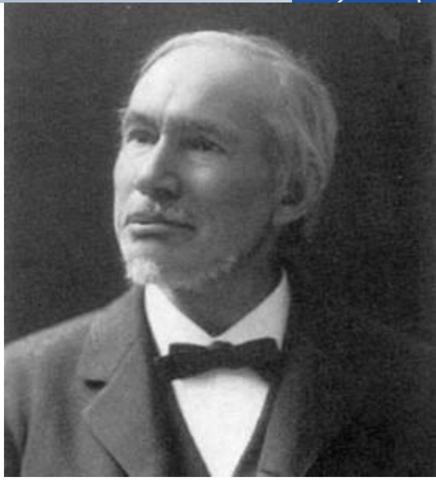
From linear to non-linear
or
from matrices to maps

Linear system in beam dynamics

- **Linear (uncoupled) transverse particle motion** is described by **Hill's equation**

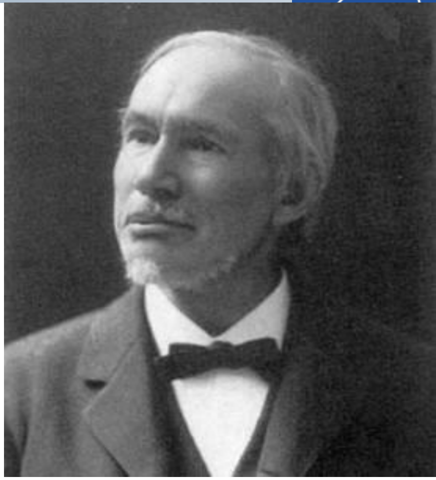
$$x'' + K_x(s)\, x = 0$$

- Linear equations with ***s*-dependent coefficients** (harmonic oscillator with time dependent frequency)



George Hill

Linear system in beam dynamics



George Hill

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- In a ring (or in transport line with symmetries), coefficients are **periodic** $K_x(s) = K_x(s+C)$
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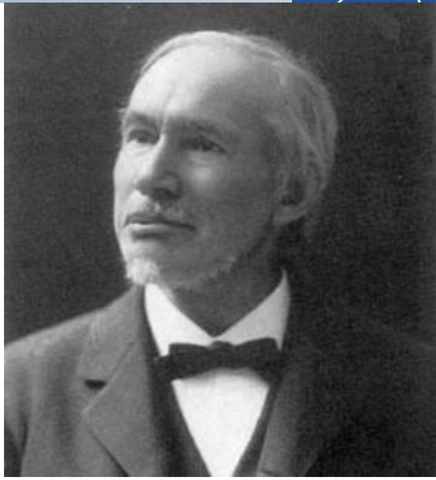

George Hill

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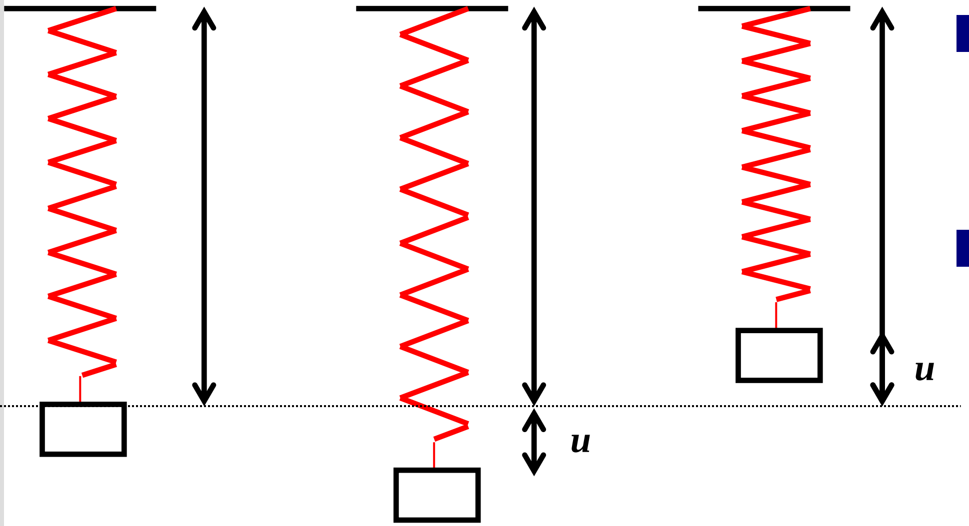
- In a ring (or in transport line with symmetries), coefficients are **periodic** $K_x(s) = K_x(s+C)$
- Not straightforward to derive closed analytical solutions for the whole accelerator…
- …but do we really **care,** in particular for a system composed by **discrete building blocks**?

Harmonic oscillator



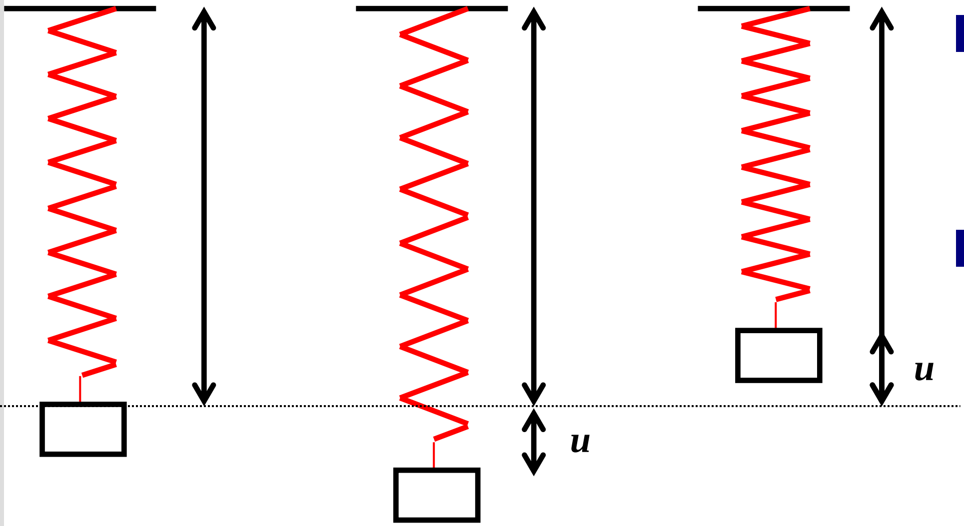
- Consider $K(s) = k_0$ = constant

$$u'' + k_0 \; u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\; u(0) + S(s)\; u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

Harmonic oscillator



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with

$$C(s) = \cos(\sqrt{k_0} s) \ , \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0} s) \quad , \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|} s) \ , \ S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|} s) \quad , \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$ which is always true for conservative systems

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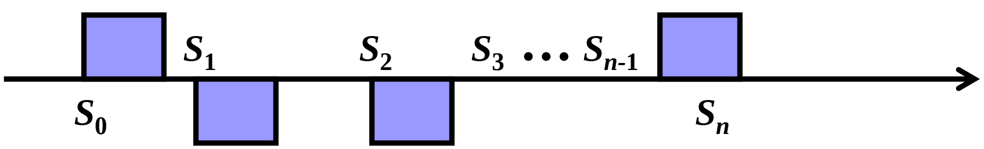
- Any line can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s_0)$$



from s_0 to s_1

from s_0 to s_2

from s_0 to s_3

from s_0 to s_n

- For a full ring, the matrix multiplication will provide the full **transfer matrix** for **1-turn**

$$\mathcal{M}_C = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Non-linear motion

- Nonlinear elements can be represented by **generalized polynomials**

$$x''K_x(s)x = \sum_{i,j} a_{ij}(s)x^i y^j$$

- For example, general magnetic fields can be represented by the **multi-pole expansion**

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

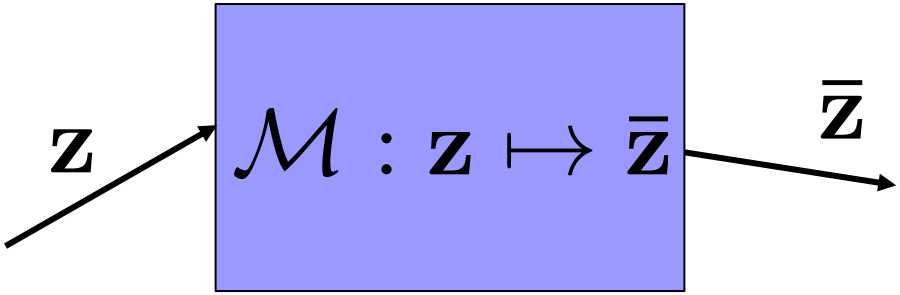
- Equations of motion in the horizontal plane become

$$x'' + K_x(s)x = -\frac{B_y(x,y,s)}{p}$$

- Closed solution does not exist, in principle!

Maps

- A generalization of the matrix (which can only describe linear systems), is a **map**, which transforms a system from some initial to some final coordinates



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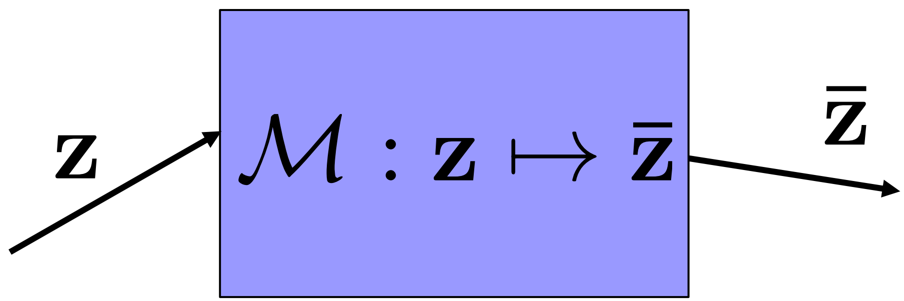

- Analyzing the map, will give useful information about the behavior of the system
- There are different ways to build the map:
  - Taylor (Power) maps
  - Lie transformations
  - Truncated Power Series Algebra (TPSA), can generate maps from straight-forward tracking
- Preservation of **symplecticity** is important

Building a non-linear map

- For a **thin quadrupole** the equivalent map can be written

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_1 \cdot x_{s_1} \\ 0 \\ k_1 \cdot y_{s_1} \end{pmatrix}$$

or through the matrix **M**, as $\vec{z}(s_2) = \mathbf{M} \cdot \vec{z}(s_1)$.

- For a **thin sextupole**, we can right the coordinate transformation as

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ \frac{1}{2}k_2 \cdot (x_{s_1}^2 - y_{s_1}^2) \\ 0 \\ k_2 \cdot (x_{s_1} \cdot y_{s_1}) \end{pmatrix}$$

or $\vec{z}(s_2) = \mathcal{M} \circ \vec{z}(s_1)$ where now $\mathcal{M}$ is a non-linear map.

Building a Taylor Map

- A general representation for the map for the horizontal position can be

$$x_{new} = \overbrace{R_{11}\cdot x + R_{12}\cdot x' + R_{21}\cdot y + R_{22}\cdot y'}^{\text{matrix part (power 1)}} +$$

$$+\overbrace{T_{111}\cdot x^2 + T_{112}\cdot xx' + T_{122}\cdot x'^2 + T_{113}\cdot xy + T_{114}\cdot xy' + \ldots\ldots}^{\text{sextupole part (power 2)}}$$

$$+\overbrace{U_{1111}\cdot x^3 + U_{1112}\cdot x^2x' + \ldots\ldots}^{\text{octupole part (power 3)}}$$

$$+\ldots\ldots\ldots$$

or, in a more compact form up to 3rd order, for $j = 1,\ldots,6$

$$z_j^{new} = \sum_{k=1}^{6} R_{jk} z_k + \sum_{k=1}^{6}\sum_{l=1}^{6} T_{jkl} z_k z_l + \sum_{k=1}^{6}\sum_{l=1}^{6}\sum_{m=1}^{6} U_{jklm} z_k z_l z_m$$

Taylor map for a sextupole

- For a sextupole in one plane, the representation is written as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{new} = \begin{pmatrix} R_{11} & R_{12} & T_{111} & T_{112} & T_{122} \\ R_{21} & R_{22} & T_{211} & T_{212} & T_{222} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ x^2 \\ xx' \\ x'^2 \end{pmatrix}$$

or in general for a sextupole of length L and strength k_2

$$\begin{aligned}
x_2 &= x_1 + Lx_1' & - k_2\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1x_1' - y_1y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right) \\
x_2' &= x_1' & - k_2\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1x_1' - y_1y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right) \\
y_2 &= y_1 + Ly_1' & + k_2\left(\tfrac{L^2}{4}x_1y_1 + \tfrac{L^3}{12}(x_1y_1' + y_1x_1') + \tfrac{L^4}{24}(x_1'y_1')\right) \\
y_2' &= y_1' & + k_2\left(\tfrac{L}{2}x_1y_1 + \tfrac{L^2}{4}(x_1y_1' + y_1x_1') + \tfrac{L^3}{6}(x_1'y_1')\right)
\end{aligned}$$

- But what about **symplecticity?**
- Need to introduce Lie formalism

Lie formalism

Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}$, $\bar{\mathbf{z}}$ which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a **map** $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$

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- The **Jacobian matrix** of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$
- The map is **symplectic** if $M^T J M = J$ where $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
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- It can be shown that $\det(M) = 1$
- It can be shown that the variables defined through a symplectic map $[\bar{z}_i, \bar{z}_j] = [z_i, z_j] = \mathcal{I}_{ij}$ which is a known relation satisfied by canonical variables
- In other words, symplectic maps **preserve** Poisson brackets

Are Taylor maps symplectic?

