Consistent EFTs in the Schwinger-Keldysh Formalism

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preliminary, in collaboration with Tobias Binder

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Introduction

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- Non-relativistic effective field theories (NREFTs) capture non-relativistic effects e.g. in QCD or for dark matter
- Schwinger-Keldysh formalism ideal for finite temperature: works in and out of equilibrium and describes dynamics in real time

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Path integral defined on a time contour.

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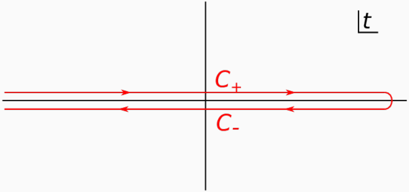
$$G(x,y) = \langle T_{\mathcal{C}} \phi(x) \phi^*(y) \rangle \,, \tag{1}$$



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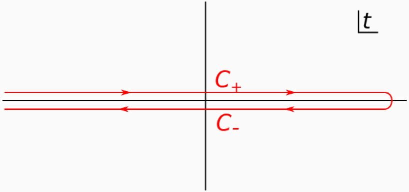

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In thermal equilibrium, have Kubo-Martin-Schwinger (KMS) relation

$$G^{-+}(x^0, y^0) = G^{+-}(x^0, y^0 - i\beta), \tag{2}$$

or

$$G^{-+}(k) = e^{\beta k^0} G^{+-}(k), \tag{3}$$

in momentum space, where $\beta = 1/T$.

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- Our work: NREFT action in Schwinger-Keldysh, **then** EoMs $\rightarrow$ no further approximations, can use full toolkit

Decays and Inverse Decays

Consider a theory with a real scalar Φ decaying into complex fields ϕ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{M^2}{2}\Phi^2 + \partial_\mu \phi^\dagger \partial^\mu \phi - \lambda \Phi \phi^\dagger \phi. \tag{4}$$

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We integrate out the field ϕ. To leading order, obtain a term

$$\frac{i\lambda^2}{2!}\int_{x^0, y^0 \in \mathcal{C}} d^4x d^4y \Phi(x) i\Pi(x, y)\Phi(y). \tag{5}$$

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In NR region

$$p^0 \approx \pm(M + \mathbf{p}^2/(2M)), \tag{6}$$

can expand

$$\Phi_{\rm NR}(x) = \frac{1}{\sqrt{2M}}(H(x)e^{-iMt} + H^\dagger(x)e^{iMt}). \tag{7}$$

Decays and Inverse Decays

To describe decays, need to compute $\mathrm{Im}\, i\Pi(x, y)$, with

$$i\Pi(x, y) = G_\phi(x, y) G_\phi(y, x). \tag{8}$$

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Using Boltzmann statistics, we find, in momentum space,

$$\mathrm{Im}\, i\Pi(k) = \frac{1}{16\pi} \begin{pmatrix} 1 + f^{\mathrm{eq}}(|k^0|) & 2(\theta(k^0) f^{\mathrm{eq}}(|k^0|) + \theta(-k^0)) \\ 2(\theta(-k^0) f^{\mathrm{eq}}(|k^0|) + \theta(k^0)) & 1 + f^{\mathrm{eq}}(|k^0|) \end{pmatrix} \tag{9}$$

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Defining $\Gamma(x,y) = \frac{\lambda^2}{M} e^{iM(x^0-y^0)} \mathrm{Im}\, i\Pi(x,y)$, and setting k on-shell, we find in coordinate space

$$\Gamma(x,y) = \frac{\lambda^2}{16\pi M} \delta(x^0 - y^0) \left(\delta^3(\mathbf{x}-\mathbf{y}) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + n^{\mathrm{eq}} e^{-(\mathbf{x}-\mathbf{y})^2 MT} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right). \tag{10}$$

Equation of Motion

From the NR effective action

$$S_{\mathrm{NR}} = \int d^4x H^\dagger(x) \left[i\partial_t + \frac{\Delta}{2M} \right] H(x) + \frac{i}{2} \int d^4x d^4y H^\dagger(x) \Gamma(x,y) H(y), \tag{11}$$