- To test the **symplecticity** of Taylor maps, we have to construct the Jacobian matrix with elements $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$
- The "thick" sextupole **Taylor map**, is written

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' && - k_2\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1x_1' - y_1y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right)\\
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\end{aligned}
$$

- All the coefficients of the Jacobian depend on initial conditions, e.g.

$$\frac{\partial y_2}{\partial y_1} = 1 + k_2\left(\frac{L^2}{4}x_1 + \frac{L^3}{12}x_1'\right)$$

and unless appropriately chosen they cannot satisfy $\det(M) = 1$

- In general, Taylor maps are **not-symplectic!**

Lie formalism

- The Poisson bracket properties satisfy what is mathematically called a **Lie** algebra
- They can be represented by (Lie) operators of the form $:f:g = [f,g]$ and $:f:^2 g = [f,[f,g]]$ etc.

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- The **(1-turn) accelerator map** can be represented by the **composition** of the maps of each element $\mathcal{M} = e^{:f_2:} e^{:f_3:} e^{:f_4:} \ldots$ where f_i (called the generator) is the Hamiltonian for each element, a polynomial of degree m in the variables $z_1, \ldots, z_n$

Lie operators for simple elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(:-\frac{1}{2}Lp^2:)$
Thin-lens Quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(:-\frac{1}{2f}x^2:)$
Thin-lens Multipole	$x = x_0$ $p = p_0 + \lambda n x^{n-1}$	$\exp(:\lambda x^n:)$
Thin-lens kick	$x = x_0$ $p = p_0 + f(x)$	$\exp(:\int_0^x f(x')dx':)$
Thick focusing quad	$x = x_0 \cos kL + \frac{p_0}{k}\sin kL$ $p = -kx_0 \sin kL + p_0 \cos kL$	$\exp[:-\frac{1}{2}L(k^2x^2 + p^2):]$
Thick defocusing quad	$x = x_0 \cosh kL + \frac{p_0}{k}\sinh kL$ $p = kx_0 \sinh kL + p_0 \cosh kL$	$\exp[:\frac{1}{2}L(k^2x^2 - p^2):]$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(:ax + bp:)$
Coordinate rotation	$x = x_0 \cos\mu + p_0 \sin\mu$ $p = -x_0 \sin\mu + p_0 \cos\mu$	$\exp[:-\frac{1}{2}\mu(x^2 + p^2):]$
Scale change	$x = e^{-\lambda}x_0$ $p = e^{\lambda}p_0$	$\exp(:\lambda xp:)$

Formulas for Lie operators

$$:a: = 0, \qquad e^{:a:} = 1$$

$$:f:a = 0, \qquad e^{:f:}a = a$$

$$:f:f = 0, \qquad e^{:f:}f = f$$

$$\{:f:, :g:\} = :[f,g]:$$

$$e^{:f:}g(X) = g(e^{:f:}X)$$

$$e^{:f:}G(:g:)e^{-:f:} = G(:e^{:f:}g:)$$

Map for quadrupole

- Consider the 1D quadrupole Hamiltonian

$$H = \frac{1}{2}(k_1 x^2 + p^2)$$

- For a quadrupole of length L, the map is written as

$$e^{\frac{L}{2}:(k_1 x^2 + p^2):}$$

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- Its application to the transverse variables is

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}x = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!}x + L\frac{(-k_1 L^2)^n}{(2n+1)!}p\right)$$

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):}p = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!}p - \sqrt{k_1}\frac{(-k_1 L^2)^n}{(2n+1)!}x\right)$$

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$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} p = \sum_{n=0}^{\infty} \left(\frac{(-k_1 L^2)^n}{(2n)!} p - \sqrt{k_1} \frac{(-k_1 L^2)^n}{(2n+1)!} x \right)$$

- This finally provides the usual quadrupole matrix

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} x = \cos(\sqrt{k_1} L) x + \frac{1}{\sqrt{k_1}} \sin(\sqrt{k_1} L) p$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} p = -\sqrt{k_1} \sin(\sqrt{k_1} L) x + \cos(\sqrt{k_1} L) p$$

Map for general monomial

- Consider a monomial in the positions and momenta $x^n p^m$
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- Its application to the transverse variables is
  - For $n \neq m$

$$e^{:\alpha x^n p^m:}x = x\left[1+\alpha(n-m)x^{n-1}p^{m-1}\right]^{\frac{m}{m-n}}$$

$$e^{:\alpha x^n p^m:}p = p\left[1+\alpha(n-m)x^{n-1}p^{m-1}\right]^{\frac{n}{n-m}}$$

  - For $n = m$

$$e^{:\alpha x^n p^n:}x = xe^{-\alpha n x^{n-1}p^{n-1}}$$

$$e^{:\alpha x^n p^n:}p = pe^{\alpha n x^{n-1}p^{n-1}}$$

Map Concatenation

- For combining together the different maps, the **Campbell-Baker-Hausdorff** formula can be used. It states that for t_1, t_2 sufficiently small, and A, B real matrices, there is a real matrix C for which

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or

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i.e. a series of Poisson bracket operations.

- Note that the **full map** is by "construction" symplectic.
- By **truncating the map** to a certain order, **symplecticity** is **lost**.

Useful form of CBH formula

- The **Campbell-Baker-Hausdorff** formula for Lie maps has another useful form, depending if the summation is done over one or the other function

$$e^{:f:}e^{:g:} = e^{:g+\left(\frac{:g:}{e^{:g:}-1}f\right)+\mathcal{O}(f^2):}$$

or

$$e^{:f:}e^{:g:} = e^{:f+\left(\frac{:f:}{1-e^{-:f:}}g\right)+\mathcal{O}(g^2):}$$

Symplectic integration

Why symplecticity is important

- **Symplecticity** guarantees that the **transformations** in phase space are **area preserving**
- To understand what deviation from symplecticity produces consider the simple case of the **quadrupole** with the general matrix written as

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

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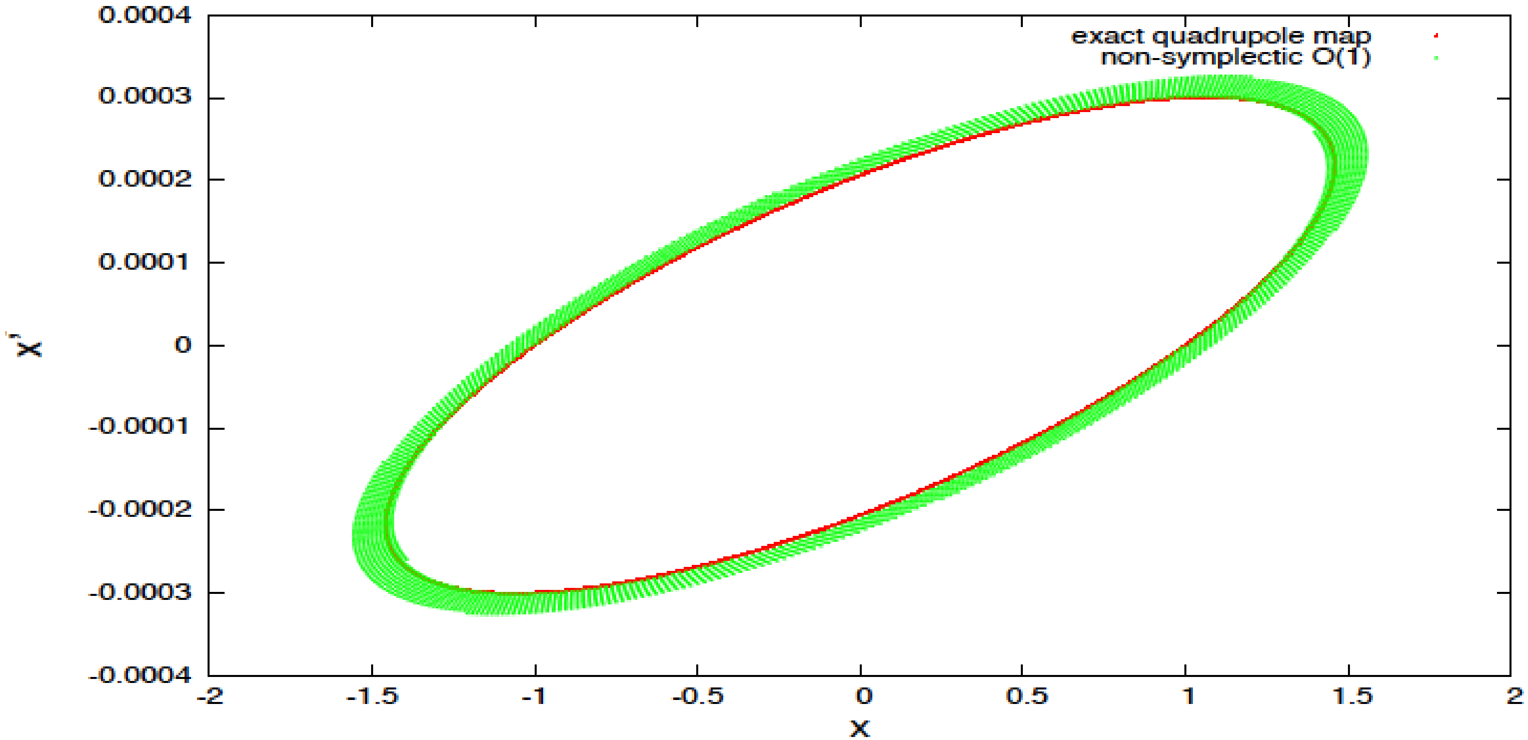
- Take the Taylor expansion for small lengths, up to first order

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & L \\ -kL & 1 \end{pmatrix} + O(L^2)$$

- This is indeed **not symplectic** as the determinant of the matrix is equal to $1 + kL^2$, i.e. there is a deviation from symplecticity at 2$^{\mathrm{nd}}$ order in the quadrupole length

Phase portrait for non-symplectic matrix

- The iterated **non-symplectic matrix** does not provide the well-know **elliptic trajectory** in phase space
- Although the trajectory is very close to the original one, it **spirals outwards towards infinity**

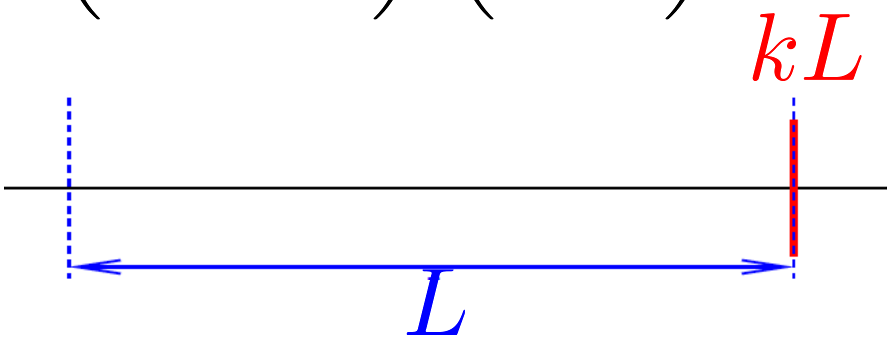


Restoring symplecticity

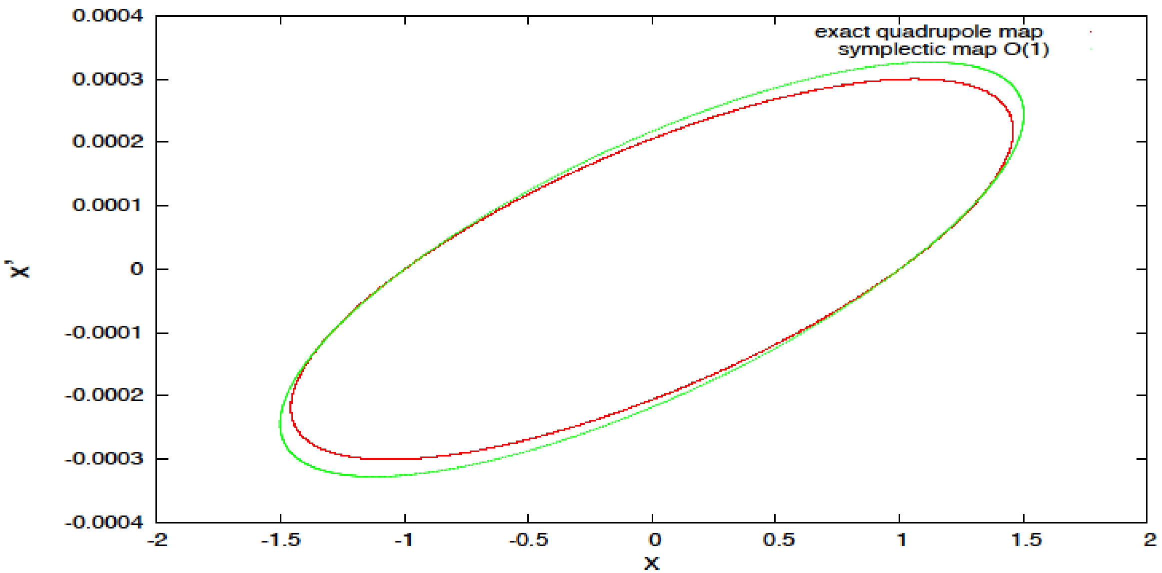
- **Symplecticity** be can **restored** by adding "artificially" a correcting term to the matrix to become

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & L \\ -kL & 1-kL^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

- In fact, the matrix now can be decomposed as a **drift** with a **thin quadrupole** at the end



- This representation, although not exact produces an **ellipse** in phase space

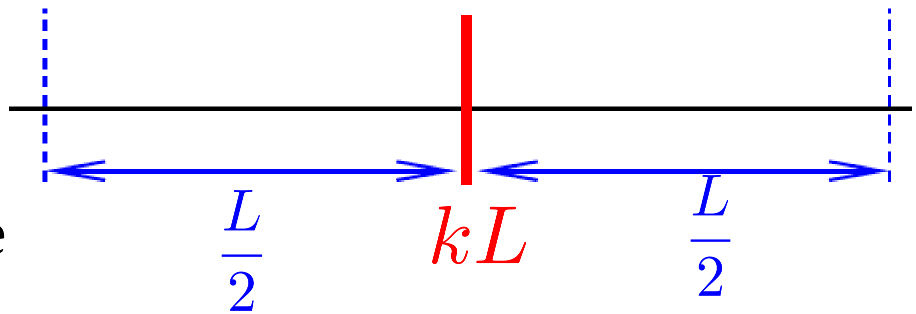


Restoring symplecticity II

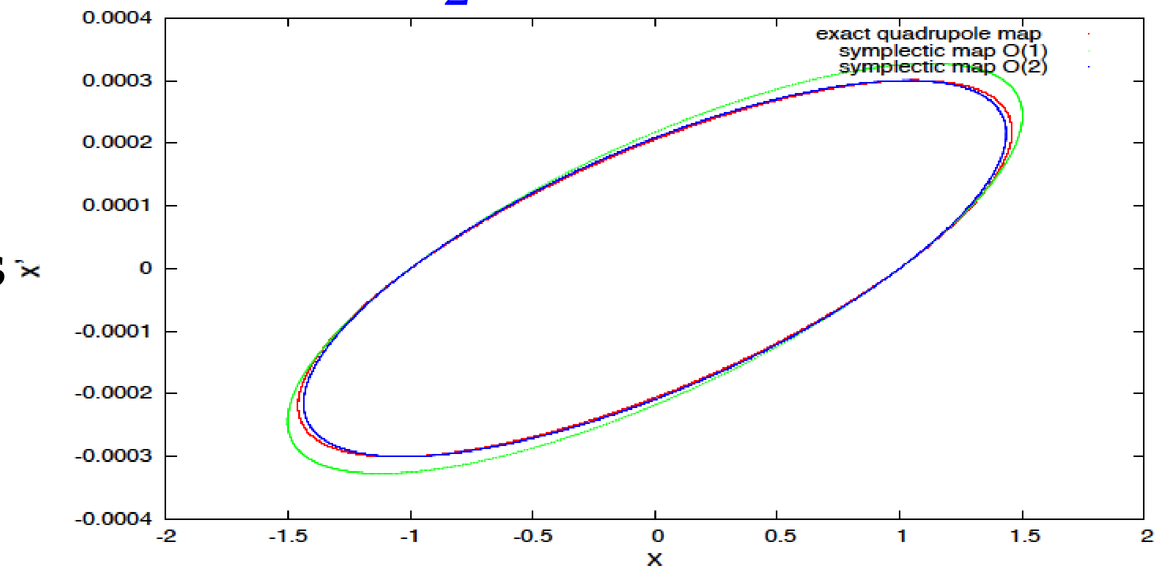
- The same approach can be continued to **2nd order** of the Taylor map, by adding a **3rd order correction**

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1-\frac{1}{2}kL^2 & L-\frac{1}{4}kL^3 \\ -kL & 1-\frac{1}{2}kL^2 \end{pmatrix} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}$$

- The matrix now can be decomposed as **two half drifts with a thin kick** at the center



- This representation is even **more exact** as the error now is at 3rd order in the length



3-kick symplectic integrator

- The idea is to distribute **three kicks with different strengths** so as to get a final map which is more accurate then the previous ones
- For the quadrupole, one can write

$$\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} 1 & d_1 L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_1 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_2 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_3 k L & 1 \end{pmatrix} \begin{pmatrix} 1 & d_4 L/2 \\ 0 & 1 \end{pmatrix}$$

which imposes $\sum d_i = \sum c_i = 1$.

- A **symmetry condition** of this form can be added

$$d_1 = d_4 \ , \quad d_2 = d_3 \ , \quad c_1 = c_3$$

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- This provides the **matrix** $\mathcal{M}_{\mathrm{Q}} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ with

$$m_{11} = m_{22} = -\frac{1}{2}kL^2 + c_1 d_2 (d_1 + \frac{c_2}{2}) k^2 L^4 - d_1 d_2^2 c_1^2 c_2 k^3 L^6$$

$$m_{12} = L - (\frac{c_2}{4} + d_1 d_2 + 2 d_1 d_2 c_1) kL^3 + 2 d_1 d_2 c_1 (d_1 d_2 + \frac{c_2}{2}) k^2 L^5 + d_1^2 d_2^2 c_1^2 c_2 k^3 L^7$$

$$m_{21} = -kL + c_1 d_2 (1 + c_2) k^2 L^3 - d_2^2 c_1^2 c_2 k^3 L^5$$

3-kick symplectic integrator

- By imposing that the **determinant** is 1, the following additional relations are obtained

$$c_1 d_2 (d_1 + \frac{c_2}{2}) = \frac{1}{24}$$

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$$d_1 = d_4 = \frac{1}{2(2-2^{1/3})}\,, \quad d_2 = d_3 = \frac{1-2^{1/3}}{2(2-2^{1/3})}$$

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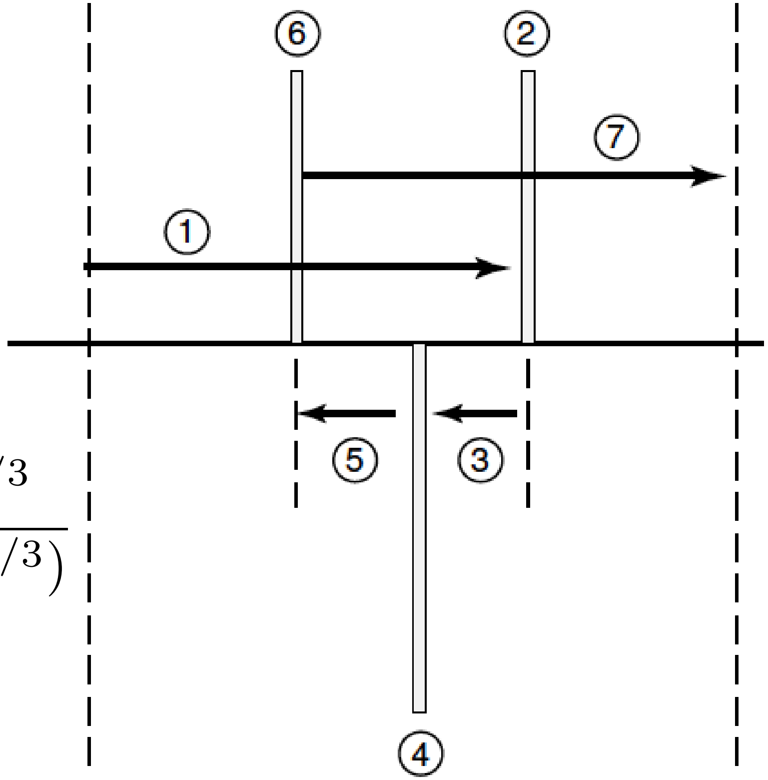

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$$d_1 = d_4 = \frac{1}{2(2 - 2^{1/3})} , \quad d_2 = d_3 = \frac{1 - 2^{1/3}}{2(2 - 2^{1/3})}$$

$$c_1 = c_3 = \frac{1}{2 - 2^{1/3}} , \quad c_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}$$

- This is actually the famous 7 step 4^{th} order symplectic integrator of **Forest**, **Ruth** and **Yoshida** (1990). It can be generalized for **any non-linear element**
- It imposes **negative drifts**…

Higher order integrators

- Yoshida has proved that a **general integrator map** of order $2k$ can be used to built a **map of order** $2k+2$

$$S_{2k+2}(t) = S_{2k}(x_1 t) \circ S_{2k}(x_0 t) \circ S_{2k}(x_1 t)$$

with $x_0 = \dfrac{-2^{\frac{1}{2k+1}}}{2-2^{\frac{1}{2k+1}}}\,,\quad x_1 = \dfrac{1}{2-2^{\frac{1}{2k+1}}}$

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- For example the **4th order scheme** can be considered as a **composition** of **three 2nd order ones** (single kicks)

$$S_4(t) = S_2(x_1 t)\circ S_2(x_0 t)\circ S_2(x_1 t)$$

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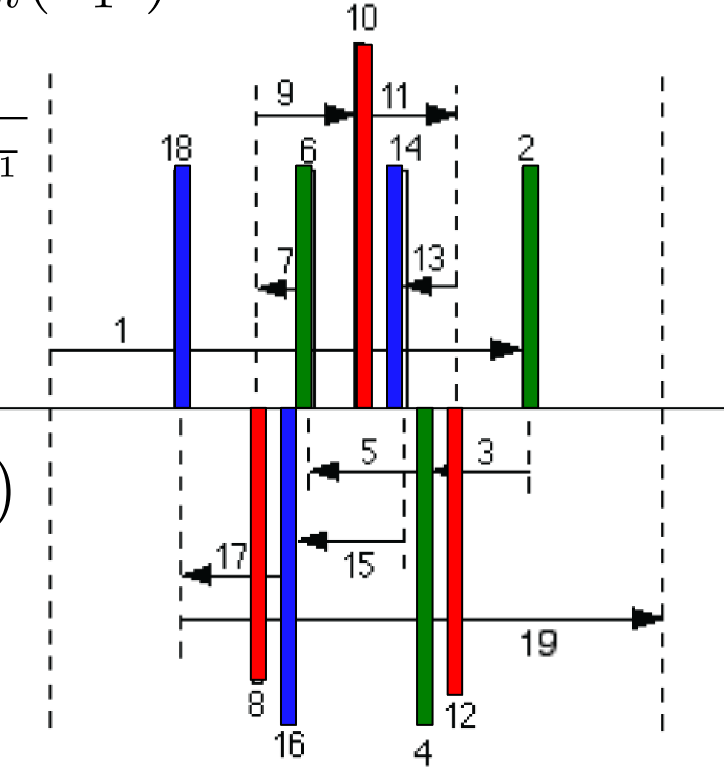

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with $$x_0 = \frac{-2^{\frac{1}{3}}}{2 - 2^{\frac{1}{3}}}, \quad x_1 = \frac{1}{2 - 2^{\frac{1}{3}}}$$

- A **6th order** integrator can be produced by **three interleaved 4th order** ones (9 kicks)

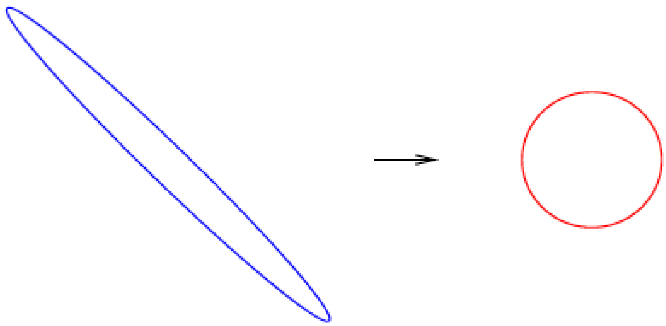
$$S_6(t) = S_4(x_1 t) \circ S_4(x_0 t) \circ S_4(x_1 t)$$

with $$x_0 = \frac{-2^{\frac{1}{5}}}{2 - 2^{\frac{1}{5}}}, \quad x_1 = \frac{1}{2 - 2^{\frac{1}{5}}}$$