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we find the equation of motion for $G^{+-}(x,x)$

$$\begin{aligned} &i\partial_t G^{+-}(x,x) = \\ &-\frac{i}{2}\int \mathrm{d}^4z\Big[(\Gamma^{++}(x,z)G^{+-}(z,x) - \Gamma^{+-}(x,z)G^{--}(z,x)) \\ &-(G^{++}(x,z)\Gamma^{+-}(z,x) + G^{+-}(x,z)\Gamma^{--}(z,x))\Big] \\ &= -\frac{i\lambda^2}{16\pi M}\int\frac{d^4k}{(2\pi)^4}\Big(G^{+-}(k) - e^{-(M+\mathbf{k}^2/(2M))/T}G^{-+}(k)\Big). \end{aligned} \tag{12}$$

First term in agreement e.g. with Binder, Covi and Mukaida (2018)

Equation of Motion

Inserting the tree-level propagators

$$G^{+-}(k) = 2\pi\delta(k^0 - \mathbf{k}^2/2M) f(k^0 + M), \tag{13a}$$

$$G^{-+}(k) = 2\pi\delta(k^0 - \mathbf{k}^2/2M), \tag{13b}$$

gives the more familiar form

$$\partial_t G^{+-}(x, x) = -\frac{\lambda^2}{16\pi M}(n - n_{\text{eq}}). \tag{14}$$

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Note: only true for on-shell k! For k arbitrary, we obtain

$$\partial_t G^{+-}(x,x) = -\frac{\lambda^2}{16\pi M}\int \frac{d^4k}{(2\pi)^4}(G^{+-}(k) - e^{-(k^0+M)/T}G^{-+}(k)) + \theta(-k^0 - M)(e^{-|k^0+M|/T} - 1)(G^{+-}(k) - e^{-(k^0+M)/T}G^{-+}(k)). \tag{15}$$

Not yet fully understood!

Coannihilation and Pair Production

We now consider Φ complex, with the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \frac{M^2}{2}\Phi^2 + \partial_\mu \phi^\dagger \partial^\mu \phi - \lambda \Phi^\dagger \Phi \phi^\dagger \phi. \tag{16}$$

We can again expand Φ as

$$\Phi_{\rm NR}(x) = \frac{1}{\sqrt{2M}}\eta(x)e^{-iMt} + \frac{1}{\sqrt{2M}}\xi(x)e^{iMt}, \tag{17}$$

with the action

$$\begin{aligned} S_{\rm NR} = &\int d^4x \eta^\dagger(x)\left(i\partial_t + \frac{\Delta}{2M}\right)\eta(x) + \xi(x)\left(i\partial_t + \frac{\Delta}{2M}\right)\xi^\dagger(x) \\ + &\frac{i\lambda^2}{8M^2}\int d^4y \big[(\eta^\dagger(x)\eta(x) + \xi(x)\xi^\dagger(x))\Pi(x,y)(\eta^\dagger(y)\eta(y) + \xi(y)\xi^\dagger(y)) \\ &+ \eta^\dagger(x)\xi(x)\Pi(x,y)e^{i2M(x^0-y^0)}\xi^\dagger(y)\eta(y) + h.c.\big], \end{aligned} \tag{18}$$

with Π as before.

Equation of Motion

Following similar steps as before, we find to leading order

$$
\begin{aligned}
i\partial_t G_\eta^{++}(x,x) = & -i\frac{\lambda^2}{16\pi M^2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\big[G_\eta^{+-}(k)G_\xi^{+-}(q) \\
& - e^{-(k^0+q^0+2M)/T}G_\eta^{-+}(k)G_\xi^{-+}(q)\big] \\
& + \theta(-k^0-q^0-2M)(e^{(k^0+q^0+2M)/T}-1)\big[G_\eta^{+-}(k)G_\xi^{+-}(q) \\
& - e^{-(k^0+q^0+2M)/T}G_\eta^{-+}(k)G_\xi^{-+}(q)\big], \qquad (19)
\end{aligned}
$$

which, on-shell, gives the familiar result

$$
\partial_t G_\eta^{++}(x,x) = -\frac{\lambda^2}{16\pi M^2}(n_\eta n_\xi - n_{\text{eq}}^2) \qquad (20)
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- Generalized to off-shell propagators
- Care must be taken to ensure consistency: Boltzmann/quantum statistics (KMS relations, correlator identities), on-/off-shell momenta during Fourier transformation, etc
- Outlook: use Schwinger-Keldysh machinery to describe Sommerfeld effect, bound-states from first principles