Normal forms

Linear normal forms

- Make a coordinate transformation so that we get a simpler form of the matrix, i.e. **ellipses** are transformed to circles (simple rotation)



$$\mathcal{M} = \mathcal{A} \circ \mathcal{R} \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R} = \mathcal{A}^{-1} \circ \mathcal{M} \circ \mathcal{A}$$

- Using linear algebra, the solution is

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s_0)} & 0 \\ -\frac{\alpha(s_0)}{\sqrt{\beta(s_0)}} & \frac{1}{\sqrt{\beta(s_0)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix}$$

- This transformation can be extended to a non-linear system

Generic normal forms

- Normal forms consists of finding a **canonical transformation** of the 1-turn map, so that it becomes simpler to analyze
- In the linear case, the Floquet transformation is a kind a normal form as it turns **ellipses** into **circles**

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- The transformation can be written formally as

$$\begin{array}{ccc} \mathbf{z} & \xrightarrow{\mathcal{M}} & \mathbf{z}' \\ \Phi^{-1}\big\downarrow & & \big\downarrow\Phi^{-1} \\ \boldsymbol{u} & \xrightarrow[\mathcal{N}]{} & \boldsymbol{u}' \end{array}$$

with the original map $\mathcal{M} = \Phi^{-1} \circ \mathcal{N} \circ \Phi$

and its normal form

$$\mathcal{N} = \Phi \circ \mathcal{M} \circ \Phi^{-1} = e^{:h_{eff}:}$$

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$$\mathcal{N} = \Phi \circ \mathcal{M} \circ \Phi^{-1} = e^{:h_{eff}:}$$

- The transformation $\Phi = e^{:F:}$ is better suited in action angle variables, i.e. $\zeta = e^{-:F_r:}$ $\mathbf{h}$ taking the system from the original action-angle $h^{\pm}_{x,y} = \sqrt{2J_{x,y}}\, e^{\mp i\phi_{x,y}}$ to a new set $\zeta^{\pm}_{x,y}(N) = \sqrt{2I_{x,y}}\, e^{\mp i\psi_{x,y}(N)}$ with the angles being just simple rotations, $\psi_{x,y}(N) = 2\pi N\nu_{x,y} + \psi_{x,y_0}$ and the new effective Hamiltonian depends only on the new actions

Effective Hamiltonian

- The generating function can be written as a polynomial in the new actions, i.e.

$$F_r = \sum_{jklm} f_{jklm}\zeta_x^{+j}\zeta_x^{-k}\zeta_y^{+l}\zeta_y^{-m} = f_{jklm}(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m}{2}}e^{-i\psi_{jklm}}$$

- There are **software tools** that built this transformation
- Once the "new" effective Hamiltonian is known, all interesting quantities can be derived
- This Hamiltonian is a function only of the new actions, and to 3rd order it is obtained as

$$\begin{aligned} h_{eff} =& \nu_x I_x + \nu_y I_y \\ &+ \frac{1}{2}\alpha_c\delta^2 + c_{x1}I_x\delta + c_{y1}I_y\delta + c_3\delta^3 \\ &+ c_{xx}I_x^2 + c_{xy}I_xI_y + c_{yy}I_y^2 + c_{x2}I_x\delta^2 + c_{y2}I_y\delta^2 c_4\delta^4 \end{aligned}$$

Effective Hamiltonian

- The correction of the tunes is given by

$$Q_x = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_x} = \frac{1}{2\pi}\left(\nu_x + 2c_{xx}I_x + c_{xy}I_y + c_{x1}\delta + c_{x2}\delta^2\right)$$

$$Q_y = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_y} = \frac{1}{2\pi}\left(\nu_y + 2c_{yy}I_y + c_{xy}I_x + c_{y1}\delta + c_{y2}\delta^2\right)$$

tunes — tune-shift with amplitude — 1st and 2nd order chromaticity

- The correction to the path length is

$$\Delta s = \frac{\partial h_{eff}}{\partial \delta} = \alpha_c\delta + c_3\delta^2 + 4c_4\delta^3 + c_{x1}I_x + c_{y1}I_y + 2c_{x2}I_x\delta + 2c_{y2}I_y\delta$$

1st, 2nd and 3rd momentum compaction

Normal form for perturbation

- Using the BCH formula, one can prove that the composition of two maps with g small can be written as (see slide 109)

$$e^{:f:}e^{:g:} = \exp\left[:f + \left(\frac{:f:}{1-e^{-:f:}}\right)g + \mathcal{O}(g^2):\right]$$

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- This can be written as

$$\begin{aligned}\mathcal{N} &= e^{:f_2:}e^{-:f_2:}e^{:F:}e^{:f_2:}e^{:f_3:}e^{:-F:} \\ &= e^{:f_2:}e^{:e^{-:f_2:}F+f_3-F:} + \dots \\ &= e^{:f_2:}e^{:(e^{-:f_2:}-1)F+f_3:} + \dots\end{aligned}$$

$$F = \frac{f_3}{1-e^{-:f_2:}}$$

- This will **transform** the new **map** to a **rotation** to leading order

Example: Octupole

- Consider a linear map followed by an octupole

$$\mathcal{M} = e^{-\frac{\nu}{2}:x^2+p^2:}e^{:\frac{x^4}{4}:} = e^{:f_2:}e^{:\frac{x^4}{4}:}$$

- The **generating function** has to be chosen such as to make the following expression simpler

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- The simplest expression is the one that the **angles** are **eliminated** and there is only dependence on the action
- We pass to the **action angle variable** (resonance basis)

$$h^{\pm} = \sqrt{2J}\, e^{\mp i\phi} = x \mp ip$$

- The perturbation is

$$x^4 = (h_+ + h_-)^4 = h^{\pm} = h_+^4 + 4h_+^3 h_- + 6h_+^2 h_-^2 + 4h_+ h_-^3 + h_-^4$$

Example: Octupole

- The **term** $6h_+^2 h_-^2 = 24J^2$ is **independent** on the **angles**. Thus we may **choose** the **generating functions** such that the **other terms are eliminated**. It takes the form

$$F = \frac{1}{16}\left(\frac{h_+^4}{1-e^{4i\nu}} + \frac{4h_+^3 h_-}{1-e^{2i\nu}} + \frac{4h_+ h_-^3}{1-e^{2i\nu}} + \frac{h_-^4}{1-e^{4i\nu}}\right)$$

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- The **new effective Hamiltonian** is depending only on the **actions** and contains the tune-shift terms
- The **generator** in the original variables is written as

$$F = -\frac{1}{64}\left[-5x^4 + 3p^4 + 6x^2p^2 + 4x^3p(2\cot(\nu) + \cot(2\nu)) + 4xp^3(2\cot(\nu) - \cot(2\nu))\right]$$

- **Constant values** of the generator describe the **trajectories** in phase space

Introduction to Truncated Power Series Algebra (TPSA)

Taylor series from tracking

- Let's consider a tracked particle at **position** α and a **small deviation** Δx. The Taylor series around this position is

$$f(a+\Delta x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}\Delta x^n$$
$$= f(a) + \frac{f'(a)}{1!}\Delta x^1 + \frac{f''(a)}{2!}\Delta x^2 + \frac{f'''(a)}{3!}\Delta x^3 + \cdots$$

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- By truncating we have $f(a+\Delta x) = f(a) + \sum_{n=1}^{m} \frac{f^{(n)}(a)}{n!}\Delta x^n$

and the function $f(x)$ can be represented by the vector $(f(\alpha), f'(\alpha), f''(\alpha), \ldots, f^{(m)}(\alpha))$

- This vector is a **Truncated Power Series Algebra**
- We need the **derivatives** $f^{(n)}(\alpha)$ of $f(x)$ at α with

$$f'(\alpha) = \frac{f(\alpha+\epsilon) - f(\alpha)}{\epsilon}$$

which is **numerically non-trivial** (small divisors, accuracy for higher orders,…)

Differential Algebra

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- Consider a pair of real numbers (q_0, q_1) and define **operations** on a pair like

$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0, q_0 \cdot r_1 + q_1 \cdot r_0)$$

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and some ordering

$$(q_0, q_1) < (r_0, r_1) \quad \text{if} \quad q_0 < r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 < r_1)$$
$$(q_0, q_1) > (r_0, r_1) \quad \text{if} \quad q_0 > r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 > r_1)$$

implying strange relations of the form

$$(0,0) < (0,1) < (r,0), \quad \forall \ r > 0$$
$$(0,1) \cdot (0,1) = (0,0) \rightarrow (0,1) = \sqrt{(0,0)}$$

Differential Algebra

- We define the **differential unit** $\epsilon \equiv (0, 1)$, which is located between 0 and any real number (infinitesimally small)
- As $(q_0, 0)$ is just a real number, we can define a **real** part and a **differential** part

$$q_0 = \mathcal{R}(q_0, q_1) \quad \text{and} \quad q_1 = \mathcal{D}(q_0, q_1)$$

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$$q_0 = \mathcal{R}(q_0, q_1) \quad \text{and} \quad q_1 = \mathcal{D}(q_0, q_1)$$

- Using the previous rules we can show

$$(1, 0) \cdot (q_0, q_1) = (q_0, q_1)$$

$$(q_0, q_1)^{-1} = (\frac{1}{q_0}, -\frac{q_1}{q_0^2})$$

- A **function** acting on a pair is $f(x) = \mathcal{R}[f(x, q_1)], \quad \forall\, q_1$
- The **differential** is

$$\mathcal{D}[f(x+\epsilon)] = \mathcal{D}[f((x, 0) + (0, 1))] = \mathcal{D}[f(x, 1)] = f'(x)$$

Differential Algebra example

- Consider the function $f(x) = x^2 + \frac{1}{x}$ with the derivative $f'(x) = 2x - \frac{1}{x^2}$. For $x = 2$, we obtain $(f(2), f'(2)) = \left(\frac{9}{2}, \frac{15}{4}\right)$

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- Let's use differential algebra, by substituting $x \to (x, 1) = (2, 1)$ to the function and use the rules

$$
\begin{aligned}
f[(2,1)] &= (2,1)^2 + (2,1)^{-1} \\
&= (4,4) + \left(\frac{1}{2}, -\frac{1}{4}\right) \\
&= \left(\frac{9}{2}, \frac{15}{4}\right) = (f(2), f'(2))
\end{aligned}
$$

- We computed exactly the derivative, only by using **algebra!**

Higher orders

- The operation can be extended to **derivatives of order** N by considering that the pair becomes

$$(q_0, 1) \to (q_0, 1, 0, 0, \ldots, 0) \quad \text{with } \epsilon = (0, 1, 0, 0, \ldots, 0)$$

- We can extend the **operations** as

$$(q_0, q_1, q_2, \ldots, q_N) + (r_0, r_1, r_2, \ldots, r_N) = (q_0 + r_0, q_1 + r_1, q_2 + r_2, \ldots, q_N + r_N)$$

$$c \cdot (q_0, q_1, q_2, \ldots, q_N) = (c \cdot q_0, c \cdot q_1, c \cdot q_2, \ldots, c \cdot q_N)$$

$$(q_0, q_1, q_2, \ldots, q_N) \cdot (r_0, r_1, r_2, \ldots, r_N) = (s_0, s_1, s_2, \ldots, s_N)$$

$$\text{with} \qquad s_i = \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} q_k r_{i-k}$$

- For example $(x, 0, 0, 0, \ldots, 0)^n = (x^n, 0, 0, 0, \ldots, 0)$

$$(0, 1, 0, 0, \ldots, 0)^n = (0, 0, 0, \ldots, \overbrace{n!}^{n+1}, \ldots, 0)$$

$$(x, 1, 0, 0, \ldots, 0)^2 = (x^2, 2x, 2, 0, \ldots, 0)$$

$$(x, 1, 0, 0, \ldots, 0)^3 = (x^3, 3x^2, 6x, 6, 0, \ldots, 0)$$

Higher dimensions

- The operation can be extended to **more variables**

$$x = (a, 1, 0, 0, 0...) \qquad \epsilon_x = (0, 1, 0, 0, 0, ..)$$

$$p_x = (b, 0, 1, 0, 0...) \qquad \epsilon_{p_x} = (0, 0, 1, 0, 0, ..)$$

- With some modified **multiplication rules**

$$(q_{00},\ q_{10},\ q_{01},\ q_{20}, ..) \cdot (r_{00},\ r_{10},\ r_{01},\ r_{20}, ..) \ = \ (s_{00},\ s_{10},\ s_{01},\ s_{20}, ..)$$

with $$s_{mn} = \sum_{k=0}^{m} \sum_{l=0}^{n} q_{kl} \cdot r_{m-k,n-l} \frac{m!\, n!}{k!\,(m-k)!\, l!\,(n-l)!}$$

providing $$f(x, p_x) = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial p_x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial p_x}, ... \right)(a, b)$$

- Using the formalism above, a **truncated Taylor map** with the desired accuracy and to **any order**, directly from **tracking data**

Summary

- **Natural way** to represent motion in an accelerator is by using **maps**
- **Powerful tools** to build them from straight-forward tracking **(TPSA)**
- **Canonical (symplectic) transformations** enable to move from variables describing a distorted phase space to something simpler (ideally circles)
- The **generating functions** passing from the old to the new variables are bounded to **diverge** in the vicinity of **resonances** (emergence of chaos, see Lectures of NLD Phenomenology)
- Calculating this generating function with **canonical perturbation theory** becomes **hopeless** for higher orders
- **Lie transformations** of **accelerator maps** enables derivation of the generating functions in an **algorithmic way**, in principle to **arbitrary order**
- For real accelerator models, we have to rely on **symplectic** integration, i.e. **particle tracking** and **methods** to analyse it (see Lectures of NLD Phenomenology)

Appendix

Lagrangian formalism

- Describe motion of particles in q_n coordinates (***n* degrees of freedom)** from time t_1 to time t_2
- It can be achieved by the **Lagrangian function** $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ with $(q_1, \ldots, q_n)$ the **generalized coordinates** and $(\dot{q}_1, \ldots, \dot{q}_n)$ the **generalized velocities**

Lagrangian formalism

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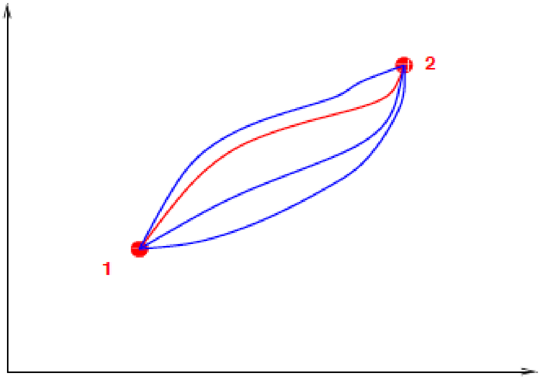
- It can be achieved by the **Lagrangian function** $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$ with $(q_1,\ldots,q_n)$ the **generalized coordinates** and $(\dot{q}_1,\ldots,\dot{q}_n)$ the **generalized velocities**
- The Lagrangian is defined as $L = T - V$, i.e. difference between **kinetic** and **potential** energy
- The integral $W = \int L(q_i,\dot{q}_i,t)dt$ defines the **action**
- **Hamilton's principle**: system evolves so as the action becomes **extremum** (principle of **stationary action**)



Euler- Lagrange equations

- By using **Hamilton's principle,** i.e. $\delta W = 0$, over some time interval t_1 and t_2 for two stationary points $\delta q(t_1) = \delta q(t_2) = 0$ (see appendix), the following differential equations for each degree of freedom are obtained, the **Euler-Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- In other words, by knowing the form of the Lagrangian, the **equations of motion** can be derived

Lagrangian mechanics

- For a simple **force law** contained in a potential function, governing motion among interacting particles, the Lagrangian is (or as Landau-Lifshitz put it "experience has shown that…")

$$L = T - V = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \ldots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i} \ ,$$

i.e. **Newton's equations.**

From Lagrangian to Hamiltonian

- Some **disadvantages** of the Lagrangian formalism:
  - **No uniqueness**: different Lagrangians can lead to same equations
  - **Physical significance** not straightforward (even its basic form given more by "experience" and the fact that it actually works that way!)
  - **Note:** Lagrangian is very useful in particle physics (invariant under Lorentz transformations)

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  - **No uniqueness**: different Lagrangians can lead to same equations
  - **Physical significance** not straightforward (even its basic form given more by "experience" and the fact that it actually works that way!)
  - **Note:** Lagrangian is very useful in particle physics (invariant under Lorentz transformations)
- Lagrangian function provides in general n second order differential equations (**coordinate space**)
- Advantage to move to system of $2n$ first order differential equations, which are more straightforward to solve (**phase space**)
- Derived by the **Hamiltonian** of the system

Derivation of Lagrange equations

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} \left(L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t)\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt$$

- Taking into account that $\delta\dot{q} = \frac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left|\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrant should also vanish, providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Derivation of Hamilton's equations

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian ("least" action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i p_i d\dot{q}_i + \dot{q}_i dp_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

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or

$$dH(q,p,t) = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q}\,, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n+2$ equations describing the motion in the **"extended" phase space** $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Examples of transformations

- The transformation $Q = -p\ ,\ \ P = q$, which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates $q = P\cos Q \; , \quad p = P\sin Q$ is not, since

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- There are actually **"polar" coordinates** that are **canonical**, given by $q = -\sqrt{2P}\cos Q \ , \quad p = \sqrt{2P}\sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P}\sin Q & \sqrt{2P}\cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

The Relativistic Hamiltonian for electromagnetic fields

Single-particle relativistic Hamiltonian

- Neglecting self fields and radiation, motion can be described by a "single-particle" Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x},t)$$

  - $\mathbf{x} = (x, y, z)$ Cartesian positions
  - $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
  - $\mathbf{A} = (A_x, A_y, A_z)$ magnetic vector potential
  - Φ electric scalar potential

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- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

with $\mathbf{v}$ the velocity vector and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor

Single-particle relativistic Hamiltonian

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- It is generally a **3 degrees of freedom** one plus time (i.e., **4 degrees of freedom**)
- The Hamiltonian represents the **total energy**

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- The Hamiltonian represents the **total energy**

$$H \equiv E = \gamma mc^2 + e\Phi$$

- The **total kinetic momentum** is

$$P = \left(\frac{H^2}{c^2} - m^2c^2\right)^{1/2}$$

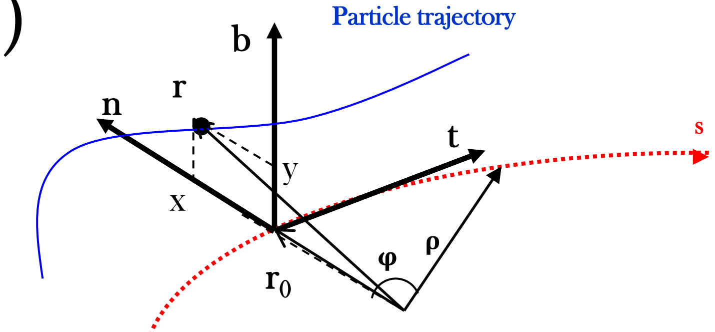
- Using **Hamilton's equations**

$$(\dot{\mathbf{x}},\dot{\mathbf{p}}) = [(\mathbf{x},\mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

From Cartesian to "curved" coordinates

- It is useful (especially for rings) to transform the Cartesian coordinate system to the **Frenet-Serret system** moving to a closed curve, with path length s



- The position coordinates in the two systems are connected by $\mathbf{r} = \mathbf{r_0}(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$
- The **Frenet-Serret unit vectors** and their derivatives are defined as $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = (\frac{d}{ds}\mathbf{r_0}(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r_0}(s), \mathbf{t} \times \mathbf{n})$

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{\rho(s)} & 0\\ 0 & 0 & \tau(s)\\ \frac{1}{\rho(s)} & 0 & -\tau(s)\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}$$

with $\rho(s)$ the radius of curvature and $\tau(s)$ the torsion which vanishes in case of planar motion

From Cartesian to "curved" variables

- We are seeking a canonical transformation between

$$(\mathbf{q},\mathbf{p}) \quad \mapsto \quad (\mathbf{Q},\mathbf{P}) \ \text{ or}$$
$$(x,y,z,p_x,p_y,p_z) \quad \mapsto \quad (X,Y,s,P_x,P_y,P_s)$$

- The generating function is

$$(\mathbf{q},\mathbf{P}) = -\left(\frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{Q}}\right)$$

- By using the relationship between the positions, the generating function is

$$F_3(\mathbf{p},\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r} + \overline{F_3}(\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r}$$

From Cartesian to "curved" variables

- for planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

- Taking into account that the **vector potential** is also transformed in the same way

$$(A_X, A_Y, A_s) = \mathbf{A} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

the **new Hamiltonian** is given by

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}, t) = c\sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1 + \frac{X}{\rho(s)})^2} + m^2c^2} + e\Phi$$

Changing of the independent variable

- It is more convenient to use the **path length** s , instead of the time as **independent variable**
- The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** and its conjugate momentum is $P_t = -\mathcal{H}$

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- The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** and its conjugate momentum is $P_t = -\mathcal{H}$
- In the same way, the new Hamiltonian with the path length as the independent variable is just $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$ with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(\frac{P_t + e\Phi}{c})^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- It can be proved that this is indeed a **canonical transformation**
- Note the existence of the **reference orbit** for zero vector potential, for which $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

Neglecting electric fields

- Due to the fact that **longitudinal** (synchrotron) motion is **much slower** than the **transverse** (betatron) one, the electric field can be set to zero and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{\underbrace{(\frac{\mathcal{H}}{c})^2 - m^2c^2}_{P^2} - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is having **2 degrees of freedom + "time"** (path length)

Momentum rescaling

- Due to the fact that **total momentum** is **much larger** than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$(\mathbf{Q},\mathbf{P}) \quad \mapsto \quad (\bar{\mathbf{q}},\bar{\mathbf{p}}) \ \text{ or}$$

$$(X,Y,t,P_X,P_Y,P_t) \quad \mapsto \quad (\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = (X,Y,-c\,t,\frac{P_X}{P_0},\frac{P_Y}{P_0},-\frac{P_t}{P_0 c})$$

- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1+\frac{\bar{x}}{\rho(s)}\right)\sqrt{\bar{p}_t^2 - \frac{m^2c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

with $\quad (\bar{A}_x,\bar{A}_y,\bar{A}_z) = \dfrac{1}{P_0\,c}(A_x,A_y,A_s)$

and $\quad \dfrac{m^2c^2}{P_0} = \dfrac{1}{\beta_0^2\gamma_0^2}$

Moving the reference frame

- Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and $\frac{d\bar{t}}{ds}\Big|_{P=P_0} = \frac{\partial \bar{H}}{\partial \bar{p}_t}\Big|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$
- It is thus useful to **move** the **reference frame** to the **reference trajectory** for which another canonical transformation is performed

$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \quad \mapsto \quad (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \;\; \text{or}$$

$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \quad \mapsto \quad (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = (\hat{x}, \hat{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \hat{p}_x, \hat{p}_y, \bar{p}_t - \frac{1}{\beta_0})$$

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- The mixed variable generating function is

$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = (\frac{\partial F_2(\bar{\mathbf{q}},\hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}},\hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}})$ providing

$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + (\bar{t} + \frac{s - s_0}{\beta_0})(\hat{p}_t + \frac{1}{\beta_0})$$

- The Hamiltonian is then

$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0}(\frac{1}{\beta_0} + \hat{p}_t) - e\hat{A}_s - \left(1 + \frac{\hat{x}}{\rho(s)}\right)\sqrt{(\hat{p}_t + \frac{1}{\beta_0})^2 - \frac{1}{\beta_0^2\gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
- In the **ultra-relativistic limit** $\beta_0 \to 1$, $\frac{1}{\beta_0^2 \gamma^2} \to 0$ and the Hamiltonian is written as

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the "hats" are dropped for simplicity

Relativistic and transverse field approximations

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
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  where the "hats" are dropped for simplicity
- If we consider **only transverse field** components, the **vector potential** has **only** a **longitudinal** component and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

High-energy, large ring approximation

- It is useful for study purposes (especially for finding an "integrable" version of the Hamiltonian) to make an extra approximation
- For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller** than **1**, i.e. the square root can be expanded.
- Considering also the large machine approximation $x << \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x(1+\delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may **not** be **a good idea**, especially for **low energy**, **small** size **rings**

Relativistic and transverse field approximations

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
- In the ultra-relativistic limit $\beta_0 \to 1$, $\frac{1}{\beta_0^2\gamma^2} \to 0$ and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1+\frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the "hats" are dropped for simplicity

- If we consider only transverse field components, the vector potential has only a longitudinal component and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1+\frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

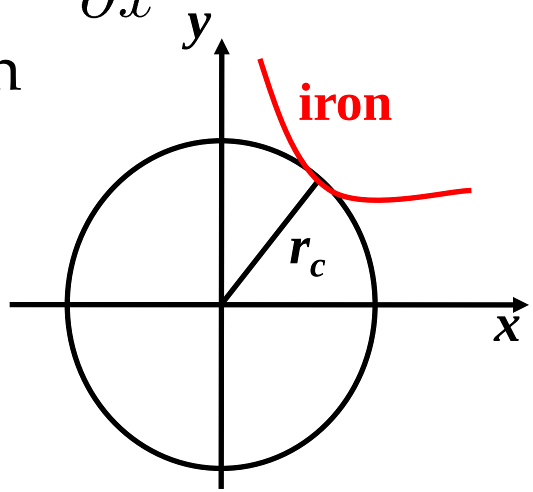
$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function

Exists complex potential of $z = x + iy$ with power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)



$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \ , \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0}r_0^{n-1} \ , \ a'_n = \frac{a_n}{10^{-4}B_0}r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)(\frac{x+iy}{r_0})^{n-1}$$

- **Note:** $n' = n - 1$ is the US convention

Canonical perturbation theory

- Expand term by term the Hamiltonian $H(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta)$ to leading order in ϵ

$$H_0(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})) = H_0(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\epsilon H_1(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta) = \epsilon H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) + \mathcal{O}(\epsilon^2)$$

- The new Hamiltonian can also be expanded in orders of ϵ

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \ldots$$

- Equating the terms of equal orders in ϵ , we obtain
  - Zero order $\bar{H}_0 = H_0(\bar{\boldsymbol{J}})$
  - First order $\bar{H}_1 = \dfrac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \dfrac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$

  where the frequency vector is $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \dfrac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$

Canonical perturbation theory

- From the first order Hamiltonian, the angles have to be eliminated. For this purpose, it can be split in two parts:
  - Average part: $\langle H_1 \rangle_{\bar{\varphi}} = \left(\frac{1}{2\pi}\right)^n \oint H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) d\bar{\boldsymbol{\varphi}}$
  - Oscillating part: $\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\varphi}}$
- The 1st order perturbation part of the Hamiltonian then becomes

$$\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

- Thus, the generating function should be chosen such that the angle dependence is eliminated, for which

$$\bar{H}_1(\bar{\boldsymbol{J}}) = \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} \quad \text{and} \quad \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} = -\{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

- The new Hamiltonian is a function of the new actions

$$\bar{H}(\bar{\boldsymbol{J}}) = H_0(\bar{\boldsymbol{J}}) + \epsilon \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \mathcal{O}(\epsilon^2)$$

with the new frequency vector

$$\bar{\boldsymbol{\omega}}(\bar{\boldsymbol{J}}) = \frac{\partial \bar{H}(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} = \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}}}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

Form of the generating function

- The question that remains to be answered is whether a generating function can be found that eliminates the angle dependence
- The oscillating part of the perturbation and the generating function can be expanded in Fourier series

$$\{H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\} = \sum_{\boldsymbol{k},p} H_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} \quad S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta) = \sum_{\boldsymbol{k},p} S_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)}$$

with $\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} = k_1\bar{\varphi}_1 + \cdots + k_n\bar{\varphi}_n$

- Following the relationship for the angle elimination, the Fourier coefficients of the generating function should satisfy

$$S_{1k}(\bar{\boldsymbol{J}}) = i\frac{H_{1k}(\bar{\boldsymbol{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p} \quad \text{with} \quad \boldsymbol{k},p \neq \mathbf{0}$$

- Then, the generating function can be written as

$$S(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}}\cdot\bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k}\neq\mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p} e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} + \mathcal{O}(\epsilon^2)$$

Second order sextupole tune-shift

- It can be shown that at second order in perturbation theory the Hamiltonian depending only on the actions can be written $$\bar{H}_2(\bar{J}) = \langle \frac{1}{2} \frac{\partial^2 H_0}{\partial \bar{J}^2} \left(\frac{\partial S_1}{\partial \phi} \right)^2 + \frac{\partial H_1}{\partial \bar{J}} \frac{\partial S_1}{\partial \phi} \rangle_\phi$$
- This can be simplified to $\bar{H}_2(\bar{J}) = \langle \frac{\partial H_1}{\partial \bar{J}} \frac{\partial S_1}{\partial \phi} \rangle_\phi$
- The two terms are $\frac{\partial H_1}{\partial \bar{J}} = \frac{K_s(s)}{2\sqrt{2}} \bar{J}^{1/2} \beta(s)^{3/2} (\cos 3\phi + 3\cos\phi)$

$$\frac{\partial S_1}{\partial \phi} = -\frac{\bar{J}^{3/2}}{2\sqrt{2}} \int_s^{s+C} K_s(s')\beta(s')^{3/2} \left[\frac{\cos(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(3\pi\nu)} \right] ds'$$

- The 2nd order Hamiltonian is given by the angle-averaged product of the last two terms.
- It is quadratic in the sextupole strength and the new action. The 2nd order tune-shift is the derivative in the action

$$\nu(\bar{J}) = \langle \frac{\partial H_2}{\partial \bar{J}} \rangle_{\phi,s} = -\frac{\bar{J}}{16\pi} \int_0^C ds K_s(s)\beta(s)^{3/2} \int_s^{s+C} K_s(s')\beta(s')^{3/2}$$
$$\times \left[\frac{\cos(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(3\pi\nu)} \right] ds'$$

Perturbation treatment for more sextupoles

- Expand both the perturbation and generating function in Fourier series of the form

$$S_1(\bar{J},\bar{\phi},\theta) = \sum_k S_{1k}(\bar{J},\theta)e^{ik\bar{\phi}} \text{ and } \{H_1(\bar{J},\bar{\phi},\theta)\} = \sum_k H_{1k}(\bar{J},\theta)e^{ik\bar{\phi}}$$

- The equation relating the amplitudes is

$$i\ k\ \nu\ S_{1k} + \frac{\partial S_{1k}}{\partial \theta} = -H_{1k}$$

which can be solved yielding

$$S_{1k} = \frac{i}{2\sin(\pi k\nu)} \int_{\theta}^{\theta+2\pi} H_{1k} e^{ik\nu(\theta'-\theta-\pi)} d\theta'$$

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- Following the canonical perturbation procedure the generating function is

$$S_1 = \sum_k \frac{i}{2\sin(\pi k\nu)} \int_\theta^{\theta+2\pi} H_{1k} e^{ik[\phi+\nu(\theta'-\theta-\pi)]} d\theta'$$

- For the sextupole, and letting $\psi(s) = \int_0^s \frac{ds'}{\beta(s')}$ we have

$$S_1 = -\frac{\bar{J}^{3/2}}{2\sqrt{2}} \int_s^{s+C} K_s(s')\beta(s')^{3/2} \left[\frac{\sin(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\sin 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{3\sin(3\pi\nu)}\right] ds'$$

Single resonance for accelerator Hamiltonian

- The single resonance accelerator Hamiltonian (Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(\nu_x J_x + \nu_y J_y) + g_{n_x,n_y}\frac{2}{R}J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

with $g_{n_x,n_y}e^{i\phi_0} = g_{j,k,l,m;p}$

- From the generating function

$F_r(\phi_x, \phi_y, \hat{J}_x, \hat{J}_y, s) = (n_x\phi_x + n_y\phi_y - p\theta)\hat{J}_x + \phi_y\hat{J}_y$

the relationships between old and new variables are

$$\hat{\phi}_x = (n_x\phi_x + n_y\phi_y - p\theta)\ , \quad J_x = n_x\hat{J}_x$$

$$\hat{\phi}_y = \phi_y\ , \quad J_y = n_y\hat{J}_x + \hat{J}_y$$

- The following Hamiltonian is obtained

$$\hat{H}(\hat{J}_x, \hat{J}_y, \hat{\phi}_x) = \frac{(n_x\nu_x + n_y\nu_y - p)\hat{J}_x + \hat{J}_y}{R} + g_{n_x,n_y}\frac{2}{R}(n_x\hat{J}_x)^{\frac{k_x}{2}}(n_y\hat{J}_x + \hat{J}_y)^{\frac{k_y}{2}}\cos(\hat{\phi}_x + \phi_0)$$

Resonance widths

- There are two integrals of motion
  - The Hamiltonian, as it is independent on "time"
  - The new action $\hat{J}_y$ as the Hamiltonian is independent on $\hat{\phi}_y$
- The two invariants in the old variables are written as:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = (\nu_x - \frac{p}{n_x + n_y})J_x + (\nu_y - \frac{p}{n_x + n_y})J_y + 2g_{n_x,n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

- Two cases can be distinguished
  - n_x , n_y have **opposite** sign, i.e. **difference** resonance, the motion is the one of an ellipse, so bounded
  - n_x , n_y have the **same** sign, i.e. **sum** resonance, the motion is the one of an hyperbola, so **not** bounded
- These are **first order** perturbation theory considerations
- The distance from the resonance is obtained as

$$\Delta = \frac{g_{n_x,n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$

General accelerator Hamiltonian

- The general accelerator Hamiltonian is written as

$$\mathcal{H}(x,y,p_x,p_y,s) = \mathcal{H}_0(x,y,p_x,p_y,s) + \sum_{k_x,k_y} h_{k_x,k_y}(s) x^{k_x} y^{k_y}$$

- The transverse coordinated can be expressed in action-angle variables as

$$u(s) = \sqrt{\frac{J_u\beta_u(s)}{2}}\left(e^{i(\phi_u(s)+\theta_u(s))} + e^{-i(\phi_u(s)+\theta_u(s))}\right)$$

- The Hamiltonian in action-angle variables is

$$\mathcal{H}'(J_x,J_y,\phi_x,\phi_y) = H_0(J_x,J_y) + H_1(J_x,J_y,\phi_x,\phi_y)$$

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  - The integrable part $H_0(J_x, J_y) = \frac{1}{R}(\nu_x J_x + \nu_y J_y)$
  - The perturbation

$$H_1(J_x, J_y, \phi_x, \phi_y; s) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x} \sum_l^{k_y} g_{j,k,l,m}(s) e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

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- The coefficients $g_{j,k,l,m}(s) = \frac{h_{k_x,k_y}(s)}{2^{\frac{j+k+l+m}{2}}} \binom{k_x}{j} \binom{k_y}{l} \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\theta_x(s) + (l-m)\theta_y(s)]}$ depend on the optics, with the indexes $k_x = j+k \; , \quad k_y = l+m$

Resonance driving terms

- As the coefficients $h_{k_x,k_y}(s)$ are periodic, the perturbation can be expanded in Fourier series

$$H_1(J_x,J_y,\phi_x,\phi_y;\theta)=\sum_{k_x,k_y} J_x^{k_x/2}J_y^{k_y/2}\sum_j^{k_x}\sum_l^{k_y}\sum_{p=-\infty}^{\infty} g_{j,k,l,m;p}e^{i[(j-k)\phi_x+(l-m)\phi_y-p\theta]}$$

with the **resonance driving terms**

$$g_{j,k,l,m;p}=\binom{k_x}{j}\binom{k_y}{l}\frac{1}{2^{\frac{j+k+l+m}{2}}}\frac{1}{2\pi}\oint h_{k_x,k_y}(s)\beta_x^{k_x/2}(s)\beta_y^{k_y/2}(s)e^{i[(j-k)\phi_x(s)+(l-m)\phi_y(s)+p\theta]}$$

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- For $n_x = j - k$, $n_y = l - m$, resonance conditions appear for $n_x\nu_x + n_y\nu_y = p$
- Goal of accelerator design and correction systems is to minimize the resonance driving terms
  - Change magnet design so that $h_{k_x,k_y}(s)$ become smaller
  - Introduce magnetic elements capable of creating a cancelling effect
  - Sort magnets or non-linear elements in a way that phase terms are minimised

Tune-shift and tune-spread

- First order correction to the tunes is computed by the derivatives with respect to the action of the average part of perturbation. For a given term, $h_{k_x,k_y}(s)x^{k_x}y^{k_y}$ the leading order correction to the tunes are

$$\delta\nu_x = \frac{J_x^{k_x/2-1}J_y^{k_y/2}}{4\pi^2}\sum_j^{k_x}\sum_l^{k_y}\bar{g}_{j,k,l,m}\oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

$$\delta\nu_y = \frac{J_x^{k_x/2}J_y^{k_y/2-1}}{4\pi^2}\sum_j^{k_x}\sum_l^{k_y}\bar{g}_{j,k,l,m}\oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

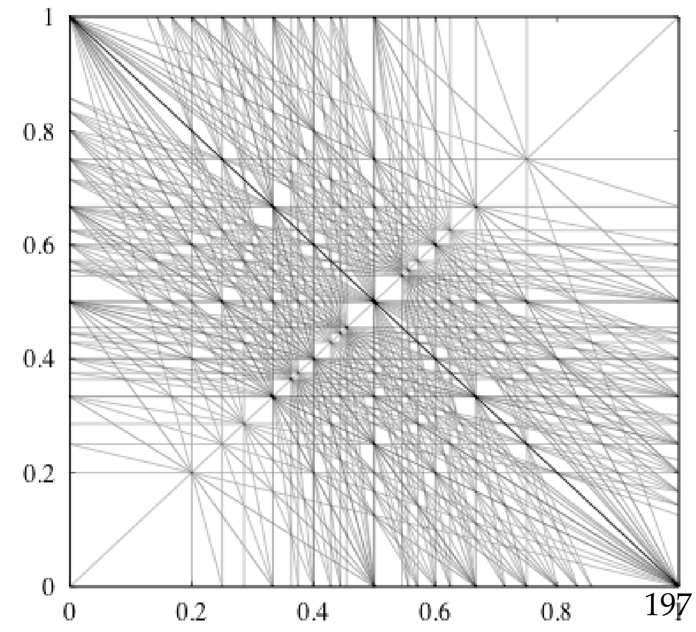
where $\bar{g}_{j,k,l,m}$ is the average of $g_{j,k,l,m}(s)$ around the ring.

- In the accelerator jargon if $\delta\nu_{x,y}$ is independent of the action, it is referred to as **tune-shift**, whereas, if it depends on the action, it is called **tune-spread** (or amplitude detuning)
- At first order, $\delta\nu_{x,y} = 0$, for **odd multi-poles** $k_x = j+k$, $k_y = l+m$ (trigonometric functions give zero averages).

Resonance classification

General resonance conditions

- The general resonance conditions is $n_x\nu_x + n_y\nu_y = p$ with order $n_x + n_y$
- For all the polynomial field terms of a $2m$-pole, the excited resonances (**at first order**) satisfy the condition $n_x + n_y = m$ but there are also **sub-resonances** for which $n_x + n_y < m$
- For **normal** (erect) multi-poles, the resonances (**at first order**) are $(n_x, n_y) = (m, 0),\ (m-2, \pm 2), \ldots$ whereas for **skew** multi-poles $(n_x, n_y) = (m-1, \pm 1),\ (m-3, \pm 3), \ldots$



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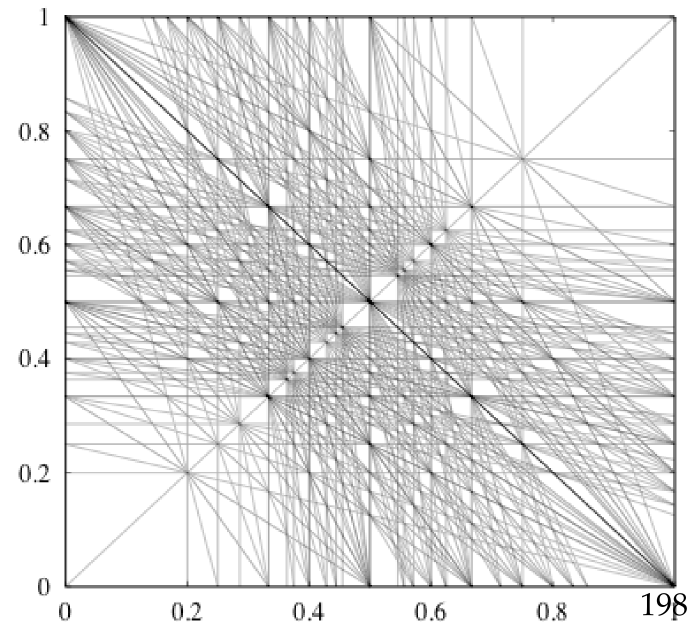
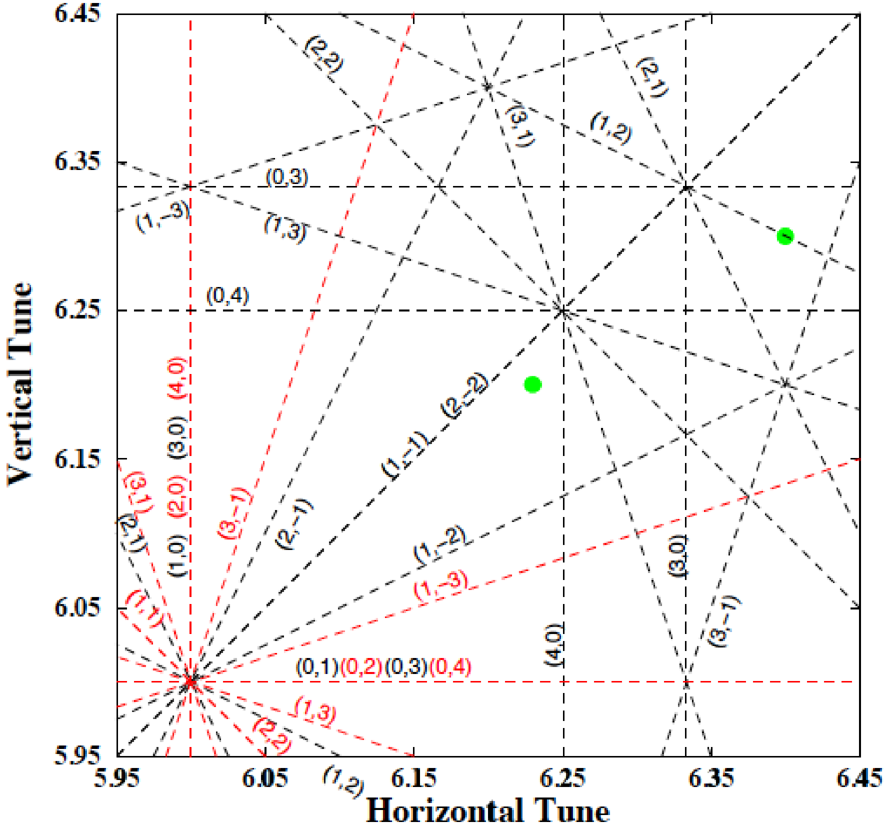
- If perturbation is large, **all** resonances can be potentially excited
- The **resonance conditions form lines** in frequency space and fill it up as the **order grows** (the rational numbers form a dense set inside the real numbers), but Fourier amplitudes should also decrease



Systematic and random resonances

- If lattice is made out of N identical cells, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $n_x\nu_x + n_y\nu_y = jN$
- These are called **systematic** resonances
- Practically, any (linear) lattice perturbation breaks super-periodicity and any **random** resonance can be excited
- Careful choice of the working point is necessary

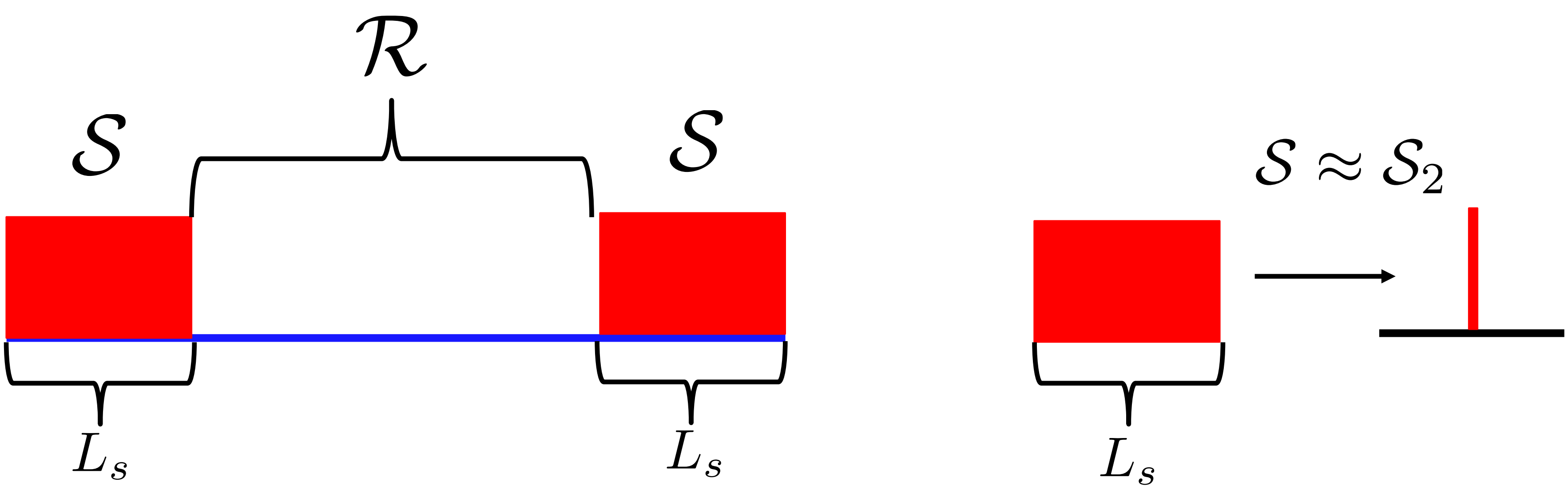


Example: The $-I$ transformer

- Consider two identical sextupoles separated by a beam line represented by a map $\mathcal{R}$
- The **sextupole map** can be represented at **second order** as

$$\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

with the **sextupole** effective **Hamiltonian** $H_s = \frac{1}{6}k_2(x^3 - 3xy^2)$ and H_d the **drift Hamiltonian**



Example: The $-I$ transformer

- Consider two identical sextupoles separated by a beam line represented by a map $\mathcal{R}$
- The **sextupole map** can be represented at **second order** as

$$\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

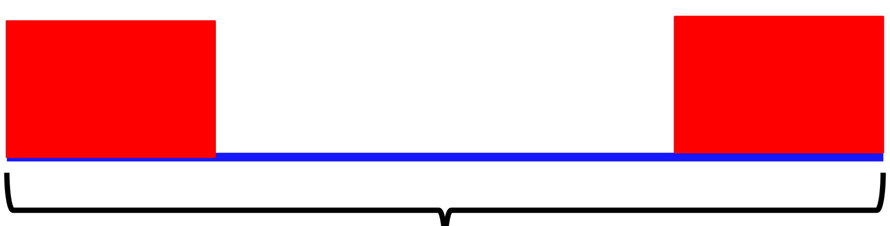
with the **sextupole** effective **Hamiltonian** $H_s = \frac{1}{6}k_2(x^3 - 3xy^2)$ and H_d the **drift Hamiltonian**

- The **total map** can be approximated at 2nd order by

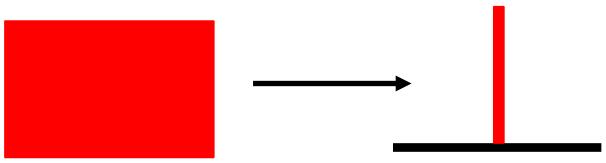
$$\mathcal{M} = \mathcal{SRS} \approx \mathcal{S}_2\mathcal{R}\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}\bar{\mathcal{R}}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

with the map $\bar{\mathcal{R}} = e^{-\frac{1}{2}L_s:H_d:}\mathcal{R}e^{-\frac{1}{2}L_s:H_d:}$

$\mathcal{S} \approx \mathcal{S}_2$



$\mathcal{M} = \mathcal{SRS}$



Example: The $-I$ transformer

- Inserting the identity $\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1} = \mathcal{I}$, we have

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

- The **similarity transformation** can be used

$$\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}} = e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}$$

- The map is then rewritten as

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

Example: The $-I$ transformer

- Inserting the identity $\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1} = \mathcal{I}$, we have

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

- The **similarity transformation** can be used

$$\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}} = e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}$$

- The map is then rewritten as

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}e^{-L_s\boxed{:\bar{\mathcal{R}}^{-1}H_s:}}e^{-L_s\boxed{:H_s:}}e^{-\frac{1}{2}L_s:H_d:}$$

- If the map $\bar{\mathcal{R}}$ is chosen such that $-\bar{\mathcal{R}}^{-1}H_s = H_s$ or $\bar{\mathcal{R}}H_s = -H_s$ so that

$$e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}e^{-L_s:H_s:} = e^{L_s:H_s:}e^{-L_s:H_s:} = \mathcal{I}$$

- In that way, the **sextupole non-linearity** is getting **eliminated** in the final map

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}e^{-\frac{1}{2}L_s:H_d:} = e^{-L_s:H_d:}\mathcal{R}e^{-L_s:H_d:}$$

Example: The $-I$ transformer

- Inspecting the form of H_s (odd in x and even in y), this can be achieved if the map is such that

$$\bar{\mathcal{R}}x = -x, \qquad \bar{\mathcal{R}}p_x = -p_x, \qquad \bar{\mathcal{R}}y = \pm y, \qquad \bar{\mathcal{R}}p_y = \pm p_y$$

- In matrix form this can be written as

$$\bar{\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_x + a_x \sin\mu_x & b_x \sin\mu_x & 0 & 0 \\ -c_x \sin\mu_x & \cos\mu_x - a_x \sin\mu_x & 0 & 0 \\ 0 & 0 & \cos\mu_y + a_y \sin\mu_y & b_y \sin\mu_y \\ 0 & 0 & -c_y \sin\mu_y & \cos\mu_y - a_y \sin\mu_y \end{pmatrix}$$

- The horizontal part of the matrix is $-\mathcal{I}_2$ and the vertical part is $\pm\mathcal{I}_2$, which is obtained for phase advances

$$\mu_x = (2n_x + 1)\pi, \qquad \mu_y = n_y \pi$$

- This is why this beam line is called a ***-I*-transformer**

Modern symplectic integration schemes

- Symplectic integrators with **positive** steps for Hamiltonian systems $H = A + \epsilon B$ with both A and B **integrable** were proposed by **McLachlan** (1995).
- **Laskar** and **Robutel** (2001) derived all orders of such integrators
- Consider the formal solution of the Hamiltonian system written in the Lie representation

$$\vec{x}(t) = \sum_{n\geq 0} \frac{t^n}{n!} L_H^n \vec{x}(0) = e^{tL_H} \vec{x}(0).$$

- A symplectic integrator of order n from t to $t+\tau$ consists of approximating the Lie map $e^{\tau L_H} = e^{\tau(L_A + L_{\epsilon B})}$ by products of $e^{c_i \tau L_A}$ and $e^{d_i \tau L_{\epsilon B}}$, $i = 1, \ldots, n$ which integrate exactly A and B over the time-spans $c_i \tau$ and $d_i \tau$
- The constants c_i and d_i are chosen to reduce the error

SABA$_2$C integrator

- The SABA$_2$ integrator is written as

$$\mathrm{SABA}_2 = e^{c_1\tau L_A} e^{d_1\tau L_{\epsilon B}} e^{c_2\tau L_A} e^{d_1\tau L_{\epsilon B}} e^{c_1\tau L_A},$$

with $c_1 = \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right)$, $c_2 = \frac{1}{\sqrt{3}}$, $d_1 = \frac{1}{2}$.

- When $\{\{A,B\},B\}$ is integrable, e.g. when A is quadratic in momenta and B depends only in positions, the accuracy of the integrator is improved by two small negative steps

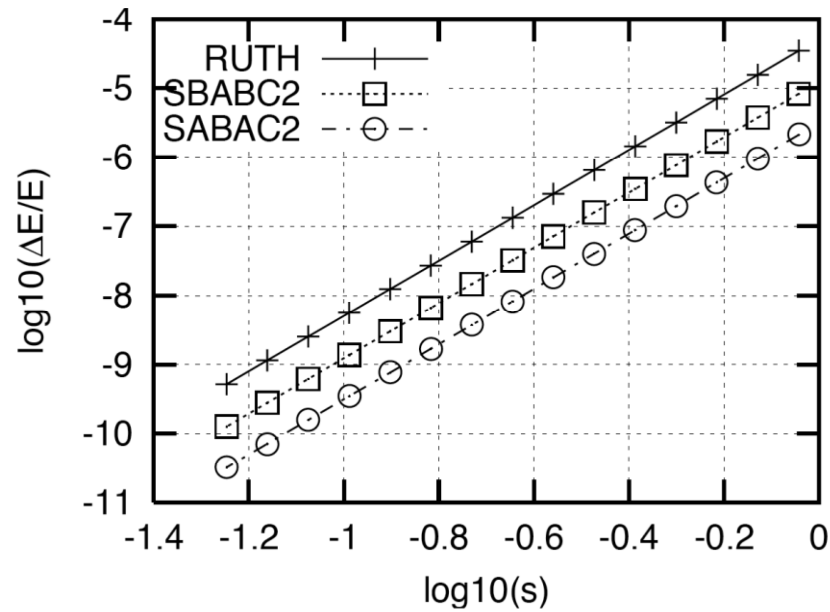
$$\mathrm{SABA}_2\mathrm{C} = e^{-\tau^3\epsilon^2\frac{c}{2}L_{\{\{A,B\},B\}}}\left(\mathrm{SABA}_2\right)e^{-\tau^3\epsilon^2\frac{c}{2}L_{\{\{A,B\},B\}}}$$

with $c = (2-\sqrt{3})/24$

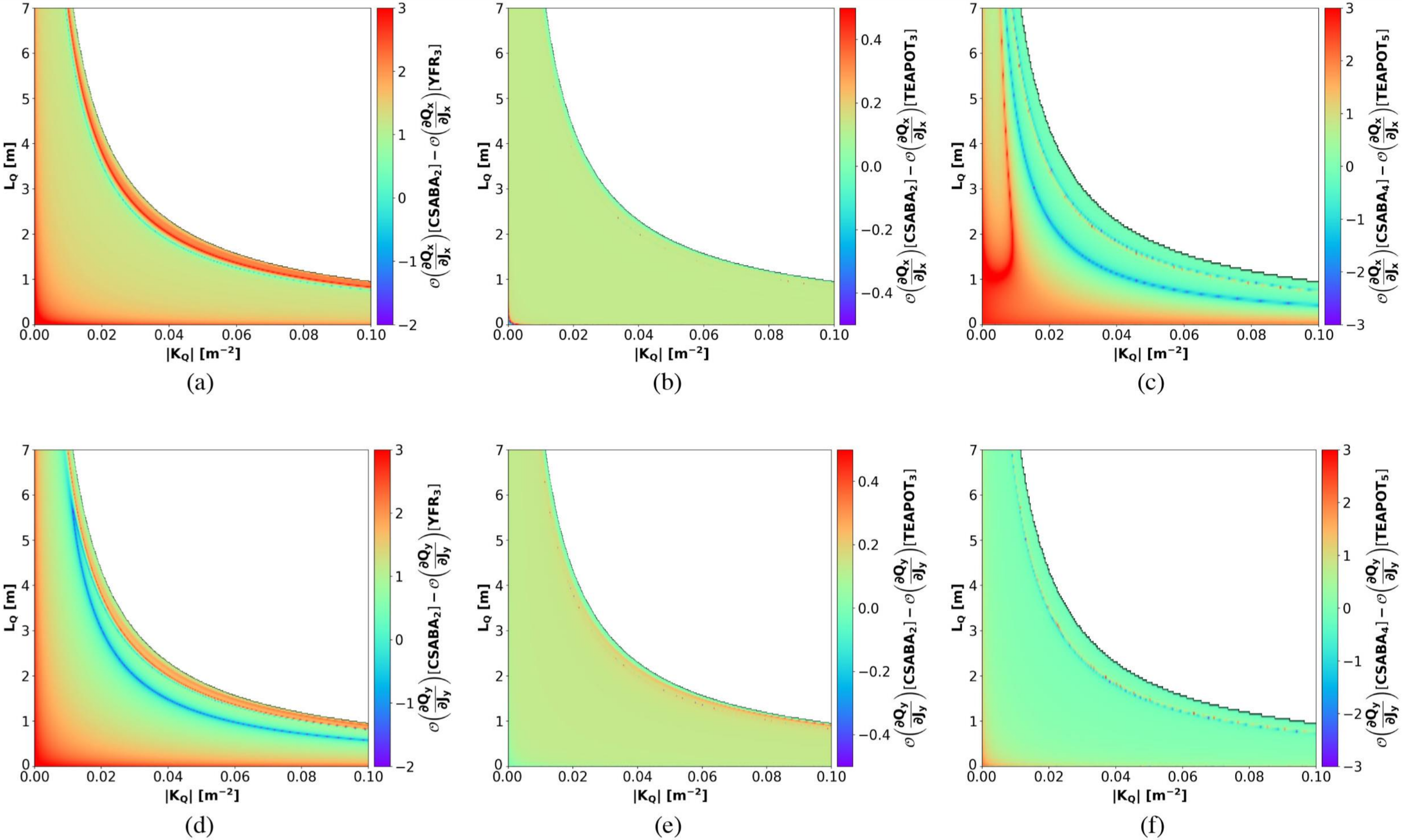
- The accuracy of SABA$_2$C is one order of magnitude higher than the Forest-Ruth 4th order scheme

- The usual "drift-kick" scheme corresponds to the 2nd order integ

$$\mathrm{SABA}_1 = e^{\frac{\tau}{2}L_A} e^{\tau L_{\epsilon B}} e^{\frac{\tau}{2}L_A},$$



SABA$_2$C integrator



- From 1 to several orders of magnitude better precision of SABA$_n$C with respect to classical integrators

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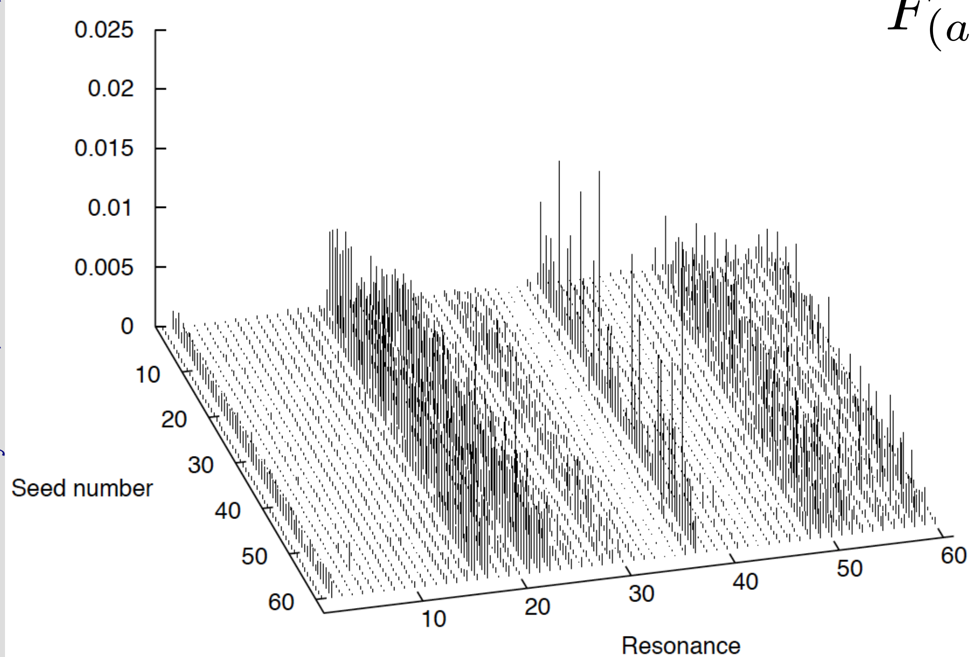
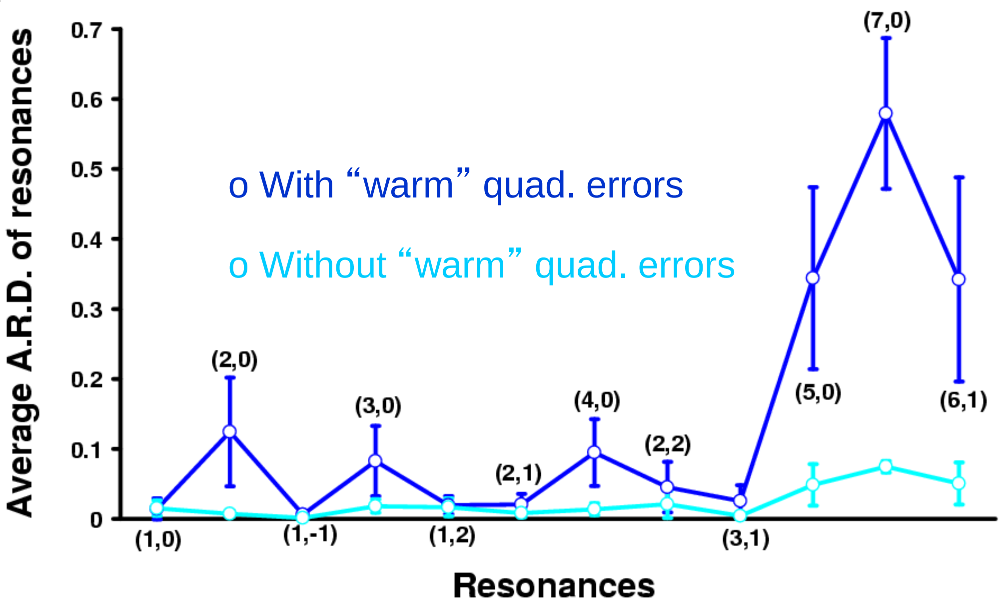
Graphical resonance representation

- It is possible by constructing the one turn map to built the generating (sometimes called "**distortion**") function $F_r \approx \sum_{jklm} f_{jklm} \, J_x^{\frac{j+k}{2}} \, J_y^{\frac{l+m}{2}} \, e^{-i\psi_{jklm}}$
- For any resonance $a\nu_x + bq_y = c$, and setting $\psi_{jklm} = 0$, the associated part of the functions is

$$F_{(a,b)} \approx \sum_{\substack{jklm \\ j+k+l+m \leq n \\ j+k=a \;,\; l+m=b}} f_{jklm} \, J_x^{\frac{a}{2}} \, J_y^{\frac{b}{2}}$$



Normal forms for LHC models



Phase	Type	DA (σ)	LHC Version		
			4	5	
				Nominal	Target
15°	Warm Quads switched ON	Average	10.0	9.1	10.4
		Minimum	8.5	7.4	8.6
	Warm Quads switched OFF	Average	10.7	11.6	12.4
		Minimum	9.6	10.3	11.3
45°	Warm Quads switched ON	Average	11.1	11.3	12.8
		Minimum	9.5	9.2	11.4
	Warm Quads switched OFF	Average	11.4	12.4	13.8
		Minimum	10.1	10.7	12.3

- In the LHC at injection (450 GeV), beam stability is necessary over a very large number of turns (10^7)
- Stability is reduced from random multi-pole imperfections mainly in the super-conducting magnets
- Area of stability (Dynamic aperture - DA) computed with particle tracking for a large number of random magnet error distributions
- Numerical tool based on normal form analysis (GRR) permitted identification of DA reduction reason (errors in the “warm” quadrupoles)